Existence and long-time behaviour for gradient flows of non convex functionals

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Problem:
Which notion of subdifferential “∂”ϕ?

Gradient flows of convex functionals: results

When ϕ is **convex**, "∂"ϕ is the subdifferential of ϕ **in the sense of convex analysis**

Gradient flows of convex functionals: results

Given $u \in D(\phi)$,

$$\xi \in \partial\phi(u) \quad \Leftrightarrow \phi(w) - \phi(u) \geq \langle \xi, w - u \rangle \quad \forall\, w \in \mathcal{H}$$

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Well-established literature:

- existence, uniqueness, approximation of solutions (**[Kōmura'67, Crandall-Pazy'69, Brézis'73]**),
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How extend (some of) these results to the non convex case?

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Idea:

Let's have a deeper look at the convex case!

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- **Approximate solutions**: interpolants on $(0, T)$ of $\{\mathrm{u}_\tau^k\}_{k=1}^n$:
  - $\{\overline{\mathrm{u}}_\tau\}$ piecewise constant;
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- **Approximate energy inequality**:

$$\int_0^t |\widehat{\mathrm{u}}_\tau'(s)|^2 \,\mathrm{d}s + \phi(\overline{\mathrm{u}}_\tau(t)) \leq \phi(u_0).$$

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$$u_n \to u, \quad \xi_n \rightharpoonup \xi, \quad \xi_n \in \partial\phi(u_n)\ \forall n \in \mathbb{N} \quad \Rightarrow \quad \xi \in \partial\phi(u)$$

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- **Passage to the limit:** u is the (unique!) solution!

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Gradient flows of convex functionals: energy identity

Since $\partial\phi$ fulfils the **chain rule**

$$\begin{cases} u \in H^1(0,T;\mathcal{H}), \\ \xi \in L^2(0,T;\mathcal{H}), \\ \xi(t) \in \partial\phi(u(t)) \text{ a.e. in } (0,T) \end{cases} \Rightarrow \quad \frac{d}{dt}\phi(u(t)) = \langle \xi(t), u'(t) \rangle \quad \text{a.e. in } (0,T)$$

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$\Rightarrow$ the solution u satisfies the **energy identity**:

$$\int_s^t |u'(r)|^2 \,\mathrm{d}r + \phi(u(t)) = \phi(u(s)) \quad \forall 0 \le s \le t \le T.$$

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has a solution if ϕ is **coercive** (for instance, $\phi(\cdot) + |\cdot|^2$ has compact sublevels)

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- **Discrete equation:** $\partial_F \phi$ the **Fréchet subdifferential** of ϕ

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- $\partial_F \phi \equiv \partial \phi$ if ϕ is convex
- $\partial_F \phi(u)$ is **convex** for all $u \in D(\phi)$
- $\partial_F \phi$ is **not strongly-weakly closed** in the sense of graphs!!!

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- **Discrete solutions**: $u_\tau^0, u_\tau^1, \dots, u_\tau^N \Rightarrow$ approximate solutions
- **Approximate equation**:

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- **with some changes:**
  - ✓ approximate energy inequality
  - ✓ a priori estimates + compactness: $\exists\, u \in H^1(0, T; \mathcal{H})$

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- **But:** you can't pass to the limit in

$$-\widehat{\mathrm{u}}_\tau'(t) \in \partial_F \phi(\overline{\mathrm{u}}_\tau(t)) \quad t \in (0, T)$$

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because $\partial_F \phi$ is **not strongly-weakly closed!!**

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The limiting subdifferential

First idea:

Consider (a version of) the **strong-weak closure** of $\partial_F \phi$:

the limiting subdifferential $\partial_\ell \phi$

given $u \in D(\phi)$,

$$\xi \in \partial_\ell \phi(u) \quad \Leftrightarrow \exists \{u_n\}, \{\xi_n\} \subset \mathcal{H} : \quad \begin{cases} \xi_n \in \partial_F \phi(u_n) \ \forall\, n \in \mathbb{N}, \\ u_n \to u, \\ \xi_n \rightharpoonup \xi, \\ \sup_n \phi(u_n) < +\infty \end{cases}$$

The gradient flow equation for the limiting subdiffential

In [**R., Savaré**, *Gradient flows of non convex functionals in Hilbert spaces and applications*, ESAIM COCV, to appear]:

- The **limiting subdifferential** is our notion of subdifferential!

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- The **limiting subdifferential** is our notion of subdifferential!
- Consider the Cauchy problem

$$\begin{cases} u'(t) + \partial_\ell \phi(u(t)) \ni 0, & t \in (0, T) \\ u(0) = u_0 \end{cases}$$

(a source tem f on the RHS might be included)

From convex to non convex: heuristics

Second idea:

Instead of passing to the limit in the **pointwise** equation

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pass to the limit in the **approximate energy inequality**

$$\frac{1}{2}\int_0^t |\widehat{\mathrm{u}}_\tau'(s)|^2 \,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial_F \phi(\overline{\mathrm{u}}_\tau(s))|^2 \mathrm{d}s + \phi(\overline{\mathrm{u}}_\tau(t)) \leq \phi(u_0)$$

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obtain the **limit energy inequality**

$$\frac{1}{2}\int_0^t |u'(s)|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial_\ell \phi(u(s))|^2 \mathrm{d}s + \phi(u(t)) \leq \phi(u_0)$$

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Third idea:

If $\partial_\ell \phi$ fulfils the **chain rule**

$$u \in H^1(0,T;\mathcal{H}) \quad \Rightarrow \quad \frac{d}{dt}\phi(u(t)) = \langle \partial_\ell \phi(u(t)), u'(t) \rangle \text{ a.e. in } (0,T)$$

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From convex to non convex: heuristics

Third idea:

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and the **energy identity**

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An existence and approximation result

Theorem 1 [R., Savaré '04]

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Moreover, u fulfils the **energy identity** (due to the chain rule)

$$\int_s^t |u'(r)|^2 \, \mathrm{d}r + \phi(u(t)) = \phi(u(s)) \quad \forall\, 0 \le s \le t \le T.$$

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$\Rightarrow$ existence of **energy solutions**

$$\int_s^t |u'(r)|^2 \, \mathrm{d}r + \phi(u(t)) = \phi(u(s)) \quad \forall\, 0 \leq s \leq t \leq T.$$

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Various possibilities: **[Sell '73,'96]**, **[Chepyzhov & Vishik '02]**, **[Melnik & Valero '02]**, **[Ball '97,'04]**

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In [**R., Segatti, Stefanelli**, *Attractors for gradient flows of non convex functionals and applications*, preprint '05]:
Ball's theory of **generalized semiflows** to the study asymptotic behaviour of the non convex gradient flow!!

Generalized Semiflows: definition

Phase space: a metric space $(\mathcal{X}, d_{\mathcal{X}})$

Generalized Semiflows: definition

A **generalized semiflow** $\mathcal{S}$ on $\mathcal{X}$ is a family of maps $g : [0,+\infty) \to \mathcal{X}$ ("solutions"), s. t.

(Existence) $\forall\, g_0 \in \mathcal{X}$ $\exists$ **at least one** $g \in \mathcal{S}$ with $g(0) = g_0$,

(Translation invariance) $\forall\, g \in \mathcal{S}$ and $\tau \geq 0$, the map $g^\tau(\cdot) := g(\cdot + \tau)$ is in $\mathcal{S}$,

(Concatenation) $\forall\, g,\ h \in \mathcal{S}$ and $t \geq 0$ with $h(0) = g(t)$, then $z \in \mathcal{S}$, where

$$z(\tau) := \begin{cases} g(\tau) & \text{if } 0 \leq \tau \leq t, \\ h(\tau - t) & \text{if } t < \tau, \end{cases}$$

(U.s.c. w.r.t. initial data) If $\{g_n\} \subset \mathcal{S}$ and $g_n(0) \to g_0$, $\exists$ subsequence $\{g_{n_k}\}$ and $g \in \mathcal{S}$ s.t. $g(0) = g_0$ and $g_{n_k}(t) \to g(t)$ for all $t \geq 0$.

Generalized Semiflows: dynamical system notions

Within this framework:

- **orbit** of a solution/set
- ω-**limit** of a solution/set
- **invariance under the semiflow** of a set
- **attracting set** (w.r.t. the Hausdorff semidistance of X)

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Definition

A set $\mathcal{A} \subset \mathcal{X}$ is a **global attractor** for a **generalized semiflow** $\mathcal{S}$ if:

- ♣ $\mathcal{A}$ is **compact**
- ♣ $\mathcal{A}$ is **invariant** under the semiflow
- ♣ $\mathcal{A}$ **attracts** the **bounded** sets of $\mathcal{X}$ (w.r.t. the Hausdorff semidistance of X)

Long-time behaviour for the gradient flow equation

$$u'(t) + \partial_\ell \phi(u(t)) \ni 0 \quad \text{for a.e. } t \in (0, T), \qquad \text{(GF)}$$

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Choice of the phase space:

$$\mathcal{X} = D(\phi) \subset \mathcal{H},$$
$$d_X(u, v) := |u - v|_{\mathcal{H}} + |\phi(u) - \phi(v)| \quad \forall\, u, v \in \mathcal{X}.$$

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- ¿ Is $\mathcal{S}$ a **generalized semiflow**?
- ¿ Does $\mathcal{S}$ possess a global attracor?

Long-time behaviour for the gradient flow equation

Theorem 2 [R., Segatti, Stefanelli '05]

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Gradient flows of C^1 perturbations of convex functionals

Theorems 1 and 2 apply to functionals ϕ of the form:

- C^1-**perturbations** of convex functionals:

$$\phi = \phi_1 + \phi_2,$$

$$\phi_1 \text{ proper, l.s.c., convex,} \quad \phi_2 \in C^1(\mathcal{H}).$$

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- $\partial_\ell \phi$ fulfils the chain rule $\Rightarrow$ existence and long-time behaviour of the solutions of

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Note:

$$\partial_\ell \phi \subset \partial\psi_1 - \partial\psi_2$$

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$$-u'(t) \in \partial_\ell \phi(u(t)) \subset \partial\psi_1(u(t)) - \partial\psi_2(u(t))$$

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the gradient flow approach gives **a class** of solutions of

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Energy solutions of quasistationary phase field models

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Applications: existence and long-time behaviour of the **gradient flow** solutions of general **quasistationary evolution systems**

$$\vartheta_t + \chi_t - \Delta\vartheta = 0 \quad \text{in } \Omega \times (0, +\infty)$$
$$F'(\chi) = \vartheta \quad \text{in } \Omega \times (0, +\infty)$$
$$\partial_n \chi = 0 \quad \text{in } \partial\Omega \times (0, +\infty)$$
$$\vartheta = 0 \quad \text{in } \partial\Omega \times (0, +\infty)$$

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$$\vartheta_t + \chi_t - \Delta\vartheta = 0 \quad \text{in } \Omega \times (0, +\infty)$$
$$-\Delta\chi + \chi^3 - \chi = \vartheta \quad \text{in } \Omega \times (0, +\infty)$$
$$\partial_n \chi = 0 \quad \text{in } \partial\Omega \times (0, +\infty)$$
$$\vartheta = 0 \quad \text{in } \partial\Omega \times (0, +\infty)$$