Long-time behaviour of gradient flows in metric spaces

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in collaboration with

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- Motivation for studying gradient flows in metric spaces

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- The metric formulation of a gradient flow $\rightsquigarrow$ the notion of **curves of maximal slope**

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- Existence & uniqueness results
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- Applications in Banach spaces
- Applications in Wasserstein spaces

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- The metric formulation of a gradient flow $\rightsquigarrow$ the notion of **curves of maximal slope**
- Existence & uniqueness results
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- Applications in Banach spaces
- Applications in Wasserstein spaces
- A more general abstract result....

Evolution PDEs of diffusive type and the Wasserstein distance

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$$L = L(x, \rho, \nabla \rho) : \mathbb{R}^n \times (0,+\infty) \times \mathbb{R}^n \to \mathbb{R} \quad \text{(Lagrangian)}$$

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Evolution PDEs of diffusive type and the Wasserstein distance

$$\begin{cases} \partial_t \rho - \mathrm{div}\left(\rho \nabla (\frac{\delta \mathcal{L}}{\delta \rho})\right) = 0 \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ \rho(x,t) \geq 0, \quad \int_{\mathbb{R}^n} \rho(x,t)\,\mathrm{d}x = 1 \ \ \forall\,(x,t) \in \mathbb{R}^n \times (0,+\infty), \\ \int_{\mathbb{R}^n} |x|^2 \rho(x,t)\,\mathrm{d}x < +\infty \quad \forall\, t \geq 0, \end{cases}$$

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with the probability measure $\mu_t := \rho(\cdot, t) dx$

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then $\mathcal{L}$ can be considered as defined on $\mathscr{P}_2(\mathbb{R}^n)$

(the space of probability measures on $\mathbb{R}^n$ with finite second moment)

Evolution PDEs of diffusive type and the Wasserstein distance

$$\begin{cases} \partial_t \rho - \mathrm{div}\left(\rho \nabla\left(\frac{\delta \mathcal{L}}{\delta \rho}\right)\right) = 0 \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ \rho(x,t) \geq 0, \quad \int_{\mathbb{R}^n} \rho(x,t)\,\mathrm{d}x = 1 \ \ \forall\,(x,t) \in \mathbb{R}^n \times (0,+\infty), \\ \int_{\mathbb{R}^n} |x|^2 \rho(x,t)\,\mathrm{d}x < +\infty \quad \forall\, t \geq 0, \end{cases}$$

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OTTO, JORDAN & KINDERLEHRER and OTTO ['97–'01]
showed that this PDE can be interpreted as
the **gradient flow** of $\mathcal{L}$ in $\mathscr{P}_2(\mathbb{R}^n)$
w.r.t. the Wasserstein distance W_2 on $\mathscr{P}_2(\mathbb{R}^n)$

Examples

Ex.1: The potential energy functional

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The potential energy functional $\rightsquigarrow$ The linear transport equation

$$\mathcal{L}_1(\rho) := \int_{\mathbb{R}^n} V(x)\rho(x)\,\mathrm{d}x, \quad \begin{cases} L_1(x,\rho,\nabla\rho) = L_1(x,\rho) = \rho V(x), \\ \frac{\delta \mathcal{L}_1}{\delta\rho} = \partial_\rho L_1(x,\rho) = V(x), \end{cases}$$

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$$\mathcal{L}_2(\rho) := \int_{\mathbb{R}^n} \rho(x)\log(\rho(x))\,\mathrm{d}x, \quad \begin{cases} L_2(x,\rho,\nabla\rho) = \rho\log(\rho), \\ \frac{\delta \mathcal{L}_2}{\delta\rho} = \partial_\rho L_2(\rho) = \log(\rho) + 1, \end{cases}$$

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The entropy functional $\rightsquigarrow$ The heat equation

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Ex.3: The internal energy functional

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Ex.3: The internal energy functional

$$\mathcal{L}_3(\rho) := \int_{\mathbb{R}^n} \frac{1}{m-1}\rho^m(x)\,\mathrm{d}x, \quad m \neq 1$$

Examples

The potential energy functional $\rightsquigarrow$ The linear transport equation

$$\mathcal{L}_1(\rho) := \int_{\mathbb{R}^n} V(x)\rho(x)\,\mathrm{d}x, \quad \begin{cases} L_1(x,\rho,\nabla\rho) = L_1(x,\rho) = \rho V(x), \\ \frac{\delta \mathcal{L}_1}{\delta\rho} = \partial_\rho L_1(x,\rho) = V(x), \end{cases}$$

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Examples

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Examples

The potential energy functional $\rightsquigarrow$ The linear transport equation

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$$\partial_t \rho - \mathrm{div}(\rho \nabla V) = 0$$

The internal energy functional $\rightsquigarrow$ The porous media equation

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$$\partial_t \rho - \Delta\rho^m = 0 \quad \text{OTTO '01}$$

Examples

The potential energy functional $\rightsquigarrow$ The linear transport equation

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Ex.4: The (Entropy+ Potential) energy functional

Examples

The potential energy functional $\rightsquigarrow$ The linear transport equation

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$$\partial_t\rho - \mathrm{div}(\rho\nabla V) = 0$$

Ex.4: The (Entropy+ Potential) energy functional

$$\mathcal{L}_4(\rho) := \int_{\mathbb{R}^n} (\rho(x)\log(\rho(x)) + \rho(x)V(x))\,\mathrm{d}x,$$

Examples

The potential energy functional $\rightsquigarrow$ The linear transport equation

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$$\partial_t\rho - \mathrm{div}(\rho\nabla V) = 0$$

Ex.4: The (Entropy+ Potential) energy functional

$$\mathcal{L}_4(\rho) := \int_{\mathbb{R}^n} (\rho\log(\rho) + \rho V), \quad \begin{cases} L_4(x,\rho,\nabla\rho) = \rho\log(\rho) + \rho V(x), \\ \frac{\delta\mathcal{L}_4}{\delta\rho} = \partial_\rho L_4(x,\rho) = \log(\rho) + 1 + V(x), \end{cases}$$

Examples

The potential energy functional $\rightsquigarrow$ The linear transport equation

$$\mathcal{L}_1(\rho) := \int_{\mathbb{R}^n} V(x)\rho(x)\,\mathrm{d}x, \quad \begin{cases} L_1(x,\rho,\nabla\rho) = L_1(x,\rho) = \rho V(x), \\ \frac{\delta\mathcal{L}_1}{\delta\rho} = \partial_\rho L_1(x,\rho) = V(x), \end{cases}$$

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$$\partial_t\rho - \mathrm{div}(\rho\nabla(\log(\rho) + 1 + V)) = 0$$

Examples

The potential energy functional $\rightsquigarrow$ **The linear transport equation**

$$\mathcal{L}_1(\rho) := \int_{\mathbb{R}^n} V(x)\rho(x)\,\mathrm{d}x, \quad \begin{cases} L_1(x,\rho,\nabla\rho) = L_1(x,\rho) = \rho V(x), \\ \frac{\delta \mathcal{L}_1}{\delta \rho} = \partial_\rho L_1(x,\rho) = V(x), \end{cases}$$

$$\partial_t \rho - \mathrm{div}(\rho \nabla V) = 0$$

Entropy+Potential $\rightsquigarrow$ **The Fokker-Planck equation**

$$\mathcal{L}_4(\rho) := \int_{\mathbb{R}^n} (\rho \log(\rho) + \rho V), \quad \begin{cases} L_4(x,\rho,\nabla\rho) = \rho \log(\rho) + \rho V(x), \\ \frac{\delta \mathcal{L}_4}{\delta \rho} = \partial_\rho L_4(x,\rho) = \log(\rho) + 1 + V(x), \end{cases}$$

$$\partial_t \rho - \Delta\rho - \mathrm{div}(\rho \nabla V) = 0$$ JORDAN-KINDERLEHRER-OTTO '97

Fourth order examples

Ex.5: The Dirichlet integral

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Fourth order examples

The Dirichlet integral $\rightsquigarrow$ The thin film equation

$$\mathcal{L}_5(\rho) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\rho(x)|^2 \,\mathrm{d}x, \qquad \begin{cases} L_5(x,\rho,\nabla\rho) = L_5(\rho) = \frac{1}{2}|\nabla\rho|^2, \\ \frac{\delta\mathcal{L}_5}{\delta\rho} = -\Delta\rho, \end{cases}$$

$$\partial_t\rho + \mathrm{div}(\rho\nabla\Delta\rho) = 0 \quad \text{OTTO '98}$$

Fourth order examples

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Ex.6: The Fisher information

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$$\mathcal{L}_6(\rho) := \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla\rho(x)|^2}{\rho(x)}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\log(\rho(x))|^2\,\rho(x)\,\mathrm{d}x$$

Fourth order examples

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$$\mathcal{L}_5(\rho) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\rho(x)|^2 \,\mathrm{d}x, \quad \begin{cases} L_5(x,\rho,\nabla\rho) = L_5(\rho) = \frac{1}{2}|\nabla\rho|^2, \\ \frac{\delta\mathcal{L}_5}{\delta\rho} = -\Delta\rho, \end{cases}$$

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$$\mathcal{L}_6(\rho) := \frac{1}{2}\int |\nabla\log(\rho)|^2\rho \quad \begin{cases} L_6(x,\rho,\nabla\rho) = |\nabla\log(\rho)|^2\,\rho, \\ \frac{\delta\mathcal{L}_6}{\delta\rho} = -2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \end{cases}$$

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$$\partial_t\rho + 2\mathrm{div}\left(\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\right) = 0 \quad \text{Gianazza-Savaré-Toscani 2006}$$

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The Fisher information $\rightsquigarrow$ Quantum drift diffusion equation

$$\mathcal{L}_6(\rho) := \frac{1}{2}\int |\nabla\log(\rho)|^2\rho \quad \begin{cases} L_6(x,\rho,\nabla\rho) = |\nabla\log(\rho)|^2\,\rho, \\ \frac{\delta\mathcal{L}_6}{\delta\rho} = -2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \end{cases}$$

$$\partial_t\rho + 2\mathrm{div}\left(\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\right) = 0 \quad \text{Gianazza-Savaré-Toscani 2006}$$

New insight

- This **gradient flow approach** has brought several developments in:
  - **approximation algorithms**
  - **asymptotic behaviour** of solutions (new contraction and energy estimates) ([Otto'01]: the porous medium equation)
  - applications to **functional inequalities** (Logarithmic Sobolev inequalities $\leftrightarrow$ trends to equilibrium a class of diffusive PDEs) .....

[Agueh, Brenier, Carlen, Carrillo, Dolbeault, Gangbo, Ghoussoub, McCann, Otto, Vazquez, Villani..]

Wasserstein spaces

- the space of Borel probability measures on $\mathbb{R}^n$ with finite **second moment**

$$\mathscr{P}_2(\mathbb{R}^n) = \left\{ \mu \text{ probability measures on } \mathbb{R}^n \, : \, \int_{\mathbb{R}^n} |x|^2 \, \mathrm{d}\mu(x) < +\infty \right\}$$

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- Given μ_1, $\mu_2 \in \mathscr{P}_2(\mathbb{R}^n)$, a **transport plan** between μ_1 and μ_2 is a measure $\boldsymbol{\mu} \in \mathscr{P}_2(\mathbb{R}^n \times \mathbb{R}^n)$ with marginals μ_1 and μ_2, i.e.

$$\pi_{1\sharp}\boldsymbol{\mu} = \mu_1, \quad \pi_{2\sharp}\boldsymbol{\mu} = \mu_2$$

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- The (squared) **Wasserstein distance** between μ_1 and μ_2 is

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- the Wasserstein distance is tightly related with the Monge-Kantorovich **optimal mass transportation** problem.

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Metric spaces are a **suitable framework** for rigorously interpreting diffusion PDE as gradient flows in the Wasserstein spaces **in the full generality**.

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Gradient Flows in Metric Spaces

Approach based on the theory of **Minimizing Movements & Curves of Maximal Slope** [DE GIORGI, MARINO, TOSQUES, DEGIOVANNI, AMBROSIO.. '80∼'90]

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Gradient Flows in Metric Spaces

• The applications of these results to gradient flows in Wasserstein spaces are made rigorous through development of a **"differential/metric calculus"** in Wasserstein spaces:

- notion of tangent space and of (sub)differential of a functional on $\mathscr{P}_p(\mathbb{R}^n)$
- calculus rules
- link between the weak formulation of evolution PDEs and their formulation as a gradient flow in $\mathscr{P}_p(\mathbb{R}^n)$

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• In [R., SAVARÉ, SEGATTI, STEFANELLI'06]: **complement** the AMBROSIO, GIGLI, SAVARÉ's results on the **long-time behaviour** of Curves of Maximal Slope

Gradient flows in metric spaces: heuristics

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To get some insight, let us go back to the euclidean case...

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We introduce suitable **"surrogates" of (the modulus of) derivatives**.

Metric derivatives

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Metric derivative & geodesics

Given an absolutely continuous curve $u : (0, T) \to X$ ($u \in \mathrm{AC}(0, T; X)$), its **metric derivative** is defined by

$$|u'|(t) := \lim_{h \to 0} \frac{d(u(t), u(t+h))}{|h|} \quad \text{for a.e. } t \in (0, T),$$

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A curve u is a (constant speed) **geodesic** if

$$d(u(s), u(t)) = |t - s||u'| \quad \forall\, s, t \in [0, 1].$$

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Local slope

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To fix ideas

Suppose that X is a Banach space B, and $\phi : B \to (-\infty, +\infty]$ is l.s.c. and **convex** (or a C^1-perturbation of a convex functional), with **subdifferential** (in the sense of Convex Analysis) $\partial\phi$. Then

$$|\partial\phi|(u) = \min\{\|\xi\|_{B'} \ : \ \xi \in \partial\phi(u)\} \quad \forall\, u \in D(\phi).$$

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Definition: chain rule

The local slope satisfies the chain rule if for any absolutely continuous curve $v : (0, T) \to D(\phi)$ the map $t \mapsto (\phi\circ)v(t)$ is **absolutely continuous** and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(v(t)) \geq -|v'|(t)\, |\partial\phi|(v(t)) \quad \text{for a.e. } t \in (0, T).$$

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We say that an **absolutely continuous** curve $u : (0, T) \to X$ is a **(2-)curve of maximal slope** for ϕ (w.r.t. the local slope) if

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To fix ideas...

- 2-curves of maximal slope in $\mathscr{P}_2(\mathbb{R}^n)$ lead (for a suitable ϕ) to the **linear** transport equation

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- p-curves of maximal slope in $\mathscr{P}_p(\mathbb{R}^n)$ lead (for a suitable ϕ) to a **nonlinear version** of the transport equation

$$\partial_t \rho - \nabla \cdot (\rho j_q(\nabla V)) = 0$$

$$j_q(r) := \begin{cases} |r|^{q-2} r & r \neq 0, \\ 0 & r = 0, \end{cases} \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Approximation of $p-$curves of maximal slope

Given an **initial datum** $u_0 \in X$, does there exist a $p-$curve of maximal slope u on $(0, T)$ fulfilling $u(0) = u_0$?

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- **Discrete solutions** $\mathrm{u}_\tau^0, \mathrm{u}_\tau^1, \ldots, \mathrm{u}_\tau^N$: solve recursively

$$\mathrm{u}_\tau^n \in \mathrm{Argmin}_{u \in X} \{ \frac{1}{p\tau} d^p(u, \mathrm{u}_\tau^{n-1}) + \phi(u) \}, \quad \mathrm{u}_\tau^0 := u_0$$

For simplicity, we take $p = 2$.

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- **Discrete solutions** $\mathrm{u}_\tau^0, \mathrm{u}_\tau^1, \ldots, \mathrm{u}_\tau^N$: solve recursively

$$\mathrm{u}_\tau^n \in \mathrm{Argmin}_{u \in X} \{ \frac{1}{p\tau} d^p(u, \mathrm{u}_\tau^{n-1}) + \phi(u) \}, \quad \mathrm{u}_\tau^0 := u_0$$

For simplicity, we take $p = 2$.

This **variational formulation** of the implicit Euler scheme still makes sense in a **purely metric** framework

Approximation of $p-$curves of maximal slope

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This **variational formulation** of the implicit Euler scheme still makes sense in a **purely metric** framework Sufficient conditions on ϕ for the minimization problem:

- ϕ lower semicontinuous;
- ϕ coercive (ϕ has compact sublevels)

Passage to the limit

- **Approximate solutions**: piecewise constant interpolants u_τ of $\{\mathrm{u}_\tau^n\}_{n=0}^N$ on $\mathscr{P}_\tau$
- **Approximate energy inequality**:

$$\frac{1}{2}\int_0^t |\mathrm{u}_\tau'|(s)^2\,\mathrm{d}s+\frac{1}{2}\int_0^t |\partial\phi|^2(\mathrm{u}_\tau(s))\,\mathrm{d}s+\phi(\mathrm{u}_\tau(t))\leq\phi(u_0)\quad\forall\, t\in[0,T].$$

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- whence
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It is natural to introduce the **relaxed slope**

$$|\partial^-\phi|(u) := \inf\left\{ \liminf_{n\uparrow\infty} |\partial\phi|(u_n) \ : \ u_n \to u,\ \sup_n \phi(u_n) < +\infty \right\}$$

i.e. the **lower semicontinuous envelope** of the local slope.

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Suppose that the **relaxed slope** $|\partial^- \phi|$ satisfies the **chain rule**

$$-\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) \leq |u'|(t) \left|\partial^- \phi\right| (u(t)) \quad \text{for a.e. } t \in (0, T).$$

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Then

$$\frac{1}{2}\int_0^t |u'|(s)^2 \, \mathrm{d}s + \frac{1}{2}\int_0^t |\partial^- \phi|^2 (u(s)) \, \mathrm{d}s \leq \phi(u_0) - \phi(u(t)) \leq \int_0^t |u'|(s) \left|\partial^- \phi\right| (u(s)) \, \mathrm{d}s,$$

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$$\frac{d}{dt}\phi(u(t)) = -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial^- \phi|^2(u(t)) \quad \text{a.e. in } (0, T),$$

i.e. u is a curve of maximal slope w.r.t. $|\partial^- \phi|$.

An existence result

Theorem [Ambrosio-Gigli-Savaré '05]

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Then, for all $u_0 \in D(\phi)$ **there exists** a p-curve of maximal slope u for ϕ (w.r.t. the relaxed slope $|\partial^- \phi|$), fulfilling $u(0) = u_0$.

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Definition: λ-geodesic convexity

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λ-geodesic convexity implies the chain rule

If $\phi : X \to (-\infty, +\infty]$ is λ-**geodesically convex**, for some $\lambda \in \mathbb{R}$, and lower semicontinuous, then

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Reasonable: if $X = B$ Banach space and $\phi : B \to (-\infty, +\infty]$ is convex and l.s.c., the convex subdifferential $\partial\phi$ is strongly-weakly closed.

Uniqueness for 2-curves of maximal slope

- **Uniqueness** proved **only for** $p = 2$!

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Theorem [Ambrosio-Gigli-Savaré '05]

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- (X, d) is the Wasserstein space $(\mathscr{P}_2(\mathbb{R}^d), W_2)$

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Theorem [Ambrosio-Gigli-Savaré '05]

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Long-time behaviour for 2-curves of maximal slope

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Theorem [Ambrosio-Gigli-Savaré ’05]

- $\lambda > 0$:
  exponential convergence of the solution as $t \to +\infty$ **to the unique minimum point** $\bar{u}$ of ϕ:

$$d(u(t), \bar{u}) \leq e^{-\lambda t} d(u_0, \bar{u}) \quad \forall\, t \geq 0$$

- $\lambda = 0 \quad + \quad \phi$ has compact sublevels:
  convergence to (an) equilibrium as $t \to +\infty$

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"Fill in the gaps" in the study of the long-time behaviour of p-curves of maximal slope

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Namely, we comprise the cases:

1. $p = 2$, $\lambda < 0$ $\rightsquigarrow$ uniqueness: **YES**
2. $p \neq 2$, $\lambda \in \mathbb{R}$ $\rightsquigarrow$ uniqueness: **NO**

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In [**R., Savaré, Segatti, Stefanelli**, *Global attractors for curves of maximal slope*, in preparation]: Ball's theory of **generalized semiflows**

Generalized Semiflows: definition

Phase space: a metric space $(\mathcal{X}, d_{\mathcal{X}})$

A **generalized semiflow** $\mathcal{S}$ on $\mathcal{X}$ is a family of maps $g : [0, +\infty) \to \mathcal{X}$ ("solutions"), s. t.

(Existence) $\forall g_0 \in \mathcal{X}$ $\exists$ **at least one** $g \in \mathcal{S}$ with $g(0) = g_0$,

(Translation invariance) $\forall g \in \mathcal{S}$ and $\tau \geq 0$, the map $g^\tau(\cdot) := g(\cdot + \tau)$ is in $\mathcal{S}$,

(Concatenation) $\forall g, h \in \mathcal{S}$ and $t \geq 0$ with $h(0) = g(t)$, then $z \in \mathcal{S}$, where

$$z(\tau) := \begin{cases} g(\tau) & \text{if } 0 \leq \tau \leq t, \\ h(\tau - t) & \text{if } t < \tau, \end{cases}$$

(U.s.c. w.r.t. initial data) If $\{g_n\} \subset \mathcal{S}$ and $g_n(0) \to g_0$, $\exists$ subsequence $\{g_{n_k}\}$ and $g \in \mathcal{S}$ s.t. $g(0) = g_0$ and $g_{n_k}(t) \to g(t)$ for all $t \geq 0$.

Generalized Semiflows: dynamical system notions

Within this framework:

- **orbit** of a solution/set
- ω-**limit** of a solution/set
- **invariance under the semiflow** of a set
- **attracting set** (w.r.t. the Hausdorff semidistance of $\mathcal{X}$)

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Definition

A set $\mathcal{A} \subset \mathcal{X}$ is a **global attractor** for a **generalized semiflow** $\mathcal{S}$ if:

- ♣ $\mathcal{A}$ is **compact**
- ♣ $\mathcal{A}$ is **invariant** under the semiflow
- ♣ $\mathcal{A}$ **attracts** the **bounded** sets of $\mathcal{X}$ (w.r.t. the Hausdorff semidistance of $\mathcal{X}$)

Long-time behaviour for p-curves of maximal slope

$$\frac{d}{dt}\phi(u(t)) = -\frac{1}{p}|u'|^{p}(t) - \frac{1}{q}|\partial^{-}\phi|^{q}(u(t)) \quad \text{for a.e. } t \in (0, T),$$

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Choice of the phase space:

$$\mathcal{X} = D(\phi) \subset X,$$
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- ¿ Is $\mathcal{S}$ a **generalized semiflow**?
- ¿ Does $\mathcal{S}$ **possess a global attractor**?

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Theorem 1 [R., Savaré, Segatti, Stefanelli '06]

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Idea of the proof: to check the u.s.c. w.r.t. initial data, fix a sequence $\{u_0^n\}_n \subset D(\phi)$ s. t. $d_{\mathcal{X}}(u_0^n, u_0) = d(u_0^n, u_0) + |\phi(u_0^n) - \phi(u_0)| \to 0$.

$$\frac{1}{p}\int_0^t |u_n'|(r)\,\mathrm{d}r + \frac{1}{q}\int_0^t |\partial^-\phi|(u(r))\,\mathrm{d}r + \phi(u_n(t)) = \phi(u_0)$$

Energy identity $\Rightarrow$ a priori estimates for $\{u_n\}$; compactness and $\exists$ of a limit curve, passage to the limit like in the existence proof.

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$$Z(\mathcal{S}) = \{\bar{u} \in D(\phi) \ : \ |\partial\phi|(\bar{u}) = 0\}$$

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- ϕ is continuous along sequences with bounded energies and slopes
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  is bounded in $(\mathcal{X}, d_{\mathcal{X}})$.

Then,

$$\mathcal{S} \text{ admits a global attractor } \mathcal{A}.$$

Long-time behaviour for p-curves of maximal slope

Theorem 2 [R., Savaré, Segatti, Stefanelli '06]

Suppose that

- ϕ is lower semicontinuous
- ϕ is coercive
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Idea of the proof:

- the generalized semiflow $\mathcal{S}$ is **compact**
- $\mathcal{S}$ has a Lyapunov functional

Applications in Banach spaces

- $X = B$ Banach space,
- $\phi : B \to (-\infty, +\infty]$ l.s.c.,

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Hence, p-curves of maximal slope for ϕ (w.r.t. $|\partial^-\phi|$) lead to solutions of the doubly nonlinear equation

$$\Im_p(u'(t)) + \partial\phi(u(t)) \ni 0 \quad \text{in } B' \quad \text{for a.e. } t \in (0, T)$$

($\Im_p : B \to B'$ the p-duality map)

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Under suitable coercivity assumptions, our long-time behaviour results give the **existence of a global attractor** for the **"metric solutions"** of

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thus recovering some results in [SEGATTI '06].

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$$\xi \in \partial_\ell \phi(u) \quad \Leftrightarrow \exists \{u_n\}, \{\xi_n\} \subset B : \quad \begin{cases} \xi_n \in \partial\phi(u_n) \ \forall\, n \in \mathbb{N}, \\ u_n \to u, \\ \xi_n \rightharpoonup^* \xi \quad \text{in } B', \\ \sup_n \phi(u_n) < +\infty \end{cases}$$

a version of the strong-weak* closure of $\partial\phi$ ([Mordhukhovich '84]).

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It can be proved that for all $u \in D(\phi)$

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Our results yield the **existence of a global attractor** for the **"metric solutions"** of

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thus extending some results by [Rossi-Segatti-Stefanelli '05].

Applications in Wasserstein spaces

Consider the functional $\phi : \mathscr{P}_p(\mathbb{R}^n) \to (-\infty, +\infty]$

$$\phi(\mu) := \int_{\mathbb{R}^n} F(\rho)\,\mathrm{d}x + \int_{\mathbb{R}^n} V\,\mathrm{d}\mu + \frac{1}{2}\int_{\mathbb{R}^n \times \mathbb{R}^n} W\,\mathrm{d}(\mu \otimes \mu) \quad \text{if } \mu = \rho\,\mathrm{d}x$$

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- $F \rightsquigarrow$ **internal energy**
- $V \rightsquigarrow$ **potential energy** ("confinement potential")
- $W \rightsquigarrow$ **interaction energy**

proposed by [Carrillo, McCann, Villani '03,'04] in the framework of **kinetic models** for equilibration velocities in **granular media**.

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Now, p-curves of maximal slope for ϕ yield solutions to the **drift-diffusion equation with nonlocal term**

$$\partial_t\rho - \mathrm{div}\left(\rho j_q\left(\frac{\nabla L_F(\rho)}{\rho} + \nabla V + (\nabla W)\star\rho\right)\right) = 0 \text{ in } \mathbb{R}^n\times(0,T),$$

where $L_F(\rho) = \rho F'(\rho) - F(\rho)$, such that

$$\begin{cases} \rho(x,t) \geq 0, \quad \int_{\mathbb{R}^n}\rho(x,t)\,\mathrm{d}x = 1 \ \ \forall\,(x,t)\in\mathbb{R}^n\times(0,+\infty), \\ \int_{\mathbb{R}^n}|x|^p\rho(x,t)\,\mathrm{d}x < +\infty \quad \forall\, t\geq 0. \end{cases}$$

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- In [Ambrosio-Gigli-Savaré '05]: an **existence result** via the approach of p-curves of maximal slope
- **No general uniqueness** result is known

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In the case $W \equiv 0$, under suitable λ-**convexity** assumptions **on** V, **growth & convexity** assumptions **on** F, [Agueh '03] has proved the exponential decay of solutions to equilibrium for $t \to +\infty$, with explicit rates of convergence, by refined Logarithmic Sonbolev inequalities

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In the general case, [Carrillo, McCann, Villani '03,'04] have proved in the case $q = 2$ uniqueness, contraction estimates, and the exponential decay of solutions to equilibrium for $t \to +\infty$, with explicit rates of convergence (recovered in the general case by [Ambrosio-Gigli-Savaré '05])

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under suitable λ-**convexity** assumptions **on** V, **growth & convexity** assumptions **on** F, **convexity & a doubling condition on** W.

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For $W = 0$, our conditions are **partially weaker** than AGUEH's, but the results too are weaker (at our best, we obtain that the attractor consists of a unique equilibrium, but no explicit rates of decay).

Towards a chain-rule free approach

- It would be crucial to **drop the λ-convexity assumption** on V $\rightsquigarrow$ methods based Logarithmic-Sobolev inequalities do not work any more $\rightsquigarrow$ the existence of a global attractor is a meaningful information..

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- We pass to the limit in the approximate energy inequality

$$\frac{1}{2}\int_s^t |\mathrm{u}'_{\tau_k}|(r)^2 \,\mathrm{d}r + \frac{1}{2}\int_s^t |\partial\phi|^2(\mathrm{u}_{\tau_k}(r))\,\mathrm{d}r + \phi(\mathrm{u}_{\tau_k}(t)) \leq \phi(\mathrm{u}_{\tau_k}(s))$$

$\forall\, 0 \leq s \leq t \leq T$ arguing

  - on the left-hand side: by lower semicontinuity
  - on the right-hand side: by monotonicity, which gives that

$$\exists \varphi(s) := \lim_{k\uparrow\infty} \phi(\mathrm{u}_{\tau_k}(s)) \geq \phi(u(s)) \quad \forall\, s \in [0, T]$$

Towards a chain-rule free approach

Let us revise the proof of the general existence theorem (for $p = 2$):

- a priori estimates & the compactness argument do not need the chain rule
- In the limit we find a **non-decreasing** function $\varphi : [0, T] \to \mathbb{R}$ such that $\forall\, 0 \le s \le t \le T$

$$\frac{1}{2}\int_s^t |u'|(r)^2 \,\mathrm{d}r + \frac{1}{2}\int_s^t |\partial^- \phi|^2(u(r)) \,\mathrm{d}r + \varphi(t) \le \varphi(s)$$

and

$$\varphi(t) \ge \phi(u(t)) \quad \forall\, t \in [0, T].$$

Towards a chain-rule free approach

Let us revise the proof of the general existence theorem (for $p = 2$):

- **Note**: the chain rule for $|\partial^- \phi|$ is used just to obtain

$$\varphi(t) = \phi(u(t)) \quad \forall\, t \in [0, T]$$

and conclude that u is a curve of maximal slope for ϕ.

- In the limit we find a **non-decreasing** function $\varphi : [0, T] \to \mathbb{R}$ such that $\forall\, 0 \leq s \leq t \leq T$

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We switch from

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to a new solution notion

$$\mathcal{S}_{\mathsf{new}} = \Big\{ (u, \varphi) \, : \, u \in \mathrm{AC}_{\mathsf{loc}}(0, +\infty; X), \\ \varphi : [0, +\infty) \to \mathbb{R} \text{ is non increasing, and (1)-(2) hold} \Big\}$$

where for all $0 \leq s \leq t \leq T$

$$\frac{1}{2}\int_s^t |u'|(r)^2 \, \mathrm{d}r + \frac{1}{2}\int_s^t |\partial^- \phi|^2(u(r)) \, \mathrm{d}r + \varphi(t) \leq \varphi(s) \tag{1}$$

$$\varphi(t) \geq \phi(u(t)) \quad \forall \, t \in [0, T]. \tag{2}$$

A new phase space & a new result

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$$\mathcal{X}_{\mathrm{old}} = D(\phi) \text{ with the distance } d_{\mathcal{X}_{\mathrm{old}}}(u, u') = d(u, u') + |\phi(u) - \phi(u')|$$

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A new result

Suppose that

- ϕ is lower semicontinuous
- ϕ is coercive
- The set of rest point for $\mathcal{S}_{\text{new}}$ is bounded.

Then, $\mathcal{S}_{\text{new}}$ is a generalized semiflow in $(\mathcal{X}_{\text{new}}, d_{\text{new}})$, and it admits a global attractor.

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Application: any evolution problem arising as limit of a “steepest descent” approximation scheme, under the “minimal” assumptions to get existence...