Analysis of a model for adhesive contact with thermal effects

Riccarda Rossi
(Università di Brescia)

joint work with
Elena Bonetti (Università di Pavia),
Giovanna Bonfanti (Università di Brescia),

Mathematical Models and Analytical Problems for Special Materials, Brescia, July 09–11 2009

Frémond's modeling of adhesive contact

Setting

A viscoelastic body $\Omega \subset \mathbb{R}^3$ in **adhesive contact** with a rigid support on a **(flat)** part Γ_{Cont} of its boundary $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$.

Related contributions

.... Andrews, Cangémi, Chau, Cocou, Eck, Fernández, Figuereido, Han, Jarušek, Klarbring, Krbec, Kuttler, Martins, Muñoz-Rivera, Point, Racke, Raous, Shi, Shillor, Sofonea, Telega, Trabucho, Wright.....

Frémond's modeling of adhesive contact

Setting

A viscoelastic body $\Omega \subset \mathbb{R}^3$ in **adhesive contact** with a rigid support on a **(flat)** part Γ_{Cont} of its boundary $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$.

Related contributions

.... Andrews, Cangémi, Chau, Cocou, Eck, Fernández, Figuereido, Han, Jarušek, Klarbring, Krbec, Kuttler, Martins, Muñoz-Rivera, Point, Racke, Raous, Shi, Shillor, Sofonea, Telega, Trabucho, Wright.....

Frémond's approach

Based on **[M. Frémond, *Non-smooth Thermomechanics*, 2002]**

Account for **microscopic motions** in the **macroscopic predictive theory**

Frémond's modeling of adhesive contact

Setting

A viscoelastic body $\Omega \subset \mathbb{R}^3$ in **adhesive contact** with a rigid support on a **(flat)** part Γ_{Cont} of its boundary $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$.

Related contributions

.... Andrews, Cangémi, Chau, Cocou, Eck, Fernández, Figuereido, Han, Jarušek, Klarbring, Krbec, Kuttler, Martins, Muñoz-Rivera, Point, Racke, Raous, Shi, Shillor, Sofonea, Telega, Trabucho, Wright.....

Frémond's approach

Based on **[M. Frémond,** ***Non-smooth Thermomechanics*****, 2002]**

Account for **microscopic motions** in the **macroscopic predictive theory**

- microscopic bonds are responsible for the adhesion, microscopic motions lead to rupture

Frémond's modeling of adhesive contact

Setting

A viscoelastic body $\Omega \subset \mathbb{R}^3$ in **adhesive contact** with a rigid support on a **(flat)** part Γ_{Cont} of its boundary $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$.

Related contributions

.... Andrews, Cangémi, Chau, Cocou, Eck, Fernández, Figuereido, Han, Jarušek, Klarbring, Krbec, Kuttler, Martins, Muñoz-Rivera, Point, Racke, Raous, Shi, Shillor, Sofonea, Telega, Trabucho, Wright.....

Frémond's approach

Based on **[M. Frémond,** *Non-smooth Thermomechanics*, **2002]**

Account for **microscopic motions** in the **macroscopic predictive theory**

- microscopic bonds are responsible for the adhesion, microscopic motions lead to rupture
- account for the **power of the microscopic motions** in the power of the interior forces

State variables

In the **isothermal case** **[Bonetti-Bonfanti-R. '07,'08]**

- in the **volume domain** Ω:
  - **small deformation** ($\varepsilon(\mathbf{u})$ symm. linear. strain tensor) **(small perturbation assumption)**
  -
- on the **contact surface** Γ_{Cont}:
  - **adhesion** (χ "phase parameter" related to the active bonds of the adhesion $\rightsquigarrow$ "damage parameter")
  - **effects of displacement** ($\mathbf{u}_{|\Gamma_{\mathrm{Cont}}}$ trace of the displacement)
  -

State variables

To account for **thermal effects:** [Bonetti-Bonfanti-R. preprint'08]

- in the **volume domain** Ω:
  - **small deformation** ($\varepsilon(\mathbf{u})$ symm. linear. strain tensor) **(small perturbation assumption)**
  - **thermal effects (θ absolute temperature)**
- on the **contact surface** Γ_{Cont}:
  - **adhesion** (χ "phase parameter" related to the active bonds of the adhesion $\rightsquigarrow$ "damage parameter")
  - **effects of displacement** ($\mathbf{u}_{|\Gamma_{\mathrm{Cont}}}$ trace of the displacement)
  - **thermal effects (θ_{s} absolute temperature)**

The equations for u and χ

From the **principle of virtual power** (interior & exterior forces, no acceleration forces)

- **momentum balance:**

$$\begin{cases} -\operatorname{div}\Sigma = \mathbf{f} \quad \text{in } \Omega \times (0,T), \\ \begin{cases} \Sigma\mathbf{n} = \mathbf{R} \quad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\mathrm{Dir}} \times (0,T), \\ \Sigma\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\mathrm{Neu}} \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} \Sigma \text{ stress tensor} \\ \mathbf{R} \text{ reaction on the contact surface} \\ \mathbf{f} \text{ volume force, } \mathbf{g} \text{ traction} \end{cases}$$

The equations for u and χ

From the **principle of virtual power** (interior & exterior forces, no acceleration forces)

- **momentum balance:**

$$\begin{cases} -\operatorname{div}\Sigma = \mathbf{f} & \text{in } \Omega \times (0,T), \\ \begin{cases} \Sigma\mathbf{n} = \mathbf{R} & \text{in } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \mathbf{u} = \mathbf{0} & \text{in } \Gamma_{\mathrm{Dir}} \times (0,T), \\ \Sigma\mathbf{n} = \mathbf{g} & \text{in } \Gamma_{\mathrm{Neu}} \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} \Sigma \text{ stress tensor} \\ \mathbf{R} \text{ reaction on the contact surface} \\ \mathbf{f} \text{ volume force, } \mathbf{g} \text{ traction} \end{cases}$$

- **equation for the microscopic motions:**

$$\begin{cases} B - \operatorname{div}\mathbf{H} = 0 & \text{in } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \mathbf{H}\cdot\mathbf{n}_{\mathrm{s}} = 0 & \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T), \end{cases}$$

$$\begin{cases} B \text{ interior microscopic work} \\ \mathbf{H} \text{ microscopic work flux vector} \end{cases}$$

The equations for θ and θ_{s}

Entropy balance for θ and θ_{s}:

- for θ:

$$\begin{cases} s_t + \operatorname{div} \mathbf{Q} = h \quad \text{in } \Omega \times (0,T), \\ \begin{cases} \mathbf{Q} \cdot \mathbf{n} = F \ \text{ on } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \quad \mathbf{Q} \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} s \text{ internal entropy} \\ \mathbf{Q} \text{ entropy flux vector} \\ h \text{ entropy source,} \\ F \text{ entropy flux through } \Gamma_{\mathrm{Cont}} \end{cases}$$

The equations for θ and θ_{s}

Entropy balance for θ and θ_{s}:

- for θ:

$$\begin{cases} s_t + \operatorname{div} \mathbf{Q} = h \quad \text{in } \Omega \times (0,T), \\ \begin{cases} \mathbf{Q} \cdot \mathbf{n} = F \ \text{ on } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \quad \mathbf{Q} \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} s \text{ internal entropy} \\ \mathbf{Q} \text{ entropy flux vector} \\ h \text{ entropy source,} \\ F \text{ entropy flux through } \Gamma_{\mathrm{Cont}} \end{cases}$$

- for θ_{s} :

$$\begin{cases} \partial_t s_{\mathrm{s}} + \operatorname{div} \mathbf{Q}_{\mathrm{s}} = F \quad \text{in } \Omega \times (0,T), \\ \mathbf{Q}_{\mathrm{s}} \cdot \mathbf{n}_{\mathrm{s}} = 0 \ \text{ on } \partial\Gamma_{\mathrm{Cont}} \times (0,T), \end{cases}$$

$$\begin{cases} s_{\mathrm{s}} \text{ contact surface entropy} \\ \mathbf{Q}_{\mathrm{s}} \text{ surface entropy flux vector} \end{cases}$$

The equations for θ and θ_s

Entropy balance for θ and θ_s:

- for θ:

$$\begin{cases} s_t + \operatorname{div} \mathbf{Q} = h & \text{in } \Omega \times (0,T), \\ \begin{cases} \mathbf{Q} \cdot \mathbf{n} = F & \text{on } \Gamma_{\mathrm{Cont}} \times (0,T), \\ \quad \mathbf{Q} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} s \text{ internal entropy} \\ \mathbf{Q} \text{ entropy flux vector} \\ h \text{ entropy source,} \\ F \text{ entropy flux through } \Gamma_{\mathrm{Cont}} \end{cases}$$

- for θ_s :

$$\begin{cases} \partial_t s_s + \operatorname{div} \mathbf{Q}_s = F & \text{in } \Omega \times (0,T), \\ \mathbf{Q}_s \cdot \mathbf{n}_s = 0 & \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T), \end{cases}$$

$$\begin{cases} s_s \text{ contact surface entropy} \\ \mathbf{Q}_s \text{ surface entropy flux vector} \end{cases}$$

Entropy balance: obtained by rescaling the internal energy balance (under small perturbation assumpt.): see **[Bonetti-Frémond'03, Bonetti-Colli-Frémond'03, Bonetti'06, Bonetti-Colli-Fabrizio-Gilardi'06,'07,08, Bonetti-Rocca-Frémond'07]**

Constitutive laws

Constitutive relations for

$$\Sigma, \mathbf{R}, s, \mathbf{Q}, F, B, \mathbf{H}, s_{\mathrm{s}}, \mathbf{Q}_{\mathrm{s}}$$

derive from the **volume & surface free energies**

$$\Psi_\Omega = \Psi_\Omega(\mathbf{u}, \theta), \qquad \Psi_{\Gamma_{\mathrm{Cont}}} = \Psi_{\Gamma_{\mathrm{Cont}}}(\mathbf{u}_{|\Gamma_{\mathrm{Cont}}}, \chi, \theta_{\mathrm{s}})$$

and the **pseudo-potentials of dissipation**

$$\Phi_\Omega = \Phi_\Omega(\nabla\theta, \varepsilon(\mathbf{u}_t)), \qquad \Phi_{\Gamma_{\mathrm{Cont}}} = \Phi_{\Gamma_{\mathrm{Cont}}}(\nabla\theta_{\mathrm{s}}, \chi_t, \theta_{|\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}})$$

Constitutive laws

Constitutive relations for

$$\mathbf{\Sigma}, \mathbf{R}, s, \mathbf{Q}, F, B, \mathbf{H}, s_{\mathrm{s}}, \mathbf{Q}_{\mathrm{s}}$$

derive from the **volume & surface free energies**

$$\Psi_\Omega = \Psi_\Omega(\mathbf{u}, \theta), \qquad \Psi_{\Gamma_{\mathrm{Cont}}} = \Psi_{\Gamma_{\mathrm{Cont}}}(\mathbf{u}_{|\Gamma_{\mathrm{Cont}}}, \chi, \theta_{\mathrm{s}})$$

and the **pseudo-potentials of dissipation**

$$\Phi_\Omega = \Phi_\Omega(\nabla\theta, \varepsilon(\mathbf{u}_t)), \qquad \Phi_{\Gamma_{\mathrm{Cont}}} = \Phi_{\Gamma_{\mathrm{Cont}}}(\nabla\theta_{\mathrm{s}}, \chi_t, \theta_{|\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}})$$

The energy potentials include **constraints on the variables** for physical consistency: $\rightsquigarrow$ **nonsmooth (multivalued) operators** in the constitutive eqns

Constitutive laws

Constitutive relations for

$$\Sigma, \mathbf{R}, s, \mathbf{Q}, F, B, \mathbf{H}, s_{\mathrm{s}}, \mathbf{Q}_{\mathrm{s}}$$

derive from the **volume & surface free energies**

$$\Psi_\Omega = \Psi_\Omega(\mathbf{u}, \theta), \qquad \Psi_{\Gamma_{\mathrm{Cont}}} = \Psi_{\Gamma_{\mathrm{Cont}}}(\mathbf{u}_{|\Gamma_{\mathrm{Cont}}}, \chi, \theta_{\mathrm{s}})$$

and the **pseudo-potentials of dissipation**

$$\Phi_\Omega = \Phi_\Omega(\nabla\theta, \varepsilon(\mathbf{u}_t)), \qquad \Phi_{\Gamma_{\mathrm{Cont}}} = \Phi_{\Gamma_{\mathrm{Cont}}}(\nabla\theta_{\mathrm{s}}, \chi_t, \theta_{|\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}})$$

The energy potentials include **constraints on the variables** for physical consistency: $\rightsquigarrow$ **nonsmooth (multivalued) operators** in the constitutive eqns

Constraints

- admissible values for χ and (possibly) irreversibility
- impenetrability condition between the body and the support
- positivity of the absolute temperatures θ and θ_{s}

The adhesion phenomenon

The "damage parameter" χ denotes the **fraction of active glue fibers** at each point of the contact surface

- $\chi = 0$ no adhesion (completely broken bonds)
- $\chi = 1$ active adhesion (unbroken bonds)

The adhesion phenomenon

The "damage parameter" χ denotes the **fraction of active glue fibers** at each point of the contact surface

- $\chi = 0$ no adhesion (completely broken bonds)
- $\chi = 1$ active adhesion (unbroken bonds)

Physical constraints

- $\chi \in [0, 1]$
- $\chi_t \leq 0$ (irreversible phenomenon)

The adhesion phenomenon

The "damage parameter" χ denotes the **fraction of active glue fibers** at each point of the contact surface

- $\chi = 0$ no adhesion (completely broken bonds)
- $\chi = 1$ active adhesion (unbroken bonds)

Physical constraints

- $\chi \in [0, 1]$

As a first step, neglect irreversibility ($\sim$ fresh, liquid glue..)

Unilateral conditions

The **impenetrability condition**

$$\mathbf{u}_{|\Gamma_{\mathrm{Cont}}} \cdot \mathbf{n} \leq \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}.$$

is ensured by the reaction on the contact surface

Unilateral conditions

The **impenetrability condition**

$$\mathbf{u}_{|\Gamma_{\mathrm{Cont}}} \cdot \mathbf{n} \leq \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}.$$

is ensured by the reaction on the contact surface
If $\chi = 0$ (no adhesion) the reaction on Γ_{Cont} is

$$\mathbf{R} = \qquad -\partial I_{]-\infty,0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$$

(omitting traces) where

$$\partial I_{(-\infty,0]}(\mathbf{u} \cdot \mathbf{n}) = \begin{cases} 0 & \text{if } \mathbf{u} \cdot \mathbf{n} < \mathbf{0} \\ [0, +\infty[& \text{if } \mathbf{u} \cdot \mathbf{n} = \mathbf{0} \end{cases}$$

- $\mathbf{R}$ is normal to Γ_{Cont}
- $\mathbf{R} = \mathbf{0}$ if $\mathbf{u} \cdot \mathbf{n} < 0$
- $\mathbf{R} = \gamma\mathbf{n}$, $\gamma \leq 0$ if $\mathbf{u} \cdot \mathbf{n} = 0$

This is in agreement with the **Signorini conditions**

Unilateral conditions

The **impenetrability condition**

$$\mathbf{u}_{|\Gamma_{\mathrm{Cont}}} \cdot \mathbf{n} \leq \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}.$$

is ensured by the reaction on the contact surface
In the case the adhesion is active $\chi > 0$

$$\mathbf{R} = -\chi\mathbf{u} - \partial\mathbf{I}_{]-\infty,\mathbf{0}]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$$

i.e., there is a reaction (with rigidity $\sim \chi$) **counteracting separation**:

$$\mathbf{R}\cdot\mathbf{n} = -\chi\ \mathbf{u}\cdot\mathbf{n} > 0 \text{ if } \mathbf{u}\cdot\mathbf{n} < 0$$

The PDE system: the momentum balance

Recall $\Omega \subset \mathbb{R}^3$ smooth, bounded and $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$

- The momentum balance

$$-\mathrm{div}\,(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0, T)$$

where: K elasticity tensor, K_v viscosity tensor, $\theta\mathbf{1}$ $\leftrightarrow$ **thermal deformation**

The PDE system: the momentum balance

Recall $\Omega \subset \mathbb{R}^3$ smooth, bounded and $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Gamma_{\mathrm{Cont}}$

- The momentum balance

$$-\mathrm{div}\,(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0, T)$$

  where: K elasticity tensor, K_v viscosity tensor, $\theta\mathbf{1} \leftrightarrow$ **thermal deformation**

- the **boundary conditions**

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Dir}}, \quad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{Neu}} \times (0, T),$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}} \times (0, T),$$

  where $-\chi\mathbf{u} - \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ on Γ_{Cont} is the reaction

The PDE system: the evolution of the adhesion

We consider on Γ_{Cont}

$$\chi_t - \Delta\chi + \partial \mathbf{I}_{[0,1]}(\chi) \ni -\lambda'(\chi)(\theta_{\mathrm{s}} - \theta_{\mathbf{eq}}) - \frac{1}{2}|\mathbf{u}|^2 \qquad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T)$$

$$\partial_n \chi = 0, \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T)$$

- $\partial \mathbf{I}_{[0,1]}(\chi) \Rightarrow \chi \in [0,1]$ **(physical consistency)**

$$\partial I_{[0,1]}(\chi) = \begin{cases}]-\infty, 0] & \text{if } \chi = 0 \\ 0 & \text{if } 0 < \chi < 1 \\ [0, +\infty[& \text{if } \chi = 1 \end{cases}$$

- λ (quadratic) function, related to the latent heat
- θ_{s} temperature of the glue, θ_{eq} constant
- $-\frac{1}{2}|\mathbf{u}|^2$ source of damage due to macroscopic movements

The PDE system: the temperature equations

- The **entropy equation** (rescaled energy balance) for θ in Ω

$$\partial_t(\log\theta) - \mathrm{div}\,\mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T) \\ -\chi(\theta_{|\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}} \times (0,T) \end{cases}$$

The PDE system: the temperature equations

- The **entropy equation** (rescaled energy balance) for θ in Ω

$$\partial_t(\log\theta) - \operatorname{div} \mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T) \\ -\chi(\theta|_{\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}} \times (0,T) \end{cases}$$

- The **entropy equation** for θ_{s} on Γ_{Cont} is

$$\partial_t(\log\theta_{\mathrm{s}}) - \lambda'(\chi)\chi_t - \Delta\theta_{\mathrm{s}} = \chi(\theta|_{\Gamma_{\mathrm{Cont}}} - \theta_{\mathrm{s}}) \quad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T)$$

$$\partial_n\theta_{\mathrm{s}} = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T)$$

The complete PDE system: difficulties

$$
\begin{aligned}
& -\operatorname{div}\left(K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1}\right)=\mathbf{f} \quad \text{in } \Omega\times(0,T),\\
& \mathbf{u}=\mathbf{0} \quad \text{on } \Gamma_{\mathrm{Dir}}\times(0,T), \quad (K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1})\mathbf{n}=\mathbf{g} \quad \text{on } \Gamma_{\mathrm{Neu}}\times(0,T),\\
& (K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1})\mathbf{n}+\chi\mathbf{u}+\partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\ni\mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}}\times(0,T),\\
& \chi_t-\Delta\chi+\partial I_{[0,1]}(\chi)\ni-\lambda'(\chi)(\theta_s-\theta_{eq})-\frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),\\
& \partial_n\chi=0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T)\\
& \partial_t(\log\theta)-\operatorname{div}\mathbf{u}_t-\Delta\theta=h \quad \text{in } \Omega\times(0,T),\\
& \partial_n\theta=\begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T),\\ -\chi(\theta-\theta_s) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T),\end{cases}\\
& \partial_t(\log\theta_s)-\lambda'(\chi)\chi_t-\Delta\theta_s=\chi(\theta-\theta_s) \quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),\\
& \partial_n\theta_s=0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T), \qquad +\text{Cauchy conditions}
\end{aligned}
$$

→ **singular character of the θ, θ_s-equations** (θ-**equation is coupled with a third type boundary condition**)
→ we deduce directly $\theta, \theta_s > 0$, crucial for **thermodynamical consistency**!

The complete PDE system: difficulties

$$
\begin{aligned}
&-\operatorname{div}\,(K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1})=\mathbf{f}\quad \text{in } \Omega\times(0,T),\\
&\mathbf{u}=\mathbf{0}\quad \text{in } \Gamma_{\mathrm{Dir}}\times(0,T),\quad (K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1})\mathbf{n}=\mathbf{g}\quad \text{in } \Gamma_{\mathrm{Neu}}\times(0,T),\\
&(K\varepsilon(\mathbf{u})+K_v\varepsilon(\mathbf{u}_t)+\theta\mathbf{1})\mathbf{n}+\chi\mathbf{u}+\partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\ni\mathbf{0}\quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),\\
&\chi_t-\Delta\chi+\partial I_{[0,1]}(\chi)\ni-\lambda'(\chi)(\theta_s-\theta_{eq})-\frac{1}{2}|\mathbf{u}|^2\quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),\\
&\partial_n\chi=0\quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T)\\
&\partial_t(\log\theta)-\operatorname{div}\,\mathbf{u}_t-\Delta\theta=h\quad \text{in } \Omega\times(0,T),\\
&\partial_n\theta=\begin{cases}0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T),\\ -\chi(\theta-\theta_s) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T),\end{cases}\\
&\partial_t(\log\theta_s)-\lambda'(\chi)\chi_t-\Delta\theta_s=\chi(\theta-\theta_s)\quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T)\\
&\partial_n\theta_s=0\quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T)\qquad +\text{Cauchy conditions}
\end{aligned}
$$

→ **(quadratic) coupling** terms on the boundary

→ we need **sufficient regularity** on θ and $\mathbf{u}$ to control their traces.

The complete PDE system: difficulties

$$
\begin{aligned}
& -\operatorname{div}\left(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}\right) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \\
& \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Dir}} \times (0,T), \quad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{Neu}} \times (0,T), \\
& (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Cont}} \times (0,T), \\
& \chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni -\lambda'(\chi)(\theta_s - \theta_{eq}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T), \\
& \partial_n\chi = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T), \\
& \partial_t(\log\theta) - \operatorname{div}\mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T), \\
& \partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T), \\ -\chi(\theta-\theta_s) & \text{on } \Gamma_{\mathrm{Cont}} \times (0,T), \end{cases} \\
& \partial_t(\log\theta_s) - \lambda'(\chi)\chi_t - \Delta\theta_s = \chi(\theta-\theta_s) \quad \text{in } \Gamma_{\mathrm{Cont}} \times (0,T), \\
& \partial_n\theta_s = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}} \times (0,T), \qquad + \text{Cauchy conditions}
\end{aligned}
$$

$\rightarrow$ **non-smooth (multivalued) constraints** on χ and $\mathbf{u}\cdot\mathbf{n}$

The existence theorem

Theorem. Given initial data

$$\begin{cases} \theta_0 \in L^1(\Omega), \quad \log(\theta_0) \in L^2(\Omega), \quad \theta_{s,0} \in L^1(\Gamma_{\mathrm{Cont}}), \quad \log(\theta_{s,0}) \in L^2(\Gamma_{\mathrm{Cont}}), \\ \mathbf{u}_0 \in H^1(\Omega)^3, \quad \chi_0 \in H^1(\Gamma_{\mathrm{Cont}}) \end{cases}$$

there exist $(\mathbf{u}, \chi, \theta, \theta_s)$ solving the **weak, variational** formulation of the **IBVP**

$$-\operatorname{div}(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0,T),$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_c \times (0,T),$$

$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni -\lambda'(\chi)(\theta_s - \theta_{eq}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_c \times (0,T),$$

$$\partial_t(\log\theta) - \operatorname{div}\mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_c \times (0,T), \\ -\chi(\theta - \theta_s) & \text{on } \Gamma_c \times (0,T), \end{cases}$$

$$\partial_t(\log\theta_s) - \lambda'(\chi)\chi_t - \Delta\theta_s = \chi(\theta - \theta_s) \quad \text{in } \Gamma_c \times (0,T),$$

The existence theorem

Theorem. Given initial data

$$\begin{cases} \theta_0 \in L^1(\Omega), \quad \log(\theta_0) \in L^2(\Omega), \quad \theta_{s,0} \in L^1(\Gamma_{\mathrm{Cont}}), \quad \log(\theta_{s,0}) \in L^2(\Gamma_{\mathrm{Cont}}), \\ \mathbf{u}_0 \in H^1(\Omega)^3, \quad \chi_0 \in H^1(\Gamma_{\mathrm{Cont}}) \end{cases}$$

there exist $(\mathbf{u}, \chi, \theta, \theta_{\mathrm{s}})$ solving the **weak, variational** formulation of the **IBVP**

$$-\operatorname{div}\,(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0,T),$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_c \times (0,T),$$

$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni -\lambda'(\chi)(\theta_s - \theta_{eq}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_c \times (0,T),$$

$$\partial_t(\log\theta) - \operatorname{div}\mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_c \times (0,T), \\ -\chi(\theta - \theta_s) & \text{on } \Gamma_c \times (0,T), \end{cases}$$

$$\partial_t(\log\theta_s) - \lambda'(\chi)\chi_t - \Delta\theta_s = \chi(\theta - \theta_s) \quad \text{in } \Gamma_c \times (0,T),$$

with $\mathbf{u} \in H^1(0,T; H^1(\Omega)^3)$

$$\chi \in L^2(0,T;H^2(\Gamma_{\mathrm{Cont}})) \cap L^\infty(0,T;H^1(\Gamma_{\mathrm{Cont}})) \cap H^1(0,T;L^2(\Gamma_{\mathrm{Cont}}))$$

$$\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \quad \log\theta \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)')$$

$$\theta_{\mathrm{s}} \in L^2(0,T;H^1(\Gamma_{\mathrm{Cont}})) \cap L^\infty(0,T;L^1(\Gamma_{\mathrm{Cont}})),$$

$$\log\theta_{\mathrm{s}} \in L^\infty(0,T;L^2(\Gamma_{\mathrm{Cont}})) \cap H^1(0,T;H^1(\Gamma_{\mathrm{Cont}})')$$

The existence theorem

Theorem. Given initial data

$$\begin{cases} \theta_0 \in L^1(\Omega), \quad \log(\theta_0) \in L^2(\Omega), \quad \theta_{s,0} \in L^1(\Gamma_{\mathrm{Cont}}), \quad \log(\theta_{s,0}) \in L^2(\Gamma_{\mathrm{Cont}}), \\ \mathbf{u}_0 \in H^1(\Omega)^3, \quad \chi_0 \in H^1(\Gamma_{\mathrm{Cont}}) \end{cases}$$

there exist $(\mathbf{u}, \chi, \theta, \theta_{\mathrm{s}})$ solving the **weak, variational** formulation of the **IBVP**

$$-\operatorname{div}\left(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1}\right) = \mathbf{f} \quad \text{in } \Omega \times (0,T),$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{on } \Gamma_c \times (0,T),$$

$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni -\lambda'(\chi)(\theta_s - \theta_{eq}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_c \times (0,T),$$

$$\partial_t(\log\theta) - \operatorname{div}\mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_c \times (0,T), \\ -\chi(\theta - \theta_s) & \text{on } \Gamma_c \times (0,T), \end{cases}$$

$$\partial_t(\log\theta_s) - \lambda'(\chi)\chi_t - \Delta\theta_s = \chi(\theta - \theta_s) \quad \text{in } \Gamma_c \times (0,T),$$

with $\mathbf{u} \in H^1(0,T;H^1(\Omega)^3)$

$\chi \in L^2(0,T;H^2(\Gamma_{\mathrm{Cont}})) \cap L^\infty(0,T;H^1(\Gamma_{\mathrm{Cont}})) \cap H^1(0,T;L^2(\Gamma_{\mathrm{Cont}}))$

$\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \quad \log\theta \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)')$

$\theta_{\mathrm{s}} \in L^2(0,T;H^1(\Gamma_{\mathrm{Cont}})) \cap L^\infty(0,T;L^1(\Gamma_{\mathrm{Cont}})),$

$\log\theta_{\mathrm{s}} \in L^\infty(0,T;L^2(\Gamma_{\mathrm{Cont}})) \cap H^1(0,T;H^1(\Gamma_{\mathrm{Cont}})')$

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

$$\partial_t(\log\theta) - \operatorname{div} \mathbf{u}_t - \Delta\theta = h \quad \text{in } \Omega \times (0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_{\mathrm{Cont}} \times (0,T) \\ -\chi(\theta-\theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}} \times (0,T) \end{cases}$$

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

$$\varepsilon\partial_t\theta+\partial_t(\log_\varepsilon\theta) - \mathrm{div}\,\mathbf{u}_t - \Delta\theta - \varepsilon\Delta\left(\partial_t\theta\right) = h \quad \text{in } \Omega\times(0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T) \\ -\chi(\theta-\theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T) \end{cases}$$

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

$$\varepsilon\partial_t\theta+\partial_t(\log_\varepsilon\theta) - \operatorname{div}\mathbf{u}_t - \Delta\theta - \varepsilon\Delta(\partial_t\theta) = h \quad \text{in } \Omega\times(0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T) \\ -\chi(\theta-\theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T) \end{cases}$$

where $\log_\varepsilon$ denotes a **Yosida-type** approximation of log

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

$$\varepsilon\partial_t\theta+\partial_t(\log_\varepsilon\theta) - \mathrm{div}\,\mathbf{u}_t - \Delta\theta-\varepsilon\Delta\left(\partial_t\theta\right) = h \quad \text{in } \Omega\times(0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T) \\ -\chi(\theta-\theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T) \end{cases}$$

where $\log_\varepsilon$ denotes a **Yosida-type** approximation of log

- Analogously for θ_s:

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

$$\varepsilon\partial_t\theta + \partial_t(\log_\varepsilon\theta) - \operatorname{div}\mathbf{u}_t - \Delta\theta - \varepsilon\Delta\left(\partial_t\theta\right) = h \quad \text{in } \Omega\times(0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T) \\ -\chi(\theta-\theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T) \end{cases}$$

where $\log_\varepsilon$ denotes a **Yosida-type** approximation of log

- Analogously for θ_s:

$$\partial_t(\log\theta_{\mathrm{s}}) - \lambda'(\chi)\chi_t - \Delta\theta_{\mathrm{s}} = \chi(\theta-\theta_{\mathrm{s}}) \quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),$$

$$\partial_n\theta_{\mathrm{s}} = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T)$$

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

$$\varepsilon\partial_t\theta + \partial_t(\log_\varepsilon\theta) - \operatorname{div}\mathbf{u}_t - \Delta\theta - \varepsilon\Delta\left(\partial_t\theta\right) = h \quad \text{in } \Omega\times(0,T),$$
$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T) \\ -\chi(\theta-\theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T) \end{cases}$$

where $\log_\varepsilon$ denotes a **Yosida-type** approximation of log

- Analogously for θ_s:

$$\varepsilon\partial_t\theta_{\mathrm{s}} + \partial_t(\log_\varepsilon\theta_{\mathrm{s}}) - \lambda'(\chi)\chi_t - \Delta\theta_{\mathrm{s}} - \varepsilon\Delta\left(\partial_t\theta_{\mathrm{s}}\right) = \chi(\theta-\theta_{\mathrm{s}}) \quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),$$
$$\partial_n\theta_{\mathrm{s}} = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T)$$

First step of the proof: approximation

- **Main difficulty:** lack of regularity for θ (presence of $\log\theta$ and boundary conditions)

$$\varepsilon\partial_t\theta+\partial_t(\log_\varepsilon\theta) - \mathrm{div}\,\mathbf{u}_t - \Delta\theta-\varepsilon\Delta\left(\partial_t\theta\right) = h \quad \text{in } \Omega\times(0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_{\mathrm{Cont}}\times(0,T) \\ -\chi(\theta-\theta_{\mathrm{s}}) & \text{on } \Gamma_{\mathrm{Cont}}\times(0,T) \end{cases}$$

where $\log_\varepsilon$ denotes a **Yosida-type** approximation of log

- Analogously for θ_s:

$$\varepsilon\partial_t\theta_{\mathrm{s}}+\partial_t(\log_\varepsilon\theta_{\mathrm{s}}) - \lambda'(\chi)\chi_t - \Delta\theta_{\mathrm{s}}-\varepsilon\Delta\left(\partial_t\theta_{\mathrm{s}}\right) = \chi(\theta-\theta_{\mathrm{s}}) \quad \text{in } \Gamma_{\mathrm{Cont}}\times(0,T),$$
$$\partial_n\theta_{\mathrm{s}} = 0 \quad \text{on } \partial\Gamma_{\mathrm{Cont}}\times(0,T)$$

- we do not approximate $\partial I_{[0,1]}(\chi)$, $\partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})$ in the eq.'s for $\mathbf{u}$ and χ

Outline of the proof

Existence:

- approximating procedure

Outline of the proof

Existence:

- **approximating** procedure
- **Schauder theorem** to solve the approximated problem
  - theory of evolution equations with **maximal monotone operators** for the equations for θ and θ_s
  - use previous results of **[Bonetti, Bonfanti, R. '07, '08]** for the system in $\mathbf{u}$ and χ

Outline of the proof

Existence:

- **approximating** procedure
- **Schauder theorem** to solve the approximated problem
  - theory of evolution equations with **maximal monotone operators** for the equations for θ and θ_s
  - use previous results of **[Bonetti, Bonfanti, R. '07, '08]** for the system in $\mathbf{u}$ and χ
- uniform **a priori estimates** (... some cancellations work, positivity of χ...)

Outline of the proof

Existence:

- **approximating** procedure
- **Schauder theorem** to solve the approximated problem
  - theory of evolution equations with **maximal monotone operators** for the equations for θ and θ_{s}
  - use previous results of **[Bonetti, Bonfanti, R. '07, '08]** for the system in $\mathbf{u}$ and χ
- uniform **a priori estimates** (... some cancellations work, positivity of χ...)
- passage to the limit by **compactness and semicontinuity** arguments

Outline of the proof

Existence:

- **approximating** procedure
- **Schauder theorem** to solve the approximated problem
  - theory of evolution equations with **maximal monotone operators** for the equations for θ and θ_s
  - use previous results of **[Bonetti, Bonfanti, R. '07, '08]** for the system in $\mathbf{u}$ and χ
- uniform **a priori estimates** (... some cancellations work, positivity of χ...)
- passage to the limit by **compactness and semicontinuity** arguments

Uniqueness?

- **NOT expected** due to the nonlinear structure of the equations, the lack of regularity of θ and the boundary conditions
- holds for the **approximating problem**

Trajectories on $(0, +\infty)$

There exists a solution in $(0, T)$ for any T: we can suitably define the notion of solution on $(0, +\infty)$

Trajectories on $(0, +\infty)$

There exists a solution in $(0, T)$ for any T: we can suitably define the notion of solution on $(0, +\infty)$

How the trajectories $(\mathbf{u}(t), \chi(t), \theta(t), \theta_s(t))$ behave as $t \to +\infty$?

Trajectories on $(0, +\infty)$

There exists a solution in $(0, T)$ for any T: we can suitably define the notion of solution on $(0, +\infty)$

How the trajectories $(\mathbf{u}(t), \chi(t), \theta(t), \theta_s(t))$ behave as $t \to +\infty$?

- need **uniform estimates** on the solutions independent of the final time T
- **further requirement** on the entropy flux through Γ_c

$$\partial_n \theta = -(\chi + c)(\theta_{|\Gamma_c} - \theta_s), \quad c > 0 \text{ on } \Gamma_c$$

$\rightsquigarrow$ residual flux even if $\chi = 0$
(crucial to obtain a L^2-estimate for $\theta_{|\Gamma_c} - \theta_s$)

Long-time a priori estimates

Due to the **dissipative character of the system**

$$
\begin{aligned}
&|\theta|_{L^\infty(1,+\infty;H^1(\Omega))} + |\nabla\theta|_{L^2(0,+\infty;L^2(\Omega))} \leq C \\
&|\theta_t|_{L^2(1,+\infty;L^p(\Omega))} \leq C \quad \text{for } p \leq 12/7 \\
&|\theta_s|_{L^\infty(1,+\infty;H^1(\Gamma_{\mathrm{Cont}}))} + |\nabla\theta_s|_{L^2(0,+\infty;L^2(\Gamma_{\mathrm{Cont}}))} \leq C \\
&|\partial_t\theta_s|_{L^2(1,+\infty;L^q(\Gamma_{\mathrm{Cont}}))} \leq C, \quad \text{for } q < 2 \\
&|\theta_{|\Gamma_{\mathrm{Cont}}} - \theta_s|_{L^2(0,+\infty;L^2(\Gamma_{\mathrm{Cont}}))} \leq C \\
&|\chi|_{L^\infty(1,+\infty;H^2(\Gamma_{\mathrm{Cont}}))} + |\chi_t|_{L^2(1,+\infty;H^1(\Gamma_{\mathrm{Cont}}))} \leq C \\
&|\mathbf{u}|_{L^\infty(0,+\infty;H^1(\Omega)^3)} + |\mathbf{u}_t|_{L^2(0,+\infty;H^1(\Omega)^3)} \leq C \\
&|\partial_t(\log\theta)|_{L^2(0,+\infty;H^1(\Omega)')} + |\partial_t(\log\theta_s)|_{L^2(0,+\infty;H^1(\Gamma_{\mathrm{Cont}})')} \leq C
\end{aligned}
$$

$\rightsquigarrow$ the **"energy"** and the **"dissipation"** are **uniformly bounded**

$\rightsquigarrow$ the solutions trajectories **converge** in a suitable sense to some cluster points as $t\infty$

The ω-limit set

The set of the **possible cluster points** $(\mathbf{u}_\infty, \chi_\infty, \theta_\infty, \theta_{s\infty})$ of the solutions trajectories

- is non-empty, connected and compact (w.r.t. to a suitable topology)

The ω-limit set

The set of the **possible cluster points** $(\mathbf{u}_\infty, \chi_\infty, \theta_\infty, \theta_{s\infty})$ of the solutions trajectories

- is non-empty, connected and compact (w.r.t. to a suitable topology)
- its elements $(\mathbf{u}_\infty, \chi_\infty, \theta_\infty, \theta_{s\infty})$ solve the **stationary problem**

$$\mathrm{div}(K\varepsilon(\mathbf{u}_\infty) + \theta_\infty \mathbf{1}) = \mathbf{f}_\infty \text{ in } \Omega \quad + \text{boundary conditions}$$

$$-\Delta\chi_\infty + \partial I_{[0,1]}(\chi_\infty) \ni -\lambda'(\chi_\infty)(\theta_{s\infty} - \theta_{eq}) - \frac{1}{2}|\mathbf{u}_\infty|^2 \text{ in } \Gamma_{\mathrm{Cont}} \quad + \text{b.c.}$$

$$\exists\ \bar{\theta}_\infty \geq 0 : \theta_\infty \equiv \bar{\theta}_\infty \text{ in } \Omega\,, \quad \theta_{s\infty} \equiv \bar{\theta}_\infty \text{ in } \Gamma_{\mathrm{Cont}}$$

The ω-limit set

The set of the **possible cluster points** $(\mathbf{u}_\infty, \chi_\infty, \theta_\infty, \theta_{s\infty})$ of the solutions trajectories

- is non-empty, connected and compact (w.r.t. to a suitable topology)
- its elements $(\mathbf{u}_\infty, \chi_\infty, \theta_\infty, \theta_{s\infty})$ solve the **stationary problem**

$$\mathrm{div}(K\varepsilon(\mathbf{u}_\infty) + \theta_\infty \mathbf{1}) = \mathbf{f}_\infty \text{ in } \Omega \quad + \text{boundary conditions}$$

$$-\Delta\chi_\infty + \partial I_{[0,1]}(\chi_\infty) \ni -\lambda'(\chi_\infty)(\theta_{s\infty} - \theta_{eq}) - \frac{1}{2}|\mathbf{u}_\infty|^2 \text{ in } \Gamma_{\mathrm{Cont}} \quad + \text{b.c.}$$

$$\exists\ \bar{\theta}_\infty \geq 0 : \theta_\infty \equiv \bar{\theta}_\infty \text{ in } \Omega\,, \quad \theta_{s\infty} \equiv \bar{\theta}_\infty \text{ in } \Gamma_{\mathrm{Cont}}$$

- thermomechanical equilibrium (**no dissipation**) in the limit $t \to \infty$.