On the Cahn-Hilliard equation with a chemical potential dependent mobility

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joint work with
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The equation

We consider the

generalized (viscous) Cahn-Hilliard equation:

$$\chi_t - \Delta(\alpha(\delta\chi_t - \Delta\chi + \phi(\chi))) = 0 \quad \text{in } \Omega \times (0, T),$$

- $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, a bdd smooth domain, $(0, T)$ a time interval;
- $\alpha : D(\alpha) \subset \mathbb{R} \to \mathbb{R}$ is strictly increasing and differentiable;
- $\delta \geq 0$ a constant;
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Both for $\delta = 0$ and $\delta > 0$: wide literature on **well-posedness** (for various variants of the model), **long-time behaviour**, dynamics of pattern formation.

Gurtin's generalized Cahn-Hilliard equation

- **M.E. Gurtin** **[Phys. D '96]** proposed a novel derivation of the Cahn-Hilliard equations, thus obtaining the

generalized viscous Cahn-Hilliard equation

$$\begin{cases} \chi_t - \mathrm{div}(\mathbf{M}(Z)\nabla w) = 0 \\ w = \delta(Z)\chi_t - \Delta\chi + \phi(\chi) \end{cases} \qquad \text{(GVCHE)}$$

- w **chemical potential**
- $\mathbf{M}$ **mobility tensor** (symmetric, positive definite)
- $\mathbf{M} = \mathbf{M}(Z)$, $\delta = \delta(Z)$, with

constitutive variables: $Z = (\chi, \nabla\chi, \chi_t, w, \nabla w)$!

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- Several results **[Miranville & Bonfoh, Carrive, Cherfils, Grasselli, Piétrus, Rakotoson, Rougirel, Schimperna, Zelik..]**: well-posedness and long-time behaviour for variants of (GVCHE) (also in the **anisotropic** case) with periodic and Neumann b.c., and $\mathbf{M}(Z) = \mathbf{M}$, $\mathbf{M}(\chi)$, constant δ.

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- Well-posedness and long-time behaviour for the standard Cahn-Hilliard eq. (viscous and non-viscous), with a **concentration**-dependent mobility tensor: **[Barrett, Blowey, Bonetti, Colli, Dreyer, Gilardi, Elliott, Novick-Cohen, Garcke, Schimperna, Sprekels..]**.

A chemical potential dependent mobility

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is a particular case of (GVCHE):

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- **no-flux boundary** conditions for χ and w: **mass conservation** for χ
- **[R.'05 & '06]** well-posedness and long-time behaviour results for **two different boundary value problems** corresponding to **two choices** of the **mobility law** α, in the case $\phi(\chi) = \chi^3 - \chi$.

Analytical difficulties

Decouple the system

$$\begin{cases} \chi_t - \Delta\alpha(w) = 0 \\ \delta\chi_t - \Delta\chi + \phi(\chi) = w \end{cases} \quad \text{in } \Omega \times (0, T)$$

$+$ no flux boundary conditions

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Aim: **generalize** the choices for α and ϕ in **[R.'05 & '06]**

Outlook

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$$\alpha : \mathbb{R} \to \mathbb{R} \text{ strictly increasing, differentiable,}$$

$$\exists\, p \geq 0, \quad \exists\, C_1,\, C_2 > 0 : \quad \forall\, r \in \mathbb{R} \qquad C_1\left(|r|^{2p} + 1\right) \leq \alpha'(r) \leq C_2\left(|r|^{2p} + 1\right);$$

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- A priori estimates and existence result for $\delta = 0$ and $\delta > 0$
- Global attractor for $\delta > 0$
- Uniqueness, regularizing effect, and exponential attractor for $\delta > 0$

A priori estimates (I)

Assumptions:

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p} \quad |\phi(r)| \leq C\left(\widehat{\phi}(r)+1\right), \quad \phi'(r) \geq -C$$

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$$(\chi_t + A\alpha(w) = 0) \quad \times \mathbf{w}$$
$$(\boldsymbol{\delta}\chi_t + A\chi + \phi(\chi) = w) \quad \times \chi_t$$

Energy estimate:

$$\int_0^t \int_\Omega \alpha'(w)|\nabla w|^2 + \boldsymbol{\delta} \int_0^t \|\chi_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\chi(t)\|^2_{L^2(\Omega)} + \int_\Omega \widehat{\phi}(\chi(t)) = \frac{1}{2}\|\nabla\chi(0)\|^2_{L^2(\Omega)} + \int_\Omega \widehat{\phi}(\chi(0))$$

whence

$$\boldsymbol{\delta}^{1/2}\|\chi_t\|_{L^2(0,T;L^2(\Omega))} + \|\nabla w\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\chi\|_{L^\infty(0,T;L^2(\Omega))} + \|\widehat{\phi}(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \leq C$$

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$$\begin{cases} \frac{1}{|\Omega|}\int_\Omega \chi_t = 0 \;\Rightarrow\; \frac{1}{|\Omega|}\int_\Omega \chi(t) = \frac{1}{|\Omega|}\int_\Omega \chi(0) \\ \frac{1}{|\Omega|}\int_\Omega \phi(\chi(t)) \equiv \frac{1}{|\Omega|}\int_\Omega w(t) \end{cases}$$

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Full estimate for χ:

$$\left(\|\nabla\chi\|_{L^\infty(0,T;L^2(\Omega))} + |m(\chi(t))|\right) \ \Rightarrow \ \|\chi\|_{L^\infty(0,T;H^1(\Omega))}$$

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Full estimate for w: need estimate for $|m(w)| = |m(\phi(\chi))|$. Use $|\phi(\chi)| \le C(\widehat{\phi}(\chi)+1)$, hence

$$\|\widehat{\phi}(\chi)\|_{L^\infty(0,T;L^1(\Omega)))} \le C \Rightarrow \|m(w)\|_{L^\infty(0,T)} = \|m(\phi(\chi)\|_{L^\infty(0,T)} \le C$$
$$\Rightarrow \|w\|_{L^2(0,T;H^1(\Omega))} \le C.$$

A priori estimates (II)

Assumptions:

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p} \quad |\phi(r)| \leq C\left(\widehat{\phi}(r) + 1\right), \quad \phi'(r) \geq -C$$

PDE system:

$$\chi_t + A\alpha(w) = 0$$
$$\delta\chi_t + A\chi + \phi(\chi) = w$$

Known estimates:

$$\begin{cases} \delta^{1/2}\|\chi_t\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;H^1(\Omega))} + \|\chi\|_{L^\infty(0,T;H^1(\Omega))} + \|\widehat{\phi}(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\ \|m(w)\|_{L^\infty(0,T)} \leq C \end{cases}$$

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Elliptic regularity estimate: from $\phi'(r) \geq -C$, we have $\phi(\chi) = \overbrace{\beta(\chi)}^{\text{monotone}} + \overbrace{\sigma(\chi)}^{\text{Lipschitz}}$

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PDE system:

$$\chi_t + A\alpha(w) = 0$$
$$\delta\chi_t + A\chi + \beta(\chi) = w - \sigma(\chi)$$

Known estimates:

$$\begin{cases} \delta^{1/2}\|\chi_t\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;H^1(\Omega))} + \|\chi\|_{L^\infty(0,T;H^1(\Omega))} + \|\widehat{\phi}(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\ \|m(w)\|_{L^\infty(0,T)} \leq C \end{cases}$$

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A priori estimates (III)

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Estimate for χ_t in the case $\delta = 0$: estimate χ_t arguing by comparison. Need estimate for $\alpha(w) \approx w^{2p+1}$.

$$\left.\begin{array}{r}\text{Energy estimate } \int_0^t \int_\Omega \alpha'(w)|\nabla w|^2 \leq C \\ \alpha'(w) \approx w^{2p}\end{array}\right\} \Rightarrow \|\nabla(w^{p+1})\|_{L^2(0,T;L^2(\Omega))} \leq C$$

Since $\|m(w)\|_{L^\infty(0,T)} \leq C$, we have $\|w^{p+1}\|_{L^2(0,T;H^1(\Omega))} \leq C$, hence $\|w^{p+1}\|_{L^2(0,T;L^6(\Omega))} \leq C$, hence

$$\|\alpha(w)\|_{L^{\rho_p}(0,T;L^{\kappa_p}(\Omega))} \leq C, \quad \text{with } \rho_p = \frac{2p+2}{2p+1}, \quad \kappa_p = \frac{6p+6}{2p+1}.$$

A priori estimates (IV)

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From a comparison we thus have

$$\|\chi_t\|_{L^{\rho_p}(0,T;W^{-2,\kappa_p}(\Omega))} \le C, \quad \rho_p > 1!!$$

The case $\delta > 0$: passage to the limit

$$\chi_t + A\alpha(w) = 0$$
$$\delta\chi_t + A\chi + \phi(\chi) = w$$

A priori estimates for $\delta > 0$:

$$\begin{cases} \|\chi_t\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;H^1(\Omega))} + \|\chi\|_{L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} \leq C, \\ \|\widehat{\phi}(\chi)\|_{L^\infty(0,T;L^1(\Omega))} + \|\phi(\chi)\|_{L^2(0,T;L^2(\Omega))} + \|\alpha(w)\|_{L^{\rho p}(0,T;L^{\kappa p}(\Omega))} \leq C \end{cases}$$

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Approximate problem:

$$\chi_t + A(\alpha_m(w)) = 0 \qquad \alpha_m \text{ truncation of } \alpha$$
$$\delta\chi_t + A\chi + \phi_\mu(\chi) = w \qquad \phi_\mu \text{ truncation of } \phi$$

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$$\chi_t + A\alpha(w) = 0$$
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In the passage to the limit as $m \to \infty$ and $\mu \to \infty$, identification of the **weak** limit of $\alpha(w_{m,\mu})$ via a monotonicity argument.

The case $\delta > 0$: existence result

Theorem I

Under assumptions

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p}, \quad |\phi(r)| \leq C\left(\widehat{\phi}(r) + 1\right), \quad \phi'(r) \geq -C$$

there exists a solution (χ, w) to the (Cauchy problem) for

$$\chi_t + A\alpha(w) = 0 \quad \text{a.e. in } \Omega \times (0, T)$$
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fulfilling the **energy identity** for all $0 \leq s \leq t \leq T$

$$\delta \int_s^t \int_\Omega |\chi_t|^2 + \int_s^t \int_\Omega \alpha'(w)|\nabla w|^2 + \overbrace{\frac{1}{2}\int_\Omega |\nabla \chi(t)|^2 + \int_\Omega \widehat{\phi}(\chi(t))}^{\mathcal{E}(\chi(t))}$$
$$= \overbrace{\frac{1}{2}\int_\Omega |\nabla \chi(s)|^2 + \int_\Omega \widehat{\phi}(\chi(s))}^{\mathcal{E}(\chi(s))}.$$

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The case $\delta = 0$: Existence result

A priori estimates for $\boldsymbol{\delta} = 0$:

$$\begin{cases} \|w\|_{L^2(0,T;H^1(\Omega))} + \|\chi\|_{L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} + \|\widehat{\phi}(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\ \|\chi_t\|_{L^{\rho_p}(0,T;W^{-2,\kappa_p}(\Omega))} + \|\alpha(w)\|_{L^{\rho_p}(0,T;L^{\kappa_p}(\Omega))} \leq C \end{cases}$$

Passage to the limit as $\boldsymbol{\delta} \searrow 0$ and $m \to \infty$, $\mu \to \infty$ in

$$\chi_t + A(\alpha_m(w)) = 0 \qquad \alpha_m \text{ **truncation of** } \alpha$$

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Theorem II

Under assumptions

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p}, \quad |\phi(r)| \leq C\left(\widehat{\phi}(r) + 1\right), \quad \phi'(r) \geq -C$$

$\exists$ a solution (χ, w) to the (Cauchy problem) for the **weak formulation**

$$_{W^{-2,\kappa_p}}\langle \chi_t, v \rangle_{W^{2,\kappa_p'}} + {}_{W^{-2,\kappa_p}}\langle A(\alpha(w)), v \rangle_{W^{2,\kappa_p'}} = 0 \quad \text{for all } v \in W^{2,\kappa_p'}(\Omega) \text{ a.e. in } (0,T),$$

$$A\chi + \phi(\chi) = w \qquad \text{a.e. in } \Omega \times (0,T).$$

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- **Very weak** formulation: it's not possible to prove energy identity.

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- **Very weak** formulation: it's not possible to prove energy identity.
- Conditions on ϕ might be slightly weakened.

Back to the viscous problem

Existence of a solution (χ, w) to

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fulfilling the **energy identity** for all $0 \le s \le t \le T$

$$\delta \int_s^t \int_\Omega |\chi_t|^2 + \int_s^t \int_\Omega \alpha'(w)|\nabla w|^2 + \mathcal{E}(\chi(t)) = \mathcal{E}(\chi(s)).$$

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- **Various possibilities:** **[Sell '73,'96], [Chepyzhov & Vishik '02], [Melnik & Valero '02], [Ball '97,'04]**

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We have used Ball's theory of **generalized semiflows**.

Generalized Semiflows: definition

Phase space: a metric space $(\mathcal{X}, d_{\mathcal{X}})$

A **generalized semiflow** $\mathcal{S}$ on $\mathcal{X}$ is a family of maps $g : [0, +\infty) \to \mathcal{X}$ (**"solutions"**), s. t.

$\forall\, g_0 \in \mathcal{X}\ \exists$ **at least one** $g \in \mathcal{S}$ with $g(0) = g_0$ **(Existence)**

$\forall\, g \in \mathcal{S}$ and $\tau \geq 0$, the map $g^\tau(\cdot) := g(\cdot + \tau) \in \mathcal{S}$ **(Translation invar.)**

$\forall\, g,\ h \in \mathcal{S}$ and $t \geq 0$ with $h(0) = g(t)$, then $z \in \mathcal{S}$, where
$$z(\tau) := \begin{cases} g(\tau) & \text{if } 0 \leq \tau \leq t, \\ h(\tau - t) & \text{if } t < \tau, \end{cases}$$
(Concatenation)

If $\{g_n\} \subset \mathcal{S}$ and $g_n(0) \to g_0$, $\exists$ subsequence $\{g_{n_k}\}$ and $g \in \mathcal{S}$ s.t. $g(0) = g_0$ and $g_{n_k}(t) \to g(t)$ for all $t \geq 0$. **(U.s.c. w.r.t. init. data)**

Global attractor for generalized semiflows

Definition

A set $\mathcal{A} \subset \mathcal{X}$ is a **global attractor** for a **generalized semiflow** $\mathcal{S}$ if:

- $\mathcal{A}$ is **compact**
- $\mathcal{A}$ is **invariant** under the semiflow
- $\mathcal{A}$ **attracts** the **bounded** sets of $\mathcal{X}$ (w.r.t. the Hausdorff semidistance of $\mathcal{X}$)

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Theorem (Ball'97)

Assume that

- $\mathcal{S}$ is **asymptotically compact**, i.e. for all $\{g_n\} \subset \mathcal{S}$ with $\{g_n(0)\}$ bounded and for all $t_n \to \infty$, $\exists$ a converging subsequence $\{g_{n_k}(t_{n_k})\}$;
- $\mathcal{S}$ has a **Lyapunov functional**;
- the set of the **stationary points** of $\mathcal{S}$ is bounded in $(\mathcal{X}, d_{\mathcal{X}})$.

Then, $\mathcal{S}$ has a (unique) global attractor $\mathcal{A}$.

The case $\delta > 0$: set-up for the long-time analysis

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$$\boldsymbol{\delta}\chi_t + A\chi + \phi(\chi) = w \quad \text{a.e. in } \Omega \times (0, T)$$

fulfilling the **energy identity** for all $0 \leq s \leq t \leq T$

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The case $\delta > 0$: set-up for the long-time analysis

Phase space:

$$\mathcal{X} = D(\mathcal{E}) = \{\chi \in H^1(\Omega) \ : \ \widehat{\phi}(\chi) \in L^1(\Omega)\}$$

$$d_{\mathcal{X}}(\chi_1, \chi_2) := \|\chi_1 - \chi_2\|_{H^1(\Omega)} + \|\widehat{\phi}(\chi_1) - \widehat{\phi}(\chi_2)\|_{L^1(\Omega)}$$

Existence of a solution (χ, w) to

$$\begin{aligned} &\chi_t + A\alpha(w) = 0 \quad \text{a.e. in } \Omega \times (0, T) \\ &\boldsymbol{\delta}\chi_t + A\chi + \phi(\chi) = w \quad \text{a.e. in } \Omega \times (0, T) \end{aligned}$$

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Solution notion: $\mathcal{S}$ is the set of all χ's s.t. $\exists\, w$ with

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and **energy identity** for all $0 \leq s \leq t \leq T$

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The case $\delta > 0$: long-time analysis

Facts

Under assumptions

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p}, \quad |\phi(r)| \leq C\left(\widehat{\phi}(r) + 1\right), \quad \phi'(r) \geq -C$$

$\exists\, C_\alpha$ s.t. the map $w \mapsto \alpha(w)w - C_\alpha |r|^{2p+2}$ is convex

then

- $\mathcal{S}$ is a **generalized semiflow** (proof of upper semicontinuity uses the technical assumption on α to pass to the limit in the energy identity)

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then

- $\mathcal{S}$ is a **generalized semiflow** (proof of upper semicontinuity uses the technical assumption on α to pass to the limit in the energy identity)
- $\mathcal{S}$ is **asymptotically compact**
- the energy $\mathcal{E}$ is a **Lyapunov function** for $\mathcal{S}$.

The case $\delta > 0$: existence of the global attractor

Theorem (III)

Under assumptions

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$$\lim_{r\to+\infty} \phi'(r) = \lim_{r\to-\infty} \phi'(r) = +\infty$$

$\mathcal{S}$ admits a **global attractor**.

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$\mathcal{S}$ admits a **global attractor**.

Ideas of the proof

Assumptions:

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p} \quad |\phi(r)| \leq C\left(\widehat{\phi}(r) + 1\right),$$

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Proof: show boundedness in $(\mathcal{X}, d_{\mathcal{X}})$ of set of stationary points, i.e. sol.'s of

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such that

$$m(\bar{\chi}) \leq m_0.$$

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$$\int_\Omega |\phi(\bar{\chi})| \leq C_{1,m_0} \int_\Omega \phi(\bar{\chi})(\bar{\chi} - m(\bar{\chi})) + C_{2,m_0}$$

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We prove $\|\widehat{\phi}(\bar{\chi})\|_{L^1(\Omega)} \leq C$, hence $\bar{\chi}$ is in a bounded set in the phase space $(\mathcal{X}, d_{\mathcal{X}})$.

Enhanced regularity and uniqueness under more restrictive conditions

Theorem (IV)

Under assumptions

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p}, \quad \text{for } p \in [0,1]$$
$$|\phi(r)| \leq C\left(\widehat{\phi}(r) + 1\right),$$
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- **enhanced regularity** of solutions

$$\chi \in L^\infty(\tau, T; H^2(\Omega)) \cap H^1(\tau, T; H^1(\Omega)) \quad \text{for all } \tau > 0$$

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- **uniqueness** from initial conditions $\chi_0 \in H^2(\Omega)$: semiflow $\rightsquigarrow$ semigroup on $H^1(\Omega)$

Exponential attractors

Exponential attractor [Eden-Foias-Nicolaenko-Temam'94]

A set $\mathcal{M} \subset H^1(\Omega)$ is an **exponential attractor** for a **semigroup** $\mathcal{S}$ if:

- $\mathcal{M}$ is **compact**
- $\mathcal{M}$ has **finite fractal dimension**
- $\mathcal{M}$ is **POSITIVELY invariant** under the semigroup
- $\mathcal{M}$ **attracts** the **bounded** sets of $H^1(\Omega)$ **EXPONENTIALLY fast**

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Exponential attractor for $\delta > 0$ under more restrictive conditions

Theorem (V)

Assume

$$\alpha \text{ increasing}, \quad \alpha'(r) \approx |r|^{2p}, \quad \text{for } p \in [0,1]$$

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then the dynamical system $(H^1(\Omega), \mathcal{S})$ has an **exponential attractor $\mathcal{M}$.**

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then the dynamical system $(H^1(\Omega), \mathcal{S})$ has an **exponential attractor $\mathcal{M}$.**

Proof: uses the method of ℓ-trajectories **[Málek-Pražák'02]**.