Some results on the vanishing viscosity approximation of rate-independent problems

Riccarda Rossi
(Università di Brescia)

joint work (in progress) with
Alexander Mielke (WIAS & Humboldt-Universität – Berlin),
Giuseppe Savaré (Università di Pavia),

Modèles mathématiques en science des matériaux

Poitiers, 12.06.2008

Doubly nonlinear evolution equations

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T), \qquad \text{(DNE)}$$

- B is a (separable) Banach space;
- $\Psi : B \to [0,+\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0,T] \times B \to (-\infty,+\infty]$ is **smooth** w.r.t. $t \in (0,T)$
- ∂_u is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**:

Doubly nonlinear evolution equations

$$\partial\Psi(u'(t)) + \mathrm{D}\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T), \qquad \text{(DNE)}$$

- B is a (separable) Banach space;
- $\Psi : B \to [0,+\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0,T] \times B \to (-\infty,+\infty]$ is **smooth** w.r.t. $t \in (0,T)$
- ∂_u is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**: $\mathrm{D}\mathcal{E}$ if $u \mapsto \mathcal{E}(t,u)$ is **smooth**,

Doubly nonlinear evolution equations

$$\partial\Psi(u'(t)) + \partial\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T), \tag{DNE}$$

- B is a (separable) Banach space;
- $\Psi : B \to [0,+\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0,T] \times B \to (-\infty,+\infty]$ is **smooth** w.r.t. $t \in (0,T)$
- ∂_u is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**: $\mathrm{D}\mathcal{E}$ if $u \mapsto \mathcal{E}(t,u)$ is **smooth**, $\partial\mathcal{E}$ if $u \mapsto \mathcal{E}(t,u)$ is **convex and l.s.c.**

Doubly nonlinear evolution equations

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T), \qquad \text{(DNE)}$$

- B is a (separable) Banach space;
- $\Psi : B \to [0,+\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ∂ **convex analysis** subdifferential;
- $\mathcal{E} : [0,T] \times B \to (-\infty,+\infty]$ is **smooth** w.r.t. $t \in (0,T)$
- ∂_u is the "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**: $\mathrm{D}\mathcal{E}$ if $u \mapsto \mathcal{E}(t,u)$ is **smooth**, $\partial\mathcal{E}$ if $u \mapsto \mathcal{E}(t,u)$ is **convex and l.s.c.**

Physical interpretation

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \tag{DNE}$$

is a generalized **balance law** in **Thermomechanics**:

- $\Psi \sim$ **dissipation** potential
- $\mathcal{E} \sim$ **energy** functional $(\mathcal{E}(\cdot, u) \sim$ (power of) **external forces**)

Physical interpretation

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T), \qquad \text{(DNE)}$$

is a generalized **balance law** in **Thermomechanics**:

- Ψ $\sim$ **dissipation** potential
- $\mathcal{E}$ $\sim$ **energy** functional $\;(\mathcal{E}(\cdot,u) \sim$ (power of) **external forces**)

Two cases:

- Ψ has **superlinear growth** $\leftrightarrow$ dissipation with **viscosity** effects

$$\lim_{\|v\|\to+\infty} \frac{\Psi(v)}{\|v\|} = +\infty$$

Physical interpretation

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T), \qquad \text{(DNE)}$$

is a generalized **balance law** in **Thermomechanics**:

- Ψ $\sim$ **dissipation** potential
- $\mathcal{E}$ $\sim$ **energy** functional $(\mathcal{E}(\cdot,u) \sim$ (power of) **external forces**)

Two cases:

- Ψ has **superlinear growth** $\leftrightarrow$ dissipation with **viscosity** effects

$$\lim_{\|v\|\to+\infty} \frac{\Psi(v)}{\|v\|} = +\infty$$

- Ψ has **linear growth** and is **positively 1-homogeneous**

$$\Psi(\lambda v) = \lambda\Psi(v) \qquad \forall\,\lambda \geq 0\ \forall\, v \in B$$

$\leftrightarrow$ **rate-independent** models

The superlinear case

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \qquad \text{(DNE)}$$

Ψ with **superlinear growth**

Applications

- elasto-visco-plasticity...
- phase transitions..
- ...

Existence results

$\mathcal{E}$ **convex** ($\mathcal{E}$ C^1 perturbation of a convex functional), B **reflexive**:

- existence & approximation of solutions: [Barbu '75], [Arai '79], [Semba '86], [Colli-Visintin '90], [Colli '92]

The superlinear case

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \qquad \text{(DNE)}$$

Ψ with **superlinear growth**

Applications

- elasto-visco-plasticity...
- phase transitions..
- ...

Existence results

$\mathcal{E}$ **convex** ($\mathcal{E}$ C^1 perturbation of a convex functional), B **reflexive**:

- existence & approximation of solutions: [Barbu '75], [Arai '79], [Semba '86], [Colli-Visintin '90], [Colli '92]
- **superlinear growth** of Ψ gives control of u'

The superlinear case

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \qquad \text{(DNE)}$$

Ψ with **superlinear growth**

Applications

- elasto-visco-plasticity...
- phase transitions..
- ...

Existence results

$\mathcal{E}$ **convex** ($\mathcal{E}$ C^1 perturbation of a convex functional), B **reflexive**:

- existence & approximation of solutions: [Barbu '75], [Arai '79], [Semba '86], [Colli-Visintin '90], [Colli '92]
- **superlinear growth** of Ψ gives control of u' $\Rightarrow$ at least $u \in W^{1,1}(0, T; B)$

The rate-independent case

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \qquad \text{(DNE)}$$

Ψ **1-positively homogeneous**

Applications

Suitable choices of Ψ and $\mathcal{E}$ lead to applications in

1. **quasistatic solid-solid phase transformations (in SMA)**: [Mielke-Theil-Levitas '02], [Mielke-Roubíček '03]
2. **quasistatic elastoplasticity**: [Dal Maso-De Simone-Mora '06], [Dal Maso-De Simone-Mora-Morini '06], [Mielke et al. '02, '03, '04],..
3. **quasistatic crack propagation**: [Mainik-Mielke '04], [Dal Maso-Francfort-Toader '05] [Francfort-Mielke '05]...
4. **damage**: [Mielke-Roubíček '06]...
5. **delamination problems**: [Kočvara-Mielke-Roubíček '03]
6. **ferromagnetism, ferroelectricity**: [Mielke-Timofte '05]....

Rate-independence

The simplest ODE example:

$$\begin{cases} B = \mathbb{R}\,, \qquad \Psi(v) = |v| \quad \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \frac{1}{2}|u|^2 - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

($\ell \in \mathrm{C}^1([0,T])$ $\sim$ external loading)

Rate-independence

The simplest ODE example:

$$\begin{cases} B = \mathbb{R}\,, \qquad \Psi(v) = |v| \ \ \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \frac{1}{2}|u|^2 - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

($\ell \in \mathrm{C}^1([0,T]) \sim$ external loading)

$$\mathrm{Sign}(u'(t)) + u(t) \ni \ell(t), \quad t \in (0,T) \qquad \text{(ES)}$$

Rate-independence

The simplest ODE example:

$$\begin{cases} B = \mathbb{R}, \qquad \Psi(v) = |v| \ \ \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \frac{1}{2}|u|^2 - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

($\ell \in \mathrm{C}^1([0,T])$ $\sim$ external loading)

$$\mathrm{Sign}(u'(t)) + u(t) \ni \ell(t), \quad t \in (0,T) \qquad \text{(ES)}$$

Remark:

- **u is solution of (ES) if and only if $u \circ \alpha$ is solution of (ES)** for every strictly increasing reparametrization α

Rate-independence

The simplest ODE example:

$$\begin{cases} B = \mathbb{R}\,, \qquad \Psi(v) = |v| \quad \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \frac{1}{2}|u|^2 - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

($\ell \in \mathrm{C}^1([0,T])$ $\sim$ external loading)

$$\mathrm{Sign}(u'(t)) + u(t) \ni \ell(t), \quad t \in (0,T) \tag{ES}$$

Remark:

- **u is solution of (ES) if and only if $u \circ \alpha$ is solution of (ES)** for every strictly increasing reparametrization α
- The output u responds to the input ℓ invariantly for time rescalings, possibly with **hysteresis effects**

Two time scales

In general, in rate-independent systems:

$$\partial\Psi(u'(t)) + \partial_u \mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T) \tag{DNE}$$

is **invariant under time rescalings**

Two time scales

In general, in rate-independent systems:

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T) \tag{DNE}$$

is **invariant under time rescalings**

We are modelling systems with two **time scales**:

- a scale **intrinsic** to the system, **fast time scale**
- the **slow time scale** of the **external loading** $\sim \partial_t\mathcal{E}$ (**dominating scale**)

Two time scales

In general, in rate-independent systems:

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \qquad t \in (0, T) \qquad \text{(DNE)}$$

is **invariant under time rescalings**

We are modelling systems with two **time scales**:

- a scale **intrinsic** to the system, **fast time scale**
- the **slow time scale** of the **external loading** $\sim \partial_t\mathcal{E}$ (**dominating scale**)

viscous dissipation is negligible!

Two time scales

In general, in rate-independent systems:

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \qquad t \in (0, T) \qquad \text{(DNE)}$$

is **invariant under time rescalings**

We are modelling systems with two **time scales**:

- a scale **intrinsic** to the system, **fast time scale**
- the **slow time scale** of the **external loading** $\sim \partial_t\mathcal{E}$ (**dominating scale**)

viscous dissipation is negligible!

$$\mathrm{Sign}(u'(t)) + u(t) \ni \ell(t), \quad t \in (0, T) \qquad \text{(ES)}$$

Two time scales

In general, in rate-independent systems:

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T) \qquad \text{(DNE)}$$

is **invariant under time rescalings**

We are modelling systems with two **time scales**:

- a scale **intrinsic** to the system, **fast time scale**
- the **slow time scale** of the **external loading** $\sim \partial_t\mathcal{E}$ (**dominating scale**)

viscous dissipation is negligible!

$$\varepsilon u'(t) + \mathrm{Sign}(u'(t)) + u(t) \ni \ell(t), \quad t \in (0,T) \qquad \text{as } \varepsilon \downarrow 0 \qquad (\mathrm{ES}_\varepsilon)$$

Results in "good" Banach spaces, for convex energies

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad t \in (0,T), \qquad \text{(DNE)}$$

Ψ **1-positively homogeneous**

Existence, approximation results [Mielke-Theil'04, Mielke-R.'07] if:

- B is reflexive
- $\mathcal{E} \in \mathrm{C}^1([0,T] \times B)$

Results in “good” Banach spaces, for convex energies

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \tag{DNE}$$

Ψ **1-positively homogeneous**

Existence, approximation results [Mielke-Theil’04, Mielke-R.’07] if:

- B is reflexive
- $\mathcal{E} \in \mathrm{C}^1([0, T] \times B)$

Uniqueness, continuous dependence on the initial data [Mielke-Theil’04, Mielke-R.’07] if:

- B is reflexive
- $\mathcal{E} \in \mathrm{C}^1([0, T] \times B)$
- $u \mapsto \mathcal{E}(t, u)$ is uniformly convex

Non reflexive spaces, non-smooth energies, jumping solutions

In **general** rate-independent problems:

- B may be **non-reflexive** (e.g., L^1 in phase transitions in SMA),
- B need **not have a linear structure** (e.g., in crack propagation)

Non reflexive spaces, non-smooth energies, jumping solutions

In **general** rate-independent problems:

- B may be **non-reflexive** (e.g., L^1 in phase transitions in SMA),
- B need **not have a linear structure** (e.g., in crack propagation)
- $\mathcal{E}$ may be **non-smooth**
- $\mathcal{E}$ may be **non-convex** ($\Rightarrow$ NO UNIQUENESS!)

Non reflexive spaces, non-smooth energies, jumping solutions

In **general** rate-independent problems:

- B may be **non-reflexive** (e.g., L^1 in phase transitions in SMA),
- B need **not have a linear structure** (e.g., in crack propagation)
- $\mathcal{E}$ may be **non-smooth**
- $\mathcal{E}$ may be **non-convex** ($\Rightarrow$ NO UNIQUENESS!)
- Ψ has a **linear growth** at ∞ $\rightsquigarrow$

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \qquad t \in (0, T),$$

$\rightsquigarrow$ standard regularity for u is $u \in \mathrm{BV}(0, T; B)$ (u may have **jumps!!!**)

Mielke's Global Energetic formulation

Global Energetic solutions [Mielke-Theil'99,'04], [Mielke-Theil-Levitas'02]

$u : [0, T] \to B$ satisfying **global stability condition** & **energy balance**

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, z) + \mathcal{D}(u(t), z) \qquad \forall\, z \in B\,,$$

$$\mathcal{E}(t, u(t)) + \mathrm{Diss}_{\mathcal{D}}(u, [0, t]) = \mathcal{E}(t, u(0)) + \int_0^t \partial_t \mathcal{E}(r, u(r))\,\mathrm{d}r\,.$$

where

- $\mathcal{D}$ is a dissipation distance defined from Ψ
- $\mathrm{Diss}_{\mathcal{D}}$ is a global dissipation functional defined from Ψ

Mielke's Global Energetic formulation

Global Energetic solutions [Mielke-Theil'99,'04], [Mielke-Theil-Levitas'02]

$u : [0, T] \to B$ satisfying **global stability condition** & **energy balance**

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, z) + \mathcal{D}(u(t), z) \qquad \forall\, z \in B\,,$$

$$\mathcal{E}(t, u(t)) + \mathrm{Diss}_{\mathcal{D}}(u, [0, t]) = \mathcal{E}(t, u(0)) + \int_0^t \partial_t \mathcal{E}(r, u(r))\,\mathrm{d}r\,.$$

Pro's

- ✓ Completely **derivative-free** $\rightsquigarrow$ adaptable to more general ambient spaces (general topological spaces [Mainik-Mielke'05])
- ✓ **equivalence** with the **differential** formulation (DNE) if $\mathcal{E}$ **convex**

Mielke's Global Energetic formulation

Global Energetic solutions [Mielke-Theil'99,'04], [Mielke-Theil-Levitas'02]

$u : [0, T] \to B$ satisfying **global stability condition** & **energy balance**

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, z) + \mathcal{D}(u(t), z) \qquad \forall\, z \in B\,,$$

$$\mathcal{E}(t, u(t)) + \mathrm{Diss}_{\mathcal{D}}(u, [0, t]) = \mathcal{E}(t, u(0)) + \int_0^t \partial_t \mathcal{E}(r, u(r))\, \mathrm{d}r\,.$$

Pro's

- ✓ Completely **derivative-free** $\rightsquigarrow$ adaptable to more general ambient spaces (general topological spaces [Mainik-Mielke'05])
- ✓ **equivalence** with the **differential** formulation (DNE) if $\mathcal{E}$ **convex**

BUT (in the **non convex** case), **global stability** forces energetic solutions to **jump too early** to avoid energy losses

Bad Vs. Good jumps

The simplest non convex case

$$\begin{cases} B = \mathbb{R}, \qquad \Psi(v) = |v| \ \ \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \mathcal{W}(u) - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

- $\mathcal{W}$ **double well** potential
- $\ell \in \mathrm{C}^1([0,T]) \sim$ external loading

$$\mathrm{Sign}(u'(t)) + \mathcal{W}'(u(t)) \ni \ell(t), \quad t \in (0,T)$$

Bad Vs. Good jumps

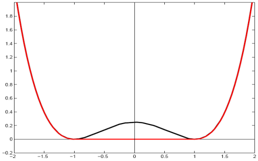
The simplest non convex case

$$\begin{cases} B = \mathbb{R}, \qquad \Psi(v) = |v| \quad \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \mathcal{W}(u) - \ell(t)u \qquad \forall\, (t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

- $\mathcal{W}$ **double well** potential
- $\ell \in \mathrm{C}^1([0,T]) \sim$ external loading

$$\mathrm{Sign}(u'(t)) + \mathcal{W}'(u(t)) \ni \ell(t), \quad t \in (0,T)$$

Convexification $\mathcal{W}^{**}$ of $\mathcal{W}$



Bad Vs. Good jumps

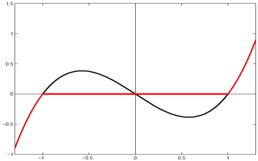
The simplest non convex case

$$\begin{cases} B = \mathbb{R}, \qquad \Psi(v) = |v| \ \ \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \mathcal{W}(u) - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

- $\mathcal{W}$ **double well** potential
- $\ell \in \mathrm{C}^1([0,T]) \sim$ external loading

$$\mathrm{Sign}(u'(t)) + \mathcal{W}'(u(t)) \ni \ell(t), \quad t \in (0,T)$$

Global solutions are given by $u(t) = (D\mathcal{W}^{**})^{-1}(\ell(t) - 1)$: jumping too early!



Bad Vs. Good jumps

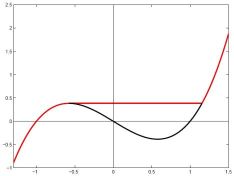
The simplest non convex case

$$\begin{cases} B = \mathbb{R}, \qquad \Psi(v) = |v| \quad \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \mathcal{W}(u) - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

- $\mathcal{W}$ **double well** potential
- $\ell \in \mathrm{C}^1([0,T]) \sim$ external loading

$$\mathrm{Sign}(u'(t)) + \mathcal{W}'(u(t)) \ni \ell(t), \quad t \in (0,T)$$

We aim to model the "right" hysteresis dynamics



The vanishing viscosity approach

Aims

- **model** ("natural") **jumps** (due to $u \in \mathrm{BV}(0, T; B)$)
- obtain **solutions jumping later** (than global energetic solutions)

The vanishing viscosity approach

Aims

- **model** ("natural") **jumps** (due to $u \in \mathrm{BV}(0, T; B)$)
- obtain **solutions jumping later** (than global energetic solutions)

Approach

Consider solutions arising as limits of viscous regularizations for **vanishing viscosity**: selection criterion for mechanically feasible jumps

The vanishing viscosity approach

Aims

- **model** ("natural") **jumps** (due to $u \in \mathrm{BV}(0, T; B)$)
- obtain **solutions jumping later** (than global energetic solutions)

Approach

Consider solutions arising as limits of viscous regularizations for **vanishing viscosity**: selection criterion for mechanically feasible jumps

Vanishing viscosity in the applications

- quasistatic evolution of fractures: [Toader-Zanini'06], [Cagnetti'07], [Cagnetti-Toader'07], [Knees-Mielke-Zanini'07], leading to **local stability**-oriented formulations: [Dal Maso-Toader'02], [Negri-Ortner'07], [Garroni-Larsen07] **(threshold evolutions in damage)**..
- plasticity with softening: [Dal Maso-DeSimone-Mora-Morini'06]

The vanishing viscosity analysis by Efendiev & Mielke

Problem

In the vanishing viscosity limit:

- local stability
- energy inequality

may not be enough for controlling jumps. ¿ Which further conditions better describe them?

The vanishing viscosity analysis by Efendiev & Mielke

Problem

In the vanishing viscosity limit:

- local stability
- energy inequality

may not be enough for controlling jumps. ¿ Which further conditions better describe them?

The approach by Efendiev-Mielke

- Jumps in the vanishing viscosity limit correspond to **viscous transitions** between stable states
- To capture the viscous transition path: **NOT SHRINK** jumps at a point, look at curves with their **arc length parametrization**
- Asymptotic analysis of (reparametrized) trajectories in an extended phase space

The vanishing viscosity analysis by Efendiev & Mielke

[Efendiev-Mielke, J. Convex Anal.'06]

Setting:

- B **finite dimensional** space
- $\mathcal{E} \in \mathrm{C}^1([0,T] \times B)$
- $\Psi(u) \sim \|u\| \quad \forall\, u \in B$

The vanishing viscosity analysis by Efendiev & Mielke

[Efendiev-Mielke, J. Convex Anal.'06]

Setting:

- B **finite dimensional** space
- $\mathcal{E} \in \mathrm{C}^1([0,T] \times B)$
- $\Psi(u) \sim \|u\| \quad \forall\, u \in B$

The **viscous regularization** of Ψ:

$$\Psi_\varepsilon(u) := \Psi(u) + \frac{\varepsilon}{2}\|u\|^2 \quad \forall\, \varepsilon > 0.$$

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be the family of solutions of the Cauchy problem

$$\begin{cases} \partial\Psi_\varepsilon(u_\varepsilon{}'(t)) + \mathrm{D}\mathcal{E}(t, u_\varepsilon(t)) \ni 0 & t \in (0,T), \\ u_\varepsilon(0) = u_0. \end{cases}$$

Problem: limit behaviour of $\{u_\varepsilon\}$ as $\varepsilon \searrow 0$

A rescaling technique

- **Arc length parametrization** of the graph $\{(t, u_\varepsilon(t)) \,:\, t \in [0,T]\}$:

$$s_\varepsilon(t) := t + \int_0^t \|u_\varepsilon{}'(s)\| \mathrm{d}s$$

$\{s_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(0,T)$: up to a subseq. $s_\varepsilon(T) \to \widehat{T}$.

- Introduce the rescaled functions

$$\widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad \forall s \in [0, s_\varepsilon(T)]$$

- From the **normalization condition**

$$\widehat{t}_\varepsilon'(s) + \|\widehat{u}_\varepsilon'(s)\| = 1 \quad \text{per q.o. } s \in (0, s_\varepsilon(T))$$

$\Rightarrow$ a priori estimates for $\{\widehat{t}_\varepsilon\}$, $\{\widehat{u}_\varepsilon\}$

- Ascoli-Arzelà + finite dimension

$$\widehat{t}_\varepsilon \to \widehat{t}, \quad \widehat{u}_\varepsilon \to \widehat{u} \quad \text{uniformly on } [0, \widehat{T}]$$

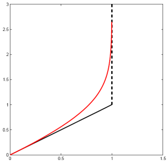
A rescaling technique

The **limit problem** solved by $(\widehat{t}, \widehat{u})$

$$\begin{cases} \partial\widehat{\Psi}(\widehat{u}'(s)) + \mathrm{D}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) \ni 0 & s \in (0, \widehat{T}) \\ \widehat{u}(0) = u_0, \quad \widehat{t}(0) = 0, \quad \widehat{t}(\widehat{T}) = T, \\ \widehat{t}'(s) + \|\widehat{u}'(s)\| = 1 & s \in (0, \widehat{T}) \end{cases}$$

where

$$\widehat{\Psi}(u') := \begin{cases} \Psi(u') & \|u'\| \leq 1, \\ +\infty & \|u'\| > 1 \end{cases}$$



Vanishing viscosity limit: sliding vs. viscous slips

$$\begin{cases} \partial\widehat{\Psi}(\widehat{u}'(s)) + \mathrm{D}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) \ni 0, \\ \widehat{u}(0) = u_0, \quad \widehat{t}(0) = 0, \quad \widehat{t}(\widehat{T}) = T, \\ \widehat{t}'(s) + \|\widehat{u}'(s)\| = 1 \end{cases}$$

$\widehat{\Psi}$ is NOT 1-homogeneous $\Rightarrow$ the problem is NOT rate-independent!

Vanishing viscosity limit: sliding vs. viscous slips

$$\begin{cases} \partial\widehat{\Psi}(\widehat{u}'(s)) + \mathrm{D}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) \ni 0, \\ \widehat{u}(0) = u_0, \quad \widehat{t}(0) = 0, \quad \widehat{t}(\widehat{T}) = T, \\ \widehat{t}'(s) + \|\widehat{u}'(s)\| = 1 \end{cases}$$

$\widehat{\Psi}$ is NOT 1-homogeneous $\Rightarrow$ the problem is NOT rate-independent!

"Sliding vs. viscous slips"

Three regimes

$\|\widehat{u}'(s)\| = 0 \Leftrightarrow \widehat{t}'(s) = 1$	STICKING
$0 < \|\widehat{u}'(s)\| < 1 \Leftrightarrow \widehat{t}'(s) \in (0, 1)$	SLIDING
$\|\widehat{u}'(s)\| = 1 \Leftrightarrow \widehat{t}'(s) = 0$	VISCOUS SLIP

Vanishing viscosity limit: sliding vs. viscous slips

$$\begin{cases} \partial\widehat{\Psi}(\widehat{u}'(s)) + \mathrm{D}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) \ni 0, \\ \widehat{u}(0) = u_0, \quad \widehat{t}(0) = 0, \quad \widehat{t}(\widehat{T}) = T, \\ \widehat{t}'(s) + \|\widehat{u}'(s)\| = 1 \end{cases}$$

$\widehat{\Psi}$ is NOT 1-homogeneous $\Rightarrow$ the problem is NOT rate-independent!

"Sliding vs. viscous slips"

Three regimes

1. for $\|\widehat{u}'(s)\| = 0$ the system is stationary
2. for $0 < \|\widehat{u}'(s)\| < 1$ the system is driven by rate-independent dissipation: reparametrizing $\widehat{u}$ leads to a standard rate-independent problem
3. $\|\widehat{u}'(s)\| = 1$ corresponds to viscous transition between stable states ("instantaneous" w.r.t. the slow time scale, whence $\widehat{t}'(s) = 0$); viscous path described by a gradient flow

Towards metric spaces

In rate-independent applications

- $\mathcal{E}$ is **non-smooth**
- $\mathcal{E}$ is **non-convex**

Towards metric spaces

In rate-independent applications

- $\mathcal{E}$ is **non-smooth**
- $\mathcal{E}$ is **non-convex**
- B does **not have the Radon-Nikodým property** (e.g., L^1 in phase transitions in SMA): absolutely continuous curves in L^1 need not be differentiable a.e.;
- B need **not have a linear structure**

Outlook

Our goal:

- Extend the Efendiev-Mielke vanishing viscosity analysis to a **metric setting**

Outlook

Our goal:

- Extend the Efendiev-Mielke vanishing viscosity analysis to a **metric setting**
- Obtain a **new notion of rate-independent evolution** in a metric setting

Outlook

Our goal:

- Extend the Efendiev-Mielke vanishing viscosity analysis to a **metric setting**
- Obtain a **new notion of rate-independent evolution** in a metric setting
- The metric framework will lead to **local, rather than global** stability!

Outlook

Our goal:

- Extend the Efendiev-Mielke vanishing viscosity analysis to a **metric setting**
- Obtain a **new notion of rate-independent evolution** in a metric setting
- The metric framework will lead to **local, rather than global** stability!

Program

In a **metric framework**:

1. **Approximate** rate-independent evolutions with **viscous evolutions** **[Mielke, R., Savaré, quasi-preprint'08]**

Outlook

Our goal:

- Extend the Efendiev-Mielke vanishing viscosity analysis to a **metric setting**
- Obtain a **new notion of rate-independent evolution** in a metric setting
- The metric framework will lead to **local, rather than global** stability!

Program

In a **metric framework**:

1. **Approximate** rate-independent evolutions with **viscous evolutions** **[Mielke, R., Savaré, quasi-preprint'08]**
2. Analysis of doubly nonlinear evolution equations where **dissipation with superlinear growth**: existence & approximation of solutions **[Mielke, R., Savaré, Annali SNS Pisa'08]**

Doubly nonlinear evolutions in metric spaces

Analysis of

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad t \in (0, T) \qquad \text{(DNE)}$$

Ψ with superlinear growth

in the framework of a **metric space** (X, d).

Relying on: theory of **gradient flows** in metric spaces (i.e. **quadratic** Ψ):

- **De Giorgi, Marino, Saccon, Tosques, Degiovanni, Ambrosio** '80 $\sim$ '90 $\rightsquigarrow$ theory of **Curves of Maximal Slope** and **Minimizing Movements**
- [*Gradient flows in metric spaces*, **Ambrosio-Gigli-Savaré** 2005] $\rightsquigarrow$ systematic theory of existence, approximation & uniqueness of solutions of metric gradient flows, with applications to gradient flows in Wasserstein spaces.

Towards the metric formulation

Problem:
How to formulate

$$\text{“ } \partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) = 0, \quad t \in (0, T) \text{ ”}$$

without **a linear/differential structure** on X?

Towards the metric formulation

Problem:
How to formulate

$$\text{“}\ \partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) = 0, \quad t \in (0,T)\ \text{”}$$

without **a linear/differential structure** on X?

Heuristics:
If the **chain rule** holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = \langle \partial_u\mathcal{E}(t,u(t)), u'(t)\rangle$$

then (DNE) is equivalent to

$$\Psi(u'(t))) + \Psi^*(-\partial_u\mathcal{E}(t,u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = 0 \quad t \in (0,T)$$

(abuse of notation: $\partial_u\mathcal{E}(t,u(t)) \sim$ singleton...)

Towards the metric formulation

In the particular case

$$\Psi(x) := \frac{|x|^p}{p}, \quad 1 < p < \infty, \qquad \Psi^*(x) := \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\Psi(u'(t)) + \Psi^*(-\partial_u \mathcal{E}(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

Towards the metric formulation

In the particular case

$$\Psi(x) := \frac{|x|^p}{p}, \quad 1 < p < \infty, \qquad \Psi^*(x) := \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{p} \left|u'(t)\right|^p + \frac{1}{q} \left|-\partial_u \mathcal{E}(t, u(t))\right|^q + \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

Towards the metric formulation

In the particular case

$$\Psi(x) := \frac{|x|^p}{p}, \quad 1 < p < \infty, \qquad \Psi^*(x) := \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{p}|u'(t)|^p + \frac{1}{q}|-\partial_u \mathcal{E}(t, u(t))|^q + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

New formulation features the **modulus of derivatives,** rather than derivatives!

Towards the metric formulation

In the particular case

$$\Psi(x) := \frac{|x|^p}{p}, \quad 1 < p < \infty, \qquad \Psi^*(x) := \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{p}\left|u'(t)\right|^p + \frac{1}{q}\left|-\partial_u \mathcal{E}(t, u(t))\right|^q + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

New formulation features the **modulus of derivatives,** rather than derivatives!

Adaptable to metric spaces upon introducing suitable **"metric surrogates"** of **"modulus of derivatives"**.

The metric derivative

- **Setting:** (X, d) complete metric space

Metric derivative

- We say that a curve $u : [0, T] \to X$ is **absolutely continuous** if

$$\exists m \in L^1(0,T) \ : \quad d(u(t),u(s)) \leq \int_s^t m(r)\,\mathrm{d}r \quad \forall\, 0 \leq s \leq t \leq T.$$

- Given $u \in \mathrm{AC}(0,T;X)$, its **metric derivative**

$$|u'|(t) := \lim_{h \to 0} \frac{d(u(t),u(t+h))}{|h|} \quad \text{for a.e. } t \in (0,T)$$

$$\|u'(t)\| \rightsquigarrow |u'|(t)$$

Slope & Chain rule

- **Setting:** (X, d) complete metric space

Local slope & Chain rule

- Given $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ and $u \in D(\mathcal{E}(t, \cdot))$, the **local slope** of $\mathcal{E}(t, \cdot)$ at u is

$$|\partial\mathcal{E}|\,(t, u) := \limsup_{v \to u} \frac{(\mathcal{E}(t, u) - \mathcal{E}(t, v))^+}{d(u, v)}$$

$$\| - \partial_u \mathcal{E}(t, u)\| \ \rightsquigarrow \ |\partial\mathcal{E}|\,(t, u)$$

Slope & Chain rule

- **Setting:** (X, d) complete metric space

Local slope & Chain rule

- Given $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ and $u \in D(\mathcal{E}(t, \cdot))$, the **local slope** of $\mathcal{E}(t, \cdot)$ at u is

$$|\partial \mathcal{E}|(t, u) := \limsup_{v \to u} \frac{(\mathcal{E}(t, u) - \mathcal{E}(t, v))^+}{d(u, v)}$$

$$\| - \partial_u \mathcal{E}(t, u)\| \rightsquigarrow |\partial \mathcal{E}|(t, u)$$

- $\mathcal{E}$ complies with the **chain rule** w.r.t. $|\partial \mathcal{E}|$ if $\forall v \in \mathrm{AC}(0, T; D(\mathcal{E}))$ the map $t \mapsto \mathcal{E}(t, v(t))$ is **absolutely continuous** and

$$\partial_t \mathcal{E}(t, v(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t, v(t)) \leq |v'|(t)\, |\partial \mathcal{E}|(t, v(t)) \quad \text{per q.o. } t \in (0, T).$$

The metric formulation

- **Basic setting**:
  - (X, d) complete metric space
  - **Energy** $\rightsquigarrow$ $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ l.s.c., complying with the **chain rule** w.r.t. $|\partial\mathcal{E}|$
  - **Dissipation** $\rightsquigarrow$ $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ l.s.c., convex, $\psi(0) = 0$, with

$$\lim_{x \to +\infty} \frac{\psi(x)}{x} = +\infty$$

Metric formulation

A curve $u \in \mathrm{AC}(0, T; X)$ satisfies the **metric formulation** of

The metric formulation

- **Basic setting**:
  - (X, d) complete metric space
  - **Energy** $\rightsquigarrow$ $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ l.s.c., complying with the **chain rule** w.r.t. $|\partial\mathcal{E}|$
  - **Dissipation** $\rightsquigarrow$ $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ l.s.c., convex, $\psi(0) = 0$, with

$$\lim_{x \to +\infty} \frac{\psi(x)}{x} = +\infty$$

Metric formulation

A curve $u \in \mathrm{AC}(0, T; X)$ satisfies the **metric formulation** of

$$\text{“}\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)\text{”}$$

The metric formulation

- **Basic setting**:
  - (X, d) complete metric space
  - **Energy** $\rightsquigarrow$ $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ l.s.c., complying with the **chain rule** w.r.t. $|\partial \mathcal{E}|$
  - **Dissipation** $\rightsquigarrow$ $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ l.s.c., convex, $\psi(0) = 0$, with

$$\lim_{x \to +\infty} \frac{\psi(x)}{x} = +\infty$$

Metric formulation

A curve $u \in \mathrm{AC}(0, T; X)$ satisfies the **metric formulation** of

$$\Psi(u'(t)) + \Psi^*(-\partial_u \mathcal{E}(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

The metric formulation

- **Basic setting**:
  - (X, d) complete metric space
  - **Energy** $\rightsquigarrow$ $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ l.s.c., complying with the **chain rule** w.r.t. $|\partial \mathcal{E}|$
  - **Dissipation** $\rightsquigarrow$ $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ l.s.c., convex, $\psi(0) = 0$, with

$$\lim_{x \to +\infty} \frac{\psi(x)}{x} = +\infty$$

Metric formulation

A curve $u \in \mathrm{AC}(0, T; X)$ satisfies the **metric formulation** of

$$\Psi(u'(t)) + \Psi^*(-\partial_u \mathcal{E}(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

if for a.e. $t \in (0, T)$

$$\psi(|u'|(t)) + \psi^*(|\partial \mathcal{E}|(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0$$

An existence result

Theorem [Mielke, R., Savaré, Annali SNS Pisa'08]

- $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ convex, l.s.c., $\psi(0) = 0$, superlinear growth
- $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ smooth w.r.t. $t \in [0, T]$
- $\mathcal{E}$ l.s.c. and coercive w.r.t. $u \in X$, **chain rule** w.r.t. $|\partial\mathcal{E}|$
- $u \mapsto |\partial\mathcal{E}|(t, u)$ is l.s.c. (along bounded energy sequences)

Then, for all $u_0 \in D(\mathcal{E})$ **there exists** a curve $u \in \mathrm{AC}(0, T; X)$ such that $u(0) = u_0$ and

$$\psi(|u'|(t)) + \psi^* \left(|\partial\mathcal{E}|(t, u(t))\right) = \partial_t\mathcal{E}(t, u(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) \quad \text{for a.e. } t \in (0, T) \,.$$

An existence result

Theorem [Mielke, R., Savaré, Annali SNS Pisa'08]

- $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ convex, l.s.c., $\psi(0) = 0$, superlinear growth
- $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ smooth w.r.t. $t \in [0, T]$
- $\mathcal{E}$ l.s.c. and coercive w.r.t. $u \in X$, **chain rule** w.r.t. $|\partial\mathcal{E}|$
- $u \mapsto |\partial\mathcal{E}|(t, u)$ is l.s.c. (along bounded energy sequences)

Then, for all $u_0 \in D(\mathcal{E})$ **there exists** a curve $u \in \mathrm{AC}(0, T; X)$ such that $u(0) = u_0$ and

$$\psi(|u'|(t)) + \psi^* \left(|\partial\mathcal{E}|(t, u(t))\right) = \partial_t\mathcal{E}(t, u(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) \quad \text{for a.e. } t \in (0, T)\,.$$

Applications: existence results for doubly nonlinear evolution equations in (possibly non reflexive) spaces

Approximation of rate-independent problems with viscous evolutions

Second step

In the metric space (X, d), approximate

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \tag{DNE}$$

Ψ **1-positively homogeneous**

with the viscous evolution

$$\varepsilon u'(t) + \partial\Psi(u'(t)) + \partial_u\mathcal{E}(t, u(t)) \ni 0 \quad t \in (0, T), \qquad \text{as } \varepsilon \searrow 0$$

Approximation of rate-independent problems with viscous evolutions

In the metric setting

- (X, d) metric space
- $\mathcal{E} : [0, T] \times X \to \mathbb{R} \cup \{+\infty\}$: assumptions for $\exists$ + **Chain rule**
- $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ convex **1-positively homogeneous** ($\psi(r) = r \ \forall r \in \mathbb{R}^+$)
- Viscous regularization of ψ: $\psi_\varepsilon(x) := x + \frac{\varepsilon}{2}x^2 \ \forall x \geq 0 \ \forall \varepsilon > 0.$
- $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathrm{AC}(0, T; X)$: **metric solutions** of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u_\varepsilon(t)) - \partial_t\mathcal{E}(t, u_\varepsilon(t)) = \\ \qquad - \psi_\varepsilon(|u_\varepsilon{}'|(t)) - \psi_\varepsilon{}^*(|\partial\mathcal{E}|\,(t, u_\varepsilon(t))) \ \text{ per q.o. } t \in (0, T) \\ u_\varepsilon(0) = u_0. \end{cases}$$

- **Problem:** ¿ limit of $\{u_\varepsilon\}$ as $\varepsilon \searrow 0$?

Vanishing viscosity revisited

Extend the Mielke-Efendiev technique to the metric setting:

♣ reparametrize by the **arc length** of $\{(t, u_\varepsilon(t)) : t \in [0, T]\}$:

$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

Vanishing viscosity revisited

Extend the Mielke-Efendiev technique to the metric setting:

♣ reparametrize by the **arc length** of $\{(t, u_\varepsilon(t)) :\ t \in [0, T]\}$:

$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon{}'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

♣ you pass from

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u_\varepsilon(t)) - \partial_t \mathcal{E}(t, u_\varepsilon(t)) = \\ \qquad - \psi_\varepsilon(|u_\varepsilon{}'|(t)) - \psi_\varepsilon{}^*(|\partial\mathcal{E}|\,(t, u_\varepsilon(t))) \qquad t \in (0, T) \\ u_\varepsilon(0) = u_0. \end{cases}$$

Vanishing viscosity revisited

Extend the Mielke-Efendiev technique to the metric setting:

♣ reparametrize by the **arc length** of $\{(t, u_\varepsilon(t)) : \ t \in [0,T]\}$:

$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon{}'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

♣ to

$$\begin{cases} \widehat{t}_\varepsilon(0) = 0 \qquad \widehat{t}_\varepsilon(s_\varepsilon(T)) = T \\ \widehat{t}_\varepsilon'(s) + |\widehat{u}_\varepsilon'|(s) = 1 \quad \text{per q.o. } s \in (0, s_\varepsilon(T)) \\ \text{rescaled metric formulation of (DNE) } (\psi_\varepsilon, \mathcal{E}) \end{cases}$$

Problem: ¿ asymptotic analysis of $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ as $\varepsilon \searrow 0$?

The asymptotic analysis result

Let

$$\widehat{\psi}(r) := \begin{cases} r & r \in [0,1]\,, \\ +\infty & r > 1\,, \end{cases} \qquad \widehat{T} := \lim_{\varepsilon \downarrow 0} s_\varepsilon(T)\,.$$

Theorem [Mielke, R., Savaré]

Assumptions: like for $\exists$ of metric solution (in particular, **chain rule**).

Then, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ converges as $\varepsilon \searrow 0$ to $(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X)$, which satisfies

$$\begin{cases} \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T \\ \widehat{t}'(s) + |\widehat{u}'|(s) = 1 \quad \text{per q.o. } s \in (0,\widehat{T}) \end{cases}$$

and the **"rescaled metric formulation"**

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s)$$
$$= -\widehat{\psi}(|\widehat{u}'|(s)) - \widehat{\psi}^*\left(|\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)\right) \quad s \in (0,\widehat{T})\,.$$

More insight into the vanishing viscosity limit

$$(\widehat{t}, \widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0, T] \times X) \qquad \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T$$

$$\widehat{t}'(s) + \widehat{u}'(s) = 1 \quad \text{per q.o. } s \in (0, \widehat{T})$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) = -\widehat{\psi}(|\widehat{u}'|(s)) - \widehat{\psi}^* \left(|\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s))\right)$$

$$\text{per q.o. } s \in (0, \widehat{T})$$

More insight into the vanishing viscosity limit

$$(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s)=1 \quad \text{per q.o. } s\in(0,\widehat{T})$$

$$\begin{cases} \textbf{En. id.:} & \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \\ \textbf{Constraint:} & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \ \text{ per q.o. } s\in(0,\widehat{T}) \end{cases}$$

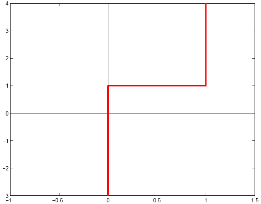
More insight into the vanishing viscosity limit

$$(\widehat{t}, \widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0, T] \times X) \qquad \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T$$

$$\widehat{t}'(s) + \widehat{u}'(s) = 1 \quad \text{per q.o. } s \in (0, \widehat{T})$$

$$\begin{cases} \textbf{En. id.:} & \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\, |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \\ \textbf{Constraint:} & |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \ \text{ per q.o. } s \in (0, \widehat{T}) \end{cases}$$

$$|\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \ \text{ per q.o. } s \in (0, \widehat{T})$$



More insight into the vanishing viscosity limit

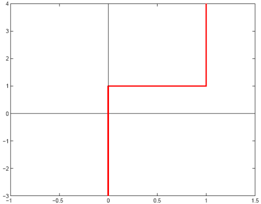
$$(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s)=1 \quad \text{per q.o. } s\in(0,\widehat{T})$$

En. id.: $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$$

Three regimes:
$$\begin{cases} |\widehat{u}'|(s)=1 \ (\Leftrightarrow \widehat{t}'(s)=0) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \\ |\widehat{u}'|(s)\in(0,1) \ (\Leftrightarrow \widehat{t}'(s)\in(0,1)) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ |\widehat{u}'|(s)=0 \ (\Leftrightarrow \widehat{t}'(s)=1) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \end{cases}$$

$$|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \quad \text{per q.o. } s\in(0,\widehat{T})$$



Towards Parametrized Rate-Independent Flows

These are the properties to retain:

$$(\widehat{t}, \widehat{u}) \in \mathrm{AC}(0, \widehat{T}; [0, T] \times X) \qquad \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T$$

$$\widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id.: $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\; |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s))$$

Three regimes: $$\begin{cases} \widehat{t}'(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow \ |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \end{cases}$$

Parametrized Rate-Independent Flows

Definition

A pair $(\widehat{t}, \widehat{u}) \in \mathrm{AC}(0, \widehat{T}; [0, T] \times X)$ is a **parametrized rate-independent flow** if

1. $\widehat{t}$ is non-decreasing, with $\widehat{t}(0) = 0$ and $\widehat{t}(\widehat{T}) = T$
2. there holds
$$\widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$
3. the map $s \in [0, \widehat{T}] \mapsto \mathcal{E}(\widehat{t}(s), \widehat{u}(s))$ is absolutely continuous and

$$\textbf{En. id.:} \ \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right)$$

$$\textbf{Three regimes:} \quad \begin{cases} \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right) \leq 1 \end{cases}$$

Parametrized Rate-Independent Flows: rate-invariance

Parametrized Rate-Independent Flow

En. id.: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$

Three regimes: $\begin{cases} \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \end{cases}$

Main features

- Approximable (via vanishing viscosity) solutions are PRIFs.

Parametrized Rate-Independent Flows: rate-invariance

Parametrized Rate-Independent Flow

En. id.: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)$

Three regimes: $\begin{cases} \widehat{t}'(s) = 0 & \Rightarrow\; |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\; |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow\; |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right) \leq 1 \end{cases}$

Main features

- Approximable (via vanishing viscosity) solutions are PRIFs.
- The class of PRIF is **invariant** for (strictly increasing) reparametrizations $\Rightarrow$ PRIF is a **truly rate-independent** notion.

Parametrized Rate-Independent Flows: flow regimes

Parametrized Rate-Independent Flows

En. id.: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$

Three regimes: $\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \end{cases}$

Mechanical interpretation

- **sticking** $\leftrightarrow$ $|\widehat{u}'|(s) = 0$

Parametrized Rate-Independent Flows: flow regimes

Parametrized Rate-Independent Flows

En. id.: $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right)$$

Three regimes: $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\left(\widehat{t}(s),\widehat{u}(s)\right) \geq 1 \end{cases}$$

Mechanical interpretation

- **sticking** $\leftrightarrow$ $|\widehat{u}'|(s) = 0$
- **sliding** $\leftrightarrow$ $\widehat{t}'(s)|\widehat{u}'|(s) > 0$

Parametrized Rate-Independent Flows: flow regimes

Parametrized Rate-Independent Flows

En. id.: $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$

Three regimes:
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \end{cases}$$

Mechanical interpretation

- **sticking** $\leftrightarrow$ $|\widehat{u}'|(s) = 0$
- **sliding** $\leftrightarrow$ $\widehat{t}'(s)|\widehat{u}'|(s) > 0$
- **viscous slip** $\leftrightarrow$ $\widehat{t}'(s) = 0.$

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id:
$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\; |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions:
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Sticking:

- $|\widehat{u}'|(s_0) = 0 \ \Rightarrow \ \widehat{t}'(s_0) > 0$ and $|\partial \mathcal{E}|\, (\widehat{t}(s_0), \widehat{u}(s_0)) \leq 1$ (**local stability**)

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id: $$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\; |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions: $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Sticking:

- $|\widehat{u}'|(s_0) = 0 \Rightarrow \widehat{t}'(s_0) > 0$ and $|\partial \mathcal{E}|\, (\widehat{t}(s_0), \widehat{u}(s_0)) \leq 1$ (**local stability**)
- in a neighb. $I(s_0)$ we have $\widehat{u}(s) \equiv \widehat{u}(s_0)$ and the **energy identity**

$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_0)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_0)) = \int_{s_1}^{s_2} \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s_0))\, \widehat{t}'(s)\, \mathrm{d}s \ \ \forall\, s_1 \leq s_2 \in I(s_0)\,.$$

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id:
$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) - |\widehat{u}'|(s)\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s))\Big)\,\mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions:
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Sliding:

- $\widehat{t}'(s_0) > 0$ & $|\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\,(\widehat{t}(s_0), \widehat{u}(s_0)) = 1$ (**local stability**)

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id:
$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\ |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions:
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Sliding:

- $\widehat{t}'(s_0) > 0$ & $|\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\, (\widehat{t}(s_0), \widehat{u}(s_0)) = 1$ (**local stability**)
- in a neighb. $I(s_0)$ the **energy identity** reads $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s) \Big)\, \mathrm{d}s$$

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id: $$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\ |\partial \mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions: $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Viscous slip at jumps:

- $\widehat{t}'(s_0) = 0 \Rightarrow |\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\,(\widehat{t}(s_0), \widehat{u}(s_0)) \geq 1$

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id: $$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\, |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions: $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow \ |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow \ |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow \ |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Viscous slip at jumps:

- $\widehat{t}'(s_0) = 0 \Rightarrow |\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\, (\widehat{t}(s_0), \widehat{u}(s_0)) \geq 1$
- in a neighb. $I(s_0)$ $\widehat{t}(s) \equiv \widehat{t}(s_0)$ & **energy identity** $\forall\, s_1 \leq s_2 \in I(s_0)$

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id:

$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\; |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s))\Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions:

$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Viscous slip at jumps:

- $\widehat{t}'(s_0) = 0 \Rightarrow |\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\,(\widehat{t}(s_0), \widehat{u}(s_0)) \geq 1$
- in a neighb. $I(s_0)$ $\widehat{t}(s) \equiv \widehat{t}(s_0)$ & **energy identity** $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\widehat{t}(s_0), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_0), \widehat{u}(s_1)) = -\int_{s_1}^{s_2} |\partial\mathcal{E}|\,(\widehat{t}(s_0), \widehat{u}(s))\, |\widehat{u}'|(s)\Big)\, \mathrm{d}s$$

which is a "generalized gradient flow"

Flow regimes

Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{per q.o. } s \in (0, \widehat{T})$$

En. id:
$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) - |\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s))\Big)\,\mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

Differential conditions:
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow \ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow \ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow \ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

Viscous slip at jumps:

- $\widehat{t}'(s_0) = 0 \Rightarrow |\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\,(\widehat{t}(s_0), \widehat{u}(s_0)) \geq 1$
- in a neighb. $I(s_0)$ $\widehat{t}(s) \equiv \widehat{t}(s_0)$ & **energy identity** $\forall\, s_1 \leq s_2 \in I(s_0)$

the viscous transition path followed by a system at a jump
is described by a "generalized gradient flow"

Conclusions

Applications

Use the notion of **parametrized rate-independent flow** to give a

- **finer description** of rate-independent evolutions
- in Banach spaces
- in metric spaces

Conclusions

Applications

Use the notion of **parametrized rate-independent flow** to give a

- **finer description** of rate-independent evolutions
- in Banach spaces
- in metric spaces

Remark

Parametrized rate-independent evolutions enforce

local, rather than global stability!

because our notion of slope is **local**

Local vs. Global Slope

- **Setting:** (X, d) complete metric space

Global slope

Given $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ and $u \in D(\mathcal{E}(t, \cdot))$, the **global slope** of $\mathcal{E}(t, \cdot)$ at u is

$$|\mathcal{G}\ell(\mathcal{E})| (t, u) := \sup_{v \neq u} \frac{(\mathcal{E}(t, u) - \mathcal{E}(t, v))^+}{d(u, v)}$$

Suppose that $\mathcal{E}(t, \cdot)$ is λ–(geodesically) convex, $\lambda \geq 0$. Then

$$|\partial \mathcal{E}| (t, u) = \mathcal{G}\ell(\mathcal{E})(t, u)$$

Comparison with the energetic formulation

During **sliding regime** for PRIF we have

$$|\partial\mathcal{E}|\,(t,u(t)) = 1 \qquad \textbf{local stability}$$

For **global energetic (metric) solutions** we have

$$|\mathcal{G}\ell(\mathcal{E})|(t,u(t)) = 1 \qquad \textbf{global stability}$$

Comparison with the energetic formulation

During **sliding regime** for PRIF we have

$$|\partial\mathcal{E}|\,(t,u(t)) = 1 \qquad \textbf{local stability}$$

For **global energetic (metric) solutions** we have

$$|\mathcal{G}\ell(\mathcal{E})|(t,u(t)) = 1 \qquad \textbf{global stability}$$

In the case:

$$\begin{cases} B = \mathbb{R}\,, \qquad \Psi(v) = |v| \quad \forall\, v \in \mathbb{R} \\ \mathcal{E}(t,u) = \mathcal{W}(u) - \ell(t)u \qquad \forall\,(t,u) \in [0,T] \times \mathbb{R} \end{cases}$$

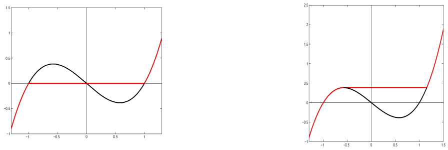


Figure: Global energetic vs. PRIF