Well-posedness and asymptotic analysis for a Penrose-Fife type phase-field system

Buona positura e analisi asintotica per un
sistema di phase field di tipo Penrose-Fife

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NOTATION

- $\Omega \subset \mathbf{R}^N$ a bounded, connected domain in $\mathbb{R}^N$, $N \leq 3$,

with smooth boundary $\partial\Omega$,

- $T > 0$ final time.

Variables

ϑ the absolute temperature of the system,

ϑ_c the phase change temperature,

χ order parameter (e.g., local proportion of solid/liquid phase in melting, fraction of pointing up spins in Ising ferromagnets)

The Penrose-Fife phase field system (I)

$$\varepsilon\vartheta_t + \lambda\chi_t - \Delta(-\frac{1}{\vartheta}) = f \quad \text{in } \Omega \times (0,T),$$

$$\delta\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) + \frac{\lambda}{\vartheta} - \frac{\lambda}{\vartheta_c} \ni 0 \quad \text{in } \Omega \times (0,T),$$

$\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a maximal monotone graph, $\beta = \partial\hat{\beta}$ for *convex* $\hat{\beta}$,

$\sigma' : \mathbb{R} \to \mathbb{R}$ a Lipschitz function,

ε, δ *relaxation parameters.*

The Penrose-Fife phase field system (II)

- Let $e := \vartheta + \chi$ be the INTERNAL ENERGY , and let $\mathbf{q}$

$$\mathbf{q} = -\nabla(\ell(\vartheta)), \quad \ell(\vartheta) = -\frac{1}{\vartheta}$$

be the heat flux: $\Longrightarrow$ the first equation is indeed

$$\partial_t e + \operatorname{div} \mathbf{q} = f,$$

the BALANCE LAW for the internal energy.

- The second equation is a Cahn-Allen type dynamics for χ, in which e.g., $\beta(\chi) + \sigma'(\chi) = \chi^3 - \chi$, derivative of the double well potential $\mathcal{W}(\chi) = \frac{(\chi^2-1)^2}{4}$.

Singular limit as $\varepsilon \downarrow 0$ (formal)

$$\begin{cases} \varepsilon\vartheta_t + \chi_t - \Delta(-\frac{1}{\vartheta}) = f, \\ \delta\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) \ni -\frac{1}{\vartheta}, \\ \partial_n\chi = \partial_n(-\frac{1}{\vartheta}) = 0 \quad \text{on } \partial\Omega \times (0,T), \end{cases}$$

$$\downarrow \ \varepsilon = 0$$

$$\begin{cases} \chi_t - \Delta(\delta\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi)) \ni f, \\ \partial_n\chi = \partial_n(-\Delta\chi + \xi + \sigma'(\chi)) = 0 \quad \xi \in \beta(\chi), \quad \text{on } \partial\Omega \times (0,T). \end{cases}$$

In the limit $\varepsilon \downarrow 0$, we formally obtain the

viscous Cahn Hilliard equation

with source term and nonlinearities.

SINGULAR LIMIT AS $\varepsilon, \delta \downarrow 0$ (FORMAL)

$$\begin{cases} \varepsilon\vartheta_t + \chi_t - \Delta(-\frac{1}{\vartheta}) = f, \\ \delta\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) \ni -\frac{1}{\vartheta}, \\ \partial_n\chi = \partial_n(-\frac{1}{\vartheta}) = 0 \quad \text{on } \partial\Omega \times (0,T), \end{cases}$$

$$\downarrow \varepsilon = \delta = 0$$

$$\begin{cases} \chi_t - \Delta(-\Delta\chi + \beta(\chi) + \sigma'(\chi)) \ni f, \\ \partial_n\chi = \partial_n(-\Delta\chi + \xi + \sigma'(\chi)) = 0 \quad \xi \in \beta(\chi), \quad \text{on } \partial\Omega \times (0,T). \end{cases}$$

In the limit $\varepsilon, \delta \downarrow 0$, we formally obtain the

CAHN HILLIARD EQUATION

WITH SOURCE TERM AND NONLINEARITIES.

The Cahn Hilliard equation

$$\chi_t - \Delta(-\Delta\chi + \chi^3 - \chi) = f \quad \text{a.e. in } \Omega \times (0,T), \quad \chi(\cdot, 0) = \chi^0$$

- (CH) models phase separation : χ is the concentration of one of the two components in a binary alloy.
- Homogeneous Neumann boundary conditions on χ and $\Delta\chi$, source term f spatially homogeneous, i.e.

$$\frac{1}{|\Omega|}\int_\Omega f(x,t)dx = 0 \quad \text{for a.e. } t \in (0,T),$$

$\Longrightarrow$ χ is a *conserved parameter* , i.e.

$$m(\chi(t)) := \frac{1}{|\Omega|}\int_\Omega \chi(x,t)\,dx = m(\chi^0) \quad \forall t \in [0,T].$$

The *viscous* Cahn Hilliard equation

$$\chi_t - \Delta(\chi_t - \Delta\chi + \chi^3 - \chi) = f \quad \text{a.e. in } \Omega \times (0,T), \quad \chi(\cdot,0) = \chi^0$$

- (VCH) was introduced by [**Novick-Cohen '88**] to model viscosity effects in the phase separation of polymeric systems; derived by [**Gurtin '96**] in a model accounting for working of internal microforces , see also [**Miranville '00, '02**]**...**
- Homogeneous Neumann b.c. and spatial homogeneity of f, $\Longrightarrow$ χ is a *conserved parameter.*
- The maximal monotone graph β ([**Blowey-Elliott '91**], [**Kenmochi-Niezgódka '95**] for (CH)) accounts for, e.g., a constraint on the values of χ.

MOTIVATIONS FOR THE ASYMPTOTIC ANALYSES

- Taking the limits $\varepsilon \downarrow 0$ (physically: small specific heat density) and ε, $\delta \downarrow 0$:

$\Rightarrow$ passage from a non-conserved dynamics to a conserved dynamics;

- Proving convergence results for $\varepsilon \downarrow 0$:

$\Rightarrow$ obtain existence results for the viscous Cahn-Hilliard equation with nonlinearities , NEVER obtained so far.

- Analogy with a similar asymptotic analysis for the Caginalp phase field model [**Caginalp '90, Stoth '95, R. '03**], [**Laurençot *et al.*: attractors**].

A BAD APPROXIMATION

Reformulate (VCH) in terms of the *chemical potential* u . See that the asymptotic analysis

$$\begin{cases} \varepsilon\vartheta_{\varepsilon t} + \chi_{\varepsilon t} - \Delta(-\frac{1}{\vartheta_\varepsilon}) = f, \\ \delta\chi_{\varepsilon t} - \Delta\chi_\varepsilon + \xi_\varepsilon + \sigma'(\chi_\varepsilon) \ni -\frac{1}{\vartheta_\varepsilon}, \quad \xi_\varepsilon \in \beta(\chi_\varepsilon) \end{cases}$$

$$\downarrow \varepsilon \downarrow 0$$

$$\begin{cases} \chi_t - \Delta u = f, \\ \delta\chi_t - \Delta\chi + \xi + \sigma'(\chi) = u, \quad \xi \in \beta(\chi), \end{cases}$$

is ill-posed (& same considerations for the (CH)):

- poor estimates, no bounds on ϑ_ε!
- if $u = \lim_{\varepsilon\downarrow 0} -\frac{1}{\vartheta_\varepsilon}$ in a "reasonable" topology, then $u \leq 0$ a.e. in: a sign constraint does not pertain to the (VCH)!!

A new approximating system (I)

- **Colli & Laurençot '98**: an alternative heat flux law , better for large temperatures :

$$\ell(\vartheta) = -\frac{1}{\vartheta} \rightsquigarrow \alpha(\vartheta) \sim \vartheta - \frac{1}{\vartheta},$$

and α increasing.

- The new approximating system for $\varepsilon \downarrow 0$: replace $\ell(\vartheta) \rightsquigarrow \alpha_\varepsilon(\vartheta) = \varepsilon^{1/2}\vartheta - \frac{1}{\vartheta}$ in *each* equation: you obtain **Problem $\mathbf{P}^\varepsilon$:**

$$\varepsilon\vartheta_t + \chi_t - \Delta(\varepsilon^{1/2}\vartheta - \frac{1}{\vartheta}) = f,$$

$$\delta\chi_t - \Delta\chi + \xi + \sigma'(\chi) = \varepsilon^{1/2}\vartheta - \frac{1}{\vartheta}, \quad \xi \in \beta(\chi).$$

$+$ hom. N.B.C. on χ and $\alpha_\varepsilon(\vartheta)$ and I.C. on χ and ϑ.

A new approximating system (II)

The previous difficulties are overcome:

- the term $\varepsilon^{1/2}\vartheta$ allows for estimates on the approximate sequence ϑ_ε;
- No more sign constraints on $u = \lim_{\varepsilon\downarrow 0} \alpha_\varepsilon(\vartheta_\varepsilon) = \varepsilon^{1/2}\vartheta_\varepsilon - \frac{1}{\vartheta_\varepsilon}$: α_ε ranges over the whole of $\mathbb{R}$!

A PHASE FIELD MODEL WITH *double* NONLINEARITY (I)

In general , we investigate the phase field system

$$\vartheta_t + \chi_t - \Delta u = f, \quad u \in \alpha(\vartheta), \quad \text{in } \Omega \times (0,T),$$

$$\chi_t - \Delta \chi + \xi + \sigma'(\chi) = u, \quad \xi \in \beta(\chi), \quad \text{in } \Omega \times (0,T),$$

- α & β *maximal monotone graphs* on $\mathbb{R}^2$,
- with the initial conditions

$$\vartheta(\cdot,0) = \vartheta^0, \quad \chi(\cdot,0) = \chi^0 \quad \text{a.e. in } \Omega,$$

- and the boundary conditions

$$\partial_n \chi = \partial_n u = 0 \quad \text{in } \partial\Omega \times (0,T).$$

PROBLEM: WELL-POSEDNESS?

Analytical difficulties

- The double nonlinearity of α & β;
- The homogeneous Neumann boundary conditions on both χ and u. Usually, third type boundary conditions for u

$$\partial_n u + \gamma u = \gamma h, \qquad \gamma > 0,\ h \in L^2(\partial\Omega \times (0,T))$$

  are given: they allow to recover a $H^1(\Omega)$-bound on u from the first equation.
- How to deal with homogeneous N.B.C. ?

Previous contributions

- Kenmochi-Kubo '99: OK double nonlinearity; third type b.c. on u;
- Zheng '92: $\alpha(\vartheta) = -\frac{1}{\vartheta}$, OK for N.B.C. on u in $1D$.
- Ito-Kenmochi-Kubo '02: $\alpha(\vartheta) = -\frac{1}{\vartheta}$, OK for N.B.C. on u under additional constraints.

Fill the gap?

General setting

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \left\{ v \in H^2(\Omega) \;:\; \partial_n v = 0 \right\},$$

with dense and compact embeddings $W \subset V \subset H \cong H' \subset V' \subset W'$.

- Consider the realization of the Laplace operator with homog. N.B.C., i.e. the operator $A : V \to V'$ defined by

$$\langle Au, v\rangle := \int_\Omega \nabla u \nabla v \, dx \quad \forall u, v \in V.$$

- The inverse operator $\mathcal{N}$ is defined for the elements $v \in V'$ of *zero* mean value $m(v)$. Take on V and V' the equivalent norms:

$$\|u\|_V^2 := \langle Au, u\rangle + (u, m(u)) \quad \forall u \in V$$

$$\|v\|_{V'}^2 := \langle v, \mathcal{N}(v - m(v))\rangle + (v, m(v)) \quad \forall v \in V'.$$

A phase field model with *double* nonlinearity (II)

Variational formulation

Problem P *Given* $\chi_0 \in V$ $\vartheta_0 \in H$ *satisfying suitable conditions, find* $\vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H)$ *and* $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ *such that* $\vartheta \in D(\alpha)$, $\chi \in D(\beta)$ a.e. in Q,

$$\partial_t \vartheta + \partial_t \chi + Au = f \quad \text{in } V' \text{ for a.e. } t \in (0,T),$$

$$\text{for } u \in L^2(0,T;V) \text{ with } u \in \alpha(\vartheta) \text{ a.e. in } Q,$$

$$\partial_t \chi + A\chi + \xi + \sigma'(\chi) = u \quad \text{in } H \text{ for a.e. } t \in (0,T),$$

$$\text{for } \xi \in L^2(0,T;H) \text{ with } \xi \in \beta(\chi) \text{ a.e. in } Q,$$

$$\chi(\cdot,0) = \chi_0, \quad \vartheta(\cdot,0) = \vartheta_0.$$

A double approximation procedure (I)

As in [**Ito-Kenmochi-Kubo'02**], a first approximate problem:

Problem $\mathbf{P}^{\nu}$. *Find ϑ and χ such that the initial conditions hold and*

$$\partial_t \vartheta + \partial_t \chi + \nu u + A u = f,\ u \in \alpha(\vartheta) \quad \text{in } V' \text{ for a.e. } t \in (0,T),$$

$$\partial_t \chi + A\chi + \xi + \sigma'(\chi) = u,\ \xi \in \beta(\chi) \quad \text{in } H \text{ for a.e. } t \in (0,T),$$

- $\mathbf{P}^{\nu}$ is **coercive:** consider the equivalent scalar product $(\!(\cdot\,,\,\cdot)\!)$ on V

$$(\!(v,w)\!) := \nu \int_\Omega v w \, dx + \int_\Omega \nabla v \nabla w \, dx \quad \forall v, w \in V.$$

Then you recover the full V-norm of u from the first equation.

$\Longrightarrow$ OK for the boundary conditions!

A DOUBLE APPROXIMATION PROCEDURE (II)

EXISTENCE FOR $\mathbf{P}^\nu$

- A SUBDIFFERENTIAL APPROACH [**Kenmochi-Kubo'99**]:

Reformulate the first equation as a subdifferential inclusion in V' by means of a proper, l.s.c, convex functional φ on V':

$$\partial_t \vartheta + \partial_t \chi + \nu u + Au = f \rightsquigarrow \partial_t \vartheta + \partial_t \chi + \partial_{V'} \varphi(\vartheta) \ni f$$

- A FURTHER REGULARIZATION: Approximate the maximal monotone graph α with $\{\alpha_n\}$, α_n increasing, bi-Lipschitz continuous, $\alpha_n \rightsquigarrow \varphi_n$; approximate β with its *Yosida regularization* β_n.
- Solve the approximate system

$$(\text{subdifferential inclusion for } \varphi_n) + (\text{Cahn-Allen eq. with } \beta_n)$$

- Passage to the limit as $n \uparrow \infty$ (standard):

$\Longrightarrow$ Well-posedness for $\mathbf{P}^\nu$!!

A double approximation procedure (III)

Passage to the limit for $\nu \downarrow 0$

$$\partial_t \vartheta + \partial_t \chi + \nu u + A u = f, \quad u \in \alpha(\vartheta),$$
$$\partial_t \chi + A\chi + \xi + \sigma'(\chi) = u \quad \xi \in \beta(\chi),$$

- Problem: lack of a V-bound for u (now ν tends to 0 !!)
- Additional assumption on β [**Colli-Gilardi-Rocca-Schimperna '03**]:

$$\exists\, M_\beta \geq 0 \,\text{s. t.} \quad \xi \leq M_\beta(1+\hat{\beta}(r)) \quad \forall \xi \in \beta(r), \quad \forall r \in \mathbb{R}.$$

- $\Rightarrow$ Then you can pass to the limit:

the solutions of $\mathbf{P}^\nu$ converge as $\nu \downarrow 0$ to the *unique* solution of **P**!

Main Existence Result

Theorem 1. **[R., '03]** *Problem* **P** *admits a unique solution* (χ, ϑ).

- In particular, for every $\varepsilon > 0$ there exists a unique solution $(\chi_\varepsilon, \vartheta_\varepsilon)$ to the Problem $\mathbf{P}^\varepsilon$:

$$\varepsilon \partial_t \vartheta_\varepsilon + \partial_t \chi_\varepsilon - \Delta(\varepsilon^{1/2} \vartheta_\varepsilon - \frac{1}{\vartheta_\varepsilon}) = f$$

$$\delta \partial_t \chi_\varepsilon - \Delta \chi_\varepsilon + \xi_\varepsilon + \sigma'(\chi_\varepsilon) = \varepsilon^{1/2} \vartheta_\varepsilon - \frac{1}{\vartheta_\varepsilon}, \quad \xi_\varepsilon \in \beta(\chi_\varepsilon)$$

$$+ \text{ hom. N.B.C. on } \chi_\varepsilon \text{ and } \alpha_\varepsilon(\vartheta_\varepsilon) + \text{ I.C. on } \chi_\varepsilon \text{ and } \vartheta_\varepsilon.$$

  Set $u_\varepsilon := \alpha_\varepsilon(\vartheta_\varepsilon) = \varepsilon^{1/2} \vartheta_\varepsilon - \frac{1}{\vartheta_\varepsilon}$.

- Passage to the limit for $\varepsilon \downarrow 0 \;\rightsquigarrow\;$ (VCH)?

Variational formulation of the Neumann problem for the viscous Cahn-Hilliard equation

Problem $\mathbf{P}^\delta$. *Given the data*

$$\chi^0 \in V, \quad \widehat{\beta}(\chi^0) \in L^1(\Omega),$$
$$f \in L^2(0,T;V'), \quad \frac{1}{|\Omega|}\int_\Omega f(x,t)dx = 0 \quad \textit{for a.e. } t \in (0,T),$$

find $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$, $u \in L^2(0,T;V)$ *s.t.*

$$\begin{cases} \partial_t \chi + Au = f \quad \text{in } V', \text{ a.e. in } (0,T), \\ \delta\partial_t\chi + A\chi + \xi + \sigma'(\chi) = u \quad \text{in } H, \text{ a.e. in } (0,T) \\ \text{for } \xi \in L^2(0,T;H), \quad \xi \in \beta(\chi) \text{ a.e. in } Q. \\ \chi(\cdot\,,0) = \chi^0. \end{cases}$$

Continuous dependence on the data holds for $\mathbf{P}^\delta$: $\Rightarrow$ uniqueness of the solution χ; existence via approximation.

Approximation

- Approximating data: Given the data χ^0 and f of Problem $\mathbf{P}^\delta$, consider the approximating data $\{\chi_\varepsilon{}^0\}$, $\{\vartheta_\varepsilon{}^0\}$, and $\{f_\varepsilon\}$ fulfilling

$$f_\varepsilon \in L^2(0,T;H),\ f_\varepsilon \to f \text{ in } L^2(0,T;V') \text{ as } \varepsilon \downarrow 0$$
$$\chi_\varepsilon{}^0 \to \chi^0 \text{ in } H \text{ as } \varepsilon \downarrow 0$$

and suitable boundedness conditions .

- Approximate solutions: Let $\{(\chi_\varepsilon, \vartheta_\varepsilon)\}$ be the sequence of solutions to $\mathbf{P}^\varepsilon$ supplemented with the data $\{\chi_\varepsilon{}^0\}$, $\{\vartheta_\varepsilon{}^0\}$ and $\{f_\varepsilon\}$.

Asymptotic behaviour of $\mathbf{P}^{\varepsilon}$ as $\varepsilon \downarrow 0$ and existence for Problem $\mathbf{P}^{\delta}$.

Theorem 2. **[R.,'03]** *There exists a triplet* (χ, u, ξ) *such that the following convergences hold as* $\varepsilon \downarrow 0$, *and along a subsequence* $\{\varepsilon_k :\}$

$$\chi_\varepsilon \rightharpoonup^* \chi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),$$

$$\chi_\varepsilon \to \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V),$$

$$\varepsilon\vartheta_\varepsilon \to 0 \text{ in } L^\infty(0,T;H), \quad \varepsilon\vartheta_\varepsilon \rightharpoonup 0 \text{ in } H^1(0,T;V'),$$

$$u_{\varepsilon_k} \rightharpoonup u \text{ as } k \uparrow \infty, \quad \text{in } L^2(0,T;V),$$

$$\xi_{\varepsilon_k} \rightharpoonup \xi \text{ as } k \uparrow \infty, \text{ in } L^2(0,T;V), \quad \text{and } \xi \in \beta(\chi) \text{ a.e. in } \Omega \times (0,T).$$

Moreover, the triplet (χ, u, ξ) *solves Problem* $\mathbf{P}^{\delta}$.

Error estimates for $\varepsilon \downarrow 0$

There exists a constant $C_{\rm err} \geq 0$, *depending on* T, $|\Omega|$ *and* L *only, such that the error estimates*

$$\|\chi_\varepsilon - \chi\|_{C^0([0,T];H)\cap L^2(0,T;V)}$$

$$\leq C_{\rm err}\left(\varepsilon^{1/8} + \|\chi_\varepsilon{}^0 - \chi^0\|_{V'}^{1/2} + |\chi^0 - \chi_\varepsilon{}^0|_H + \|f - f_\varepsilon\|_{L^2(0,T;V')}^{1/2}\right),$$

$$\|\varepsilon\vartheta_\varepsilon\|_{L^\infty(0,T;H)} \leq C\varepsilon^{1/4}$$

hold for every $\varepsilon \in (0,1)$.

Asymptotic analysis for ε and $\delta \downarrow 0$

- We investigate the asymptotic behaviour as ε and $\delta \downarrow 0$ of the solutions $\chi_{\varepsilon\delta}$, $\vartheta_{\varepsilon\delta}$ to

$$\varepsilon \partial_t \vartheta_{\varepsilon\delta} + \partial_t \chi_{\varepsilon\delta} - \Delta(\varepsilon^{1/2}\vartheta_{\varepsilon\delta} - \frac{1}{\vartheta_{\varepsilon\delta}}) = f,$$

$$\delta \partial_t \chi_{\varepsilon\delta} - \Delta \chi_{\varepsilon\delta} + {\chi_{\varepsilon\delta}}^3 - \chi_{\varepsilon\delta} = \varepsilon^{1/2}\vartheta_{\varepsilon\delta} - \frac{1}{\vartheta_{\varepsilon\delta}},$$

$+$ hom. N.B.C. on $\chi_{\varepsilon\delta}$ and $\alpha_\varepsilon(\vartheta_{\varepsilon\delta})$ + I.C. on $\chi_{\varepsilon\delta}$ and $\vartheta_{\varepsilon\delta}$.

  Let $u_{\varepsilon\delta} := \alpha_\varepsilon(\vartheta_{\varepsilon\delta}) = \varepsilon^{1/2}\vartheta_{\varepsilon\delta} - \frac{1}{\vartheta_{\varepsilon\delta}}$.
  We refer to this problem as Problem $\mathbf{P}^{\varepsilon\delta}$.

- The limiting problem is the *standard* Cahn Hilliard equation with source term f.

APPROXIMATION OF THE CAHN HILLIARD EQUATION

Variational formulation of the limit problem. *Given* $\chi^0 \in V$ *and* $f \in L^2(0,T;V')$ *with null mean value, find* $\chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W)$, $u \in L^2(0,T;V)$ *s.t.*

$$\partial_t \chi + Au = f \quad \text{in } V', \text{ for a.e. } t \in (0,T),$$
$$A\chi + \chi^3 - \chi = u \quad \text{in } H, \text{ for a.e. } t \in (0,T).$$
$$\chi(\,\cdot\,,0) = \chi^0.$$

- **Approximation of the initial data** χ^0 and f: consider the sequences $\{\chi^0_{\varepsilon\delta}\} \subset V$, $\{\vartheta^0_{\varepsilon\delta}\} \subset H$ and $\{f_{\varepsilon\delta}\} \subset L^2(0,T;H)$

$$\chi^0_{\varepsilon\delta} \to \chi^0 \text{ in } H, \quad f_{\varepsilon\delta} \to f \text{ in } L^2(0,T;V')$$

- **Approximate solutions:** for every ε, $\delta > 0$ consider the pair $(\chi_{\varepsilon\delta}, \vartheta_{\varepsilon\delta})$ solving $\mathbf{P}^{\varepsilon\delta}$ with the data $\chi^0_{\varepsilon\delta}$, $\vartheta^0_{\varepsilon\delta}$ and $f_{\varepsilon\delta}$.

Asymptotic behaviour of $\mathbf{P}^{\varepsilon\delta}$ as ε **and** $\delta \downarrow 0$

Theorem 3. **[R.,'03]** *Under analogous assumptions on the approximating initial data* $\{\chi^0_{\varepsilon\delta}\}$, $\{\vartheta^0_{\varepsilon\delta}\}$, *and* $\{f_{\varepsilon\delta}\}$, *there exists a pair* (χ, u) *such that the following convergences hold as* ε, $\delta \downarrow 0$*:*

$$\chi_{\varepsilon\delta} \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;V) \cap L^2(0,T;W),$$

$$\chi_{\varepsilon\delta} \to \chi \quad \text{in } C^0([0,T];V') \cap L^2(0,T;V),$$

$$u_{\varepsilon\delta} \rightharpoonup u \quad \text{in } L^2(0,T;V),$$

$$\varepsilon\vartheta_{\varepsilon\delta} \to 0 \quad \text{in } L^2(0,T;H),$$

$$\delta\partial_t\chi_{\varepsilon\delta} \to 0 \quad \text{in } L^2(0,T;H),$$

Furthermore, $\chi \in C^0([0,T];H)$ *and it is the unique solution for the Neumann problem for the Cahn-Hilliard equation.*

Error estimates for $\varepsilon, \delta \downarrow 0$

There exists a constant $M_{err} \geq 0$*, only depending on* T *and* $|\Omega|$*, such that the error estimates*

$$\|\chi - \chi_{\varepsilon\delta}\|_{C^0([0,T];V')\cap L^2(0,T;V)}$$
$$\leq M_{\rm err}\left(\|\chi^0 - \chi^0_{\varepsilon\delta}\|_{V'} + \delta^{1/2}|\chi^0 - \chi^0_{\varepsilon\delta}|_H\right)$$
$$+ M_{\rm err}\left(\|f - f_{\varepsilon\delta}\|_{L^2(0,T;V')} + \varepsilon^{1/8} + \delta\right)$$
$$\|\varepsilon\vartheta_\varepsilon\|_{L^\infty(0,T;H)} \leq C\varepsilon^{1/4}$$

hold for every ε*,* $\delta \in (0,1)$*.*

Open problem

- Asymptotic analysis as $\varepsilon \downarrow 0$ for the Penrose-Fife phase field system with special heat flux law:

$$\varepsilon\vartheta_t + \chi_t - \Delta(\vartheta - \frac{1}{\vartheta}) = f \quad \text{in } \Omega \times (0,T),$$

$$\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) \ni -\frac{1}{\vartheta} \quad \text{in } \Omega \times (0,T).$$

Difficult!

Work in progress...

Existence for the limit problem:

$$\chi_t - \Delta(\vartheta - \frac{1}{\vartheta}) = f \quad \text{in } \Omega \times (0,T),$$

$$\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) \ni -\frac{1}{\vartheta} \quad \text{in } \Omega \times (0,T).$$