Existence and uniqueness results for a class of quasivariational problems

Risultati di esistenza e unicità per una classe di problemi quasivariazionali

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Moreau's sweeping process

Given a separable Hilbert space H, with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, a final time $T > 0$, a set-valued mapping

$$C : [0, T] \longrightarrow 2^H$$

with $C(t) \neq \emptyset$, **closed**, and **convex** for all $t \in [0, T]$,

and $u_0 \in C(0)$,

we look for $u : [0, T] \longrightarrow H$, $u \in W^{1,1}(0, T; H)$ fulfilling

$$\begin{cases} u'(t) + \partial I_{C(t)}(u(t)) \ni 0 \quad \text{for a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases} \qquad \text{(SP)}$$

Moreau's sweeping process

• A special case of (SP) is the **evolution variational inequality**

$$\text{find } v : [0, T] \to H \text{ with } v(0) = v_0 \text{ and for a.e. } t \in (0, T)$$
$$v(t) \in C', \quad \langle v'(t), v(t) - w \rangle \le \langle f(t), v(t) - w \rangle \quad \forall w \in C'.$$

• Let $N_{C(t)}(u(t))$ be the outward **normal cone** to $C(t)$ at $u(t)$: then (SP) may be rephrased as

$$-u'(t) \in N_{C(t)}(u(t)) \quad \text{for a.e. } t \in (0, T).$$

Applications: in NON-SMOOTH MECHANICS, (e.g., elastoplasticity), CONVEX OPTIMIZATION, MATHEMATICAL ECONOMICS.

Uniqueness for (SP)

Let u_1, $u_2 \in W^{1,1}(0,T;H)$ be two solutions (SP) with data ${u_1}^0$ and ${u_2}^0$: then for a.e. $t \in (0,T)$,

$$u_1(t) \in C(t), \quad \langle {u_1}'(t), v - u_1(t) \rangle \geq 0 \quad \forall v \in C(t), \qquad (1)$$

$$u_2(t) \in C(t), \quad \langle {u_2}'(t), w - u_2(t) \rangle \geq 0 \quad \forall w \in C(t). \qquad (2)$$

Choose $v = u_2(t)$ in (1), $w = u_1(t)$ in (2), sum them up and integrate in time: then,

$$\int_0^t \langle {u_1}'(s) - {u_2}'(s), u_1(s) - u_2(s) \rangle ds \leq 0,$$

whence the **continuous dependence estimate** ($\Rightarrow$ **uniqueness**)

$$|u_1(t) - u_2(t)|^2 \leq |{u_1}^0 - {u_2}^0| \quad \text{for a.e. } t \in (0,T).$$

Approximation: the catching-up algorithm

Notation: Given $x \in H$ and $A, B \subset H$,

$$\mathrm{proj}(x, A) \quad \text{is the projection of } x \text{ on } A,$$

$$e(A, B) := \sup_{a \in A} \inf_{b \in B} |a - b| \quad \text{the Hausdorff semidistance of } A \text{ and } B,$$

$$d_{\mathcal{H}}(A, B) := \max\{e(A, B), e(B, A)\} \quad \text{their Hausdorff distance.}$$

Approximation: Fix a time step τ and a partition of $(0, T)$
$t_0 := 0 < t_1 < \ldots < t_n < \ldots t_N := T$, $t_n - t_{n-1} = \tau$.
Define recursively $\{U^n\}_{n=0}^N$ by

$$U^0 := u_0, \quad U^{n+1} := \mathrm{proj}(U^n, C(t_{n+1})) \in C(t_{n+1})$$

and consider the piecewise constant interpolants

$$\overline{U}_\tau(t) = U^n, \quad t_{n-1} < t \leq t_n, \quad n = 1, \ldots, N.$$

Existence for (SP)

Assume that $C : [0,T] \to 2^H$ has **finite retraction** on $[0,T]$, i.e. for every $[s,t] \subset [0,T]$

$$\mathrm{ret}(C; s,t) := \sup \left\{ \sum_{i=1}^{k} e(C(s_{i-1}), C(s_i)) : \right.$$

$$\left. s_0 := s < \ldots < s_k := t \right\} < +\infty.$$

E.g., when C is **Lipschitz** w.r.t the **Hausdorff distance**

$$d_{\mathcal{H}}(C(t), C(s)) \leq L|t-s| \quad \forall s,t, \in [0,T].$$

Let $r : [0,T] \to [0,+\infty)$ be the **non-decreasing** function fulfilling

$$r(t) - r(s) := \mathrm{ret}(C; s,t) \quad \forall s,t \in [0,T], \; s \leq t.$$

Existence for (SP)

A priori estimates: under these assumptions, we have

$$\|\overline{U}_\tau\|_{L^\infty(0,t)} + \mathrm{Var}_{[0,t]}(\overline{U}_\tau) \le C_1 r(t) + C_2 \le C \quad \forall t \in [0,T], \quad \forall \tau > 0.$$

Compactness of the approximate solutions: there exist a subsequence $\{\overline{U}_{\tau_k}\}$ and $u \in BV([0,T];H)$ such that

$$\overline{U}_{\tau_k}(t) \to u(t) \quad \text{weakly in } H \text{ for every } t \in [0,T],$$
$$\mathrm{Var}_{[s,t]}(u) \le r(t) - r(s) \quad \forall\, [s,t] \subset [0,T].$$

Theorem [Moreau]. If the retraction function

r is **absolutely continuous** on $[0,T]$,

then $u \in W^{1,1}(0,T;H)$ and it is the **unique solution** to (SP).

From a variational to a quasivariational problem

Given a set-valued funtion

$K : [0,T] \times H \to 2^H$ with non-empty, **convex**, and **closed** values, and $u_0 \in K(0, u_0)$,

let us consider the **quasivariational sweeping process**

$$\begin{cases} u'(t) + \partial I_{K(t,u(t))}(u(t)) \ni 0 & \text{for a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{QSP}$$

• (QSP) arises in QUASISTATICAL EVOLUTION PROBLEMS with friction, MICRO-MECHANICAL DAMAGE MODELS, and the evolution of SHAPE MEMORY ALLOYS, and includes **quasivariational evolution inequalities** as special cases.

Existence for (QSP)

Main difficulty: the moving set $K(t, u(t))$ also depends on the current state $u(t)$: it is a state-dependent process.

A first existence result: [Kunze-Monteiro Marques, '98]. Assume that K is **Lipschitz continuous w.r.t the Hausdorff distance**, i.e. $\exists L_1, L_2 > 0$ s.t. $\forall s, t \in [0, T], u, v \in H$

$$d_{\mathcal{H}}(K(t,u), K(s,v)) \leq L_1|t-s| + L_2|u-v| \quad \mathbf{L_2 < 1},$$

and fulfils a **compactness assumption**.
Then, (QSP) has a Lipschitz continuous **solution** $u : [0, T] \to H$, which is the limit of the approximate solutions yielded by the **implicit catching-up algorithm**

$$U^0 := u_0, \quad U^{n+1} := \mathrm{proj}(U^n, K(t_{n+1}, U^{n+1})).$$

Counterexample to existence for (QSP)

If $\mathbf{L_2 \geq 1}$, , (QSP) may have no (absolutely continuous) solutions!

Consider the problem $H := \mathbb{R}$, find $w : [0,1] \longrightarrow \mathbb{R}$ s.t.

$$w'(t) + \partial I_{K'(w(t))}(w(t)) \ni 1, \quad t \in (0,1], \quad w(0) = 0,$$

where $K'(w) := [\psi(w), +\infty)$, with $\psi(w) := (2w - 1/2)^+$, $w \in \mathbb{R}$.

Counterexample to existence for (QSP)

Facts:
$\overline{w}(t) := t$ is the **unique solution** on $[0, 1/2]$;
any solution w fulfils $w' \geq 1$,
whence, for $t > 1/2$, $1/2 < w(t) \leq 1$:
then $w(t) < \psi(w(t))$ and
$w(t) \notin K'(w(t))$! **Absurd!**

- Define $K(t, u) := K'(u + t) - t$: K is **uniformly Lipschitz continuous** with constant $\mathbf{L_2 = 2}$, and the related quasivariational problem (QSP) has **no absolutely continuous solutions** on $[0, 1]$.

Uniqueness for (QSP)

• Due to its **quasivariational** character, (QSP) **loses uniqueness** of solutions (counterexamples even for $L_2 < 1$!): let u_1, $u_2 \in W^{1,1}(0,T;H)$ solve (QSP) with data ${u_1}^0$ and ${u_2}^0$: then,

$$u_1(t) \in K(t, u_1(t)), \quad \langle {u_1}'(t), v - u_1(t) \rangle \geq 0 \quad \forall v \in K(t, u_1(t)),$$
$$u_2(t) \in K(t, u_2(t)), \quad \langle {u_2}'(t), w - u_2(t) \rangle \geq 0 \quad \forall w \in K(t, u_2(t)).$$

No more possible to choose $v := u_2(t)$ $(w := u_1(t))$: in general, $u_2(t) \notin K(t, u_1(t))$ $(u_1(t) \notin K(t, u_2(t)))$!

• **First well-posedness result for (QSP): [Brokate-Krejci-Schnabel, '03].** Assuming that $K : [0,T] \times H \to 2^H$ is "smooth" and strengthening the Lipschitz continuity assumptions (still with numerical restrictions on the Lipschitz constants): **uniqueness** (and existence) is obtained via a fixed point technique!

Outlook

- ¿ Possible to obtain existence for (QSP) without compactness and Lipschitz continuity assumptions on K? $\rightsquigarrow$ Switch to **monotonicity** assumptions on K:
  $\rightsquigarrow$ existence for (QSP) is deduced from existence for (SP)!

- ¿ Possible to obtain uniqueness for (QSP) without smoothness and Lipschitz continuity of K? $\rightsquigarrow$ Switch to **monotonicity** assumptions on K:
  $\rightsquigarrow$ they compensate the quasivariational character of (QSP) and enforce uniqueness!

AN ORDER APPROACH IN $H = \mathbb{R}$

In $H = \mathbb{R}$, the convex-valued function $K : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ fulfils

$\mathbf{K(t,u) = [K_*(t,u), K^*(t,u)]}$ for some $K_*(t,u), K^*(t,u) \in \mathbb{R}$.

Let us suppose that

$$\text{the maps } u \mapsto K^*(t,u),\ \ u \mapsto K_*(t,u)$$

are continuous and non-increasing for every $t \in [0,T]$.

Then, for every $t \in [0,T]$ there exists a unique pair $(c^*(t), c_*(t))$

$$c^*(t) = K^*(t, c^*(t)), \quad c_*(t) = K_*(t, c_*(t)), \quad c_*(t) \le c^*(t).$$

The sweeping process (SP) for set-valued function $C^* : [0,T] \to 2^{\mathbb{R}}$

$$C^*(t) := [c_*(t), c^*(t)] \quad \forall\, t \in [0,T]$$

encodes the quasivariational evolution (QSP).

An order approach in $H = \mathbb{R}$

$$C^*(t) := [c_*(t), c^*(t)],$$

$$c^*(t) = K^*(t, c^*(t)),$$

$$c_*(t) = K_*(t, c_*(t)).$$

Crucial fact: The set-valued function $C^* : [0,T] \to 2^{\mathbb{R}}$ fulfils:

$$C^*(t) = [c_*(t), c^*(t)] = \{u \in H \ : u \in K(t,u)\} \quad \forall\, t \in [0,T],$$

$$C^*(t) \subset K(t,u) \quad \text{for every } u \text{ such that } u \in K(t,u) \quad \forall\, t \in [0,T].$$

Uniqueness for (QSP)

Crucial fact: let u be a solution to (QSP), with $u(0) = u_0$. Let v be a solution to (SP), with $v(0) = u_0$. Then

$$u(t) = v(t) \quad \forall t \in [0, T].$$

Idea: u and v fulfil

$u(t) \in K(t, u(t)), \quad \langle u'(t), z - u(t) \rangle \geq 0 \quad \forall z \in K(t, u(t)),$

$v(t) \in C^*(t), \quad \langle v'(t), w - v(t) \rangle \geq 0 \quad \forall w \in C^*(t),$

Now we **can** choose

$z := v(t)$ (for $v(t) \in K(t, u(t))$!)

$w := u(t)$ (for $u(t) \in C^*(t)$!), whence

$|v(t) - u(t)|^2 \leq |v(0) - u(0)|^2 = 0 \ \forall t \in [0, T].$

Corollary: (QSP) has a **unique solution.**

Existence for (QSP) in $H = \mathbb{R}$

Assume that that there exist $R^*, R_* : [0,T] \to \mathbb{R}$,

$$R^*,\ R_* \text{ absolutely continuous on } [0,T], \text{ s.t.}$$
$$|K^*(t,u) - K^*(s,u)| \leq |R^*(t) - R^*(s)|,$$
$$|K_*(t,u) - K_*(s,u)| \leq |R_*(t) - R_*(s)|$$
$$\text{for all } s,t \in [0,T],\ u \in \mathbb{R}.$$

Then, the associated $C^* : [0,T] \to 2^{\mathbb{R}}$ has **finite retraction** on $[0,T]$, and the **retraction function**

$$r \text{ is absolutely continuous on } [0,T].$$

By Moreau's well-posedness result,

the sweeping process (SP) for to C^* has a unique solution v.

Existence for (QSP) in $H = \mathbb{R}$

Crucial fact: Let $v \in W^{1,1}(0,T)$ be the solution to (SP) for the multifunction C^*. Then, v is also (the unique) solution to (QSP).
Idea: We have to show that

$$v'(t)(z - v(t)) \geq 0 \quad \forall z \in K(t, v(t)), \quad \text{for a.e. } t \in (0,T).$$

Trivial case: $c_*(t) = c^*(t)$ (and then $C^*(t) \equiv K(t, v(t))$); suppose that or $c_*(t) < c^*(t)$: then, for a.e. $t \in (0,T)$

$$v'(t) \begin{cases} \in (-\infty, 0] & \text{if } v(t) = c^*(t), \\ = 0 & \text{if } c_*(t) < v(t) < c^*(t), \\ \in [0, +\infty) & \text{if } v(t) = c_*(t), \end{cases}$$

E.g., if $v(t) = c^*(t)$, then $v(t) = K^*(t, c^*(t)) = K^*(t, v(t))$, so that $z \leq v(t)$ for every $z \in K(t, v(t))$; on the other hand, $v'(t) \leq 0$, and we conclude.

ORDERS IN HILBERT SPACES

Hilbert pseudo-lattices: Given a non-empty subset $P \subset H$ s.t.

$$P = \{u \in H : \langle u, v\rangle \geq 0 \quad \forall v \in P\}$$

(i.e., P is a closed *strict cone*), the relation $\leq$ given by

$$u \leq v \ \text{ iff } \ v - u \in P \quad \forall u, v \in H,$$

is an **order** on H; the pair (H, P) is a **Hilbert pseudolattice**.
Examples: • $(\mathbb{R}, [0, +\infty))$; $(\mathbb{R}^N, Q_N)$, where $Q_N = \{(x_1, \ldots, x_N) \in \mathbb{R}^m : \ x_i \geq 0, \ i = 1, \ldots, N\}$ ($x \leq x'$ iff $x_i \leq x_i{}' \ \forall i = 1, \ldots, N$).
• on $L^2(\Omega)$ ($\Omega \subset \mathbb{R}^N$ be a bounded domain), the **essential pointwise order**, induced by the cone $\mathcal{P} = \{f \in L^2(\Omega) : \ f(x) \geq 0$ for a.e. $x \in \Omega\}$, i.e.,

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for a.e.} \ \ x \in X.$$

Orders and monotonicity in Hilbert spaces

Given a Hilbert pseudolattice (H, P), we introduce

$$u^+ := \mathrm{proj}(u, P), \quad u^- := \mathrm{proj}(-u, P) = (-u)^+ \quad \forall u \in H.$$

Definition Let $F : H \to 2^H$. We say that F is **monotone** iff

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \forall u_1, u_2,\ v_i \in F(u_i),\ i = 1, 2.$$

F is **T-monotone** ([Brezis-Stampacchia, '68]) iff

$$\langle v_1 - v_2, (u_1 - u_2)^+ \rangle \geq 0 \quad \forall u_1, u_2,\ v_i \in F(u_i),\ i = 1, 2.$$

We say that F is **non-decreasing** iff it is single-valued and

$$u_1 \leq u_2 \Rightarrow F(u_1) \leq F(u_2) \quad \forall u_1, u_2 \in D(F).$$

These properties are **equivalent only** in the case $(\mathbb{R}, [0, +\infty))$.

A GENERAL UNIQUENESS RESULT FOR (QSP)

Consider the quasivariational sweeping process (QSP)

$$u'(t) + \partial I_{K(t,u(t))}(u(t)) \ni 0 \quad \text{for a.e. } t \in (0,T), \quad u(0) = u_0,$$

for a $K : [0,T] \times H \to 2^H$ taking **interval values**:

$$K(t,u) = [K_*(t,u), K^*(t,u)] \quad \text{for some} \quad K_*(t,u), K^*(t,u) \in H.$$

Assume that $\forall t \in (0,T)$ the operators $-K^*(t,\cdot),\ -K_*(t,\cdot)$ are

maximal (for graph inclusion within **monotone** operators),

T-monotone,

non-decreasing.

Crucial: for every $t \in [0,T]$ there exists a **unique pair** $(c^*(t), c_*(t))$

$$c^*(t) = K^*(t, c^*(t)), \quad c_*(t) = K_*(t, c_*(t)), \quad c_*(t) \le c^*(t).$$

A general uniqueness result for (QSP)

Define $C^* : [0,T] \to 2^H$ by

$$C^*(t) := [c_*(t), c^*(t)]. \quad \forall t \in [0,T].$$

Crucial: Then, for every $t \in [0,T]$

$$C^*(t) = [c_*(t), c^*(t)] = \{u \in H \ : u \in K(t,u)\},$$
$$C^*(t) \subset K(t,u) \quad \text{for every } u \text{ such that } u \in K(t,u).$$

Theorem 1 [R.-Stefanelli, '04]. Let $u \in W^{1,1}(0,T;H)$ be a solution to (QSP), and $v \in W^{1,1}(0,T;H)$ be a solution to (SP) for the set-valued function C^*, with $v(0) = u_0$. Then

$$u(t) = v(t) \quad \forall t \in [0,T].$$

In particular, (QSP) admits a **unique** solution.

AN EXISTENCE RESULT FOR (QSP) IN THE CASE $H = L^2(\Omega)$

- Consider two functions $f^*,\ f_* : [0,T] \times \Omega \times \mathbb{R} \to [-M, M]$ s.t. for every $t \in [0,T]$ and for almost every $x \in \Omega$ the real functions

$$w \mapsto f^*(t,x,w) \quad \text{and} \quad w \mapsto f_*(t,x,w) \quad \text{are continuous,}$$
$$y \mapsto f^*(t,x,y) \quad \text{and} \quad y \mapsto f_*(t,x,y) \quad \text{are non-increasing.}$$

- Define $\mathcal{K} : [0,T] \times L^2(\Omega) \to 2^{L^2(\Omega)}$ by

$$\mathcal{K}(t,w) := \{z \in L^2(\Omega) : \ f_*(t,x,w(x)) \leq z(x) \leq f^*(t,x,w(x)) \text{ for a.e. } x \in \Omega\}, \quad t \in [0,T],\ w \in L^2(\Omega).$$

$\mathcal{K}(t,w)$ is an interval w.r.t. the pointwise order on $L^2(\Omega)$, with

$$\mathcal{K}^*(t,w)(x) := f^*(t,x,w(x)), \quad \mathcal{K}_*(t,w)(x) := f_*(t,x,w(x)).$$

- Consider the (QSP) driven by $\mathcal{K}$, with initial datum $u_0 \in L^2(\Omega)$.

An existence result for (QSP) in the case $H = L^2(\Omega)$

- Then, there exists a unique pair c^*, $c_* \in L^\infty(0,T;L^2(\Omega))$ s.t.

$$f^*(t,x,c^*(t,x)) = c^*(t,x), \quad f_*(t,x,c_*(t,x)) = c_*(t,x), \quad \text{and}$$
$$c_*(t,x) \le c^*(t,x) \quad \text{for a.e. } x \in \Omega \quad \forall t \in [0,T].$$

- This defines a set-valued function $\mathcal{C} : [0,T] \to 2^{L^2(\Omega)}$ by

$$\mathcal{C}(t)(x) := [c_*(t,x), c^*(t,x)] \quad \text{for a.e. } x \in \Omega \quad \forall t \in [0,T]$$

- Assume that there exist R^*, $R_* \in W^{1,1}([0,T])$ such that

$$|f^*(t,x,w) - f^*(s,x,w)| \le |R^*(t) - R^*(s)|,$$
$$|f_*(t,x,w) - f_*(s,x,w)| \le |R_*(t) - R_*(s)|$$

for all $s,t \in [0,T]$ and $w \in \mathbb{R}$, and almost every $x \in \Omega$.

E.g., f_* and f^* are Lipschitz continuous in t, uniformly w.r.t x, w.

An existence result for (QSP) in the case $H = L^2(\Omega)$

Theorem 2 [R.-Stefanelli, '04]. The sweeping process (SP) for the set-valued function $\mathcal{C}$

$$u'(t) + \partial I_{\mathcal{C}(t)}(u(t)) \ni 0 \quad \text{for a.e. } t \in (0,T), \quad u(0) = u_0,$$

admits a (unique) solution v, which is the unique solution of the quasivariational sweeping process (QSP) for $\mathcal{K}$

$$u'(t) + \partial I_{\mathcal{K}(t,u(t)))}(u(t)) \ni 0 \quad \text{for a.e. } t \in (0,T), \quad u(0) = u_0.$$

This is also an **approximation result** for (QSP)!

Applications: modelization of the **"super-elastic"** effect in **shape memory alloys** [Auricchio-Stefanelli].