Limiti di successioni

Sia $(a_n)_{n\in\mathbb{N}}$ una successione in $\mathbb{R}$ e sia $a \in \mathbb{R}$

- $a_n \to a \quad \Leftrightarrow \quad \forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : \; n > n_\epsilon \; \Rightarrow \; |a_n - a| < \epsilon$
- $a_n \to +\infty \quad \Leftrightarrow \quad \forall M > 0, \exists n_M \in \mathbb{N} : \; n > n_M \; \Rightarrow \; a_n > M$
- $a_n \to -\infty \quad \Leftrightarrow \quad \forall M > 0, \exists n_M \in \mathbb{N} : \; n > n_M \; \Rightarrow \; a_n < -M$

Operazioni sui limiti - Forme d'indecisione

Siano $a, b \in \overline{\mathbb{R}}$, siano $a_n \to a$ e $b_n \to b$. Allora,
se tutte le operazioni coinvolte sono ben definite, si ha:

- $a_n \pm b_n \to a \pm b$;
- $a_n \cdot b_n \to a \cdot b$;
- $\frac{a_n}{b_n} \to \frac{a}{b}$;
- $a_n^{b_n} \to a^b$;

Non sono operazioni ben definite le Forme d'indecisione (o *forme indeterminate*).

$$\infty - \infty, \quad 0 \cdot \infty, \quad \frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^\infty, \quad \infty^0, \quad 0^0$$

Verifica di limiti

1. $\lim_{n\to\infty} \frac{1}{n} = 0$
2. $\lim_{n\to\infty} e^n = \infty$
3. $\lim_{n\to\infty} e^{\frac{1}{n}} = 1$
4. $\lim_{n\to\infty} e^{-n} = 0$
5. $\lim_{n\to\infty} \log n = \infty$
6. $\lim_{n\to\infty} \log\left(\frac{1}{n}\right) = -\infty$
7. $\lim_{n\to\infty} \tan\left(\frac{\pi}{2} - \frac{1}{n}\right) = \infty$
8. $\lim_{n\to\infty} \tan\left(-\frac{\pi}{2} + \frac{1}{n}\right) = -\infty$
9. $\lim_{n\to\infty} \cot\left(\frac{1}{n}\right) = \infty$
10. $\lim_{n\to\infty} \cot\left(\pi - \frac{1}{n}\right) = -\infty$
11. $\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2}$
12. $\lim_{n\to\infty} \frac{n}{2n+5} \neq 1$
13. $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Successioni ottenute da funzioni elementari.

Sia $f : dom(f) \subset \mathbb{R} \to \mathbb{R}$ una funzione elementare,

siano $(a_n)_{n\in\mathbb{N}}$ una successione contenuta in $dom(f)$, $A \in dom(f)$, allora:

$$a_n \to A \quad \Rightarrow f(a_n) \to f(A)$$

Dimostrare i seguenti risultati (per i limiti $1-6$ si intende che A vari nel dominio della funzione in questione):

1. $\lim_{n\to\infty} e^{a_n} = e^A$
2. $\lim_{n\to\infty} \log a_n = \log A$
3. $\lim_{n\to\infty} \sin a_n = \sin A$
4. $\lim_{n\to\infty} \cos a_n = \cos A$
5. $\lim_{n\to\infty} \tan a_n = \tan(A)$
6. $\lim_{n\to\infty} \cot a_n = \cot A$
7. $\lim_{n\to\infty} \sin(\frac{1}{n}) = 0$
8. $\lim_{n\to\infty} \log(cos(\frac{1}{n})) = 0$
9. $\lim_{n\to\infty} \sqrt{\frac{n}{2n+5}} = \frac{1}{\sqrt{2}}$

Limiti notevoli per le successioni

1. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
2. $\lim_{n\to\infty} \sqrt[n]{n^b} = 1$ con $b \in \mathbb{R}$
3. $\lim_{n\to\infty} \sqrt[n]{a} = 1$, con $a > 0$
4. $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$
5. $\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$
6. $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$

Esercizi:

10. $\lim_{n\to\infty} \sqrt[n]{n^3}$ [= 1]
11. $\lim_{n\to\infty} \sqrt[n]{2 + \frac{1}{n}}$ [= +∞]
12. $\lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n}\right)}{\log \sqrt[n]{n}}$ [= 0]
13. $\lim_{n\to\infty} \frac{\log(1+n)}{\log \frac{1}{\sqrt[n]{n}}}$ [= 1]

Scala degli infiniti

Siano $a > 1, b > 0$, allora vale la seguente gerarchia di infiniti::

$$\log_a n \quad \prec \quad n^b \quad \prec \quad a^n \quad \prec \quad n! \quad \prec \quad n^n$$

$$\lim_{n\to\infty} \frac{\log_a n}{n^b} = 0, \qquad \lim_{n\to\infty} \frac{n^b}{a^n} = 0 \qquad \lim_{n\to\infty} \frac{a^n}{n!} = 0 \qquad \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Elevamenti a potenze positive non modificano la scala degli infiniti.

Esercizi:

14. $\lim_{n\to\infty} \frac{(\log n)^{100\sqrt{2}}}{n^\pi}$ $[= 0]$
15. $\lim_{n\to\infty} \frac{n^2}{(\log n)^{2019}}$ $[= +\infty]$
16. $\lim_{n\to\infty} \frac{e^{100n}}{log^7 n}$ $[= +\infty]$
17. $\lim_{n\to\infty} \frac{log_2^{20} n}{3^{2\pi n}}$ $[= 0]$
18. $\lim_{n\to\infty} \frac{2^{1000n}}{n^n}$ $[= 0]$
19. $\lim_{n\to\infty} \frac{n^n}{e^{7n}}$ $[= +\infty]$

Limiti notevoli con $\epsilon_n \to 0$

Sia $\epsilon_n \to 0$ con $\epsilon_n \neq 0$ definitivamente (e opportune restrizioni nel caso del limite 2),

1. $(1 + x \cdot \epsilon_n)^{\frac{1}{\epsilon_n}} \to e^x, \quad \forall x \in \mathbb{R}$
2. $\frac{\log(1+\epsilon_n)}{\epsilon_n} \to 1$
3. $\frac{x^{\epsilon_n}-1}{\epsilon_n} \to \log x, \quad \forall x > 0$
4. $\frac{(1+\epsilon_n)^x-1}{\epsilon_n} \to x, \quad \forall x \in \mathbb{R}$
5. $\lim_{n\to\infty} \frac{\sin \epsilon_n}{\epsilon_n} = 1$
6. $\lim_{n\to\infty} \frac{1-\cos \epsilon_n}{\epsilon_n^2} = \frac{1}{2}$
7. $\lim_{n\to\infty} \frac{e^{\epsilon_n}-1}{\epsilon_n} = 1$

Esercizi:

20. $\lim_{n\to\infty} \frac{\sinh \epsilon_n}{\epsilon_n} \qquad [= 1]$
21. $\lim_{n\to\infty} \frac{\tan \epsilon_n}{\epsilon_n} \qquad [= 1]$
22. $\lim_{n\to\infty} n^2 \left(\log(n+1) + \log \frac{1}{n}\right) \sin \frac{1}{n}$ $[= 1]$
23. $\lim_{n\to\infty} \frac{n^5 \log n - n^5 \log(n+1)}{2n^4 + n^5 \sin \frac{1}{n} + n^6 \sin \frac{1}{n^2}}$ $\left[= -\frac{1}{4}\right]$
24. $\lim_{n\to\infty} \frac{\sqrt{n+3}-\sqrt{n}}{e^{n^{-1/2}}-1} \qquad \left[= \frac{3}{2}\right]$
25. $\lim_{n\to\infty} \frac{2n^{7/2}\left[\sqrt{n^3-\frac{1}{n}}-\sqrt{n^3}\right]}{\log\left[\left(1+\frac{7}{n}\right)^{2n^2}\right]}$ $\left[= -\frac{7}{2}\right]$

Limiti notevoli con $a_n \to \infty$

1. $\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e$

2. $\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{b_n} = e^{\lim_{n\to\infty}\frac{b_n}{a_n}}$

Esercizi:

26. $|a_n| \to +\infty \Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e$

27. $\lim_{n\to\infty}\left(\log(n^2+1)-\log n^2\right) n^2 = 1$

28. $\lim_{n\to\infty}\left(1-\sin\frac{1}{n}\right)^n = e$

29. $\lim_{n\to\infty}\left(\cos\frac{1}{n}+\tan\frac{1}{n}\right)^n = e$

30. $\lim_{n\to\infty}\left(1+\frac{1}{n!}\right)^{n^n}$ $[= +\infty]$

31. $\lim_{n\to\infty}\left(\frac{n^2+2n+7}{n^2-n}\right)^n$ $[= e^3]$

Esercizi

32. $\lim_{n\to\infty} \sin(\frac{1}{n}) + \log(cos(\frac{1}{n}))$ $[= 0]$
33. $\lim_{n\to\infty} \left(\sqrt{\frac{n}{2n+5}}\right)^{\frac{(-1)^n}{n}}$ $[= 1]$
34. $\lim_{n\to\infty} \sqrt{\frac{3n}{2n^2+5}} \cdot \frac{(-1)^n}{n}$ $[= 0]$
35. $\lim_{n\to\infty} \frac{n^2+3}{n^3+3n^2+n}$ $[= 0]$
36. $\lim_{n\to\infty} \frac{\sqrt{n^6+4n^2}}{n^3+5n}$ $[=1]$
37. $\lim_{n\to\infty} \frac{n^5+7n^3+n}{n^4+5n}$ $[= +\infty]$
38. $\lim_{n\to\infty} n! \sin(n!\pi)$ $[= 0]$
39. $\lim_{n\to\infty} \sqrt{n^3+5} - \sqrt{9n^3+1}$ $[= -\infty]$
40. $\lim_{n\to\infty} \frac{3n-1}{n+3}$ $[= 3]$
41. $\lim_{n\to\infty} \frac{n+1}{n^2+3}$ $[= 0]$
42. $\lim_{n\to\infty} \frac{1-n}{\sqrt{n+1}}$ $[= -\infty]$

Esercizi

43. $\lim_{n\to\infty} \frac{n+(-1)^n}{n-(-1)^n}$ $[= 1]$

44. $\lim_{n\to\infty} \frac{e^{3\sqrt{\ln^2 n+\ln n-1}}}{n^3}$; $[= 1]$

45. $\lim_{n\to\infty} e^{n^2} \log[cos(n(n+1)\pi)]$ $[= 0]$

46. $\lim_{n\to\infty} \sqrt{n-2} - \sqrt{n+1}$ $[= 0]$

47. $\lim_{n\to\infty} \sqrt{n^2+3} - \sqrt{n+n^2+1}$ $[= -\frac{1}{2}]$

Esercizi

48. $\lim_{n\to\infty} n \sin\left(\frac{n(n+1)\pi}{2}\right)$ [$= 0$]
49. $\lim_{n\to\infty} 3^n \cos\left(\frac{2n+1}{2}\pi\right)$ [$= 0$]
50. $\lim_{n\to\infty} (-1)^{n!}$ [$= 1$]
51. $\lim_{n\to\infty} \cos\frac{n\pi}{2}$ [$\nexists$]
52. $\lim_{n\to\infty} n^{(-1)^n}$ [$\nexists$]
53. $\lim_{n\to\infty} \frac{n+\sin n}{\arctan n - n}$ [$= -1$]
54. $\lim_{n\to\infty} \sqrt{n^2+4n+3} - n$
[$= 2$]
55. $\lim_{n\to\infty} \frac{(2n)^n \sin n - n^{2n}}{(n+3)^{2n}+6n!}$ [$= -\frac{1}{e^6}$]
56. $\lim_{n\to\infty} \frac{\left(n+\frac{1}{2}\right)^n+3n^n}{n^n+7n!+2^n}$
[$= e^{1/2}+3$]
57. $\lim_{n\to\infty} \frac{n^{n-3}+(n-3)^n}{6n^n+7n!}$ [$= \frac{e^{-3}}{6}$]
58. $\lim_{n\to\infty} \left(\frac{n^3+3n^2+6}{n^3-2n^2+6}\right)^n$ [$= e^5$]
59. $\lim_{n\to\infty} \frac{\log[(n+3)!]-\log n!}{\log 2n^6}$ [$= \frac{1}{2}$]

Esercizi

60. $\lim_{n\to\infty} \log n \cdot \log\left(1+\frac{3}{n^2}\right) = 0$

61. $\lim_{n\to\infty} \frac{\frac{1}{n!}-\frac{1}{(n+1)!}}{\log(n!+5)-\log(n!)} = \frac{1}{5}$

62. $\lim_{n\to\infty} \frac{n-n^{1+\frac{1}{n}}}{\log(n^5)} = -\frac{1}{5}$

63. $\lim_{n\to\infty} \frac{n\left(n^{\frac{1}{7n}}-1\right)}{\log[(n+3)!]-\log[(n+1)!]} = \frac{1}{14}$

64. $\lim_{n\to\infty} \log(5+4e^n) - n = \log 4$

65. $\lim_{n\to\infty} \frac{n!(n+1)^n \sin\left(\frac{7n}{(n+1)!}\right)}{2^n+(n+2)^n} = 7e^{-1}$

66. $\lim_{n\to\infty} \frac{\sin\left(\frac{3}{n}\right)\log\left(1+\frac{2}{n}\right)^n}{\sqrt{n^2+2}-\sqrt{n^2-1}} = 4$

67. $\lim_{n\to\infty} \frac{\log(7+n)-\log n}{n-\sqrt{n^2-1}} = 14$

Esercizi

68. $$\lim_{n\to\infty} \frac{(n!)^{n-1}-[(n-1)!]^n}{[(n-1)!(n-2)]^{n-1}} + \frac{(-1)^n \sin\left(\frac{1}{n+2}\right)}{n!} = \frac{1}{e^2}$$

69. $$\lim_{n\to\infty} \frac{7n^2+\arctan n}{n+\sin(n!)} \frac{n^{n+1}(n-1)!}{(n+2)^n(n+1)!} = 7e^{-2}$$

70. $$\lim_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)^n - 1\right]^{\sqrt[3]{n}} = 0$$

71. $$\lim_{n\to\infty} \frac{(n+2)!\left(e^{\frac{1}{3(n-1)!}}-1\right)}{\sqrt{\frac{n^6}{4}+7n^4+\log^2(n+1)}} = \frac{2}{3}$$

72. $$\lim_{n\to\infty} \left(1+\frac{7}{n}\right)^n \frac{n^{2n}\sin\left(n^{-n}\right)}{\sqrt{n^3+n^{2n}}-\sqrt{n^3}} = e^7$$

73. $$\lim_{n\to\infty} \frac{[(n+2)!-\sin(n^n)]\left[n^{\frac{2}{n}}-1\right]}{(n+1)!\log(n^3+1)+7n^{-n}} = \frac{2}{3}$$

74. $$\lim_{n\to\infty} \frac{\left(e^{\frac{1}{2n!}}-1\right)((n+1)!+2^n)n^n}{(n+1)^{n+1}} = \frac{1}{2e}$$

Limiti dipendenti da parametri

75.

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & |q| < 1 \\ 1 & q = 1 \\ +\infty & q > 1 \\ \nexists & q \leq -1 \end{cases}$$

76.

$$\lim_{n\to\infty} n^a = \begin{cases} 0 & a < 0 \\ 1 & a = 0 \\ +\infty & a > 0 \end{cases}$$

77. Determinare per quali $a \in \mathbb{R}$, esiste finito il

$$\lim_{n\to\infty} \frac{9\log^{a-4} n + 4\arctan n^2}{\log[(n+2)!] - \log n!} \qquad [a \leq 5]$$

Limiti dipendenti da parametri

78. Determinare per quali $a \in \mathbb{R}$, esiste finito il

$$\lim_{n\to\infty} \frac{3e^{n+\log n} + n^{a+1}}{(n+7)(2a)^n} \qquad [a \geq \frac{e}{2}]$$

79. Determinare per quali $a \in \mathbb{R}$, esiste finito il

$$\lim_{n\to\infty} \frac{n^a\sqrt{n}}{n^3\sqrt[3]{n}+1} \qquad [a \leq \frac{17}{6}]$$

80. Determinare per quali $a \in \mathbb{R}$

$$\lim_{n\to\infty} n^a \log(n^2+7) - 2n^a \log n = 7 \qquad [a = 2]$$

81. Determinare il limite al variare di $a \in \mathbb{R}$

$$\lim_{n\to\infty} \left(1 - \cos\frac{7}{n}\right)\sqrt{1+n^{4a}}\log\left(1+\frac{2}{n^2}\right)$$

$$[= 0 \ a < 2; \ = 49 \ a = 2; \ +\infty \ a > 2]$$

Limiti dipendenti da parametri

82. Determinare il limite al variare di $a \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{n^a - 1}{n^5\sqrt{n} + 3}; \qquad [= +\infty \text{ se } a > \tfrac{5}{2}; = 1 \text{ se } a = \tfrac{5}{2}; = 0 \text{ se } a < \tfrac{5}{2}]$$

83. Determinare $a > 0$ tale che

$$\lim_{n\to\infty} \left(\frac{n+a}{n-a}\right)^n = 4 \qquad [a = \log 2]$$

84. Determinare il limite al variare di $a \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{e^{n+a\log n} + 3}{e^{n+7} + n^3} \sin\frac{2}{n^2} \qquad [0\ \ a < 2;\ \frac{2}{e^7}\ \ a = 2; +\infty\ \ a > 2]$$

85. Determinare il limite al variare di $a \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{n + \log n}{n^{a+4}\log n^n + n!} \qquad [0\ \ a \geq -4; +\infty\ \ a < -4]$$

Limiti dipendenti da parametri

86. Determinare il limite al variare di $a > 0$

$$\lim_{n\to\infty} \frac{a^n + \arctan(n!) + 1 - \cos\frac{1}{n^2}}{7^{\log n} + 2^n + \log n^3}$$

$$[0 \ \ 0 < a < 2; 1 \ \ a = 2; +\infty \ \ a > 2]$$

87. Determinare il limite al variare di $a \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{\left(1 - \cos\frac{1}{n}\right)^2 \log\left(e^3 + \frac{1}{n}\right)^n}{\left(\sqrt{n+4}\right)^a}$$

$$[0 \ \ a > -6; \frac{3}{4} \ \ a = -6; +\infty \ \ a < -6]$$

Formula di Stirling

$$n! = n^n e^{-n} \sqrt{2\pi n} e^{\frac{\theta_n}{12n}}$$

con

$$\frac{1}{12n+1} < \frac{\theta_n}{12n} < \frac{1}{12n}$$

quindi:

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$$

Esercizi:

$$\lim_{n\to\infty} \frac{n!(2e)^n \sqrt{n^2+1}}{n^3/2(2n)^n} = \sqrt{2\pi} \qquad \lim_{n\to\infty} \frac{(2n)!}{(n!)^2} = +\infty$$