Esercizi svolti
tratti dai temi
d'esame sulle
successioni

Es. 2 del 01/04/2015

$\beta \in \mathbb{R}$. Calcolare

$$\lim_{n\to+\infty}\left[\frac{(n+1)!-n!}{3n^{n+1}}+\frac{\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n^2}\right)}{3n^{\beta}}\right]$$

Studio il limite delle successioni addende.

$$\lim_{n\to+\infty}\frac{(n+1)!-n!}{3n^{n+1}}=\lim_{n\to+\infty}\frac{(n+1)!\left(1-\frac{1}{n+1}\right)}{3n^{n+1}}$$

$$=\lim_{n\to+\infty}\frac{(n+1)!\left(1-\frac{1}{n+1}\right)}{3(n+1)^{n+1}}\cdot\left(\frac{n+1}{n}\right)^{n+1}$$

ora: $1-\frac{1}{n+1}\longrightarrow 1$

$$\left(\frac{n+1}{n}\right)^{n+1}=\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right)\longrightarrow e$$

mentre $\frac{n+1}{(n+1)^{n+1}}\longrightarrow 0$ (gerarchia degli infiniti)

Quindi il limite della ~~le~~ successione è 0.

Per la seconda, ricordo che

$$\lim_{t\to 0} \frac{\cos(t)-1}{t^2} = -\frac{1}{2}$$

e quindi

$$\cos(t)-1 \sim -\frac{1}{2}t^2$$

applico il limite notevole a $t=\frac{1}{n}$ e $t=\frac{1}{n^2}$ e ottengo

$$\lim_{n\to+\infty} \frac{\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n^2}\right)}{3n^{\beta}} = \lim_{n\to+\infty} \frac{\cos\left(\frac{1}{n}\right)-1-\cos\left(\frac{1}{n^2}\right)+1}{3n^{\beta}}$$

$$= \lim_{n\to+\infty} \frac{-\frac{1}{2n^2}+\frac{1}{2n^4}}{3n^{\beta}} =$$

$$= \lim_{n\to+\infty} \frac{1}{6}\left(\frac{\frac{1}{n^4}-\frac{1}{n^2}}{n^{\beta}}\right)$$

$$= \lim_{n\to+\infty} \frac{1}{6} \frac{1-n^2}{n^{4+\beta}}$$

$$= \begin{cases} 0 & \text{se } 4+\beta > 2 \iff \beta > -2 \\ -1/6 & \text{se } 4+\beta = 2 \iff \beta = -2 \\ -\infty & \text{se } 4+\beta < 2 \iff \beta < -2 \end{cases}$$

Risposta $\boxed{B}$

Es. 2 del 02/02/2015

$\alpha \in \mathbb{R}$. Calcolare

$$\lim_{n\to+\infty} \frac{\left[\sqrt{n^2+3} - n\right]\cos(n) + 3n^\alpha}{\sqrt{n^5+3} + \log(n)}$$

$$L_1 = \lim_{n\to+\infty} \frac{\left(\sqrt{n^2+3} - n\right)\cos(n)}{\sqrt{n^5+3} + \log(n)} =$$

$$= \lim_{n\to+\infty} \frac{n\left(\sqrt{1+\frac{3}{n^2}}-1\right)\cos(n)}{n^{5/2}\left(\sqrt{1+3/n^5}+\frac{\log(n)}{n^{5/2}}\right)}$$

ora moltiplico e divido per $\left[\sqrt{1+\frac{3}{n^2}}+1\right]$

$$= \lim_{n\to+\infty} \frac{\cos(n)\cdot n\cdot\left(\cancel{1}+\frac{3}{n^2}-\cancel{1}\right)}{n^{5/2}\left[\underbrace{\sqrt{1+\frac{3}{n^2}}+1}_{\downarrow 2}\right]\cdot\left(\underbrace{\sqrt{1+\frac{3}{n^5}}}_{\downarrow 1}+\underbrace{\frac{\log(n)}{n^{5/2}}}_{\downarrow 0}\right)}$$

$$= \lim_{n\to+\infty} \frac{\cos(n)}{n^{5/2}\cdot n\underbrace{[\;]}_{\downarrow 2}\underbrace{[\;]}_{\downarrow 1}} = 0$$

$$L_2 = \lim_{n\to+\infty} \frac{3n^{\alpha}}{\sqrt{n^5+3}+\log(n)} =$$

$$= \lim_{n\to+\infty} \frac{3}{n^{5/2-\alpha}\left(\sqrt{1+\frac{3}{n^5}}+\frac{\log(n)}{n^{5/2}}\right)}$$

$$= \begin{cases} 0 & \text{se } 5/2-\alpha>0. \\ 3 & \text{se } 5/2-\alpha=0. \\ +\infty & \text{se } 5/2-\alpha<0. \end{cases}$$

Risposta $\boxed{B}$

ES. 2 del 12/01/2015

$$\lim_{n\to+\infty} \frac{\sin\left(e^{-n}+\frac{2}{n!}\right)\left(\frac{1}{7^n}+2^n\right)+3}{n\left(\log(n+2)-\log(n)\right)} =$$

Esamino separatamente i fattori

AL NUMERATORE:

- $e^{-n}+\frac{2}{n!} = e^{-n}\left(1+\frac{2e^n}{n!}\right) \sim e^{-n}$

perché $\lim_{n\to+\infty} \frac{e^n}{n!} = 0$

(gerarchia di infiniti).

Quindi $\sin\left(e^{-n} + \frac{2}{n!}\right) \sim e^{-n}$

(limite notevole $\sin(t) \sim t$ per $t \to 0$)

- $\frac{1}{7^n} + 2^n = 2^n\left(1 + \frac{1}{(14)^n}\right) \sim 2^n$

Quindi $L = \lim_{n\to\infty} \frac{e^{-n} \cdot 2^n + 3}{n \log\left(\frac{n+2}{n}\right)}$

$$= \lim_{n\to+\infty} \frac{\left(\frac{2}{e}\right)^n + 3}{n \log\left(1 + \frac{2}{n}\right)}$$

<u>ora</u> $\log\left(1 + \frac{2}{n}\right) \sim \frac{2}{n}$, mentre $\left(\frac{2}{e}\right)^n \to 0$

$$\Rightarrow L = \lim_{n\to+\infty} \frac{3}{n \cdot \frac{2}{n}} = \frac{3}{2}$$

Risposta $\boxed{B}$

ES. 4 del 10/11/2014 (1ª prova intermedia)

$$\lim_{n\to+\infty} \frac{\log\left(1+\frac{2}{n^n}\right)\left[(n+1)^n - n^n\right]}{\cos\left(\frac{7}{n}\right)\left[\left(\frac{1}{7}\right)^n + \frac{en^2-1}{n^2+3n+1} - \frac{n!}{n^n}\right]}$$

Al denominatore:

- $\left(\frac{1}{7}\right)^n \to 0$
- $\frac{n!}{n^n} \to 0$ (gerarchia di infiniti)
- $\frac{en^2-1}{n^2+3n+1} \sim e$

• $\cos\left(\frac{7}{n}\right) \longrightarrow 1$ per $n \to +\infty$

Al numeratore: $\log\left(1+\frac{2}{n^n}\right) \sim \frac{2}{n^n}$.

• $(n+1)^n - n^n = n^n\left(\left(\frac{n+1}{n}\right)^n - 1\right) =$

e quindi il numeratore

$$\sim \frac{2}{\cancel{n^n}} \cancel{n^n}\left(\left(\frac{n+1}{n}\right)^n - 1\right)$$

Rimettendo tutto insieme troviamo

$$L = \lim_{n \to +\infty} \frac{2 \cdot \left(\left(\frac{n+1}{n}\right)^n - 1\right)}{e}$$

$$= \frac{2(e-1)}{e} \longrightarrow \text{Risposta } \boxed{C}$$

es. 2 del 13/11/2009

$$\lim_{n\to+\infty} \sqrt[n]{\frac{n^3+1}{n^2-1}} \left(1+\frac{1}{2n}\right)^{3n} =: L$$

- $$\sqrt[n]{\frac{n^3+1}{n^2-1}} =$$

$$= \exp\left(\frac{1}{n}\cdot \log\left(\frac{n^3+1}{n^2-1}\right)\right) \sim \underbrace{\exp\left(\frac{1}{n}\log(n)\right)}_{\to e^0=1 \text{ per } n\to\infty}$$

(Ricordo che $\frac{\log(n)}{n} \to 0$ per la gerarchia degli infiniti)

- $$\left(1+\frac{1}{2n}\right)^{3n} = \exp\left(3n\log\left(1+\frac{1}{2n}\right)\right)$$

$$\sim \exp\left(\frac{3n}{2n}\right) = e^{3/2}.$$

$\Rightarrow L = e^{3/2}$ Risposta $\boxed{C}$

es. 2 del 07/09/2016

$$\lim_{n\to+\infty} \left(\frac{1}{2} n^{1/n} + \frac{\sin(n!)}{n}\right) \cdot \frac{\sqrt{1+n^2+7n^\alpha} - n}{\log(1+e^{n+2}) - n/2}$$

- $n^{1/n} = \exp\left(\frac{1}{n}\log(n)\right) \to e^0 = 1$ per $n\to\infty$

- $\frac{\sin(n!)}{n} \to 0$

e quindi il 1° fattore $\to 1$ per $n\to\infty$

Per il 2° fattore, ho che

$$\log(1+e^{n+2}) - \frac{n}{2} \sim \log(e^{n+2}) - \frac{n}{2}$$
$$\sim n+2-\frac{n}{2} \sim n/2+2$$

Distinguo 3 casi

$\alpha > 2 \Rightarrow$
$$\sqrt{1+n^2+7n^\alpha} - n =$$
$$= \sqrt{n^\alpha\left(\frac{1}{n^\alpha} + \frac{n^2}{n^\alpha} + 7\right)} - n$$
$$\sim n^{\alpha/2} - n$$

e allora
$$L = \lim_{n\to+\infty} \frac{n^{\alpha/2} - n}{\frac{n}{2} + 2} = +\infty$$
perché $\alpha > 2$

$\alpha = 2 \Rightarrow$
$$\sqrt{1+n^2+7n^2} - n$$
$$\sim 2\sqrt{2}\, n - n$$

Siccome il numeratore $\sim \frac{n}{2} + 2$,

concludo che il limite è finito

$\alpha < 2 \Rightarrow \sqrt{1+n^2+7n^\alpha} - n \sim n\left(\sqrt{1+\frac{1}{n^2}+\frac{7n^\alpha}{n^2}} - 1\right)$

$$L = \lim_{n\to+\infty} \frac{n\left(\sqrt{1+\frac{1}{n^2}+\frac{7}{n^{2-\alpha}}}-1\right)}{\frac{n}{2}+2}$$

$$= \lim_{n\to+\infty} \frac{\cancel{n}}{\cancel{n}\underbrace{\left(\frac{1}{2}+\frac{2}{n}\right)}_{\frac{1}{2}}} \cdot \frac{\cancel{1}+\frac{1}{n^2}+\frac{7}{n^{2-\alpha}}\cancel{-1}}{\underbrace{\sqrt{1+\frac{1}{n^2}+\frac{7}{n^{2-\alpha}}}+1}_{2}}$$

$\frac{1}{n^2}+\frac{7}{n^{2-\alpha}} \to 0$

$$= 0$$

Quindi il limite esiste finito

$\iff \alpha \leq 2 \longrightarrow \boxed{B}$

ES. 2 del 04/04/2012

$$\lim_{n\to+\infty} \frac{\left(e^{\frac{1}{2n!}}-1\right)\left((n+1)!+2^n\right)\cdot n^n}{(n+1)^{n+1}} = L$$

Uso il limite notevole

$$\lim_{t\to 0} \frac{e^t-1}{t} = 1$$

$$\Longrightarrow e^{\frac{1}{2n!}} - 1 \sim \frac{1}{2n!}$$

$$\cdot \left((n+1)! + 2^n\right) = (n+1)!\left[1+\underbrace{\frac{2^n}{(n+1)!}}_{\to 0}\right] \sim (n+1)!$$

$$\Rightarrow L = \lim_{n\to+\infty} \frac{\frac{1}{2\cdot n!}\cdot(n+1)!\; n^n}{(n+1)^n\cdot(n+1)} =$$

$$= \lim_{n\to+\infty} \frac{1}{2}\cdot\left(\frac{n}{n+1}\right)^n \frac{\cancel{(n+1)!}}{\cancel{n+1}}\cdot\frac{1}{\cancel{n!}}$$

$$= \lim_{n \to +\infty} \frac{1}{2} \cdot \left(\frac{n+1}{n} \right)^{-n} = \frac{1}{2e}$$

Risposta $\boxed{A}$