Esercizi su integrali doppi: cambiamento di variabili

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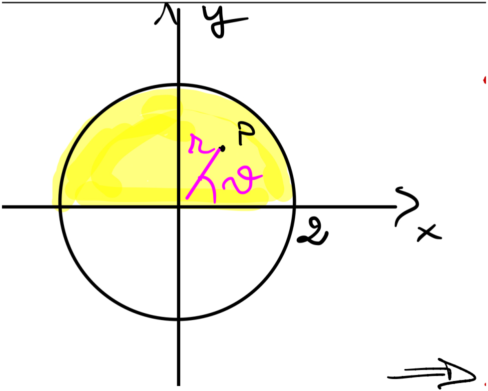
Analisi Matematica B

Es. 1.

$$I = \iint_T e^{-x^2-y^2}\,dxdy$$

ove

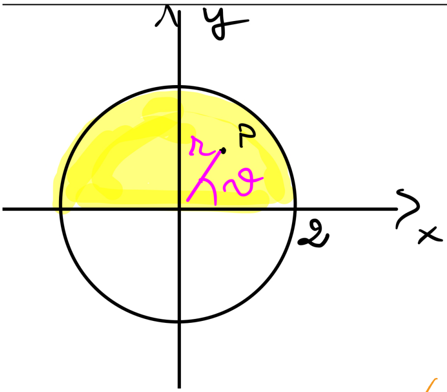
$$T = \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leq 4, \ y \geq 0\}$$



T ha

- simmetria circolare
- in $f(x,y)$ compare l'espressione x^2+y^2

$\Longrightarrow$ conviene usare le COORD. POLARI



Scrivo T in coord. polari

$$\tilde{T}: \begin{cases} r \in [0,2) \\ \vartheta \in [0,\pi] \end{cases}$$

$$= [0,2) \times [0,\pi]$$

(un semidisco è diventato un rettangolo!)

$$f(x,y) = \exp(-(x^2+y^2)) \rightsquigarrow \tilde{f}(r,\vartheta) = \exp(-r^2)$$

$$dx\,dy \rightsquigarrow r\,dr\,d\vartheta$$

$$I = \iint_{[0,2]\times[0,\pi]} \exp(-r^2)\, r \, dr\, d\theta$$

$$= \left(\int_0^2 r \exp(-r^2)\, dr\right)\left(\int_0^\pi 1 \, d\theta\right)$$

NB: sto integrando
su un rettangolo
il prodotto di una fz.
della sola r per una
fz. della sola θ

$$= \pi \left(-\frac{1}{2}\right) \int_0^2 -2\, r \exp(-r^2)\, dr$$

$$= -\frac{\pi}{2}\left[\exp(-r^2)\right]_0^2 = -\frac{\pi}{2}\left(e^{-4}-1\right)$$

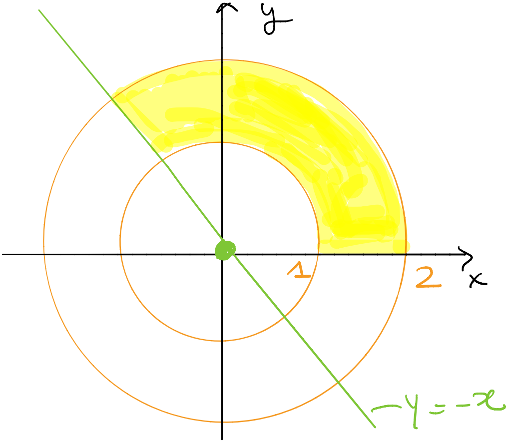
$$= \frac{\pi}{2}\left(1-e^{-4}\right)$$

Es. 2.

$$I = \iint_T y^2 \, dxdy$$

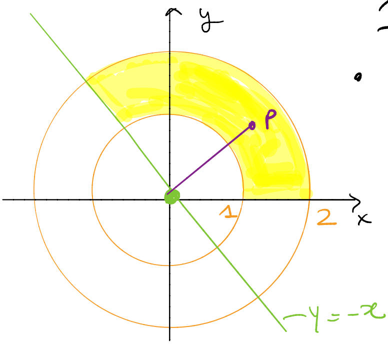
con

$$T = \left\{(x, y) \in \mathbb{R}^2 \ : \ 1 \leq x^2 + y^2 \leq 4, \ y \geq 0, \ x + y \geq 0\right\}$$



DOMINIO con simmetria circolare

$\Rightarrow$ conviene passare alle coordinate polari



- $\tilde{T}$: $1 \le r \le 2$, $\theta \in \left[0, \frac{3}{4}\pi\right]$

$$= [1,2] \times \left[0, \tfrac{3}{4}\pi\right]$$

- $f(x,y) = y^2$

$$\rightsquigarrow \tilde{f}(r,\theta) = r^2 \sin^2(\theta)$$

- $dx\,dy \rightsquigarrow r\,dr\,d\theta$

e quindi

$$I = \iint_{[1,2]\times[0,\frac{3}{4}\pi]} r^2 \sin^2(\theta)\, r\, dr\, d\theta$$

$$= \left(\int_1^2 r^3 \, dr \right) \left(\int_0^{3/4\pi} \sin^2(\vartheta) \, d\vartheta \right)$$

$$= \left[\frac{r^4}{4} \right]_1^2 \left[\frac{\vartheta - \sin(\vartheta)\cos(\vartheta)}{2} \right]_0^{3/4\pi}$$

$$= \frac{15}{4} \cdot \left(\frac{3}{8}\pi - \frac{1}{2} \sin\left(\frac{3}{4}\pi\right) \cos\left(\frac{3}{4}\pi\right) \right)$$

$$= \frac{15}{4} \left(\frac{3}{8}\pi - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \left(-\frac{\sqrt{2}}{2} \right) \right)$$

$$= \frac{15}{4} \left(\frac{3}{8}\pi + \frac{1}{4} \right)$$

Es. 3.

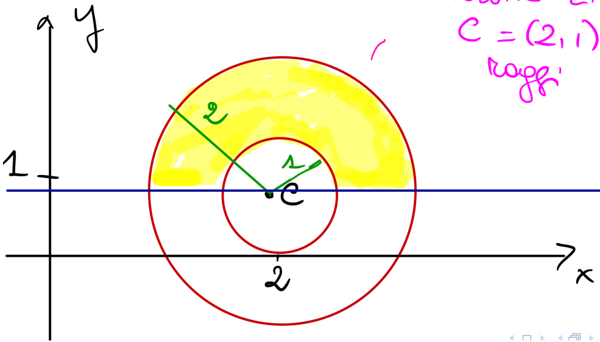
$$I = \iint_T x \, dxdy$$

ove

$$T = \{(x,y) \in \mathbb{R}^2 \ : \ y \geq 1, \ 1 \leq (x-2)^2 + (y-1)^2 \leq 4\}$$

[redacted]

corona circolare con $C=(2,1)$ con raggi 1 e 2



In questo caso il dominio di integrazione ha una simmetria circolare centrata in $C=(2,1)$

$\Rightarrow$ Le coordinate polari appropriate devono riflettere che il centro $C \neq (0,0)$

Come trovarle?

1. Coordinate polari per simmetrie circolari centrate nell'origine

$$x = r\cos(\theta) = r\cos(\theta) + 0 = r\cos(\theta) + x_0$$

$$y = r\sin(\theta) = r\sin(\theta) + 0 = r\sin(\theta) + y_0$$

con $(x_0, y_0) = (0,0)$ centro di simmetria

$\Rightarrow$ 2. coordinate polari per simmetrie circolari con centro $C=(x_0,y_0)$

$$\begin{cases} x = r\cos(\theta) + x_0 \\ y = r\sin(\theta) + y_0 \end{cases} \qquad (*)$$

[In qs. caso $C=(2,1) \Rightarrow \begin{cases} x = r\cos(\theta) + 2 \\ y = r\sin(\theta) + 1 \end{cases}$]

Quale sarà la formula per il cambiamento di variabile corrispondente a $(*)$?

$$dx\,dy \rightarrow |JG(r,\theta)|\,dr\,d\theta$$

Es: per (*), $G(r,\theta) = (r\cos(\theta) + x_0, r\sin(\theta) + y_0)$

verificare che

$$|JG(r,\theta)| = r$$

(intuitivamente, deve essere così, perché ho semplicem. fatto una traslazione)

Quindi

$$\iint_T x \, dxdy = \iint_{\tilde{T}} \tilde{f}(r,\theta) \, r \, dr d\theta$$

$$\tilde{f}(r,\theta) = r\cos(\theta) + 2$$

Come diventa T ?

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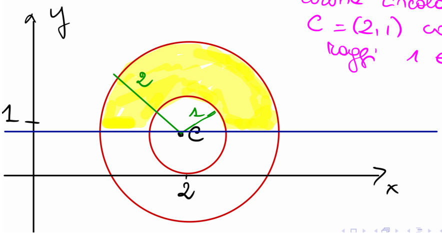


corona circolare con $C=(2,1)$ con raggi 1 e 2

$$\begin{cases} x-2 = r\cos(\theta) \\ y-1 = r\sin(\theta) \end{cases}$$

$$\tilde{T}: \{(r,\theta) \,/\, r\in[1,2],\ \theta\in[0,\pi]\}$$

$$= [1,2]\times[0,\pi]$$

$$I = \iint_{[1,2]\times[0,\pi]} (r\cos(\theta)+2)\, r\,dr\,d\theta =$$

$$= \iint_{[1,2]\times[0,\pi]} r^2\cos(\theta)\,dr\,d\theta + \iint_{[1,2]\times[0,\pi]} 2r\,dr\,d\theta$$

$$= \underbrace{\left(\int_1^2 r^2\,dr\right)\left(\int_0^\pi \cos(\theta)\,d\theta\right)} + \left(\int_1^2 2r\,dr\right)\left(\int_0^\pi 1\,d\theta\right)$$

$$= 0 + \left[r^2\right]_1^2 \cdot \pi$$

perché $\int_0^\pi \cos(\theta)\,d\theta = 0$

$$= 3\pi$$

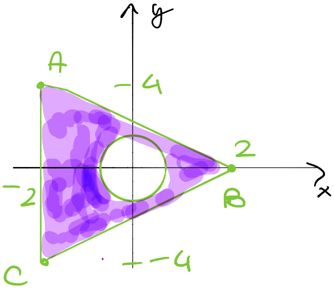
Es. 4.

Siano T il triangolo di vertici $A = (-2, 4)$, $B = (2, 0)$, e $C = (-2, -4)$ e

$$D = \{(x, y) \in T \,:\, x^2 + y^2 \geq 1\}$$

Calcolare

$$I = \iint_D x^2 \, dxdy$$



D è un triangolo bucato.

PASSO 1 : considerazioni di simmetria :

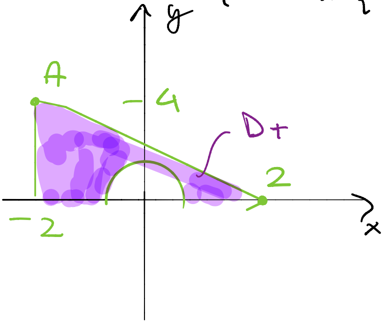
1) D è simmetrico rispetto all'asse x

2) $f(x,y) = x^2$ è PARI in y (infatti non dipende da y)

e quindi

$$I = \iint_D x^2\,dx\,dy = 2\iint_{D_+} x^2\,dx\,dy$$

con $D_+ = D \cap \{y \geq 0\}$



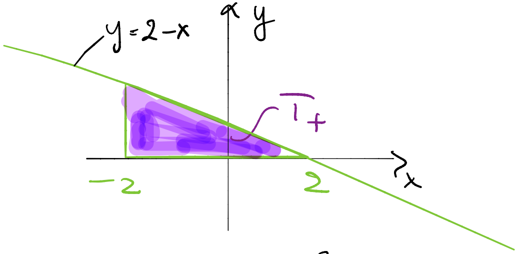
$$D_+ = T_+ \setminus C_+$$

$$\iint_{D_+} x^2\,dx\,dy =$$

$$= \iint_{T_+} x^2\,dx\,dy - \iint_{C_+} x^2\,dx\,dy$$

Calcolo i 2 integrali separatamente

- $$\iint_{T_f} x^2\,dx\,dy$$



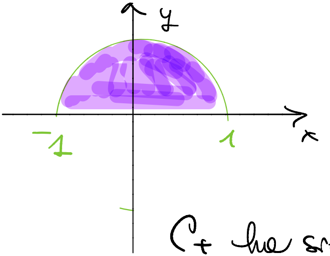
$$T_f = \{(x,y) \in \mathbb{R}^2 \,/\, -2 \le x \le 2,\ 0 \le y \le 2-x\}$$

$$\iint_{T_f} x^2\,dx\,dy = \int_{-2}^{2}\left(\int_0^{2-x} x^2\,dy\right)dx$$

$$= \int_{-2}^{2} x^2(2-x)\,dx = \qquad \int_{-2}^{2}(-x^3)\,dx = 0$$

$$= 2\int_{-2}^{2} x^2\,dx = 4\int_0^2 x^2\,dx = 4\cdot\frac{8}{3} = \frac{32}{3}$$

$$\iint_{C_+} x^2\,dx\,dy$$



C_+ ha simmetria circolare

$\Downarrow$

uso le coordinate polari

$C_+ \longrightarrow \tilde{C}_+$:

$r \in [0,1]$

$\theta \in [0,\pi]$

$$= \iint_{[0,1]\times[0,\pi]} r^2\cos^2(\theta)\, r\,dr\,d\theta$$

$$= \left(\int_0^1 r^3\,dr\right)\cdot\left(\int_0^\pi \cos^2(\theta)\,d\theta\right)$$

$$= \frac{1}{4}\left[\frac{\theta+\sin(\theta)\cos(\theta)}{2}\right]_0^\pi$$

$$= \pi/8$$

Quindi:

- $\iint_{D_+} x^2\,dx\,dy =$

$$\iint_{T_+} x^2\,dx\,dy - \iint_{C_+} x^2\,dx\,dy = \frac{32}{3} - \frac{\pi}{8}$$

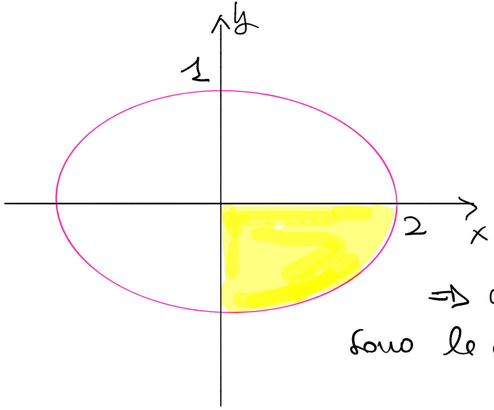
- $I = 2\iint_{D_+} x^2\,dx\,dy = \frac{64}{3} - \pi/4$

Es. 5.

$$I = \iint_D (x+y)\ dxdy$$

ove

$$D = \left\{ (x,y) \in \mathbb{R}^2\ :\ x \geq 0,\ y \leq 0,\ \frac{x^2}{4} + y^2 \leq 1 \right\}$$



Il dominio di integrazione ha simmetria ELLITTICA

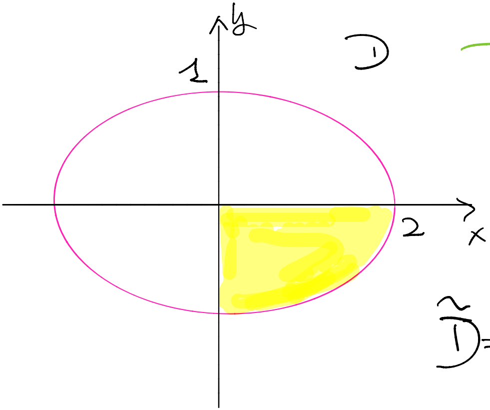
$\Rightarrow$ Le coordinate più appropriate sono le coordinate polari generalizzate, o coord. ellittiche

Per un dominio con simmetria basata su ellisse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, il cambiamento di coordinate appropriato è

$$\begin{cases} x = a r \cos(\theta) \\ y = b r \sin(\theta) \end{cases}$$

$\rightarrow dx\,dy \longrightarrow abr\,dr\,d\theta$



$\rightarrow$ Come diventa D in coordinate ellittiche

$$\begin{cases} x = 2r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

?

$$\tilde{D} = \{ (r, \theta) \ / \quad r \in [0,1], \ \theta \in [\tfrac{3}{2}\pi, 2\pi] \}$$

$$= [0,1] \times [\tfrac{3}{2}\pi, 2\pi]$$

e quindi

$$\iint_D (x+y)\,dx\,dy = \iint_{\tilde{D}} (2r\cos(\theta) + r\sin(\theta)) \cdot 2r\,dr\,d\theta$$

$$= \iint_{[0,1]\times[\frac{3}{2}\pi,\,2\pi]} 4r^2\cos(\theta)\,dr\,d\theta$$

$$+ \iint_{[0,1]\times[\frac{3}{2}\pi,\,2\pi]} 2r^2\sin(\theta)\,dr\,d\theta$$

$$= 4\int_0^1 r^2\,dr \cdot \int_{\frac{3}{2}\pi}^{2\pi} \cos(\theta)\,d\theta$$

$$+ 2\int_0^1 r^2\,dr \cdot \int_{\frac{3}{2}\pi}^{2\pi} \sin(\theta)\,d\theta$$

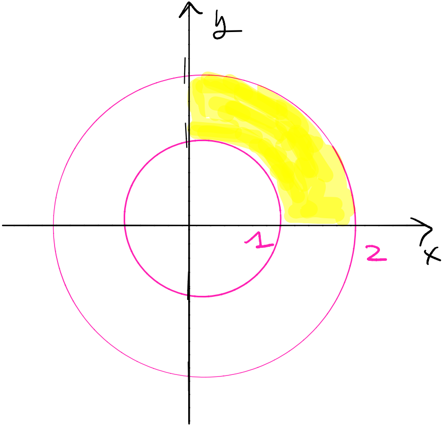
$$= \frac{4}{3}\cdot\Big[\sin(\theta)\Big]_{\frac{3}{2}\pi}^{2\pi} + \frac{2}{3}\cdot\Big[-\cos(\theta)\Big]_{\frac{3}{2}\pi}^{2\pi} = \frac{2}{3}$$

Es. 6.

$$I = 2 \iint_T xy\, e^{\frac{2y^2}{x^2+y^2}}\, dxdy\,,$$

dove

$$T = \{(x,y) \in \mathbb{R}^2 :\ x \geq 0\,,\ y \geq 0\,,\ 1 \leq x^2 + y^2 \leq 4\}$$

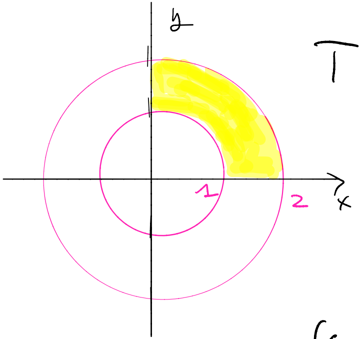


Passo alle coordinate polari!

- Il dominio ha simmetria circolare

+

- $f(x,y)$ contiene il termine x^2+y^2



$$T \longrightarrow \tilde{T} := \{(r,\theta) \,/\, 1 \le r \le 2,\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$= [1,2] \times [0, \pi/2]$$

$$f(x,y) = 2xy \exp\left(\frac{2y^2}{x^2+y^2}\right)$$

$$\downarrow$$

$$\tilde{f}(r,\theta) = 2r^2 \cos(\theta) \sin(\theta) \cdot \exp\left(\frac{2r^2 \sin^2(\theta)}{r^2}\right)$$

$$= 2r^2 \cos(\theta) \sin(\theta) \exp(2 \sin^2(\theta))$$

6

E quindi

$$I = \iint_T 2xy \exp\left(\frac{2y^2}{x^2+y^2}\right) dx\,dy$$

$$= \iint_{[1,2]\times[0,\pi/2]} 2r^2\cos(\theta)\sin(\theta)\exp(2\sin^2(\theta))\, r\,dr\,d\theta$$

$$= \left(\int_1^2 r^3\,dr\right)\left(\int_0^{\pi/2} \overbrace{2\cos(\theta)\sin(\theta)}^{=\frac{1}{2}\frac{d}{d\theta}(2\sin^2(\theta))}\exp(2\sin^2(\theta))\,d\theta\right)$$

$$= \frac{15}{4}\cdot\left[\frac{\exp(2\sin^2(\theta))}{2}\right]_0^{\pi/2}$$

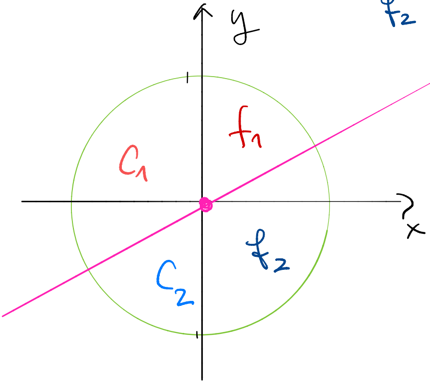
Es. 7.

$$\iint_C f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

con

$$C=\{(x,y)\in\mathbb{R}^2\ :\ x^2+y^2\le 1\},$$

$$f(x,y)=\begin{cases}\overset{f_1}{3} & \text{se } y\ge x,\\ \underbrace{x}_{f_2} & \text{se } y<x\end{cases}$$

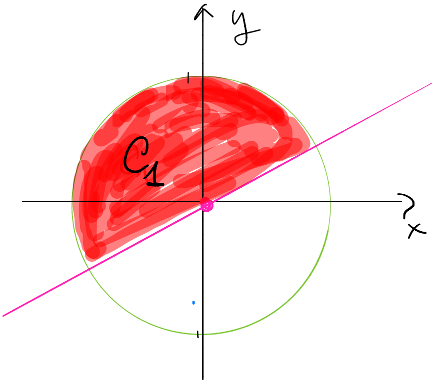


$f_1(x,y)=3$

$f_2(x,y)=x$

$$\iint_C f(x,y)\,dx\,dy =$$

$$= \iint_{C_1} 3\,dx\,dy + \iint_{C_2} x\,dx\,dy$$

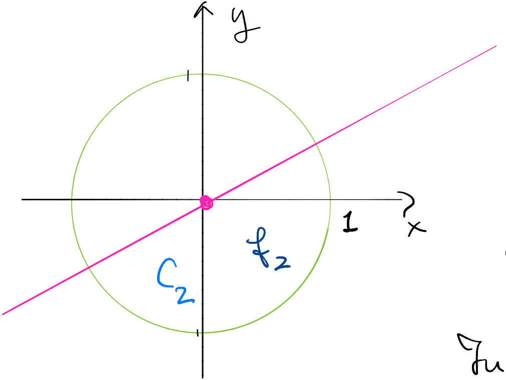


$$\iint_{C_1} 3\,dx\,dy =$$

$$= 3\,\text{area}(C_1)$$

$$= \frac{3\cdot\pi}{2}$$

(C_1 è un semidisco)



$f_2(x,y) = x$

$$I = \iint_{C_2} x \, dx\,dy$$

devo passare alle coordinate polari

In coord. polari

$C_2 \rightsquigarrow \tilde{C}_2 :$ $r \in [0,1]$

$\theta \in [\underbrace{5/4}_{\pi + \frac{\pi}{4}} \pi, \underbrace{9/4}_{2\pi + \frac{\pi}{4}} \pi]$

$= [0,1] \times [5/4 \pi, 9/4 \pi]$

E quindi $\displaystyle\iint_{C_2} x\,dx\,dy = \iint_{\tilde{C}_2} (r\cos(\theta))\, r\,dr\,d\theta$

$$= \left(\int_0^1 r^2\,dr\right) \cdot \left(\int_{5/4\pi}^{9/4\pi} \cos(\theta)\,d\theta\right)$$

$$= \frac{1}{3} \cdot \Big[\sin(\theta)\Big]_{5/4\pi}^{9/4\pi} =$$

$$= 1/3\left(+\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) = \frac{\sqrt{2}}{3}$$

Quindi

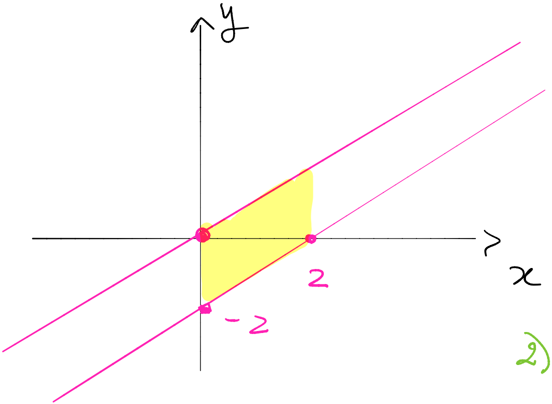
$$I = \frac{3\pi}{2} + \frac{\sqrt{2}}{3}$$

Es. 8.

$$I = \iint_T \frac{(x-y)^2}{1+(x-y)^2}\,dx\,dy$$

con

$$T = \{(x,y) \in \mathbb{R}^2 \;:\; 0 \le x \le 2,\; 0 \le x-y \le 2\}$$



$y \geq x - 2$

$y \leq x$

Osservazione:

1) f dipende da (x-y)

2) T dipende da (x-y)

È quindi passo dalle coordinate (x,y) alle coordinate (u,v) tali che

$$\begin{cases} x = u \\ x - y = v \end{cases}$$

Il cambiam. di VARIABILE:

$$\begin{cases} x = G_1(u,v) = u \\ y = G_2(u,v) = x - v = u - v \end{cases}$$

$$|JG(u,v)| = 1$$

E quindi

$$I = \iint_T \frac{(x-y)^2}{1+(x-y)^2}\,dx\,dy = \iint_{\tilde{T}} \frac{v^2}{1+v^2}\,du\,dv$$

Come diventa il dominio di integrazione?

$T = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le x \le 2,\ 0 \le x-y \le 2\}$

$x = u$

$x - y = v$ $\Rightarrow$ T diventa

$\tilde{T} = [0,2] \times [0,2]$

e quindi $$I = \iint_{[0,2]\times[0,2]} \frac{v^2}{1+v^2}\,du\,dv$$

$$= \left(\int_0^2 1\, du\right) \cdot \left(\int_0^2 \frac{v^2}{1+v^2}\, dv\right)$$

$$= 2 \cdot \left(\int_0^2 \frac{v^2+1-1}{1+v^2}\, dv\right)$$

$$= 2 \left(\int_0^2 \left(1 - \frac{1}{1+v^2}\right) dv\right)$$

$$= 2 \left[v - \arctan(v) \right]_0^2$$

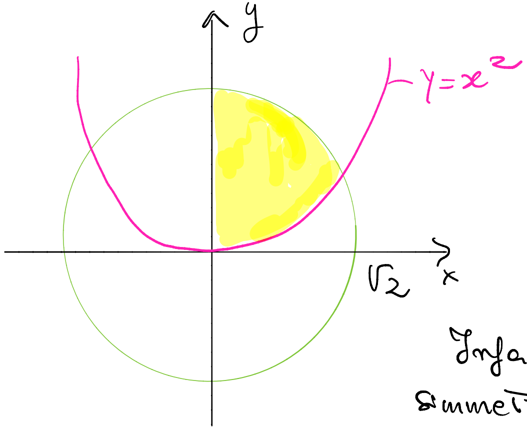
$$= 2 \left[2 - \arctan(2) \right]$$

Es. 9.

$$I = \iint_T y \, \mathrm{d}x \, \mathrm{d}y$$

con

$$T = \{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0, \ x^2 + y^2 \leq 2, \ y \geq x^2\}$$

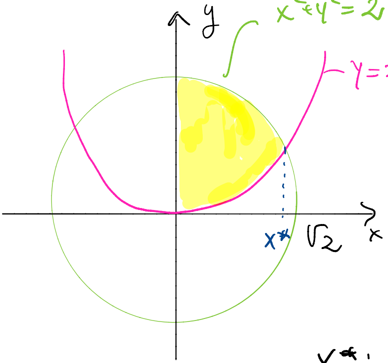


<u>NB</u>: <u>non</u> conviene passare alle coordinate polari anche se T è costruito a partire da un disco!

Infatti, T <u>non</u> ha simmetria circolare!!!

Quindi uso le formule di riduzione!

$x^2+y^2=2\,,\ y\geq 0 \Rightarrow y=\sqrt{2-x^2}$



T è un dominio normale rispetto all'asse x!

$$T:\begin{cases} 0 \leq x \leq x^* \\ x^2 \leq y \leq \sqrt{2-x^2} \end{cases}$$

$$x^*:\begin{cases} y = x^2 \\ x^2+y^2=2 \end{cases} \Rightarrow x^4+x^2-2=0$$

da cui $t^2+t-2=0 \Rightarrow t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2}$

$$= \frac{-1 \pm 3}{2} = \begin{cases} -2 \\ 1 \end{cases}$$

$t = -2$ non è accettabile (è un quadrato)

$t = 1 \Rightarrow \quad x^* = 1 \quad (x^* > 0)$

Quindi

$$T = \left\{ (x, y) \in \mathbb{R}^2 \,/\, \begin{array}{l} 0 \leq x \leq 1 \\ x^2 \leq y \leq \sqrt{2 - x^2} \end{array} \right\}$$

E quindi

$$\iint_T y \, dx dy = \int_0^1 \left(\int_{x^2}^{\sqrt{2-x^2}} y \, dy \right) dx$$

$$= \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{2-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 \left(2 - x^2 - x^4 \right) dx$$

$$= \frac{1}{2} \left[2 - \frac{1}{3} - \frac{1}{5} \right] = \frac{11}{15}$$