Esercizio 8

Calcolare il flusso del rotore del campo

$$\overrightarrow{F}(x,y,z) = y\,\overrightarrow{i}_1 + z\,\overrightarrow{i}_2 + x\,\overrightarrow{i}_3$$

attraverso la superficie

$$\mathcal{S}: \quad \begin{cases} z = 1 - x^2 - y^2, \\ z \geq 0 \end{cases}$$

Passo 1: calcolo rot($\overrightarrow{F}$) Usando formalmente la "regola del determinante" si ha

$$\text{rot}(\overrightarrow{F})(x,y,z)$$

$$= \begin{bmatrix} \overrightarrow{i}_1 & \overrightarrow{i}_2 & \overrightarrow{i}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{bmatrix}$$

$$= \overrightarrow{i}_1 \left(\frac{\partial}{\partial y}x - \frac{\partial}{\partial z}z\right) - \overrightarrow{i}_2 \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial z}y\right) + \overrightarrow{i}_3 \left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}y\right)$$

$$= -\overrightarrow{i}_1 - \overrightarrow{i}_2 - \overrightarrow{i}_3$$

Passo 2: calcolo il flusso. $\mathcal{S}$ è data in forma cartesiana, con

$$\mathcal{S}: \; z = f(x,y) = 1 - x^2 - y^2, \qquad (x,y) \in T = \Big\{(x,y) \in \mathbb{R}^2 \; : \; x^2 + y^2 \leq 1\Big\}.$$

quindi

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = -2x \\ \frac{\partial f}{\partial y}(x,y) = -2y \end{cases}$$

e trovo

$$\overrightarrow{n} = \frac{2x\overrightarrow{i}_1 + 2y\overrightarrow{i}_2 + \overrightarrow{i}_3}{\sqrt{1 + 4x^2 + 4y^2}}$$

Quindi

$$\iint_{\mathcal{S}} \overrightarrow{\text{rot}}(\overrightarrow{F}) \cdot \overrightarrow{n}\, ds$$

$$= \iint_T \overrightarrow{\text{rot}}(\overrightarrow{F}(x,y,f(x,y))) \cdot \left(-\frac{\partial f}{\partial x}(x,y)\overrightarrow{i}_1 - \frac{\partial f}{\partial y}(x,y)\overrightarrow{i}_2 + 1\overrightarrow{i}_3\right) dxdy$$

$$= \iint_T (-\overrightarrow{i}_1 - \overrightarrow{i}_2 - \overrightarrow{i}_3) \cdot (2x\overrightarrow{i}_1 + 2y\overrightarrow{i}_2 + \overrightarrow{i}_3)\, dxdy$$

$$= \iint_T (-2x - 2y - 1)\, dxdy$$

- Per simmetria
$$\iint_T (-2x)\,dxdy = 0, \qquad \iint_T (-2y)\,dxdy = 0$$
- si ha
$$-\iint_T 1\,dxdy = -\text{area}(T) = -\pi$$

Alternativamente, usando il teorema di Stokes:

$$\iint_{\mathcal{S}} \overrightarrow{\text{rot}}(\overrightarrow{F}) \cdot \overrightarrow{n}\,\mathrm{d}S = \oint_{\Gamma=\partial\mathcal{S}} \overrightarrow{F} \cdot \mathrm{d}\Gamma$$

In questo caso

$$\Gamma :\ x^2 + y^2 = 1$$

che parametrizziamo con

$\overrightarrow{r}(t) = \cos(t)\overrightarrow{i}_1 + \sin(t)\overrightarrow{i}_2 + 0\overrightarrow{i}_3,\ \ t \in [0, 2\pi].$

Quindi

$\overrightarrow{r}'(t) = -\sin(t)\overrightarrow{i}_1 + \cos(t)\overrightarrow{i}_2 + 0\overrightarrow{i}_3,\ t \in [0, 2\pi]$

da cui

$$\int_{\mathcal{S}} \overrightarrow{\text{rot}}(\overrightarrow{F}) \cdot \overrightarrow{n}\,\mathrm{d}S = \int_0^{2\pi} \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\,\mathrm{d}t$$
$$= -\int_0^{2\pi} \sin^2(t)\,\mathrm{d}t = \ldots = -\pi$$