Esercizio 3

Calcolare

$$I = \oint_{\Gamma} (e^{x^4} - y)\, dx + [x^3 + \sinh(3y^2)]\, dy$$

dove Γ è l'ellisse $9x^2 + 4y^2 = 36$ percorso <u>due volte</u> in senso <u>orario</u>.

<u>Passo 1:</u> Chiamo $\widetilde{\Gamma}$ la curva che percorre l'ellisse <u>una volta</u> in senso <u>antiorario</u>. Allora

$$I = \oint_{\Gamma} \left[(e^{x^4} - y) \overrightarrow{i}_1 + [x^3 + \sinh(3y^2)] \overrightarrow{i}_2 \right] \cdot d\Gamma$$

$$= -2 \oint_{\widetilde{\Gamma}} \left[(e^{x^4} - y) \overrightarrow{i}_1 + [x^3 + \sinh(3y^2)] \overrightarrow{i}_2 \right] \cdot d\widetilde{\Gamma}$$

<u>Passo 2</u>: Uso la formula di G.G. "al contrario".

Chiamo $E = \{(x,y) \in \mathbb{R}^2 : \ 9x^2 + 4y^2 \leq 36\}$:

$$I = -2 \iint_E \left[\frac{\partial}{\partial x}(x^3 + \sinh(3y^2)) - \frac{\partial}{\partial y}(e^{x^4} - y) \right] dxdy$$

$$= -2 \iint_E \left(3x^2 + 1\right) dxdy$$

$$= -8 \iint_{E^+} \left(3x^2 + 1\right) dxdy$$

Passo 3: Passaggio a coordinate polari

$$\begin{cases} x = 2\rho\cos(\theta) \\ y = 3\rho\sin(\theta) \end{cases} \qquad \Rightarrow \qquad \det J(\rho,\theta) = 6\rho$$

$$T \ \longrightarrow \qquad \begin{aligned} S &= \{(\rho,\theta) : \ 0 \leq \rho \leq 1 \,, \ 0 \leq \theta \leq \tfrac{\pi}{2}\} \\ &= [0,1] \times \left[0, \tfrac{\pi}{2}\right] \end{aligned}$$

Allora:

$$I = -8 \iint_S (12\rho^2 \cos^2(\theta) + 1) 6\rho \, d\rho d\theta$$

$$= -8 \cdot 72 \iint_S \rho^3 \cos^2(\theta) \, d\rho d\theta - 48 \iint_S \rho \, d\rho d\theta$$

$$= -8 \cdot 72 \int_0^1 \rho^3 \left(\int_0^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2} \, d\theta \right) \, d\rho$$

$$-48 \int_0^1 \frac{\pi}{2} \rho \, d\rho$$

$$= -8 \cdot 72 \int_0^1 \rho^3 \left[\frac{1}{2}\theta + \frac{1}{4} \sin(2\theta) \right]_0^{\frac{\pi}{2}} \, d\rho - 24\pi \left[\frac{\rho^2}{2} \right]_0^1$$

$$= -2 \cdot 72\pi \left[\frac{\rho^4}{4} \right]_0^1 - 12\pi = -48\pi$$