Nonlinear Analysis: Real World Applications 22 (2015) 473–507

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Modeling via the internal energy balance and analysis of adhesive contact with friction in thermoviscoelasticity

Elena Bonetti [a,*], Giovanna Bonfanti [b], Riccarda Rossi [b]

[a] Dipartimento di Matematica "F. Casorati", Università di Pavia, Via Ferrata 1, I-27100 Pavia, Italy
[b] DICATAM - Sezione di Matematica, Università di Brescia, Via Valotti 9, I-25133 Brescia, Italy

ARTICLE INFO

Article history:
Received 1 May 2014
Accepted 30 September 2014
Available online 6 November 2014

Keywords:
Internal energy balance
Adhesive contact
Friction
Thermoviscoelasticity
Nonsmooth internal constraints
Global-in-time existence of weak solutions

ABSTRACT

In this paper we introduce and investigate a model for adhesive contact with friction between a thermoviscoelastic body and a rigid support.

A PDE system, consisting of the evolution equations for the temperatures in the bulk domain and on the contact surface, of the momentum balance, and of the equation for the internal variable describing the state of the adhesion, is derived on the basis of a surface damage theory by M. Frémond.

The existence of global-in-time solutions to the associated initial–boundary value problem is proved by passing to the limit in a carefully tailored time-discretization scheme.

1. Introduction

The mathematical field of Contact Mechanics has extensively developed over the last decades, as illustrated, for instance, in the monographs [1–3]. The focus of this paper is on a PDE system pertaining to a subclass of contact models, namely *adhesive contact* and *delamination* models. Their analysis is relevant to a number of mechanical and engineering problems, ranging from fractures in brittle materials, to the investigation of earthquakes, to the study of layered composite structures in machine designing and manufacturing. Indeed, the interface regions between the various laminates essentially affect the strength and stability of the structural elements, and the degradation of the adhesive substance on such regions may lead to material failure.

It turns out that the damage theory can be successfully used for describing adhesive contact between solids, in terms of a suitable internal variable accounting for the state of the adhesion. This approach is in fact mainly due to M. Frémond [4], see also [5,6]. It is closely related to the theory of phase change problems in nonsmooth thermomechanics [7,8], too. The first results on the analysis of adhesive contact and delamination models à la Frémond date back to [9]. These problems have recently attracted remarkable attention, and have been widely investigated, both in the case of *rate-independent* evolution for the adhesion parameter in a series of papers by T. Roubíček, see e.g., [10,11] and, in the case of *rate-dependent*, or *viscous*, evolution, in, e.g., [12–14], and in [15–21].

In this paper, we confine the analysis to the case of a thermoviscoelastic body, located in a domain $\Omega \subset \mathbb{R}^3$, in (adhesive) contact with a *rigid support* on a part Γ_C of its boundary $\partial\Omega$. In accordance with Frémond's theory, we model the evolution of adhesion between the body and the support in terms of a *surface damage* parameter χ, related to the (local) fraction of active molecular links between the body and the support. Therefore, χ takes values in the interval [0, 1], and

* Corresponding author.
E-mail addresses: elena.bonetti@unipv.it (E. Bonetti), giovanna.bonfanti@unibs.it (G. Bonfanti), riccarda.rossi@unibs.it (R. Rossi).

http://dx.doi.org/10.1016/j.nonrwa.2014.09.020
1468-1218/

$\chi = 1$ ($\chi = 0$, *respectively*) means that the adhesive substance is fully effective (that the bonds are completely damaged, resp.), while $\chi \in (0,1)$ describes an intermediate situation. In the isothermal case, the other variable is the displacement vector $\mathbf{u}$, at small strains. In this framework, the analysis was carried out in [15,16] and in [19], where frictional effects were included.

In [17] we introduced and analyzed a model also accounting for thermal effects. Besides the balance of momentum and the adhesion parameter equations, the related PDE system consisted of two evolution equations for the (absolute) temperatures ϑ and ϑ_s in the bulk domain and on the contact surface. Hence, this model allows for different temperatures in Ω and on Γ_C: it reflects the fact that the adhesive substance on Γ_C may have different thermomechanical properties with respect to the thermoviscoelastic body in Ω. In [20], in view of applications to more realistic problems, the temperature-dependent model from [17] was extended to encompass frictional effects. We derived a PDE system for which we proved the existence of global-in-time solutions. Unidirectionality in the evolution of the internal variable χ was included in the model in [21], leading to significant analytical difficulties.

An important and original feature of the anisothermal models analyzed in [17,18,20,21] is that the evolution of the temperature variables ϑ and ϑ_s is governed by *entropy*, in place of *internal energy*, balance equations. While referring to, e.g., [22,23] for a more accurate illustration of this approach, we may mention here that the entropy equations considered in [20] are recovered by rescaling the internal energy balance equations, neglecting some higher order dissipative contributions under the small perturbation assumption. As shown in [20, Sec. 2], this leads to a thermodynamically consistent model, where the strict positivity of ϑ and ϑ_s is so to say enforced by the very form of the equations, cf. (1.7) and (1.9).

Here we aim to further contribute to the understanding of this class of adhesive contact models, by analyzing the temperature-dependent system, with frictional effects, where the equations for ϑ and ϑ_s are derived from the *internal energy*, instead of the *entropy*, balance.

The PDE system. Here and in what follows, Ω is a (sufficiently smooth) bounded domain in $\mathbb{R}^3$, in which the body is located, with $\partial\Omega = \overline{\Gamma}_{\mathrm{Dir}} \cup \overline{\Gamma}_{\mathrm{Neu}} \cup \overline{\Gamma}_{\mathrm{C}}$, and Γ_C the contact surface between the body and the rigid support. From now on, we will suppose that Γ_C is a smooth bounded surface of $\mathbb{R}^2$ (one may think of a *flat* surface). Before detailing the PDE system under study, let us fix the following

Notation 1.1. We denote by $\mathbf{n}$ the outward unit normal vector to $\partial\Omega$ and by $\mathbf{n}_s$ the outward unit normal vector to $\partial\Gamma_C$. We shall write v, in place of $v|_{\Gamma_C}$, for the trace on Γ_C of a function v defined in Ω. We shall also adopt the following notation: given a vector $\mathbf{v} \in \mathbb{R}^3$, we denote by v_N and $\mathbf{v}_T$ its normal component and its tangential part, i.e. $v_N := \mathbf{v}\cdot\mathbf{n}$ and $\mathbf{v}_T := \mathbf{v} - v_N\mathbf{n}$. Analogously, the normal component and the tangential part of the Cauchy stress tensor $\boldsymbol{\sigma}$ (while $\varepsilon(\mathbf{u})$ is the small-strain tensor), will be denoted by σ_N and $\boldsymbol{\sigma}_T$, with $\sigma_N := \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}$ and $\boldsymbol{\sigma}_T := \boldsymbol{\sigma}\mathbf{n} - \sigma_N\mathbf{n}$. Finally, we recall that the multivalued operator $\partial I_C : \mathbb{R} \rightrightarrows \mathbb{R}$ (with C the interval $[0,1]$ or the half-line $(-\infty, 0]$) is the subdifferential (in the sense of convex analysis) of the indicator function of the convex set C, viz. $I_C(u) = 0$ for $u \in C$, and $I_C(u) = +\infty$ otherwise.

A rigorous derivation of the following PDE system can be found in Section 2; here we note that a *regularized* version (cf. (1.5) and (1.6)) shall be analyzed in this paper. Let us mention in advance that, to avoid overburdening notation we shall first write (1.1) as a system of subdifferential inclusions, without specifying the elements, in the various subdifferentials, which actually fulfill the equations. In particular, it has to be understood that the same selection $\eta \in \partial I_{(-\infty,0]}(u_N)$ is considered in (1.1b), (1.1c), (1.1g) and (1.1h). Upon detailing the variational formulation of (1.1) in Section 3.3, more precise notation will be used. The system reads:

$$c_v(\vartheta)\partial_t\vartheta - \vartheta\,\mathrm{div}(\partial_t\mathbf{u}) - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = \boldsymbol{\varepsilon}(\partial_t\mathbf{u})\mathbb{V}\boldsymbol{\varepsilon}(\partial_t\mathbf{u}) + h \quad \text{in } \Omega\times(0,T), \tag{1.1a}$$

$$\mathsf{K}(\vartheta)\nabla\vartheta\mathbf{n}\begin{cases} = 0 & \text{in } (\partial\Omega\setminus\Gamma_C)\times(0,T),\\ \in -k(\chi)\vartheta(\vartheta-\vartheta_s) - \mathfrak{c}'(\vartheta-\vartheta_s)\vartheta\,\partial I_{(-\infty,0]}(u_N)|\partial_t\mathbf{u}_T| & \text{in } \Gamma_C\times(0,T),\end{cases} \tag{1.1b}$$

$$\begin{aligned} c_v(\vartheta_s)\partial_t\vartheta_s - \vartheta_s\partial_t(\lambda(\chi)) - \mathrm{div}(\mathsf{K}(\vartheta_s)\nabla\vartheta_s) &\in |\partial_t\chi|^2 + k(\chi)\vartheta_s(\vartheta-\vartheta_s)\\ &+ (\mathfrak{c}(\vartheta-\vartheta_s) + \vartheta_s\mathfrak{c}'(\vartheta-\vartheta_s))\partial I_{(-\infty,0]}(u_N)|\partial_t\mathbf{u}_T| \quad \text{in } \Gamma_C\times(0,T),\end{aligned} \tag{1.1c}$$

$$\mathsf{K}(\vartheta_s)\nabla\vartheta_s\mathbf{n}_s = 0 \quad \text{in } \partial\Gamma_C\times(0,T), \tag{1.1d}$$

$$-\mathrm{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{with } \boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbb{V}\boldsymbol{\varepsilon}(\partial_t\mathbf{u}) + \vartheta\mathbb{I} \text{ in } \Omega\times(0,T), \tag{1.1e}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\mathrm{Dir}}\times(0,T), \qquad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\mathrm{Neu}}\times(0,T), \tag{1.1f}$$

$$\sigma_N \in -\chi u_N - \partial I_{(-\infty,0]}(u_N) \quad \text{in } \Gamma_C\times(0,T), \tag{1.1g}$$

$$\boldsymbol{\sigma}_T \in -\chi\mathbf{u}_T - \mathfrak{c}(\vartheta-\vartheta_s)\partial I_{(-\infty,0]}(u_N)\partial j(\mathbf{u}_t) \quad \text{in } \Gamma_C\times(0,T), \tag{1.1h}$$

$$\partial_t\chi - \Delta\chi + \partial I_{[0,1]}(\chi) + \gamma'(\chi) \ni -\lambda'(\chi)(\vartheta_s - \vartheta_{\mathrm{eq}}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_C\times(0,T), \tag{1.1i}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{in } \partial\Gamma_C\times(0,T). \tag{1.1j}$$

Let us now get a closer look at the particular equations in system (1.1). In the equation for the bulk temperature ϑ, c_v and K are, respectively, the heat capacity and the heat conductivity. As specified in Section 3.2, they are continuous positive functions satisfying suitable growth conditions. Besides the heat source h, the right-hand side of (1.1a) features a quadratic term in $\varepsilon(\partial_t \mathbf{u})$, with $\mathbb{V} \in \mathbb{R}^{3\times3\times3\times3}$ the viscosity tensor (while $\mathbb{E} \in \mathbb{R}^{3\times3\times3\times3}$ in (1.1e) is the elasticity tensor). The coupling to the Eq. (1.1c) for the temperature on the contact surface is provided by the third-type boundary condition (1.1b) on Γ_{C}. Therein, the function k, related to thermal diffusion, is also suitably smooth. The terms $\vartheta\mathfrak{c}'(\vartheta-\vartheta_s)\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t\mathbf{u}_{\mathrm{T}}|$ and $(\mathfrak{c}(\vartheta-\vartheta_s)+\vartheta_s\mathfrak{c}'(\vartheta-\vartheta_s))\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t\mathbf{u}_{\mathrm{T}}|$ in (1.1b) and (1.1c) need to be understood as $\vartheta\mathfrak{c}'(\vartheta-\vartheta_s)\eta|\partial_t\mathbf{u}_{\mathrm{T}}|$ and $(\mathfrak{c}(\vartheta-\vartheta_s)+\vartheta_s\mathfrak{c}'(\vartheta-\vartheta_s))\eta|\partial_t\mathbf{u}_{\mathrm{T}}|$ with $\eta\in\partial I_{(-\infty,0]}(u_{\mathrm{N}})$. These terms are related to *friction*. The (sufficiently smooth) function $\mathfrak{c}$ has indeed the meaning of a *friction* coefficient, depending on the thermal gap between the temperatures ϑ and ϑ_s, cf. Remark 2.1.

Also the Eq. (1.1c) for ϑ_s features the heat capacity and heat conductivity functions c_v and K, while the function λ is related to the latent heat.

In the (quasistatic, since inertia is neglected) momentum balance (1.1e), the stress tensor $\boldsymbol{\sigma}$ also includes the thermal expansion term $\vartheta\mathbb{I}$, with $\mathbb{I}\in\mathbb{R}^{3\times3}$ the identity matrix. Further, $\mathbf{f}$ is a volume force. The body is fixed on the Dirichlet part Γ_{Dir} of the boundary and a given traction is applied on the Neumann part Γ_{Neu}. In the boundary condition (1.1g)–(1.1h) on the contact surface, the term $\partial I_{(-\infty,0]}(u_{\mathrm{N}})$ enforces the non-interpenetration between the body and the support through the inequality $u_{\mathrm{N}}\le 0$ on Γ_{C}. Indeed, (1.1g) can be recast in complementarity form as

$$u_{\mathrm{N}}\le 0,\qquad \sigma_{\mathrm{N}}+\chi u_{\mathrm{N}}\le 0,\qquad u_{\mathrm{N}}(\sigma_{\mathrm{N}}+\chi u_{\mathrm{N}})=0\quad \text{in } \Gamma_{\mathrm{C}}\times(0,T). \tag{1.2}$$

Observe that (1.2) are a generalization of the classical Signorini conditions for unilateral contact, to the case of adhesive contact. Condition (1.1h) features the subdifferential $\partial j:\mathbb{R}\rightrightarrows\mathbb{R}$ of the function $j:\mathbb{R}^3\to[0,+\infty)$ defined by $j(\mathbf{v})=|\mathbf{v}_{\mathrm{T}}|$. Hence,

$$\partial j(\mathbf{v})=\begin{cases}\dfrac{\mathbf{v}_{\mathrm{T}}}{|\mathbf{v}_{\mathrm{T}}|} & \text{if } \mathbf{v}_{\mathrm{T}}\neq\mathbf{0},\\ \{\mathbf{w}_{\mathrm{T}}:\mathbf{w}\in\overline{B}_1\} & \text{if } \mathbf{v}_{\mathrm{T}}=\mathbf{0},\end{cases} \tag{1.3}$$

with $\overline{B}_1$ the closed unit ball in $\mathbb{R}^3$. Therefore, (1.1h) rephrases in terms of the following three conditions

$$\begin{aligned}
&|\boldsymbol{\sigma}_{\mathrm{T}}+\chi\mathbf{u}_{\mathrm{T}}|\le\mathfrak{c}(\vartheta-\vartheta_s)|\sigma_{\mathrm{N}}+\chi u_{\mathrm{N}}| && \text{in } \Gamma_{\mathrm{C}}\times(0,T),\\
&|\boldsymbol{\sigma}_{\mathrm{T}}+\chi\mathbf{u}_{\mathrm{T}}|<\mathfrak{c}(\vartheta-\vartheta_s)|\sigma_{\mathrm{N}}+\chi u_{\mathrm{N}}|\Longrightarrow\partial_t\mathbf{u}_{\mathrm{T}}=\mathbf{0} && \text{in } \Gamma_{\mathrm{C}}\times(0,T),\\
&|\boldsymbol{\sigma}_{\mathrm{T}}+\chi\mathbf{u}_{\mathrm{T}}|=\mathfrak{c}(\vartheta-\vartheta_s)|\sigma_{\mathrm{N}}+\chi u_{\mathrm{N}}|\Longrightarrow\exists\,\nu\ge0:\ \partial_t\mathbf{u}_{\mathrm{T}}=-\nu(\boldsymbol{\sigma}_{\mathrm{T}}+\chi\mathbf{u}_{\mathrm{T}}) && \text{in } \Gamma_{\mathrm{C}}\times(0,T),
\end{aligned} \tag{1.4}$$

which generalize the *dry friction* Coulomb law, to the case when adhesion effects are taken into account.

Finally, in (1.1i) the subdifferential term $\partial I_{[0,1]}$ enforces the physical constraint that χ takes values in [0, 1], γ is a sufficiently smooth function, and ϑ_{eq} a critical temperature.

Our analysis. The mathematical difficulties attached to system (1.1) are of three different types:

(1) The highly nonlinear character of the equations, due to the presence of several multivalued, subdifferential operators rendering the constraints on the internal variables, the unilateral contact conditions, and the friction law. In particular, let us stress that (1.1h) involves the product of two subdifferentials.
(2) The coupling between *bulk* and *contact surface* equations, which needs sufficient spatial regularity of the variables ϑ and $\mathbf{u}$ for their traces on Γ_{C} to be defined. On the other hand, the mixed boundary conditions on ϑ and $\mathbf{u}$ do not allow for elliptic regularity estimates: in particular the H^2-regularity of ϑ, $\mathbf{u}$, and $\partial_t\mathbf{u}$ seems to be out of reach.
(3) The temperature equations (1.1a) and (1.1c) need to be carefully handled due to the presence of quadratic terms on their right-hand side, which are only estimated in $L^1(\Omega\times(0,T))$ and in $L^1(\Gamma_{\mathrm{C}}\times(0,T))$, respectively.

Handling the product of the two multivalued operators in (1.1h) appears to be an unresolved difficulty, even in the case of contact problems without adhesion and temperature. Therefore, along the lines of the pioneering paper [24] by G. Duvaut, we shall regularize (1.1h) by means of a *nonlocal version* of the Coulomb law. More precisely, we shall replace the nonlinearity in (1.1h) involving friction by the term

$$\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))|(\partial_t\mathbf{u}), \tag{1.5}$$

and, accordingly, the terms $\vartheta\mathfrak{c}'(\vartheta-\vartheta_s)\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t\mathbf{u}_{\mathrm{T}}|$ and $(\mathfrak{c}(\vartheta-\vartheta_s)+\vartheta_s\mathfrak{c}'(\vartheta-\vartheta_s))\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t\mathbf{u}_{\mathrm{T}}|$ in (1.1b) and (1.1c) by

$$\vartheta\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))|\,|\partial_t\mathbf{u}_{\mathrm{T}}|\quad\text{and}\quad(\mathfrak{c}(\vartheta-\vartheta_s)+\vartheta_s\mathfrak{c}'(\vartheta-\vartheta_s))|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))|\,|\partial_t\mathbf{u}_{\mathrm{T}}|, \tag{1.6}$$

respectively. In (1.5) and (1.6), $\mathcal{R}$ is a regularization operator with suitable properties, cf. Hypothesis (V) in Section 3.2. The regularized friction law resulting from the replacement (1.5) in (1.1h) can be interpreted as taking into account nonlocal interactions on the contact surface.

Concerning the temperature equations, in the previous publications [17,18,20,21] the (still thermodynamically consistent) PDE system tackled featured, in place of (1.1a)–(1.1d), the entropy equations

$$\partial_t(\ln(\vartheta))-\operatorname{div}(\partial_t\mathbf{u})-\Delta\vartheta=h\quad\text{in } \Omega\times(0,T), \tag{1.7}$$

$$\partial_{\mathbf{n}}\vartheta\begin{cases}=0 & \text{in } (\partial\Omega\setminus\Gamma_{\mathrm{C}})\times(0,T),\\ \in-k(\chi)(\vartheta-\vartheta_s)-\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))|\,|\partial_t\mathbf{u}_{\mathrm{T}}| & \text{in } \Gamma_{\mathrm{C}}\times(0,T),\end{cases} \tag{1.8}$$

$$\partial_t(\ln(\vartheta_s))-\partial_t(\lambda(\chi))-\Delta\vartheta_s\in k(\chi)(\vartheta-\vartheta_s)+\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))|\,|\partial_t\mathbf{u}_{\mathrm{T}}|\quad\text{in } \Gamma_{\mathrm{C}}\times(0,T), \tag{1.9}$$

$$\partial_{\mathbf{n}_s}\vartheta_s=0\quad\text{in } \partial\Gamma_{\mathrm{C}}\times(0,T), \tag{1.10}$$

where the quadratic dissipative contributions on the right-hand side of (1.1a) and (1.1c) have been neglected. In the present case, instead, these quadratic terms are kept, bringing along considerable difficulties. We will deal with them by resorting to Boccardo–Gallouët [25] type estimates, developed in the same way as in [26], where they were applied to general rate-independent processes in thermoviscoelastic bodies. For these estimates, a crucial role is played by the assumption that the heat capacity c_v and the heat conductivity K have a suitable growth, cf. (3.16)–(3.17).

We will analyze (the Cauchy problem associated with) a regularized version of system (1.1), by a carefully crafted time-discretization scheme. Along the lines of [26], in order to cope with the nonlinear character of the function c_v multiplying $\partial_t\vartheta$ and $\partial_t\vartheta_s$ in (1.1a) and (1.1c), we will use an "enthalpy" transformation, turning the terms $c_v(\vartheta)\partial_t\vartheta$ and $c_v(\vartheta_s)\partial_t\vartheta_s$ into the linear terms $\partial_t w$ and $\partial_t w_s$, with w and w_s as the enthalpy variables (3.7). In fact, we will state our main existence result, Theorem 1, for the (Cauchy problem associated with the) PDE system resulting from the enthalpy change of variable, formulated in a appropriately weak way.

As already mentioned, in the derivation of the a priori estimates on the time-discrete solutions a key role is played by Boccardo–Gallouët techniques. The passage to the limit in the time-discrete scheme hinges on suitable compactness arguments, combined with variational tools for the limit passage in the various subdifferential operators. Concerning the properties of the solution quadruplet $(w, w_s, \mathbf{u}, \chi)$, let us stress that, with a maximum principle argument we will be able to prove that, under a positivity assumption on the heat source h,

$$w \geq 0 \quad \text{a.e. in } \Omega \times (0,T), \qquad w_s \geq 0 \quad \text{a.e. in } \Gamma_C \times (0,T).$$

This will ensure the non-negativity of the original temperatures ϑ and ϑ_s. Observe that, in [17,18,20,21] the *strict* positivity of the temperature variables ϑ and ϑ_s was directly enforced by Eqs. (1.7) and (1.9), via the logarithmic terms therein. Instead, in the present case a strict positivity result seems to be open, essentially due to the nonlinear coupling between the two temperature equations, cf. Remark 4.4.

The model developed in [11] for rate-independent adhesive contact in thermoviscoelasticity is significantly different in this respect. There, Γ_C is the delamination surface between two sub-domains Ω^+ and Ω^- such that $\Omega = \Omega^+ \cup \Omega^-$, and it is not assumed that the adhesive substance on Γ_C may have thermomechanical properties of its own. That is why, the equation for the contact surface temperature is not included in the model. Essentially, the role of the thermal gap $\vartheta - \vartheta_s$ in system (1.1) is played in [11] by the jump of the (bulk) temperature across the interface. Because of this, the analysis in [11] has substantially different aspects to our own. In particular, in [11] the strict positivity of the enthalpy, hence of the temperature, results from a comparison argument which cannot be adapted to our case.

Another significant difference to [11] is that, therein the equation for the adhesion parameter has a rate-independent character. Therefore, as usual for rate-independent systems, cf. e.g. [27], it is formulated in terms of an energy balance, and of a (semi-)stability condition. Despite all the well-known difficulties attached to the analysis of rate-independent evolution, mainly due to a lack of time-regularity for the adhesion parameter, the energetic formulation has the remarkable advantage that the internal parameter equation is no longer formulated as a pointwise-in-time differential inclusion. Hence some of the difficulties related to the identification of the limit of the nonlinear terms and maximal monotone operators featuring in (1.1i) are bypassed. A consequence of this is that unidirectionality in the evolution of the adhesive substance can be accounted for in the analysis. Instead, in the present case we need to pass to the limit in (the time-discrete version of) (1.1i) as a differential inclusion. This is the reason why we are not able to encompass in our model this unidirectionality, cf. Remark 3.8. Nevertheless, we observe that the degradation process of some adhesive substances may be reversible, e.g. as in the case of some polymers or even velcro.

Plan of the paper. In Section 2, the derivation of the PDE system (1.1) is developed following the approach by Frémond. Section 3.1 is devoted to notation, preliminary results, and the setup of the enthalpy transformations, while in Section 3.2 we give the variational formulation of the problem and state Theorem 1. The time-discrete analysis is developed in Section 4, where we prove the existence of solutions to the time-discrete version of system (1.1) and derive a series of a priori estimates on the approximate solutions. Finally, in Section 5 we develop the limit passage to the time-continuous system, and conclude the proof of Theorem 1.

2. The model

We now detail the derivation of the PDE system (1.1), following the approach from [7,8].

The free energy. The thermomechanical equilibrium of the system is governed by a free energy functional, split into two contributions, defined in the domain Ω and on the contact surface Γ_C, depending on the state variables. The latter are χ, its gradient $\nabla\chi$, the strain tensor $\varepsilon(\mathbf{u})$ depending on the vector of small displacement $\mathbf{u}$ (we restrict our analysis to a small perturbation regime), the trace of $\mathbf{u}$ on the contact surface, the absolute temperature ϑ of the body, and the (possibly different) absolute temperature ϑ_s on the contact surface. Thus, the free energy (density) is defined as follows

$$\Psi = \Psi_\Omega + \Psi_{\Gamma_C}. \tag{2.1}$$

The bulk free energy (density) Ψ_Ω in Ω is given by

$$\Psi_\Omega := \widehat{\psi}(\vartheta) + \vartheta\,\mathrm{tr}(\varepsilon(\mathbf{u})) + \frac{1}{2}\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}), \tag{2.2}$$

where the function $\widehat{\psi}$ is sufficiently smooth and concave, and $\mathbb{E}$ is the (positive definite and symmetric) elasticity tensor. The surface free energy Ψ_{Γ_C} on the contact surface Γ_C reads

$$\Psi_{\Gamma_C} = \widehat{\psi}(\vartheta_s) + \lambda(\chi)(\vartheta_s - \vartheta_{eq}) + I_{[0,1]}(\chi) + \gamma(\chi) + \frac{c_N}{2}\chi(u_N)^2 + \frac{c_T}{2}\chi|\mathbf{u}_T|^2 + I_{(-\infty,0]}(u_N) + \frac{\kappa_s}{2}|\nabla\chi|^2, \tag{2.3}$$

where c_N, c_T, κ_s are positive constants. Note that c_N and c_T (which are the adhesive coefficients for the normal and tangential components, respectively) a priori may be different, due to possible anisotropy in the response of the material to stresses. However, for the sake of simplicity in what follows we let $c_N = c_T = \kappa_s = 1$. As already mentioned in the introduction, in (2.3) the terms $I_{[0,1]}(\chi)$ and $I_{(-\infty,0]}(u_N)$ ensure the physical constraint $\chi \in [0, 1]$ and the impenetrability $u_N \le 0$ between the body and the support. The function λ provides the latent heat λ'. Moreover, the term $\lambda(\chi)(\vartheta_s - \vartheta_{eq})$ leads to the contribution $-\lambda'(\chi)(\vartheta_s - \vartheta_{eq})$ on the right-hand side of (1.1i). Here, the critical temperature ϑ_{eq} may be understood as a threshold for damage. According to its sign, $-\lambda'(\chi)(\vartheta_s - \vartheta_{eq})$ is either a contribution to the cohesion of the adhesive substance, or a source of damage for it. In fact, the whole term $-\gamma'(\chi) - \lambda'(\chi)(\vartheta_s - \vartheta_{eq})$ is a (generalized) cohesion of the material, depending on the temperature.

The dissipation potential. The dissipation potential is given by

$$\Phi = \Phi_\Omega + \Phi_{\Gamma_C},$$

with Φ_Ω defined in Ω and Φ_{Γ_C} on Γ_C. For the bulk contribution Φ_Ω we have

$$\Phi_\Omega := \frac{1}{2}\varepsilon(\partial_t\mathbf{u})\mathbb{V}\varepsilon(\partial_t\mathbf{u}) + \frac{\mathsf{K}(\vartheta)}{2\vartheta}|\nabla\vartheta|^2, \tag{2.4}$$

while the contact surface contribution Φ_{Γ_C} reads

$$\Phi_{\Gamma_C} := \mathfrak{c}(\vartheta - \vartheta_s)\,|-R_N + u_N\chi|\,j(\partial_t\mathbf{u}) + \frac{1}{2}|\partial_t\chi|^2 + \frac{\mathsf{K}(\vartheta_s)}{2\vartheta_s}|\nabla\vartheta_s|^2 + \frac{1}{2}k(\chi)(\vartheta - \vartheta_s)^2. \tag{2.5}$$

Observe that Φ_{Γ_C} encompasses the dissipative effects on Γ_C due to friction. Indeed, the positive function $\mathfrak{c}$ has the meaning of a friction coefficient, R_N will be specified later (see (2.20)), and (recall that the contact surface is assumed to be flat), the function j is

$$j(\mathbf{v}) = |\mathbf{v}_T| \quad \text{for all } \mathbf{v} \in \mathbb{R}^3. \tag{2.6}$$

The 1-homogeneity of the function j in (2.6) reflects the rate-independent character of frictional dissipation.

Remark 2.1. Let us highlight that, in our approach, in addition to $\nabla\vartheta_s$ the other dissipative variable, generating heat evolution and transfer on the contact surface, is the thermal gap $\vartheta - \vartheta_s$ between the trace on the contact surface of the volume temperature ϑ, and the contact surface temperature ϑ_s.

This choice reflects the modeling approach for contact with adhesion introduced in [7, Chap. 14]. There it is assumed that, two bodies may have some non-local interactions on the parts of their boundaries that are in contact, so that the difference of their temperatures plays a role in the heat/entropy diffusion, in the same way as the power of internal forces is written for the difference of the (boundary) velocities. Here, we are considering in particular local interactions that could be obtained as the limit of non-local forces as suggested by [7, p. 434] (in a small-perturbation regime). As a consequence, our dissipation potential on the boundary depends on $\partial_t\mathbf{u}_T$ (actually here we are assuming that no displacements occur on the support) and on the difference of the temperatures. Accordingly, we have considered a friction coefficient $\mathfrak{c}$ depending on $\vartheta - \vartheta_s$.

A temperature-dependence (cf. e.g. [2,3]) of the friction coefficient $\mathfrak{c}$ reflects the modeling ansatz that *friction is related to heat*. In fact, a consequence of this, we have the contribution $\mathfrak{c}'(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_N)|\partial_t\mathbf{u}_T|$ as a source of heat on the contact surface Γ_C, both in the boundary condition (1.1b) for ϑ and in Eq. (1.1c) for ϑ_s.

Indeed, we might have allowed for $\mathfrak{c}$ to depend also on the single variables ϑ and ϑ_s, i.e. $\mathfrak{c} = \mathfrak{c}(\vartheta, \vartheta_s, \vartheta - \vartheta_s)$, upon imposing suitable conditions for the analysis. Nonetheless, observe that this would not have affected the structure of system (1.1), as the terms involving $\mathfrak{c}$ only arise from derivatives of the dissipation potential Φ_{Γ_C} with respect to the dissipative variables $\partial_t\mathbf{u}_T$ and $\vartheta - \vartheta_s$, cf. (2.18) and (2.21).

In (2.4) and (2.5), the positive function K is the thermal diffusion coefficient in the bulk domain and on the contact surface, while the positive (and sufficiently smooth) function k is also a contact surface thermal diffusion coefficient, accounting for the heat exchange between the body and the adhesive substance on Γ_C. The assumptions on all of these functions (i.e. on the dissipation potential) have to eventually guarantee that the Clausius–Duhem inequality is satisfied (see (2.24) and (2.25)). Note that here we do not impose any constraint on the sign of the time derivative $\partial_t\chi$. Hence we do not encompass unidirectionality in the degradation of the adhesive substance on Γ_C, but allow for a possibly reversible evolution of surface damage on Γ_C.

The balance equations. The equations of the temperature variables are recovered from the first principle of thermodynamics, i.e. the internal energy balance in Ω and Γ_C. In the bulk domain, the former reads

$$\partial_t e + \operatorname{div}\mathbf{q} = h + P_{int} \quad \text{in } \Omega \times (0, T), \tag{2.7}$$

$$\mathbf{q}\cdot\mathbf{n} = \vartheta F \quad \text{on } \Gamma_c \times (0, T), \qquad \mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on } (\partial\Omega \setminus \Gamma_c) \times (0, T) \tag{2.8}$$

with h an external volume heat (density) source and ϑF the heat flux through the contact surface. Here

$$F = \frac{\mathbf{q}}{\vartheta} \cdot \mathbf{n} =: \mathbf{Q} \cdot \mathbf{n} \tag{2.9}$$

denotes the entropy flux exchanged through the boundary, $e = \Psi_\Omega + s\vartheta$ (s the entropy) the internal energy, $\mathbf{q}$ the heat flux in the bulk domain, and P_{int} the actual power of interior forces. On the contact surface, the internal energy balance reads

$$\partial_t e_{\mathrm{s}} + \operatorname{div} \mathbf{q}_{\mathrm{s}} = \vartheta_{\mathrm{s}} F + P_{\mathrm{s,int}} \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \qquad \mathbf{q}_{\mathrm{s}} \cdot \mathbf{n}_{\mathrm{s}} = 0 \quad \text{on } \partial\Gamma_{\mathrm{C}} \times (0,T), \tag{2.10}$$

where we have denoted by $e_{\mathrm{s}} = \Psi_{\Gamma_{\mathrm{C}}} + \vartheta_{\mathrm{s}} s_{\mathrm{s}}$ and $\mathbf{q}_{\mathrm{s}}$ the energy and the heat flux on the contact surface, respectively (s_{s} the entropy on the contact surface), and by $P_{\mathrm{s,int}}$ the actual power of the interior forces on the contact surface. Note that, we have assumed that the entropy flux through the contact surface is an entropy source for the adhesive substance (indeed, the term F is involved both in the boundary condition (2.8) and on the right-hand side of (2.10)). We couple (2.7) and (2.10) with the generalized momentum equation for macroscopic motions in Ω and with the equation for micro-movements in Γ_{C}. In the momentum balance, inertial effects are neglected, hence we have

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{2.11}$$

$$\boldsymbol{\sigma}\mathbf{n} = -\mathbf{R} \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T), \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Dir}} \times (0,T), \qquad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{Neu}} \times (0,T), \tag{2.12}$$

where $\boldsymbol{\sigma}$ is the macroscopic stress tensor, $\mathbf{f}$ is a volume applied force, $\mathbf{g}$ is a known traction, $\mathbf{R}$ is the action of the obstacle on the solid. and

$$B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \qquad \mathbf{H} \cdot \mathbf{n}_{\mathrm{s}} = 0 \quad \text{on } \partial\Gamma_{\mathrm{C}} \times (0,T), \tag{2.13}$$

with $\mathbf{H}$ and B microscopic internal stresses, responsible for the damage of the adhesive bonds between the body and the support. As for the power of internal forces, we have

$$P_{\mathrm{int}} = \boldsymbol{\sigma}\varepsilon(\partial_t \mathbf{u}), \qquad P_{\mathrm{s,int}} = B\partial_t \chi + \mathbf{H} \cdot \nabla \partial_t \chi + \mathbf{R} \cdot \partial_t \mathbf{u}. \tag{2.14}$$

In particular, let us point out that on the contact surface we are including in the power of internal forces the macroscopic reaction $\mathbf{R}$ as well as microscopic forces B and $\mathbf{H}$, responsible for adhesion at a microscopic level.

The constitutive equations. To recover the PDE system (1.1a)–(1.1j), we have to combine (2.7)–(2.14) with suitable constitutive relations for the involved physical quantities in terms of Ψ and Φ. We have

$$s = -\frac{\partial \Psi_\Omega}{\partial \vartheta}, \qquad s_{\mathrm{s}} = -\frac{\partial \Psi_{\Gamma_{\mathrm{C}}}}{\partial \vartheta_{\mathrm{s}}}, \tag{2.15}$$

$$\mathbf{q} = -\vartheta \frac{\partial \Phi_\Omega}{\partial \nabla \vartheta}, \qquad \mathbf{q}_{\mathrm{s}} = -\vartheta_{\mathrm{s}} \frac{\partial \Phi_{\Gamma_{\mathrm{C}}}}{\partial \nabla \vartheta_{\mathrm{s}}}. \tag{2.16}$$

In particular, from (2.2)–(2.3), (2.15), and the forthcoming constitutive equations, we can relate c_v in (1.1a) and (1.1c) to $\widehat{\psi}$ in (2.2)–(2.3) by

$$c_v(x) = -x\widehat{\psi}''(x). \tag{2.17}$$

Observe that $\widehat{\psi}$ is defined only on $[0, +\infty)$ and concave. Hence, also c_v is only defined for positive temperatures, and positive, as shall be required later, cf. (3.16).

As for as the entropy flux through the boundary F, we impose on $\Gamma_{\mathrm{C}} \times (0,T)$

$$F = \frac{\partial \Phi_{\Gamma_{\mathrm{C}}}}{\partial(\vartheta - \vartheta_{\mathrm{s}})}. \tag{2.18}$$

Then, we prescribe for (the dissipative and non-dissipative contributions to) the stress tensor

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{nd}} + \boldsymbol{\sigma}^{d} = \frac{\partial \Psi_\Omega}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \Phi_\Omega}{\partial \varepsilon(\partial_t \mathbf{u})}, \tag{2.19}$$

and the reaction $\mathbf{R} = R_{\mathrm{N}}\mathbf{n} + \mathbf{R}_{\mathrm{T}}$ reads

$$R_{\mathrm{N}} = R_{\mathrm{N}}^{\mathrm{nd}} = \frac{\partial \Psi_{\Gamma_{\mathrm{C}}}}{\partial u_{\mathrm{N}}}, \tag{2.20}$$

$$\mathbf{R}_{\mathrm{T}} = \mathbf{R}_{\mathrm{T}}^{\mathrm{nd}} + \mathbf{R}_{\mathrm{T}}^{\mathrm{d}} = \frac{\partial \Psi_{\Gamma_{\mathrm{C}}}}{\partial \mathbf{u}_{\mathrm{T}}} + \frac{\partial \Phi_{\Gamma_{\mathrm{C}}}}{\partial(\partial_t \mathbf{u}_{\mathrm{T}})}. \tag{2.21}$$

Finally, B and $\mathbf{H}$ are given by

$$B = B^{\mathrm{nd}} + B^{\mathrm{d}} = \frac{\partial \Psi_{\Gamma_{\mathrm{C}}}}{\partial \chi} + \frac{\partial \Phi_{\Gamma_{\mathrm{C}}}}{\partial(\partial_t \chi)}, \tag{2.22}$$

$$\mathbf{H} = \mathbf{H}^{\mathrm{nd}} + \mathbf{H}^{\mathrm{d}} = \frac{\partial \Psi_{\Gamma_{\mathrm{C}}}}{\partial \nabla \chi} + \frac{\partial \Phi_{\Gamma_{\mathrm{C}}}}{\partial(\nabla(\partial_t \chi))}. \tag{2.23}$$

With these choices, combining (2.15)–(2.23) with (2.7)–(2.14) we derive the PDE system (1.1a)–(1.1j). A discussion of its thermodynamical consistency follows the very same lines of the considerations developed in [20, Sec. 2], to which the reader may refer. However, for the sake of completeness, let us sketch the main ideas of the argument. We first consider the Clausius–Duhem inequality in the bulk domain. Combining (2.7) with (2.15), (2.16), and (2.19), by the chain rule, we get

$$\vartheta\,\partial_t s + \vartheta\,\mathrm{div}\,\mathbf{Q} = \frac{\partial \Phi_\Omega}{\partial \varepsilon(\partial_t \mathbf{u})}\varepsilon(\partial_t \mathbf{u}) + \frac{\partial \Phi_\Omega}{\partial \nabla\vartheta}\nabla\vartheta \geq 0, \tag{2.24}$$

where the latter inequality is due to the properties of the dissipation potential Φ_Ω (2.4) (recall that the viscosity tensor $\mathbb{V}$ is positive definite, K is a strictly positive function, and the absolute temperature is strictly positive). Hence, we write the Clausius–Duhem inequality on the contact surface including the interaction of the adhesive substance with the body. It reads

$$\partial_t s_s + \mathrm{div}\,\mathbf{Q}_s + \partial_t s_{\mathrm{int}} \geq \mathbf{Q}\cdot\mathbf{n}. \tag{2.25}$$

Here, s_{int} is the entropy exchange on the contact surface, defined in terms of the variable $\frac{1}{2}(\vartheta+\vartheta_s)$, and the surface entropy $\mathbf{Q}_s$ is defined as $\mathbf{Q}_s := \mathbf{q}_s/\vartheta_s$. As we have assumed that the entropy F exchanged from the body through the contact surface coincides with the one received by the adhesive substance, we have

$$\frac{1}{2}(\vartheta+\vartheta_s)\partial_t s_{\mathrm{int}} = F(\vartheta-\vartheta_s). \tag{2.26}$$

Thus, due to (2.10), (2.15)–(2.16), (2.20)–(2.23), and (2.26), inequality (2.25) is ensured once

$$\frac{\partial \Phi_{\Gamma_c}}{\partial(\vartheta-\vartheta_s)}(\vartheta-\vartheta_s) + \frac{\partial \Phi_{\Gamma_c}}{\partial(\partial_t\chi)}\partial_t\chi + \frac{\partial \Phi_{\Gamma_c}}{\partial(\partial_t \mathbf{u}_{\mathrm{T}})}\partial_t\mathbf{u}_{\mathrm{T}} + \frac{\partial \Phi_{\Gamma_c}}{\partial(\nabla\vartheta_s)}\nabla\vartheta_s \geq 0. \tag{2.27}$$

The above inequality is verified upon prescribing that

$$\mathsf{K}\geq 0, \qquad \mathfrak{c}\geq 0, \qquad k\geq 0, \qquad \mathfrak{c}'(\vartheta-\vartheta_s)(\vartheta-\vartheta_s)\geq 0.$$

We highlight that all of the above conditions are guaranteed by the forthcoming hypotheses (3.17), (3.20), and (3.21) on K, k and $\mathfrak{c}$.

Remark 2.2. Let us point out that the PDE system tackled in this paper, featuring the non-local regularization for the Coulomb law $\mathcal{R}$, can be derived following the very same procedure described above. Indeed, it is sufficient to replace $\Phi_{\Gamma_{\mathrm{C}}}$ from (2.5) by

$$\Phi_{\Gamma_{\mathrm{C}}} := \mathfrak{c}(\vartheta-\vartheta_s)\,|\mathcal{R}(-R_{\mathrm{N}}+u_{\mathrm{N}}\chi)|\,j(\partial_t\mathbf{u}) + \frac{1}{2}|\partial_t\chi|^2 + \frac{\mathsf{K}(\vartheta_s)}{2\vartheta_s}|\nabla\vartheta_s|^2 + \frac{1}{2}k(\chi)(\vartheta-\vartheta_s)^2. \tag{2.28}$$

In fact, $\Phi_{\Gamma_{\mathrm{C}}}$ is derived only w.r.t. dissipative variables.

3. Setup and main result

Before stating the analytical problem we are solving and the corresponding existence result, we first fix the notation and the assumptions.

3.1. Setup

Throughout the paper we shall assume that

$$\begin{aligned}&\Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^3, \text{ with}\\ &\quad \partial\Omega = \overline{\Gamma}_{\mathrm{Dir}}\cup\overline{\Gamma}_{\mathrm{Neu}}\cup\overline{\Gamma}_{\mathrm{C}},\ \Gamma_{\mathrm{Dir}},\Gamma_{\mathrm{Neu}},\Gamma_{\mathrm{C}}, \text{ open disjoint subsets in the relative topology of } \partial\Omega, \text{ such that}\\ &\quad \mathcal{H}^2(\Gamma_{\mathrm{Dir}}),\ \mathcal{H}^2(\Gamma_{\mathrm{C}})>0, \text{ and } \Gamma_{\mathrm{C}}\subset\mathbb{R}^2 \text{ a sufficiently smooth } \textit{flat} \text{ surface.}\end{aligned} \tag{3.1}$$

More precisely, by *flat* we mean that Γ_{C} is a subset of a hyperplane of $\mathbb{R}^3$ and $\mathcal{H}^2(\Gamma_{\mathrm{C}}) = \mathcal{L}^2(\Gamma_{\mathrm{C}})$, $\mathcal{L}^d$ and $\mathcal{H}^d$ denoting the d-dimensional Lebesgue and Hausdorff measures, respectively. As for smoothness, we require that Γ_{C} has a $\mathrm{C}^{1,1}$-boundary.

Notation 3.1. Given a Banach space X, we denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between its dual space X^* and X itself, and by $\|\cdot\|_X$ the norm in X. However, in the case of spaces of functions with values in $\mathbb{R}^3$ such as, for instance, $L^2(\Omega;\mathbb{R}^3)$, we will often simply write $\|\cdot\|_{L^2(\Omega)}$. Moreover, we shall use the following short-hand notation for function spaces

$$\begin{aligned}&H := L^2(\Omega), \qquad V := H^1(\Omega), \qquad \mathbf{H} := L^2(\Omega;\mathbb{R}^3), \qquad \mathbf{V} := H^1(\Omega;\mathbb{R}^3),\\ &H_{\Gamma_{\mathrm{C}}} := L^2(\Gamma_{\mathrm{C}}), \qquad V_{\Gamma_{\mathrm{C}}} := H^1(\Gamma_{\mathrm{C}}), \qquad Y_{\Gamma_{\mathrm{C}}} := H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{C}}),\\ &\mathbf{W} := \{\mathbf{v}\in\mathbf{V} : \mathbf{v}=\mathbf{0} \text{ a.e. on } \Gamma_{\mathrm{Dir}}\}, \qquad \mathbf{H}_{\Gamma_{\mathrm{C}}} := L^2(\Gamma_{\mathrm{C}};\mathbb{R}^3), \qquad \mathbf{Y} := H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{C}};\mathbb{R}^3),\end{aligned}$$

where we recall that

$$H^{1/2}_{00,\Gamma_{\rm Dir}}(\Gamma_{\rm C}) = \left\{ w \in H^{1/2}(\Gamma_{\rm C}) : \exists\, \tilde{w} \in H^{1/2}(\partial\Omega) \text{ with } \tilde{w} = w \text{ in } \Gamma_{\rm C},\ \tilde{w} = 0 \text{ in } \Gamma_{\rm Dir} \right\}$$

and $H^{1/2}_{00,\Gamma_{\rm Dir}}(\Gamma_{\rm C};\mathbb{R}^3)$ is analogously defined. We will also use the space $H^{1/2}_{00,\Gamma_{\rm Dir}}(\Gamma_{\rm Neu};\mathbb{R}^3)$. The space $\mathbf{W}$ is endowed with the natural norm induced by $\mathbf{V}$. The Laplace operator with homogeneous boundary conditions shall be denoted by

$$A : V_{\Gamma_{\rm C}} \to V^*_{\Gamma_{\rm C}} \qquad \langle A\chi, w\rangle_{V_{\Gamma_{\rm C}}} := \int_{\Gamma_{\rm C}} \nabla\chi\nabla w \,{\rm d}x \quad \text{for all } \chi,\ w \in V_{\Gamma_{\rm C}}. \tag{3.2}$$

We will also make use of the notation

$$m(w) := \frac{1}{\mathcal{L}^d(A)} \int_A w \,{\rm d}x \quad \text{for } w \in L^1(A). \tag{3.3}$$

Linear viscoelasticity. We are in the framework of the linear viscoelasticity theory (see e.g. [20] for some more details). In particular, we prescribe that the fourth-order tensors $\mathbb{E}$ and $\mathbb{V}$ (denoting the elasticity and the viscosity tensor, respectively) are symmetric and positive definite. Moreover, we require that they are uniformly bounded, in such a way that the following bilinear symmetric forms $a, b : \mathbf{W} \times \mathbf{W} \to \mathbb{R}$, defined by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{v})\,{\rm d}x \qquad b(\mathbf{u},\mathbf{v}) := \int_\Omega \varepsilon(\mathbf{u})\mathbb{V}\varepsilon(\mathbf{v})\,{\rm d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{W},$$

are continuous. In particular, we have

$$\exists\, \bar{C} > 0 : \ |a(\mathbf{u},\mathbf{v})| + |b(\mathbf{u},\mathbf{v})| \le \bar{C}\|\mathbf{u}\|_{\mathbf{W}}\|\mathbf{v}\|_{\mathbf{W}} \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{W}. \tag{3.4}$$

Moreover, since $\Gamma_{\rm Dir}$ has positive measure, by Korn's inequality we deduce that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are $\mathbf{W}$-elliptic, i.e., there exist $C_a, C_b > 0$ such that

$$a(\mathbf{u},\mathbf{u}) \ge C_a\|\mathbf{u}\|^2_{\mathbf{W}}, \qquad b(\mathbf{u},\mathbf{u}) \ge C_b\|\mathbf{u}\|^2_{\mathbf{W}} \quad \text{for all } \mathbf{u} \in \mathbf{W}. \tag{3.5}$$

The Gagliardo–Nirenberg inequality. Let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We recall (cf. [28, p. 125]) that for all $r,\ q \in [1,+\infty]$, and for all $v \in W^{1,r}(O) \cap L^q(O)$ there holds

$$\|v\|_{L^s(O)} \le C_{\rm GN}\|v\|^{\theta}_{W^{1,r}(O)}\|v\|^{1-\theta}_{L^q(O)}, \quad \text{with } \frac{1}{s} = \theta\left(\frac{1}{r} - \frac{1}{d}\right) + (1-\theta)\frac{1}{q},\ 0 \le \theta \le 1, \tag{3.6}$$

the positive constant $C_{\rm GN}$ depending only on d, r, q, θ, O.

Enthalpy transformation. We now reformulate (the regularization of) system (1.1), cf. (1.5)–(1.6), in terms of the *enthalpy*, instead of the absolute temperature. More precisely, in the lines of [26], we introduce the enthalpy variables w and $w_{\rm s}$ related to the absolute temperatures ϑ and $\vartheta_{\rm s}$ via

$$w = \widehat{c_v}(\vartheta), \qquad w_{\rm s} = \widehat{c_v}(\vartheta_{\rm s}) \quad \text{with } \widehat{c_v}(r) := \int_0^r c_v(s)\,{\rm d}s. \tag{3.7}$$

Since c_v is a smooth and strictly positive function, $\widehat{c_v}$ is well-defined and strictly increasing. Thus, we may introduce the functions Θ and K defined by

$$\Theta(\omega) := \begin{cases} \widehat{c_v}^{-1}(\omega) & \text{if } \omega \ge 0, \\ 0 & \text{if } \omega < 0, \end{cases} \quad \text{and} \quad K(\omega) := \frac{\mathsf{K}(\Theta(\omega))}{c_v(\Theta(\omega))}. \tag{3.8}$$

In view of (3.7)–(3.8), we rewrite system (1.1) (regularized by means of (1.5)–(1.6)) in terms of the enthalpy variables, i.e.

$$\partial_t w - \Theta(w)\,{\rm div}(\partial_t\mathbf{u}) - {\rm div}(K(w)\nabla w) = \boldsymbol{\varepsilon}(\partial_t\mathbf{u})\mathbb{V}\boldsymbol{\varepsilon}(\partial_t\mathbf{u}) + h \quad \text{in } \Omega \times (0,T), \tag{3.9a}$$

$$K(w)\nabla w\mathbf{n} \begin{cases} = 0 & \text{in } (\partial\Omega \setminus \Gamma_{\rm C}) \times (0,T), \\ \in -k(\chi)\Theta(w)(\Theta(w) - \Theta(w_{\rm s})) & \\ \qquad -\Theta(w)\mathfrak{c}'(\Theta(w) - \Theta(w_{\rm s}))|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\rm N}))|\,|\partial_t\mathbf{u}_{\rm T}| & \text{in } \Gamma_{\rm C} \times (0,T), \end{cases} \tag{3.9b}$$

$$\begin{aligned} &\partial_t w_{\rm s} - \Theta(w_{\rm s})\partial_t(\lambda(\chi)) - {\rm div}(K(w_{\rm s})\nabla w_{\rm s}) \\ &\quad \in |\partial_t\chi|^2 + k(\chi)(\Theta(w) - \Theta(w_{\rm s}))\Theta(w_{\rm s}) \\ &\qquad + (\mathfrak{c}(\Theta(w) - \Theta(w_{\rm s})) + \Theta(w_{\rm s})\mathfrak{c}'(\Theta(w) - \Theta(w_{\rm s})))|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\rm N}))|\,|\partial_t\mathbf{u}_{\rm T}| \quad \text{in } \Gamma_{\rm C} \times (0,T), \end{aligned} \tag{3.9c}$$

$$K(w_{\rm s})\nabla w_{\rm s}\mathbf{n}_{\rm s} = 0 \quad \text{in } \partial\Gamma_{\rm C} \times (0,T), \tag{3.9d}$$

$$-{\rm div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{with } \boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbb{V}\boldsymbol{\varepsilon}(\partial_t\mathbf{u}) + \Theta(w)\mathbb{I} \text{ in } \Omega \times (0,T), \tag{3.9e}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\rm Dir} \times (0,T), \qquad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\rm Neu} \times (0,T), \tag{3.9f}$$

$$\sigma_{\mathrm{N}} \in -\chi u_{\mathrm{N}} - \partial I_{(-\infty,0]}(u_{\mathrm{N}}) \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \tag{3.9g}$$

$$\boldsymbol{\sigma}_{\mathrm{T}} \in -\chi \mathbf{u}_{\mathrm{T}} - \mathfrak{c}(\Theta(w) - \Theta(w_{\mathrm{s}}))|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))|\partial j(\mathbf{u}_t) \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \tag{3.9h}$$

$$\partial_t \chi - \Delta\chi + \partial I_{[0,1]}(\chi) + \gamma'(\chi) \ni -\lambda'(\chi)\Theta(w_{\mathrm{s}}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\mathrm{C}} \times (0,T), \tag{3.9i}$$

$$\partial_{\mathbf{n}_{\mathrm{s}}} \chi = 0 \quad \text{in } \partial\Gamma_{\mathrm{C}} \times (0,T), \tag{3.9j}$$

supplemented by the initial conditions

$$w(0) = w_0 \qquad w_{\mathrm{s}}(0) = w_{\mathrm{s}}^0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \qquad \chi(0) = \chi_0. \tag{3.10}$$

Note that, to simplify notation we have incorporated the contribution $-\lambda'(\chi)\vartheta_{\mathrm{eq}}$ occurring in (1.1i) into the term $\gamma'(\chi)$ in (3.9i), and, as for system (1.1), written the subdifferential inclusions without specifying the selections in the various subdifferentials. Of course, the same convention concerning the term $\partial I_{(-\infty,0]}(u_{\mathrm{N}})$ holds as for (1.1).

Remark 3.2. The enthalpy transformation (3.7) was proposed in [26] in order to deal with PDE systems where a quasilinear internal energy balance analogous to (1.1a) is coupled with *rate-independent* processes in thermoviscoelastic systems. The advantage of this change of variables is that the *nonlinear* term $c_v(\vartheta)\partial_t\vartheta$ in (1.1a) ($c_v(\vartheta_{\mathrm{s}})\partial_t\vartheta_{\mathrm{s}}$ in (1.1c), respectively) is replaced by the *linear* contribution $\partial_t w$ in (3.9a) ($\partial_t w_{\mathrm{s}}$ in (3.9c), respectively). We will exploit this feature, when proving the existence of solutions to the problem (3.9)–(3.10) by means of a time-discretization scheme.

3.2. Assumptions

From now on, we will focus on the analysis of system (3.9) in the enthalpy variables w and w_{s}. Hence, in what follows we are going to state our assumptions directly on the nonlinear functions Θ and K featuring in (3.9a) and (3.9c); we will translate them in terms of requirements on the original functions c_v and K in Remark 3.3. As far as the other nonlinearities are concerned, let us mention in advance that, indeed, we are going to address a slightly extended version of system (3.9), where the subdifferential operators $\partial I_{(-\infty,0]}$ and $\partial I_{[0,1]}$ are replaced by general maximal monotone operators. In particular, we will replace $I_{(-\infty,0]}$ in (2.3) with a general convex function ϕ, dropping the constraint that $\mathrm{dom}(\phi) \subset (-\infty,0]$. Let us emphasize that the *physical case*, in which the constraint $u_{\mathrm{N}} \le 0$ on Γ_{C} is enforced, occurs when $\mathrm{dom}(\phi) \subset (-\infty,0]$, and it is included in our analysis. Similarly, in place of $I_{[0,1]}$ we will consider a general convex $\widehat{\beta}$, with $\mathrm{dom}(\widehat{\beta}) \subset [0,\infty)$, encompassing the particular case $\widehat{\beta} = I_{[0,1]}$. These generalizations will allow us to highlight which properties are actually needed in the analysis of (3.9), at the same time making the structure of the a priori estimates more transparent.

We now list our assumptions:

Hypothesis (I). We consider a function

$$\phi : \mathbb{R} \to [0,+\infty] \text{ proper, convex and lower semicontinuous, with } \phi(0) = 0 \tag{3.11}$$

and effective domain $\mathrm{dom}(\phi)$. Then, we define

$$\varphi : Y_{\Gamma_{\mathrm{C}}} \to [0,+\infty] \text{ by } \varphi(v) := \begin{cases} \int_{\Gamma_{\mathrm{C}}} \phi(v)\,\mathrm{d}x & \text{if } \phi(v) \in L^1(\Gamma_{\mathrm{C}}), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.12}$$

Hence, we introduce

$$\boldsymbol{\varphi} : \mathbf{Y} \to [0,+\infty], \text{ defined by } \boldsymbol{\varphi}(\mathbf{u}) := \varphi(u_{\mathrm{N}}) \quad \text{for all } \mathbf{u} \in \mathbf{Y}. \tag{3.13}$$

Since $\boldsymbol{\varphi} : \mathbf{Y} \to [0,+\infty]$ is a proper, convex and lower semicontinuous functional on $\mathbf{Y}$, its subdifferential $\partial\boldsymbol{\varphi} : \mathbf{Y} \rightrightarrows \mathbf{Y}^*$ is a maximal monotone operator.

Hypothesis (II). We consider the multivalued operator $\partial\widehat{\beta} : \mathbb{R} \rightrightarrows \mathbb{R}$ with

$$\widehat{\beta} : \mathbb{R} \to (-\infty,+\infty] \text{ proper, convex and lower semicontinuous, such that } \mathrm{dom}(\widehat{\beta}) \subset [0,+\infty). \tag{H4}$$

In what follows, we will use the notation $\beta := \partial\widehat{\beta}$.

Hypothesis (III). As for the function Θ, we require that

$$\Theta : \mathbb{R} \to \mathbb{R} \text{ is continuous, non-decreasing and}$$

$$\exists\, \sigma > \frac{8}{5}, \exists\, c_1, c_2 > 0\ \forall\, \theta \ge 0 : \ \Theta(\theta) \le c_1\theta^{1/\sigma} + c_2, \quad \text{while } \Theta(\theta) = 0 \text{ for all } \theta < 0. \tag{3.14}$$

It follows from (3.14) that, in fact, Θ has sublinear growth.

Hypothesis (IV). Concerning the function K, we assume that

$$K : \mathbb{R} \to \mathbb{R} \text{ is continuous and } \exists\, c_3,\ c_4 > 0\ \forall\, \theta \in \mathbb{R} :\ c_3 \le K(\theta) \le c_4. \tag{3.15}$$

Remark 3.3. Observe that, if the functions Θ and K are defined from c_v and K through (3.7) and (3.8), properties (3.14) and (3.15) are guaranteed as soon as the functions c_v and K comply with

$$c_v : [0, +\infty) \to [0, +\infty) \text{ is continuous and } \exists\, \tilde{c}_1 > 0\ \forall\, \theta \in [0, +\infty) : c_v(\theta) \ge \tilde{c}_1(1+\theta)^{\sigma-1}, \tag{3.16}$$

$$\mathsf{K} : [0, +\infty) \to [0, +\infty) \text{ is continuous and } \exists\, \tilde{c}_3,\ \tilde{c}_4 > 0\ \forall\, \theta \in [0, +\infty) :\ \tilde{c}_3 c_v(\theta) \le \mathsf{K}(\theta) \le \tilde{c}_4 c_v(\theta). \tag{3.17}$$

Observe that (2.17), (3.14), (3.16) imply that we restrict our analysis to the case when the purely thermal contribution $\widehat{\psi} = \widehat{\psi}(\theta)$ to the free energies Ψ_Ω and $\Psi_{\Gamma_{\mathrm{C}}}$ fulfills the growth condition

$$-\theta \widehat{\psi}''(\theta) \ge \tilde{c}_1(1+\theta)^{3/5+\epsilon} \quad \text{with } \epsilon > 0.$$

In particular, let us point that we cannot deal with the fairly classical choice $\widehat{\psi}(\theta) = -\theta \log(\theta)$. However, many models in the literature are recovered using a polynomial form for the thermal part in the free energy, as in the present case.

Hypothesis (V). As for the regularizing operator $\mathcal{R}$, in the lines of [19,20], we require that there exists $\nu > 0$ such that $\mathcal{R} : L^2(0, T; \mathbf{Y}^*) \to L^\infty(0, T; L^{16/5+\nu}(\Gamma_{\mathrm{C}}; \mathbb{R}^3))$ is bounded, viz.

$$\exists\, S > 0\ \forall\, \boldsymbol{\eta} \in L^2(0, T; \mathbf{Y}^*) \ :\ \|\mathcal{R}(\boldsymbol{\eta})\|_{L^\infty(0,T;L^{16/5+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3))} \le S\|\boldsymbol{\eta}\|_{L^2(0,T;\mathbf{Y}^*)}. \tag{3.18}$$

Moreover, we impose that

$$\mathcal{R} : L^2(0, T; \mathbf{Y}^*) \to L^\infty(0, T; L^{16/5+\nu}(\Gamma_{\mathrm{C}}; \mathbb{R}^3)) \text{ is weakly–strongly continuous, viz.}$$

$$\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \text{ in } L^2(0, T; \mathbf{Y}^*) \ \Rightarrow\ \mathcal{R}(\boldsymbol{\eta}_n) \to \mathcal{R}(\boldsymbol{\eta}) \text{ in } L^\infty(0, T; L^{16/5+\nu}(\Gamma_{\mathrm{C}}; \mathbb{R}^3)) \tag{3.19}$$

for all $(\boldsymbol{\eta}_n),\ \boldsymbol{\eta} \in L^2(0, T; \mathbf{Y}^*)$. Clearly, if $\mathcal{R}$ is linear, then (3.19) implies the boundedness (3.18). We refer to [20, Example 3.2] for the explicit construction of an operator $\mathcal{R}$ complying with (3.18)–(3.19).

Remark 3.4. The requirement that $\mathcal{R}$ is weakly–strongly continuous with values in $L^\infty(0, T; L^{16/5+\nu}(\Gamma_{\mathrm{C}}; \mathbb{R}^3))$ ensures that the variational formulation of (3.9) is well defined, cf. the forthcoming Remark 3.6. As the discussion therein will reveal, the choice of the space $L^{16/5+\nu}(\Gamma_{\mathrm{C}}; \mathbb{R}^3)$ is tightly related to the exponent σ in (3.14), cf. (3.33) later on.

Moreover, (3.19) will play a crucial role for the limit passage in the frictional terms in (the discretized version of) system (3.29).

In fact, for the derivation of the a priori estimates on the discrete solutions, and for the passage to the limit argument developed in Section 5, we will need to resort to a stronger version of conditions (3.18)–(3.19), cf. Hypothesis 4.1 later on. Nonetheless, as we will explain in Remark 5.1, this can be bypassed via a double approximation procedure.

That is why, in the statement of Theorem 1 we will stay with the weaker conditions (3.18)–(3.19).

Hypothesis (VI). We assume that the functions k in (3.9b)–(3.9c), $\mathfrak{c}$ in (3.9b), (3.9c), and (3.9h), λ in (3.9c) and (3.9i), and γ in (3.9i) fulfill

$$k : \mathbb{R} \to [0, +\infty) \text{ is Lipschitz continuous,} \tag{3.20}$$

$$\mathfrak{c} \in \mathrm{C}^1(\mathbb{R}), \qquad \exists\, c_5,\ c_6,\ c_7 \ge 0\ \forall\, \theta \in \mathbb{R} : c_5 \le \mathfrak{c}(\theta) \le c_6,\ |\mathfrak{c}'(\theta)| \le c_7, \qquad \mathfrak{c}'(\theta)\theta \ge 0, \tag{3.21}$$

$$\lambda : \mathbb{R} \to \mathbb{R} \text{ Lipschitz continuous and } \delta\text{-concave for some } \delta \in \mathbb{R}, \tag{3.22}$$

$$\gamma \in \mathrm{C}^1(\mathbb{R}),\ \gamma\ \nu\text{-convex for some } \nu \in \mathbb{R}, \text{ and } \gamma' : \mathbb{R} \to \mathbb{R} \text{ Lipschitz continuous, and}$$

$$\exists\, c_8,\ c_9 > 0\ \forall\, r \in \mathbb{R} :\ W(r) := \widehat{\beta}(r) + \gamma(r) \ge c_8|r| - c_9. \tag{3.23}$$

Observe that the δ-concavity and the ν-convexity requirements for λ and for γ mean that

$$\text{the map } r \mapsto \lambda(r) - \frac{\delta}{2}r^2 \text{ is concave and the map } r \mapsto \gamma(r) + \frac{\nu}{2}r^2 \text{ is convex.} \tag{3.24}$$

Properties (3.24) will play a crucial role in the derivation of the *discrete total energy* inequality (4.25), cf. the proof of Lemma 4.5. In fact, if, in addition, we have that $\mathrm{dom}(\widehat{\beta})$ is a *bounded* sub-interval $[\chi_*, \chi^*]$ of $[0, +\infty)$, then the validity of (3.24) is ensured as soon as we assume that $\lambda,\ \gamma \in \mathrm{C}^2(\mathbb{R})$. Indeed, in such a case there exist $\delta,\ \nu > 0$ such that $\lambda''(r) \le \delta$ and $\gamma''(r) \ge -\nu$ for all $r \in [\chi_*, \chi^*]$.

Assumptions on the problem and on the initial data. We require that

$$h \in L^1(0, T; L^1(\Omega)), \qquad \mathbf{f} \in W^{1,1}(0, T; \mathbf{W}^*), \qquad \mathbf{g} \in W^{1,1}(0, T; H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}}; \mathbb{R}^3)^*). \tag{3.25}$$

For later convenience, we remark that, thanks to the second and third of (3.25) the function $\mathbf{F} : (0, T) \to \mathbf{W}^*$ defined by

$$\langle \mathbf{F}(t), \mathbf{v}\rangle_{\mathbf{W}} := \langle \mathbf{f}(t), \mathbf{v}\rangle_{\mathbf{W}} + \langle \mathbf{g}(t), \mathbf{v}\rangle_{H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^3)} \quad \text{for all } \mathbf{v} \in \mathbf{W} \text{ and almost all } t \in (0, T),$$

satisfies

$$\mathbf{F} \in W^{1,1}(0, T; \mathbf{W}^*). \tag{3.26}$$

For the initial data we impose that

$$w_0 \in L^1(\Omega) \text{ and } w_s^0 \in L^1(\Gamma_C), \qquad \mathbf{u}_0 \in \mathbf{W} \text{ and } \mathbf{u}_0 \in \mathrm{dom}(\boldsymbol{\varphi}), \qquad \chi_0 \in V_{\Gamma_C}, \text{ and } \widehat{\beta}(\chi_0) \in L^1(\Gamma_C). \tag{3.27}$$

3.3. Weak formulation of the problem and main result

We now detail the weak formulation of the initial–boundary value problem we are dealing with and we state the corresponding global-in-time existence result. We mention in advance that the properties $w \in L^r(0, T; W^{1,r}(\Omega))$ and $w_s \in L^\rho(0, T; W^{1,\rho}(\Gamma_C))$ will result from Boccardo–Gallouët type estimates [25] on the (discrete) enthalpy equations, combined with the Gagliardo–Nirenberg inequality, cf. the proof of the forthcoming Proposition 4.6.

Problem 3.5. Given a quadruplet of data $(w_0, w_s^0, \mathbf{u}_0, \chi_0)$ fulfilling (3.27), find $(w, w_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \boldsymbol{\mu}, \xi)$, with

$$w \in L^r(0, T; W^{1,r}(\Omega)) \cap L^\infty(0, T; L^1(\Omega)) \cap \mathrm{BV}([0, T]; W^{1,r'}(\Omega)^*) \quad \text{for every } 1 \le r < \frac{5}{4}, \tag{3.28a}$$

$$w_s \in L^\rho(0, T; W^{1,\rho}(\Gamma_C)) \cap L^\infty(0, T; L^1(\Gamma_C)) \cap \mathrm{BV}([0, T]; W^{1,\rho'}(\Gamma_C)^*) \quad \text{for every } 1 \le \rho < \frac{4}{3}, \tag{3.28b}$$

$$\mathbf{u} \in H^1(0, T; \mathbf{W}), \tag{3.28c}$$

$$\chi \in L^2(0, T; H^2(\Gamma_C)) \cap L^\infty(0, T; V_{\Gamma_C}) \cap H^1(0, T; H_{\Gamma_C}), \tag{3.28d}$$

$$\boldsymbol{\eta} \in L^2(0, T; \mathbf{Y}^*), \tag{3.28e}$$

$$\boldsymbol{\mu} \in L^2(0, T; L^{4/3}(\Gamma_C; \mathbb{R}^3)), \tag{3.28f}$$

$$\xi \in L^2(0, T; H_{\Gamma_C}), \tag{3.28g}$$

(where r' and ρ' denote the conjugate exponents of r and ρ, respectively), satisfying the initial conditions (3.10) and

$$\begin{aligned}
&\langle w(t), \varphi(t)\rangle_{W^{1,r'}(\Omega)} - \int_0^t \int_\Omega w\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_\Omega \Theta(w)\,\mathrm{div}(\partial_t\mathbf{u})\varphi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega K(w)\nabla w\nabla\varphi \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t \int_{\Gamma_C} \Big(k(\chi)\Theta(w)(\Theta(w) - \Theta(w_s)) + \Theta(w)\mathfrak{c}'(\Theta(w) - \Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|\,|\partial_t\mathbf{u}_{\mathrm{T}}|\Big)\varphi \,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \int_\Omega \boldsymbol{\varepsilon}(\partial_t\mathbf{u})\mathbb{V}\boldsymbol{\varepsilon}(\partial_t\mathbf{u})\varphi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega h\varphi \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega w_0\varphi(0)\,\mathrm{d}x \\
&\quad \text{for all } \varphi \in \mathcal{F} := \mathrm{C}^0([0, T]; W^{1,r'}(\Omega)) \cap W^{1,r'}(0, T; L^{r'}(\Omega)) \text{ and for all } t \in (0, T],
\end{aligned} \tag{3.29a}$$

$$\begin{aligned}
&\langle w_s(t), \psi(t)\rangle_{W^{1,\rho'}(\Gamma_C)} - \int_0^t \int_{\Gamma_C} w_s\partial_t\psi \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_{\Gamma_C} \Theta(w_s)\partial_t(\lambda(\chi))\psi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Gamma_C} K(w_s)\nabla w_s\nabla\psi \,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \int_{\Gamma_C} \Big(k(\chi)\Theta(w_s)(\Theta(w) - \Theta(w_s)) + (\mathfrak{c}(\Theta(w) - \Theta(w_s)) \\
&\qquad + \Theta(w_s)\mathfrak{c}'(\Theta(w) - \Theta(w_s)))|\mathcal{R}(\boldsymbol{\eta})|\,|\partial_t\mathbf{u}_{\mathrm{T}}|\Big)\psi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Gamma_C} |\partial_t\chi|^2\psi \,\mathrm{d}x\,\mathrm{d}s + \int_{\Gamma_C} w_s^0\psi(0)\,\mathrm{d}x \\
&\quad \text{for all } \psi \in \mathcal{G} := \mathrm{C}^0([0, T]; W^{1,\rho'}(\Gamma_C)) \cap W^{1,\rho'}(0, T; L^{\rho'}(\Gamma_C)) \text{ and for all } t \in (0, T],
\end{aligned} \tag{3.29b}$$

$$\begin{aligned}
&b(\partial_t\mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_\Omega \Theta(w)\,\mathrm{div}(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_C} \chi\mathbf{u}\mathbf{v}\,\mathrm{d}x + \langle\boldsymbol{\eta}, \mathbf{v}\rangle_{\mathbf{Y}} \\
&\quad + \int_{\Gamma_C} \mathfrak{c}(\Theta(w) - \Theta(w_s))\boldsymbol{\mu}\mathbf{v}\,\mathrm{d}x = \langle\mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in \mathbf{W} \text{ a.e. in } (0, T),
\end{aligned} \tag{3.29c}$$

$$\boldsymbol{\eta} \in \partial\boldsymbol{\varphi}(\mathbf{u}) \quad \text{in } \mathbf{Y}^*, \text{ a.e. in } (0,T), \tag{3.29d}$$

$$\boldsymbol{\mu} = |\mathfrak{R}(\boldsymbol{\eta})|\mathbf{z} \quad \text{with } \mathbf{z} \in \partial j(\mathbf{u}_t) \text{ a.e. in } \Gamma_C \times (0,T), \tag{3.29e}$$

$$\partial_t\chi + A\chi + \xi + \gamma'(\chi) = -\lambda'(\chi)\Theta(w_s) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_C \times (0,T), \tag{3.29f}$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } \Gamma_C \times (0,T). \tag{3.29g}$$

Remark 3.6. The variational formulation (3.29) is well defined. In particular, let us focus on the weak formulation for the enthalpy equation (3.29a); analogous considerations apply to (3.29b), where indeed we dispose of higher integrability properties for the functions involved.

Since $1 \le r < \frac{5}{4}$, we have $r' > 5$, hence for test functions $\varphi \in \mathscr{F}$ the integral terms in Ω are well defined. Moreover, it follows from the third of (3.28a) that $w(t) \in W^{1,r'}(\Omega)^*$ for all $t \in [0,T]$, which guarantees that the duality pairing $\langle w(t), \varphi(t)\rangle_{W^{1,r'}(\Omega)}$ is defined everywhere in $[0,T]$. This pairing reduces to the integral $\int_\Omega w(t)\varphi(t)\,dx$ for almost all $t \in (0,T)$, since $w \in L^\infty(0,T;L^1(\Omega))$.

As for the integrals on the contact surface in (3.29a), observe that every test function φ is in $L^\infty(\Gamma_C \times (0,T))$. On the other hand, from (3.28a) and (3.28b) we gather by trace theorems, Sobolev embeddings and interpolation arguments (cf. the proof of Proposition 4.6 later on), that

$$\begin{aligned}
&\Theta(w) \in L^{r\sigma}(0,T;L^{p\sigma}(\Gamma_C)) \quad \text{for all } 1 \le r < \frac{5}{4} \text{ and for all } 1 \le p \le \frac{2r}{3-r},\\
&\Theta(w_s) \in L^{\varrho\sigma}(0,T;L^{q\sigma}(\Gamma_C)) \quad \text{for all } 1 \le \varrho < \frac{4}{3} \text{ and for all } 1 \le q \le \frac{2\varrho}{2-\varrho}.
\end{aligned} \tag{3.30}$$

Since $\sigma > \frac{8}{5}$, it can be checked that the above integrability properties yield

$$\Theta(w) \in L^{2+\epsilon}(0,T;L^{16/7}(\Gamma_C)), \qquad \Theta(w_s) \in L^{2+\epsilon'}(0,T;L^{32/5}(\Gamma_C)) \quad \text{for } \epsilon, \epsilon' > 0. \tag{3.31}$$

Taking into account that $k(\chi) \in L^\infty(0,T;L^p(\Gamma_c))$ for all $1 \le p < \infty$ thanks to (3.28d) and (3.20), we ultimately infer that the term $\int_0^t \int_{\Gamma_C} k(\chi)\Theta(w)(\Theta(w) - \Theta(w_s))\varphi\,dx\,ds$ is well defined for all $\varphi \in \mathcal{F}$. So is

$$\int_0^t \int_{\Gamma_C} \Theta(w)\mathfrak{c}'(\Theta(w) - \Theta(w_s))|\mathfrak{R}(\boldsymbol{\eta})|\,|\partial_t\mathbf{u}_T|\varphi\,dx\,ds, \tag{3.32}$$

as $\mathfrak{c}'(\Theta(w) - \Theta(w_s)) \in L^\infty(\Gamma_C \times (0,T))$ by (3.21), $\mathfrak{R}(\boldsymbol{\eta}) \in L^\infty(0,T;L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3))$ by (3.18), $|\partial_t\mathbf{u}_T| \in L^2(0,T;L^4(\Gamma_C))$ by (3.28c) and Sobolev embeddings, and again in view of (3.31).

This discussion highlights that, indeed, the choice of the space $L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)$ in (3.18)–(3.19) is tailored to the exponent $\sigma > 8/5$ for which (3.14) holds. More precisely, for the integral in (3.32) to be well defined, for fixed $\sigma > 8/5$ it is sufficient that $\mathfrak{R}$ be weakly–strongly continuous with values in the space $L^\infty(0,T;L^{\zeta(\sigma)}(\Gamma_C;\mathbb{R}^3))$, and by the Hölder inequality $\zeta(\sigma)$ has to satisfy $\frac{3-r}{2r\sigma} + \frac{1}{\zeta(\sigma)} + \frac{1}{4} = 1$ with r as in (3.30). Therefore, since $r < 5/4$, we find $\frac{1}{\zeta(\sigma)} < \frac{3}{4} - \frac{7}{10\sigma}$. All in all, taking into account that $\zeta(\sigma)$ should be greater than $4/3$ in view of the $L^4(\Gamma_C)$-summability for the test functions in the momentum equation, we conclude

$$\zeta(\sigma) > \max\left\{\frac{20\sigma}{15\sigma - 14}, \frac{4}{3}\right\}. \tag{3.33}$$

Since for $\sigma > 8/5$ we have $\frac{20\sigma}{15\sigma-14} < 16/5$, we retrieve conditions (3.18)–(3.19).

The energy balance. Observe that every solution quadruplet $(w, w_s, \mathbf{u}, \chi)$ to system (3.29) also fulfills the total energy equality (written in terms of the enthalpy variables)

$$\begin{aligned}
&\int_\Omega w(t)\,dx + \int_{\Gamma_C} w_s(t)\,dx + \frac{1}{2}a(u(t),u(t)) + \frac{1}{2}\int_{\Gamma_C} \chi(t)|\mathbf{u}(t)|^2\,dx + \boldsymbol{\varphi}(\mathbf{u}(t))\\
&\quad + \int_s^t \int_{\Gamma_C} k(\chi)(\Theta(w) - \Theta(w_s))^2\,dx\,dr + \int_s^t \int_{\Gamma_C} \mathfrak{c}'(\Theta(w) - \Theta(w_s))(\Theta(w) - \Theta(w_s))|\mathfrak{R}(\boldsymbol{\eta})|\,|\partial_t\mathbf{u}_T|\,dx\,dr\\
&\quad + \int_{\Gamma_C} \left(\frac{1}{2}|\nabla\chi(t)|^2 + W(\chi(t))\right) dx\\
&= \int_\Omega w(s)\,dx + \int_{\Gamma_C} w_s(s)\,dx + \frac{1}{2}a(u(s),u(s)) + \frac{1}{2}\int_{\Gamma_C} \chi(s)|\mathbf{u}(s)|^2\,dx + \boldsymbol{\varphi}(\mathbf{u}(s))\\
&\quad + \int_{\Gamma_C} \left(\frac{1}{2}|\nabla\chi(s)|^2 + W(\chi(s))\right) dx + \int_s^t \left(\int_\Omega h\,dx + \langle \mathbf{F}, \partial_t\mathbf{u}\rangle_{\mathbf{W}}\right) dr,
\end{aligned} \tag{3.34}$$

for almost all $0 \le s \le t \le T$, and for $s = 0$, with $\boldsymbol{\eta}(t) \in \partial\boldsymbol{\varphi}(\mathbf{u}(t))$ for almost all $t \in (0, T)$, cf. (3.29d), and with $W(\chi) = \widehat{\beta}(\chi) + \gamma(\chi)$. In fact, (3.34) follows from testing (3.29a) by 1, (3.29b) by 1, (3.29c) by $\partial_t \mathbf{u}$, (3.29f) by $\partial_t \chi$, integrating in time, and finally adding the resulting relations. One then observes the cancelation of various coupling terms between the equations, integrates by parts, and employs the chain rule (for possibly nonsmooth but convex energies), to obtain (3.34). The latter improves to an identity holding *for all* $0 \le s \le t \le T$, if the terms $\int_\Omega w(\cdot)\,\mathrm{d}x$ and $\int_{\Gamma_C} w_s(\cdot)\,\mathrm{d}x$, which are well defined only *almost everywhere* (pointwise in time), are replaced by the duality pairings $\langle w(\cdot), 1\rangle_{W^{1,r'}(\Omega)}$ and $\langle w_s(\cdot), 1\rangle_{W^{1,\rho'}(\Gamma_C)}$, respectively, cf. Remark 3.6.

Introducing the total energy functional $\mathscr{E}$ (cf. (2.2), (2.3))

$$\mathscr{E}(w, w_s, \mathbf{u}, \chi) := \int_\Omega w\,\mathrm{d}x + \int_{\Gamma_C} w_s\,\mathrm{d}x + \frac{1}{2}a(u, u) + \frac{1}{2}\int_{\Gamma_C} \chi|\mathbf{u}|^2\,\mathrm{d}x + \boldsymbol{\varphi}(\mathbf{u}) + \int_{\Gamma_C}\left(\frac{1}{2}|\nabla\chi|^2 + W(\chi)\right)\mathrm{d}x, \tag{3.35}$$

due to the positivity of the frictional contribution (see (3.21)), from the energy balance (3.34) we deduce the following inequality

$$\mathscr{E}(w(t), w_s(t), \mathbf{u}(t), \chi(t)) \le \mathscr{E}(w(s), w_s(s), \mathbf{u}(s), \chi(s)) + \int_s^t \left(\int_\Omega h\,\mathrm{d}x + \langle \mathbf{F}, \partial_t \mathbf{u}\rangle_{\mathbf{W}}\right)\mathrm{d}r \tag{3.36}$$

for almost all $0 \le s \le t \le T$, and for $s = 0$.

With our main result, Theorem 1, we state the existence of a global solution to Problem 3.5, satisfying the additional properties (3.37)–(3.41).

Theorem 1. *Assume Hypotheses (I)–(VI) and conditions (3.25), (3.27) on the data* $(h, \mathbf{f}, \mathbf{g})$ *and on the initial data* $(w_0, w_s^0, \mathbf{u}_0, \chi_0)$. *Then,*

1. *Problem 3.5 admits a solution* $(w, w_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \boldsymbol{\mu}, \xi)$, *where* w, w_s *have the additional regularity*

$$w \in W^{1,1}(0, T; W^{1,r'}(\Omega)^*), \qquad w_s \in W^{1,1}(0, T; W^{1,\rho'}(\Gamma_C)^*), \tag{3.37}$$

and the quadruplet $(w, w_s, \mathbf{u}, \chi)$ *satisfies enhanced formulations of the enthalpy equations, namely*

$$\begin{aligned}
&\langle \partial_t w, v\rangle_{W^{1,r'}(\Omega)} - \int_\Omega \Theta(w)\,\mathrm{div}(\partial_t \mathbf{u}) v\,\mathrm{d}x + \int_\Omega K(w)\nabla w \nabla v\,\mathrm{d}x\\
&\quad + \int_{\Gamma_C}\Big(k(\chi)\Theta(w)(\Theta(w) - \Theta(w_s)) + \Theta(w)\mathfrak{c}'(\Theta(w) - \Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|\,|\partial_t \mathbf{u}_T|\Big) v\,\mathrm{d}x\\
&= \int_\Omega \boldsymbol{\varepsilon}(\partial_t \mathbf{u})\mathbb{V}\boldsymbol{\varepsilon}(\partial_t \mathbf{u}) v\,\mathrm{d}x + \int_\Omega h v\,\mathrm{d}x \quad \textit{for all } v \in W^{1,r'}(\Omega),\ \textit{a.e. in } (0, T),
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
&\langle \partial_t w_s, \psi\rangle_{W^{1,\rho'}(\Gamma_C)} - \int_{\Gamma_C} \Theta(w_s)\partial_t(\lambda(\chi))\psi\,\mathrm{d}x + \int_{\Gamma_C} K(w_s)\nabla w_s \nabla\psi\,\mathrm{d}x\\
&= \int_{\Gamma_C}\Big(k(\chi)\Theta(w_s)(\Theta(w) - \Theta(w_s)) + \Big(\mathfrak{c}(\Theta(w) - \Theta(w_s))\\
&\qquad + \Theta(w_s)\,\mathfrak{c}'(\Theta(w) - \Theta(w_s))\Big)|\mathcal{R}(\boldsymbol{\eta})|\,|\partial_t \mathbf{u}_T|\Big)\psi\,\mathrm{d}x\\
&\quad + \int_{\Gamma_C} |\partial_t \chi|^2 \psi\,\mathrm{d}x \quad \textit{for all } \psi \in W^{1,\rho'}(\Gamma_C),\ \textit{a.e. in } (0, T).
\end{aligned} \tag{3.39}$$

Finally, we also have

$$\boldsymbol{\mu} \in L^\infty(0, T; L^{16/5+\nu}(\Gamma_C; \mathbb{R}^3)) \quad \textit{with } \nu > 0 \textit{ from (3.18)}. \tag{3.40}$$

2. *Suppose moreover that*

$$h \ge 0 \quad \textit{a.e. in } \Omega \times (0, T) \quad \textit{and} \quad w_0 \ge 0 \quad \textit{a.e. in } \Omega, \quad w_s^0 \ge 0 \quad \textit{a.e. in } \Gamma_C. \tag{3.41}$$

Then, $w \ge 0$ *a.e. in* $\Omega \times (0, T)$ *and* $w_s \ge 0$ *a.e. in* $\Gamma_C \times (0, T)$.

We conclude this section with some comments on Theorem 1.

Remark 3.7. Clearly, Theorem 1 also yields the existence of a solution quadruplet $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ to (the weak formulation of) the original PDE system (1.1), regularized by means of (1.5) and (1.6), upon defining $\vartheta = \Theta(w)$ and $\vartheta_s = \Theta(w_s)$, in view of (3.7)–(3.8). Since $w \ge 0$ a.e. in $\Omega \times (0, T)$ and $w_s \ge 0$ a.e. in $\Gamma_C \times (0, T)$, we find that $\vartheta \ge 0$ and $\vartheta_s \ge 0$ a.e. in $\Omega \times (0, T)$ and $\Gamma_C \times (0, T)$, respectively.

Remark 3.8 (*Unidirectionality*). It is at the moment an open problem to encompass in the present analysis the unidirectionality of the degradation process for the adhesive substance on $\Gamma_{\rm C}$. Indeed, this would result in the following adhesion parameter equation

$$\partial_t \chi + \partial I_{(-\infty,0]}(\partial_t \chi) - \Delta\chi + \partial I_{[0,1]}(\chi) + \gamma'(\chi) \ni -\lambda'(\chi)\Theta(w_{\rm s}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\rm C} \times (0,T), \tag{3.42}$$

additionally featuring the subdifferential of the indicator function $I_{(-\infty,0]}$ evaluated at $\partial_t\chi$, whence the unidirectionality $\partial_t \chi \leq 0$ a.e. in $\Gamma_{\rm C} \times (0,T)$. Observe that Eq. (3.42) has a *doubly nonlinear* character. The main difficulty attached to its analysis relates to estimating the unbounded terms $\partial I_{(-\infty,0]}(\partial_t\chi)$ and $\partial I_{[0,1]}(\chi)$ *separately*, which would also bring to a suitable estimate for $-\Delta\chi$. A well-known technique for achieving this, dating back to [29], would involve testing (3.42) by $\partial_t(-\Delta\chi + \partial I_{[0,1]}(\chi))$. The related calculations, which could be rendered rigorously on a suitable approximate level, in fact lead to a by-part integration in time of the right-hand side of (3.42). However, in the present case this cannot be carried out due to the poor time-regularity (3.28b) of the enthalpy variable $w_{\rm s}$. That is why, the analysis of (3.42) remains open.

Finally, let us mention that, in what follows, the symbols c, c', C, C' will be used to denote a positive constant depending on data, and possibly varying from line to line. Furthermore, the symbols I_i, $i = 1, \ldots,$ will be used as place-holders for several integral terms popping in the various estimates. We warn the reader that we will not be self-consistent with the numbering so that, for instance, the symbol I_1 will occur several times with different meanings.

4. Time discretization

In this section, we set up the time discrete scheme (cf. (4.10)) approximating system (3.9) and provide an existence result for it in Lemma 4.3. We then construct a family of approximate solutions to Problem 3.5 by considering the piecewise constant and piecewise linear interpolants of the discrete solutions. Hence we show in Lemma 4.5 that these approximate solutions satisfy the discrete version (4.25) of the total energy inequality (3.36). From (4.25), in Proposition 4.6 we will deduce the first set of a priori estimates, namely the so-called *energy estimates*, on the approximate solutions. We shall combine them with other a priori bounds and exploit suitable compactness results to pass to the limit in (4.10) and conclude the proof of Theorem 1 in Section 5.

4.1. Analysis of the time discrete scheme

We consider a partition

$$\mathscr{P}_\tau := \{t_0 := 0 < t_\tau^1 < \cdots < t_\tau^j < \cdots < t_\tau^{J_\tau - 1} < t_\tau^{J_\tau} := T\} \tag{4.1}$$

of $[0,T]$, with uniform time-step $\tau > 0$, and approximate the data $\mathbf{F}$ (3.26) and h (3.25) by local means, i.e. setting for all $j = 1, \ldots, J_\tau$

$$\mathbf{F}_\tau^j := \frac{1}{\tau}\int_{t_\tau^{j-1}}^{t_\tau^j} \mathbf{F}(s)\,{\rm d}s, \qquad h_\tau^j := \frac{1}{\tau}\int_{t_\tau^{j-1}}^{t_\tau^j} h(s)\,{\rm d}s. \tag{4.2}$$

We construct the discrete solutions to system (3.9) by recursively solving an elliptic system, (4.10), where:

(1) the higher order terms

$$-\tau\,{\rm div}(|\varepsilon(\mathbf{u}_\tau^j)|^{\gamma-2}\varepsilon(\mathbf{u}_\tau^j)) \quad \text{and} \quad \tau|\chi_\tau^j|^{\gamma-2}\chi_\tau^j \quad \text{with } \gamma > 4, \tag{4.3}$$

have been added to the left-hand sides of the discrete momentum equation (4.10c) and the discrete equation for χ (4.10d), along the lines of [26]. On the time-continuous level, this corresponds to adding to the bulk free energy Ψ_Ω (2.2) the term $\frac{\tau}{\gamma}|\varepsilon(\mathbf{u})|^\gamma$ and to the contact surface free energy $\Psi_{\Gamma_{\rm C}}$ (2.3) the term $\frac{\tau}{\gamma}|\chi|^\gamma$. Accordingly, we shall consider sequences $(w_\tau^0)_\tau$, $(w_{{\rm s},\tau}^0)_\tau$, and $(\mathbf{u}_\tau^0)_\tau$ approximating the initial data w_0, $w_{\rm s}^0$, and $\mathbf{u}_0$, namely

$$\begin{cases} (w_\tau^0)_\tau \subset V, & w_\tau^0 \to w_0 \quad \text{in } L^1(\Omega), \\ (w_{{\rm s},\tau}^0)_\tau \subset V_{\Gamma_{\rm C}}, & w_{{\rm s},\tau}^0 \to w_{\rm s}^0 \quad \text{in } L^1(\Gamma_{\rm C}), \\ (\mathbf{u}_\tau^0)_\tau \subset W^{1,\gamma}_{\Gamma_{\rm Dir}}(\Omega;\mathbb{R}^3) := W^{1,\gamma}(\Omega;\mathbb{R}^3)\cap \mathbf{W}, & \mathbf{u}_\tau^0 \to \mathbf{u}_0 \quad \text{in } \mathbf{W} \end{cases} \quad \text{as } \tau \downarrow 0. \tag{4.4}$$

(2) We shall discretize the term $\boldsymbol{\eta} \in \partial\boldsymbol{\varphi}(\mathbf{u})$ in (3.29a)–(3.29d) by

$$\phi_\tau'(u_{\tau,{\rm N}}^j)\mathbf{n} =: \eta_\tau^j \mathbf{n}, \tag{4.5}$$

cf. (4.10a)–(4.10c). In (4.5), $u_{\tau,{\rm N}}^j$ is the normal component of $\mathbf{u}_\tau^j$ and

$$\phi_\tau(x) := \inf_{y\in\mathbb{R}}\left(\frac{|y-x|^2}{2\tau} + \phi(y)\right) \tag{4.6}$$

is the Moreau–Yosida approximation of the convex function ϕ in (3.11), cf. e.g. [30,31]. We recall that ϕ_τ is convex, differentiable, with $0 \le \phi_\tau(x) \le \phi(x)$, and that ϕ'_τ is the Yosida regularization of the subdifferential $\partial\phi : \mathbb{R} \rightrightarrows \mathbb{R}$.

(3) Furthermore, we will "discretize" the nonlocal operator $\mathcal{R}$ by localizing it in time. More precisely, at the jth-step we will work with the (time-independent) operator

$$\mathcal{R}^j_\tau : \mathbf{Y}^* \to L^{16/5+\nu}(\Gamma_{\rm C};\mathbb{R}^3) \text{ defined by } \mathcal{R}^j_\tau(\boldsymbol{\eta}) := \frac{1}{\tau}\int_{t^{j-1}_\tau}^{t^j_\tau} \mathcal{R}(\boldsymbol{\eta})(s)\,\mathrm{d}s \tag{4.7}$$

where in turn $\mathcal{R}(\boldsymbol{\eta})$ has to be understood as $\mathcal{R}(\widetilde{\boldsymbol{\eta}})$, with $\widetilde{\boldsymbol{\eta}} \in L^2(0,T;\mathbf{Y}^*)$ the constant function $\widetilde{\boldsymbol{\eta}}(t) \equiv \boldsymbol{\eta}$.

(4) In system (4.10), we will also use that the function λ decomposes as

$$\lambda(r) := \lambda(r) - \frac{\delta}{2}r^2 + \frac{\delta}{2}r^2 =: \lambda_\delta(r) + \frac{\delta}{2}r^2 \quad \text{with } \lambda_\delta \text{ concave}, \tag{4.8}$$

(cf. (3.24)) and analogously we will write $\gamma(r) := \gamma_\nu(r) - \frac{\nu}{2}r^2$, with γ_ν convex, by the ν-convexity of γ.

We postpone to Remark 4.2 a thorough explanation of the features of system (4.10), and of the motivation for (4.3), (4.5), and (4.8), and right away present our time-discrete scheme.

Problem 4.1 (*Time Discretization of System* (3.9))**.** Let $\gamma > 4$. Starting from the initial data

$$(w^0_\tau, w^0_{\mathrm{s},\tau}, \mathbf{u}^0_\tau, \chi_0) \tag{4.9}$$

with χ_0 from (3.27) and $(w^0_\tau, w^0_{\mathrm{s},\tau}, \mathbf{u}^0_\tau) \in V \times V_{\Gamma_{\rm C}} \times W^{1,\gamma}_{\Gamma_{\rm Dir}}(\Omega;\mathbb{R}^3)$ as in (4.4), find $(w^j_\tau, w^j_{\mathrm{s},\tau}, \mathbf{u}^j_\tau, \chi^j_\tau)_{j=1}^{J_\tau} \subset V \times V_{\Gamma_{\rm C}} \times W^{1,\gamma}_{\Gamma_{\rm Dir}}(\Omega;\mathbb{R}^3) \times V_{\Gamma_{\rm C}}$ fulfilling

$$\begin{aligned}
&\int_\Omega \frac{w^j_\tau - w^{j-1}_\tau}{\tau} v\,\mathrm{d}x - \int_\Omega \Theta(w^j_\tau)\mathrm{div}\left(\frac{\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau}{\tau}\right) v\,\mathrm{d}x + \int_\Omega K(w^{j-1}_\tau)\nabla w^j_\tau \nabla v\,\mathrm{d}x \\
&\quad + \int_{\Gamma_{\rm C}} \left(k(\chi^{j-1}_\tau)\Theta(w^j_\tau)(\Theta(w^j_\tau) - \Theta(w^j_{\mathrm{s},\tau})) + \Theta(w^j_\tau)\mathfrak{c}'(\Theta(w^j_\tau) - \Theta(w^j_{\mathrm{s},\tau}))|\mathcal{R}^j_\tau(\eta^j_\tau \mathbf{n})| \left|\frac{\mathbf{u}^j_{\tau,\mathrm{T}} - \mathbf{u}^{j-1}_{\tau,\mathrm{T}}}{\tau}\right| \right) v\,\mathrm{d}x \\
&= \int_\Omega \varepsilon\left(\frac{\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau}{\tau}\right) v\,\mathrm{d}x + \int_\Omega h^j_\tau v\,\mathrm{d}x \quad \text{for all } v \in V
\end{aligned} \tag{4.10a}$$

where $\eta^j_\tau = \phi'_\tau(u^j_{\tau,\mathrm{N}})$, cf. (4.5), and $\mathbf{u}^j_{\tau,\mathrm{T}}$, $\mathbf{u}^{j-1}_{\tau,\mathrm{T}}$ are the tangential components of $\mathbf{u}^j_\tau$, $\mathbf{u}^{j-1}_\tau$

$$\begin{aligned}
&\int_{\Gamma_{\rm C}} \frac{w^j_{\mathrm{s},\tau} - w^{j-1}_{\mathrm{s},\tau}}{\tau} v\,\mathrm{d}x - \int_{\Gamma_{\rm C}} \Theta(w^j_{\mathrm{s},\tau})\frac{\lambda(\chi^j_\tau) - \lambda(\chi^{j-1}_\tau)}{\tau} v\,\mathrm{d}x + \int_{\Gamma_{\rm C}} K(w^{j-1}_{\mathrm{s},\tau})\nabla w^j_{\mathrm{s},\tau}\nabla v\,\mathrm{d}x \\
&= \int_{\Gamma_{\rm C}} \Bigg(k(\chi^{j-1}_\tau)\Theta(w^j_{\mathrm{s},\tau})(\Theta(w^j_\tau) - \Theta(w^j_{\mathrm{s},\tau})) \\
&\qquad + \Big(\mathfrak{c}(\Theta(w^j_\tau) - \Theta(w^j_{\mathrm{s},\tau})) + \Theta(w^j_{\mathrm{s},\tau})\mathfrak{c}'(\Theta(w^j_\tau) - \Theta(w^j_{\mathrm{s},\tau}))\Big)|\mathcal{R}^j_\tau(\eta^j_\tau\mathbf{n})| \left|\frac{\mathbf{u}^j_{\tau,\mathrm{T}} - \mathbf{u}^{j-1}_{\tau,\mathrm{T}}}{\tau}\right| \Bigg) v\,\mathrm{d}x \\
&\quad + \int_{\Gamma_{\rm C}} \left|\frac{\chi^j_\tau - \chi^{j-1}_\tau}{\tau}\right|^2 v\,\mathrm{d}x \quad \text{for all } v \in V_{\Gamma_{\rm C}},
\end{aligned} \tag{4.10b}$$

$$\begin{aligned}
&b\left(\frac{\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau}{\tau}, \mathbf{v}\right) + a(\mathbf{u}^j_\tau, \mathbf{v}) + \tau\int_\Omega |\varepsilon(\mathbf{u}^j_\tau)|^{\gamma-2}\varepsilon(\mathbf{u}^j_\tau)\varepsilon(\mathbf{v})\,\mathrm{d}x + \int_\Omega \Theta(w^j_\tau)\mathrm{div}(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_{\rm C}} \chi^j_\tau \mathbf{u}^j_\tau \mathbf{v}\,\mathrm{d}x \\
&\quad + \int_{\Gamma_{\rm C}} \eta^j_\tau \mathbf{n}\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_{\rm C}} \mathfrak{c}(\Theta(w^j_\tau) - \Theta(w^j_{\mathrm{s},\tau}))\boldsymbol{\mu}^j_\tau \mathbf{v}\,\mathrm{d}x = \langle \mathbf{F}^j_\tau, \mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in W^{1,\gamma}_{\Gamma_{\rm Dir}}(\Omega;\mathbb{R}^3) \\
&\boldsymbol{\mu}^j_\tau = |\mathcal{R}^j_\tau(\eta^j_\tau\mathbf{n})|\mathbf{z}^j_\tau \text{ with } \eta^j_\tau = \phi'_\tau(u^j_{\tau,\mathrm{N}}) \text{ and } \mathbf{z}^j_\tau \in \partial j\left(\frac{\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau}{\tau}\right) \text{ a.e. in } \Gamma_{\rm C},
\end{aligned} \tag{4.10c}$$

$$\begin{aligned}
&\frac{\chi^j_\tau - \chi^{j-1}_\tau}{\tau} + A\chi^j_\tau + \xi^j_\tau + \tau|\chi^j_\tau|^{\gamma-2}\chi^j_\tau + \gamma'_\nu(\chi^j_\tau) - \nu\chi^{j-1}_\tau \\
&= -\left(\lambda'_\delta(\chi^{j-1}_\tau) + \delta\chi^j_\tau\right)\Theta(w^j_{\mathrm{s},\tau}) - \frac{1}{2}|\mathbf{u}^{j-1}_\tau|^2 \quad \text{a.e. in } \Gamma_{\rm C},\ \xi^j_\tau \in \beta(\chi^j_\tau) \text{ a.e. in } \Gamma_{\rm C}.
\end{aligned} \tag{4.10d}$$

In (4.10a)–(4.10c), $\mathcal{R}^j_\tau$ is from (4.7).

Remark 4.2 (*Features of the Time-Discretization Scheme*)**.** Observe that the scheme (4.10) is fully implicit and that the discrete equations (4.10a)–(4.10d) are coupled with each other. Therefore it is not possible to prove the existence of solutions to (4.10) by solving consecutive minimization problems in the single variables w, w_{s}, $\mathbf{u}$, and χ. In fact, following [26,11], in Lemma 4.3 we will resort to a fixed-point type result from the Leray–Schauder theory of pseudo-monotone operators.

The tight coupling between (4.10a)–(4.10d) is essentially due to the fact that the terms

$$\Theta(w_\tau^j)\mathrm{div}\left(\frac{\mathbf{u}_\tau^j-\mathbf{u}_\tau^{j-1}}{\tau}\right) \quad \text{and} \quad \Theta(w_{\mathrm{s},\tau}^j)\left(\frac{\lambda(\chi_\tau^j)-\lambda(\chi_\tau^{j-1})}{\tau}\right)$$

on the left-hand sides of (4.10a) and (4.10b) are *implicit* in w and w_{s}, respectively. Accordingly, the corresponding coupled terms in the (4.10c) and (4.10d) must be kept implicit in w and w_{s} in order to ensure the cancelations (4.26) that guarantee the validity of the discrete total energy inequality (4.25). In turn, the implicit character of the aforementioned terms in Eqs. (4.10a) and (4.10b) is necessary for the argument yielding the non-negativity (4.12) of the discrete enthalpies $(w_\tau^j)_{j=1}^{J_\tau}$ and $(w_{\mathrm{s},\tau}^j)_{j=1}^{J_\tau}$, cf. the proof of Lemma 4.3 and Remark 4.4.

It is also worthwhile to notice that

(1) the approximation of the term $\boldsymbol{\eta}$ in (3.29c) by $\phi_\tau'(u_{\tau,\mathrm{N}}^j)\mathbf{n}$ has been devised in order to have a term directed as the normal $\mathbf{n}$, and hence orthogonal to the tangential term $\boldsymbol{\mu}_\tau^j$. This will have a key role in the forthcoming *Seventh estimate* (cf. the proof of Proposition 4.6), where the bound (4.80) is derived for $\phi_\tau'(u_{\tau,\mathrm{N}}^j)\mathbf{n}$;
(2) as in [26], the role of the higher order contributions (4.3) on the left-hand sides of (4.10c)–(4.10d) is to compensate the quadratic growth of the terms

$$\varepsilon\left(\frac{\mathbf{u}_\tau^j-\mathbf{u}_\tau^{j-1}}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^j-\mathbf{u}_\tau^{j-1}}{\tau}\right), \qquad \left|\frac{\chi_\tau^j-\chi_\tau^{j-1}}{\tau}\right|^2$$

on the right-hand sides of (4.10a) and (4.10b). This will be apparent in the proof of Lemma 4.3.
(3) The scheme (4.10) has been carefully tailored in such a way as to ensure the discrete total energy inequality (4.25). In fact, it will result partly from the cancelations (4.26) of the *dissipative* terms, and partly from carefully handling some coupling terms via *convexity* (or *concavity*) inequalities. This motivates the splittings

$$\gamma_\nu'(\chi_\tau^j)-\nu\chi_\tau^{j-1} \quad \text{and} \quad \lambda_\delta'(\chi_\tau^{j-1})+\delta\chi_\tau^j \tag{4.11}$$

of the terms approximating $\gamma'(\chi)$ and $\lambda'(\chi)$ in (4.10d). Keeping the (derivative of the) convex part *implicit* and the (derivative of the) concave part *explicit* enables us to exploit the right convexity/concavity inequalities in (4.34)–(4.35).

We now establish the existence of solutions to scheme (4.10), and the non-negativity of the discrete enthalpies.

Lemma 4.3 (*Existence of Time Discrete Solutions*)**.** *Under Hypotheses (I)–(VI), conditions (3.25) on the data* $(h,\mathbf{f},\mathbf{g})$*, the third of (3.27), and (4.4) on the initial data* $(w_\tau^0, w_{\mathrm{s},\tau}^0, \mathbf{u}_\tau^0, \chi_0)$*, for every sufficiently small* $\tau > 0$ *there exists a solution* $(w_\tau^j, w_{\mathrm{s},\tau}^j, \mathbf{u}_\tau^j, \chi_\tau^j)_{j=1}^{J_\tau}$ *to Problem 4.1. If in addition (3.41) holds, then*

$$w_\tau^j \geq 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad w_{\mathrm{s},\tau}^j \geq 0 \quad \text{a.e. in } \Gamma_{\mathrm{C}} \text{ for all } j=1,\ldots,J_\tau. \tag{4.12}$$

Proof. System (4.10) rewrites as

$$\begin{aligned}
&\int_\Omega \left(w_\tau^j-\Theta(w_\tau^j)\mathrm{div}(\mathbf{u}_\tau^j-\mathbf{u}_\tau^{j-1})-\frac{1}{\tau}\varepsilon(\mathbf{u}_\tau^j)\mathbb{V}\varepsilon(\mathbf{u}_\tau^j)+\frac{2}{\tau}\varepsilon(\mathbf{u}_\tau^j)\mathbb{V}\varepsilon(\mathbf{u}_\tau^{j-1})\right)v\,\mathrm{d}x\\
&\quad+\tau\int_\Omega K(w_\tau^{j-1})\nabla w_\tau^j\nabla v\,\mathrm{d}x+\int_{\Gamma_{\mathrm{C}}}\Big(\tau k(\chi_\tau^{j-1})\Theta(w_\tau^j)(\Theta(w_\tau^j)-\Theta(w_{\mathrm{s},\tau}^j))\\
&\qquad+\Theta(w_\tau^j)\mathfrak{c}'(\Theta(w_\tau^j)-\Theta(w_{\mathrm{s},\tau}^j))|\mathcal{R}_\tau^j(\eta_\tau^j\mathbf{n})|\,\left|\mathbf{u}_{\tau,\mathrm{T}}^j-\mathbf{u}_{\tau,\mathrm{T}}^{j-1}\right|\Big)v\,\mathrm{d}x\\
&=\int_\Omega\left(w_\tau^{j-1}+\frac{1}{\tau}\varepsilon(\mathbf{u}_\tau^{j-1})\mathbb{V}\varepsilon(\mathbf{u}_\tau^{j-1})+\tau h_\tau^j\right)v\,\mathrm{d}x \quad \text{for all } v\in V,
\end{aligned} \tag{4.13a}$$

$$\begin{aligned}
&\int_{\Gamma_{\mathrm{C}}}\Big(w_{\mathrm{s},\tau}^j-\Theta(w_{\mathrm{s},\tau}^j)(\lambda(\chi_\tau^j)-\lambda(\chi_\tau^{j-1}))-\tau k(\chi_\tau^{j-1})\Theta(w_{\mathrm{s},\tau}^j)(\Theta(w_\tau^j)-\Theta(w_{\mathrm{s},\tau}^j))\\
&\quad-\Big(\mathfrak{c}(\Theta(w_\tau^j)-\Theta(w_{\mathrm{s},\tau}^j))+\Theta(w_{\mathrm{s},\tau}^j)\mathfrak{c}'(\Theta(w_\tau^j)-\Theta(w_{\mathrm{s},\tau}^j))\Big)|\mathcal{R}_\tau^j(\eta_\tau^j\mathbf{n})|\,\left|\mathbf{u}_{\tau,\mathrm{T}}^j-\mathbf{u}_{\tau,\mathrm{T}}^{j-1}\right|\\
&\quad-\frac{1}{\tau}|\chi_\tau^j|^2+\frac{2}{\tau}\chi_\tau^j\chi_\tau^{j-1}\Big)v\,\mathrm{d}x+\tau\int_{\Gamma_{\mathrm{C}}}K(w_{\mathrm{s},\tau}^{j-1})\nabla w_{\mathrm{s},\tau}^j\nabla v\,\mathrm{d}x
\end{aligned}$$

$$= \int_{\Gamma_C} \left(w_{s,\tau}^{j-1} + \frac{1}{\tau} |\chi_\tau^{j-1}|^2 \right) v \, dx \quad \text{for all } v \in V_{\Gamma_C}, \tag{4.13b}$$

$$b\left(\mathbf{u}_\tau^j, \mathbf{v}\right) + \tau a(\mathbf{u}_\tau^j, \mathbf{v}) + \tau^2 \int_\Omega |\varepsilon(\mathbf{u}_\tau^j)|^{\gamma-2} \varepsilon(\mathbf{u}_\tau^j)\varepsilon(\mathbf{v}) \, dx + \tau \int_\Omega \Theta(w_\tau^j) \mathrm{div}(\mathbf{v}) \, dx + \tau \int_{\Gamma_C} \chi_\tau^j \mathbf{u}_\tau^j \mathbf{v} \, dx$$

$$+ \tau \int_{\Gamma_C} \eta_\tau^j \mathbf{n} \mathbf{v} \, dx + \tau \int_{\Gamma_C} \mathfrak{c}(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j)) \boldsymbol{\mu}_\tau^j \mathbf{v} \, dx$$

$$= b\left(\mathbf{u}_\tau^{j-1}, \mathbf{v}\right) + \tau \left\langle \mathbf{F}_\tau^j, \mathbf{v} \right\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^3),$$

$$\boldsymbol{\mu}_\tau^j \in |\mathcal{R}_\tau^j(\phi_\tau'(u_{\tau,N}^j)\mathbf{n})| \partial j \left(\frac{\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}}{\tau} \right) \text{ a.e. in } \Gamma_C, \tag{4.13c}$$

$$\chi_\tau^j + \tau A \chi_\tau^j + \tau \xi_\tau^j + \tau^2 |\chi_\tau^j|^{\gamma-2} \chi_\tau^j + \tau \gamma_\nu'(\chi_\tau^j) + \tau \left(\lambda_\delta'(\chi_\tau^{j-1}) + \delta \chi_\tau^j \right) \Theta(w_{s,\tau}^j)$$

$$= \chi_\tau^{j-1} + \tau \nu \chi_\tau^{j-1} - \frac{\tau}{2} |\mathbf{u}_\tau^{j-1}|^2 \quad \text{a.e. in } \Gamma_C. \tag{4.13d}$$

Let us introduce the operator $\mathscr{R}_j : \mathcal{Q} \to \mathcal{Q}^*$, where we have used the place-holder $\mathcal{Q} := V \times V_{\Gamma_C} \times W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^3) \times H^1(\Gamma_C)$, defined on $q := (w, w_s, \mathbf{u}, \chi)$ by the left-hand side of system (4.13) (where $q = q_\tau^j = (w_\tau^j, w_{s,\tau}^j, \mathbf{u}_\tau^j, \chi_\tau^j)$): hence $\mathscr{R}_j$ depends on the solution quadruplet $(w_\tau^{j-1}, w_{s,\tau}^{j-1}, \mathbf{u}_\tau^{j-1}, \chi_\tau^{j-1})$ at the previous step $j-1$. Let us denote by $\mathscr{H}_j \in \mathcal{Q}^*$ the vector associated with the right-hand side terms in (4.13). Observe that $\mathscr{H}_j$ is given at the step j. Hence, (4.13) reformulates as

$$\mathscr{R}_j(w_\tau^j, w_{s,\tau}^j, \mathbf{u}_\tau^j, \chi_\tau^j) = \mathscr{H}_j. \tag{4.14}$$

The existence of a solution to (4.14) can be deduced from an abstract existence result for a wide class of elliptic systems, based on the theory of pseudo-monotone operators, cf. e.g. [32, Chap. II, Thm. 2.6].

Indeed, it can be verified that $\mathscr{R}_j$ is a pseudo-monotone operator (cf. [32, Chap. II, Def. 2.1]), and that $\mathscr{R}_j$ is coercive on $\mathcal{Q} = H^1(\Omega) \times H^1(\Gamma_C) \times W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^3) \times H^1(\Gamma_C)$, namely that

$$\lim_{\|(w, w_s, \mathbf{u}, \chi)\|_{\mathcal{Q}} \to \infty} \frac{\langle \mathscr{R}_j(w, w_s, \mathbf{u}, \chi), (w, w_s, \mathbf{u}, \chi) \rangle}{\|(w, w_s, \mathbf{u}, \chi)\|_{\mathcal{Q}}} = \infty.$$

To see this, one directly argues on system (4.13), testing (4.13a) by w_τ^j, (4.13b) by $w_{s,\tau}^j$, (4.13c) by $\mathbf{u}_\tau^j$, and (4.13d) by χ_τ^j, and adding the resulting relations. We shall not develop all the calculations in detail, referring, e.g., to the proofs of [11, Lemma 7.4] and of [33, Lemma 4.4] where the time-discrete analysis of PDE systems for adhesive contact, phase change, and damage with similar features to our own was carried out. Let us just highlight here how the most troublesome terms resulting from the test of (4.13a) by w_τ^j and of (4.13b) by $w_{s,\tau}^j$ can be dealt with, also by means of the higher order terms (4.3). Indeed, from (4.13a), tested by w_τ^j, and added to (4.13b), tested by $w_{s,\tau}^j$, one gets the term

$$\tau \int_{\Gamma_C} k(\chi_\tau^{j-1})(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j)) \left(\Theta(w_\tau^j) w_\tau^j - \Theta(w_{s,\tau}^j) w_{s,\tau}^j \right) dx \ge 0$$

the latter inequality by the monotonicity of the function $w \mapsto \Theta(w)w$. Analogously,

$$\int_{\Gamma_C} |\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})| \left| \mathbf{u}_{\tau,T}^j - \mathbf{u}_{\tau,T}^{j-1} \right| \mathfrak{c}'(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j)) \left(\Theta(w_\tau^j) w_\tau^j - \Theta(w_{s,\tau}^j) w_{s,\tau}^j \right) dx \ge 0$$

taking into account that the term $\mathfrak{c}'(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))$ has the same sign as $(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))$ in view of (3.21). Moreover, we can perform the following estimates

$$\begin{aligned}
&\left| \int_{\Gamma_C} \mathfrak{c}(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j)) w_{s,\tau}^j |\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})| \left| \mathbf{u}_{\tau,T}^j - \mathbf{u}_{\tau,T}^{j-1} \right| dx \right| \\
&\overset{(1)}{\le} C \|w_{s,\tau}^j\|_{L^q(\Gamma_C)} \|\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})\|_{L^{16/5}(\Gamma_C)} \|\mathbf{u}_{\tau,T}^j - \mathbf{u}_{\tau,T}^{j-1}\|_{L^4(\Gamma_C)} \\
&\overset{(2)}{\le} C \|w_{s,\tau}^j\|_{L^q(\Gamma_C)} \|\phi_\tau'(u_{\tau,N}^j)\|_{\mathbf{Y}^*} \left(\|\mathbf{u}_\tau^j\|_{\mathbf{W}} + 1 \right) \\
&\overset{(3)}{\le} C \|w_{s,\tau}^j\|_{V_{\Gamma_C}} \left(\|\mathbf{u}_\tau^j\|_{\mathbf{W}}^2 + 1 \right),
\end{aligned}$$

where (1) follows from the fact that $\mathfrak{c}$ is bounded, and from the Hölder inequality with $q > 1$ such that $\frac{1}{q} + \frac{5}{16} + \frac{1}{4} = 1$, (2) from (3.18) (cf. the forthcoming Lemma 4.7), and Sobolev embeddings/trace theorems. The latter also grant (3) (in particular, the fact that $V_{\Gamma_C} \subset L^q(\Gamma_C)$ for all $1 \le q < \infty$), in combination with the fact that ϕ_τ is Lipschitz continuous. Now, we are in the position to absorb the term $\|w^j_{s,\tau}\|_{V_{\Gamma_C}}$ into $\|w^j_{s,\tau}\|^2_{V_{\Gamma_C}}$ on the left-hand side. Instead, $\|\mathbf{u}^j_\tau\|^2_{\mathbf{W}}$ is estimated by $\tau^2\|\varepsilon(\mathbf{u}^j_\tau)\|^\gamma_{L^\gamma(\Omega)}$ which arises from testing the left-hand side of (4.13c) by $\mathbf{u}^j_\tau$. The term $\int_{\Gamma_C}\varepsilon\left(\frac{\mathbf{u}^j_\tau-\mathbf{u}^{j-1}_\tau}{\tau}\right)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}^j_\tau-\mathbf{u}^{j-1}_\tau}{\tau}\right)w^j_\tau\,dx$ can be estimated by similar arguments. With analogous calculations one deals with the quadratic term on the right-hand side of (4.13b), tested by $w^j_{s,\tau}$. All in all, we obtain a bound for $\|w^j_\tau\|^2_{H^1(\Omega)}$, $\|w^j_{s,\tau}\|^2_{V_{\Gamma_C}}$, $\|\mathbf{u}^j_\tau\|^\gamma_{W^{1,\gamma}(\Omega;\mathbb{R}^3)}$, and $\|\chi^j_\tau\|^2_{V_{\Gamma_C}}$. We thus conclude the desired coercivity. Therefore, [32, Chap. II, Thm. 2.6] applies, and the existence of a solution to (4.10) ensues.

Positivity of the discrete enthalpies. In order to check (4.12), we proceed by induction. Hence we suppose that $w^{j-1}_\tau \ge 0$ a.e. in Ω and $w^{j-1}_{s,\tau} \ge 0$ a.e. in Γ_C, test (4.13a) by $-(w^j_\tau)^-$, test (4.13b) by $-(w^j_{s,\tau})^-$, (where $(w)^- = -\min(w,0)$ denotes the negative part of a real number w), and add the resulting relations. Observe that $-(w^j_\tau)^- \in H^1(\Omega)$ and $-(w^j_{s,\tau})^- \in H^1(\Gamma_C)$ are admissible test functions. We then have

$$
\begin{aligned}
&\int_\Omega |(w^j_\tau)^-|^2\,dx + \int_{\Gamma_C}|(w^j_{s,\tau})^-|^2\,dx + \tau\int_\Omega K(w^{j-1}_\tau)|\nabla(w^j_\tau)^-|^2\,dx + \tau\int_{\Gamma_C}K(w^{j-1}_{s,\tau})|\nabla(w^j_{s,\tau})^-|^2\,dx\\
&\quad+\tau\int_{\Gamma_C}k(\chi^{j-1}_\tau)(\Theta(w^j_\tau)-\Theta(w^j_{s,\tau}))\left(-\Theta(w^j_\tau)(w^j_\tau)^-+\Theta(w^j_{s,\tau})(w^j_{s,\tau})^-\right)dx\\
&\quad+\tau\int_{\Gamma_C}\mathfrak{c}(\Theta(w^j_\tau)-\Theta(w^j_{s,\tau}))|\mathcal{R}^j_\tau(\eta^j_\tau\mathbf{n})|\left|\mathbf{u}^j_{\tau,\mathrm{T}}-\mathbf{u}^{j-1}_{\tau,\mathrm{T}}\right|(w^j_{s,\tau})^-\,dx\\
&\quad+\tau\int_{\Gamma_C}\left(\mathfrak{c}'(\Theta(w^j_\tau)-\Theta(w^j_{s,\tau}))|\mathcal{R}^j_\tau(\eta^j_\tau\mathbf{n})|\left|\mathbf{u}^j_{\tau,\mathrm{T}}-\mathbf{u}^{j-1}_{\tau,\mathrm{T}}\right|\right)\left(-\Theta(w^j_\tau)(w^j_\tau)^-+\Theta(w^j_{s,\tau})(w^j_\tau)^-\right)dx\\
&=-\frac{1}{\tau}\int_\Omega\varepsilon(\mathbf{u}^j_\tau-\mathbf{u}^{j-1}_\tau)\mathbb{V}\varepsilon(\mathbf{u}^j_\tau-\mathbf{u}^{j-1}_\tau)(w^j_\tau)^-\,dx-\int_\Omega(\tau h^j_\tau+w^{j-1}_{s,\tau})(w^j_\tau)^-\,dx-\frac{1}{\tau}\int_{\Gamma_C}|\chi^j_\tau-\chi^{j-1}_\tau|^2(w^j_{s,\tau})^-\,dx\\
&\quad-\int_{\Gamma_C}w^{j-1}_{s,\tau}(w^j_{s,\tau})^-\,dx\le 0.
\end{aligned}
$$

Now, the last inequality follows from the fact all the four integrals on the right-hand side are negative, also in view of (3.41) and of the previously supposed non-negativity of w^{j-1}_τ and $w^{j-1}_{s,\tau}$. On the other hand, the fifth and seventh integrals on the left-hand side are zero due to the crucial facts (cf. (3.14)) that

$$\Theta(w^j_\tau)(w^j_\tau)^- = 0\quad\text{a.e. in }\Omega\quad\text{and}\quad\Theta(w^j_{s,\tau})(w^j_{s,\tau})^-=0\quad\text{a.e. in }\Gamma_C. \tag{4.15}$$

The sixth integral on the left-hand side is non-negative thanks to the positivity of $\mathfrak{c}$. All in all, we infer

$$\int_\Omega|(w^j_\tau)^-|^2\,dx+\int_{\Gamma_C}|(w^j_{s,\tau})^-|^2\,dx+\tau\int_\Omega K(w^{j-1}_\tau)|\nabla(w^j_\tau)^-|^2\,dx+\tau\int_{\Gamma_C}K(w^{j-1}_{s,\tau})|\nabla(w^j_{s,\tau})^-|^2\,dx\le 0,$$

whence (4.12). □

Remark 4.4 (*Non-Negativity vs. Strict Positivity of the Discrete Enthalpies*). The proof of the non-negativity (4.12) of $(w^j_\tau)^{J_\tau}_{j=1}$ and $(w^j_{s,\tau})^{J_\tau}_{j=1}$ is tightly related to the implicit character of the terms $\Theta(w^j_\tau)\mathrm{div}((\mathbf{u}^j_\tau-\mathbf{u}^{j-1}_\tau)\tau)$ and $\Theta(w^j_{s,\tau})(\lambda(\chi^j_\tau)-\lambda(\chi^{j-1}_\tau))\tau$, cf. (4.15).

The proof of the *strict positivity* of $(w^j_\tau)^{J_\tau}_{j=1}$ and $(w^j_{s,\tau})^{J_\tau}_{j=1}$ seems to be open, at the moment.

Approximate solutions and approximate equations. Let $\mathcal{Y}$ be a given Banach space and $(y^j_\tau)^{J_\tau}_{j=1}\subset\mathcal{Y}$ a given J_τ-tuple. We introduce the left-continuous and right-continuous piecewise constant, and the piecewise linear interpolants of the values $\{y^j_\tau\}^{J_\tau}_{j=1}$, associated with the partition $\mathscr{P}_\tau$ (4.1), viz.

$$
\left.
\begin{array}{llll}
\bar{y}_\tau:(0,T)\to\mathcal{Y} & \text{defined by} & \bar{y}_\tau(t):=y^j_\tau, & \bar{y}_\tau(t^j_\tau)=y^j_\tau,\\
\underline{y}_\tau:(0,T)\to\mathcal{Y} & \text{defined by} & \underline{y}_\tau(t):=y^{j-1}_\tau, & \underline{y}_\tau(t^j_\tau)=y^j_\tau,\\
y_\tau:(0,T)\to\mathcal{Y} & \text{defined by} & y_\tau(t):=\dfrac{t-t^{j-1}_\tau}{\tau}y^j_\tau+\dfrac{t^j_\tau-t}{\tau}y^{j-1}_\tau &
\end{array}
\right\}\quad\text{for }t\in(t^{j-1}_\tau,t^j_\tau).
$$

Furthermore, we denote by $\bar{\mathrm{t}}_\tau$ and by $\underline{\mathrm{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition $\mathscr{P}_\tau$, i.e. $\bar{\mathrm{t}}_\tau(t):=t^j_\tau$ if $t^{j-1}_\tau<t<t^j_\tau$ and $\underline{\mathrm{t}}_\tau(t):=t^{j-1}_\tau$ if $t^{j-1}_\tau<t<t^j_\tau$, and $\bar{\mathrm{t}}_\tau(t^j_\tau)=t^j_\tau=\underline{\mathrm{t}}_\tau(t^j_\tau)$. Clearly, for every $t\in[0,T]$ we have $\bar{\mathrm{t}}_\tau(t)\downarrow t$ and $\underline{\mathrm{t}}_\tau(t)\uparrow t$ as $\tau\to 0$.

In view of (3.25), it is easy to check that the piecewise constant interpolants $(\bar{h}_\tau)_\tau$ and $(\bar{\mathbf{F}}_\tau)_\tau$ of the values $(h^j_\tau)^{J_\tau}_{j=1}$ and $(\mathbf{F}^j_\tau)^{J_\tau}_{j=1}$ (4.2) fulfill as $\tau\downarrow 0$

$$\bar{h}_\tau\to h\quad\text{in }L^1(0,T;L^1(\Omega)), \tag{4.16}$$

$$\overline{\mathbf{F}}_\tau \to \mathbf{F} \quad \text{in } L^\infty(0,T;\mathbf{W}^*), \qquad \mathbf{F}_\tau \to \mathbf{F} \quad \text{in } W^{1,1}(0,T;\mathbf{W}^*). \tag{4.17}$$

We now rewrite the discrete equations (4.10a)–(4.10d) in terms of the interpolants

$$\begin{aligned} &\overline{w}_\tau,\quad \underline{w}_\tau,\quad w_\tau,\quad \overline{w}_{s,\tau},\quad \underline{w}_{s,\tau},\quad w_{s,\tau},\quad \overline{\mathbf{u}}_\tau,\quad \underline{\mathbf{u}}_\tau,\quad \mathbf{u}_\tau,\\ &\quad \overline{\chi}_\tau,\quad \underline{\chi}_\tau,\quad \chi_\tau,\quad \overline{\eta}_\tau,\quad \underline{\eta}_\tau,\quad \overline{\boldsymbol{\mu}}_\tau,\quad \overline{\mathbf{z}}_\tau,\quad \overline{\xi}_\tau, \end{aligned} \tag{4.18}$$

of the elements $(w_\tau^j, w_{s,\tau}^j, \mathbf{u}_\tau^j, \chi_\tau^j, \eta_\tau^j, \boldsymbol{\mu}_\tau^j, \mathbf{z}_\tau^j, \xi_\tau^j)_{j=1}^{J_\tau}$ in (4.10a)–(4.10d). We will also use the notation

$$\overline{\boldsymbol{\eta}}_\tau := \overline{\eta}_\tau \mathbf{n}, \qquad \underline{\boldsymbol{\eta}}_\tau := \underline{\eta}_\tau \mathbf{n}, \tag{4.19}$$

$$\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau \mathbf{n})(t) := \mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n}) \quad \text{for } t \in (t_\tau^{j-1}, t_\tau^j), \tag{4.20}$$

and analogously for $\overline{\mathcal{R}}_\tau(\underline{\eta}_\tau \mathbf{n})$. Hence, we have for almost all $t \in (0,T)$

$$\begin{aligned} &\int_\Omega \partial_t w_\tau(t) v\,\mathrm{d}x - \int_\Omega \Theta(\overline{w}_\tau(t))\mathrm{div}\,(\partial_t \mathbf{u}_\tau(t))\, v\,\mathrm{d}x + \int_\Omega K(\underline{w}_\tau(t))\nabla \overline{w}_\tau(t)\nabla v\,\mathrm{d}x\\ &\quad + \int_{\Gamma_C} k(\underline{\chi}_\tau(t))\Theta(\overline{w}_\tau(t))(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))v\,\mathrm{d}x\\ &\quad + \int_{\Gamma_C} \Theta(\overline{w}_\tau(t))\mathfrak{c}'(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))|\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau(t)\mathbf{n})|\,\left|\partial_t \mathbf{u}_{\tau,\mathrm{T}}(t)\right| v\,\mathrm{d}x\\ &= \int_\Omega \varepsilon\,(\partial_t \mathbf{u}_\tau(t))\,\mathbb{V}\varepsilon\,(\partial_t \mathbf{u}_\tau(t))\, v\,\mathrm{d}x + \int_\Omega \overline{h}_\tau(t) v\,\mathrm{d}x \quad \text{for all } v \in V \end{aligned} \tag{4.21a}$$

$$\begin{aligned} &\int_{\Gamma_C} \partial_t w_{s,\tau}(t) v\,\mathrm{d}x - \int_{\Gamma_C} \Theta(\overline{w}_{s,\tau}(t))\frac{\lambda(\overline{\chi}_\tau(t)) - \lambda(\underline{\chi}_\tau(t))}{\tau} v\,\mathrm{d}x + \int_{\Gamma_C} K(\underline{w}_{s,\tau}(t))\nabla \overline{w}_{s,\tau}(t)\nabla v\,\mathrm{d}x\\ &= \int_{\Gamma_C} \Big(k(\underline{\chi}_\tau(t))\Theta(\overline{w}_{s,\tau}(t))(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))\\ &\quad + \left(\mathfrak{c}(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t))) + \Theta(\overline{w}_{s,\tau}(t))\mathfrak{c}'(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))\right)|\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau \mathbf{n})(t)|\,\left|\partial_t \mathbf{u}_{\tau,\mathrm{T}}(t)\right|\Big) v\\ &\quad + \int_{\Gamma_C} |\partial_t \chi_\tau(t)|^2\, v\,\mathrm{d}x \quad \text{for all } v \in V_{\Gamma_C}, \end{aligned} \tag{4.21b}$$

$$\begin{aligned} &b\,(\partial_t \mathbf{u}_\tau(t), \mathbf{v}) + a(\overline{\mathbf{u}}_\tau(t), \mathbf{v}) + \tau \int_\Omega |\varepsilon(\overline{\mathbf{u}}_\tau(t))|^{\gamma-2}\varepsilon(\overline{\mathbf{u}}_\tau(t))\varepsilon(\mathbf{v})\,\mathrm{d}x + \int_\Omega \Theta(\overline{w}_\tau(t))\mathrm{div}(\mathbf{v})\,\mathrm{d}x\\ &\quad + \int_{\Gamma_C} \overline{\chi}_\tau(t)\overline{\mathbf{u}}_\tau(t)\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_C} \overline{\eta}_\tau(t)\mathbf{n}\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_C} \mathfrak{c}(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))\overline{\boldsymbol{\mu}}_\tau(t)\mathbf{v}\,\mathrm{d}x\\ &= \left\langle \overline{\mathbf{F}}_\tau(t), \mathbf{v}\right\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^3), \end{aligned} \tag{4.21c}$$

$$\overline{\boldsymbol{\mu}}_\tau(t) = |\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau \mathbf{n})(t)|\overline{\mathbf{z}}_\tau(t) \quad \text{with } \overline{\mathbf{z}}_\tau(t) \in \partial j\,(\partial_t \mathbf{u}_\tau(t)) \text{ a.e. in } \Gamma_C, \tag{4.21d}$$

$$\begin{aligned} &\partial_t \chi_\tau(t) + A\overline{\chi}_\tau(t) + \overline{\xi}_\tau(t) + \tau|\overline{\chi}_\tau(t)|^{\gamma-2}\overline{\chi}_\tau(t) + \gamma_\nu'(\overline{\chi}_\tau(t)) - \nu\underline{\chi}_\tau(t)\\ &= -\left(\lambda_\delta'(\underline{\chi}_\tau(t)) + \delta\overline{\chi}_\tau(t)\right)\Theta(\overline{w}_{s,\tau}(t)) - \frac{1}{2}|\underline{\mathbf{u}}_\tau(t)|^2 \quad \text{a.e. in } \Gamma_C. \end{aligned} \tag{4.21e}$$

In Section 5 we will pass to the limit in system (4.21) as $\tau \downarrow 0$.

4.2. A priori estimates

We start by providing the discrete analogue of the energy balance (3.36), namely the discrete total energy inequality (4.25). It features the discrete analogue of the energy functional $\mathscr{E}$ from (3.35), viz.

$$\begin{aligned} \mathscr{E}_\tau(w, w_s, \mathbf{u}, \chi) := &\int_\Omega w\,\mathrm{d}x + \int_{\Gamma_C} w_s\,\mathrm{d}x + \frac{1}{2}a(u,u) + \frac{\tau}{\gamma}\int_\Omega |\varepsilon(\mathbf{u})|^\gamma\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_C} \chi|\mathbf{u}|^2\,\mathrm{d}x + \int_{\Gamma_C} \phi_\tau(u_{\mathrm{N}})\,\mathrm{d}x\\ &+ \frac{\tau}{\gamma}\int_{\Gamma_C} |\chi|^\gamma\,\mathrm{d}x + \int_{\Gamma_C}\left(\frac{1}{2}|\nabla\chi|^2 + W(\chi)\right)\mathrm{d}x. \end{aligned} \tag{4.22}$$

In the proof of Proposition 4.6, from (4.25) we shall deduce the first set of estimates (namely, the *energy estimates*) on the discrete solutions $(\overline{w}_\tau, \overline{w}_{s,\tau}, \overline{\mathbf{u}}_\tau, \overline{\chi}_\tau)$. They will serve as the basis for all the remaining estimates.

Let us mention in advance that, in the proof of (4.25) we will also use the following crucial inequalities:

– for any *convex* (differentiable) function $\psi : \mathbb{R} \to (-\infty, +\infty]$

$$\psi(x) - \psi(y) \le \psi'(x)(x-y) \quad \text{for all } x, y \in \mathrm{dom}(\psi), \tag{4.23}$$

– for any *concave* (differentiable) function $\varrho : \mathbb{R} \to [-\infty, +\infty)$

$$\varrho(x) - \varrho(y) \le \varrho'(y)(x-y) \quad \text{for all } x, y \in \mathrm{dom}(\varrho), \tag{4.24}$$

Lemma 4.5 (*Discrete Energy Inequality*)**.** *Under Hypotheses (I)–(VI), conditions (3.25) on the data* $(h, \mathbf{f}, \mathbf{g})$, *the third of (3.27), and (4.4) on the initial data* $(w_\tau^0, w_{s,\tau}^0, \mathbf{u}_\tau^0, \chi_0)$, *there holds*

$$\begin{aligned}
&\mathscr{E}_\tau(\overline{w}_\tau(t), \overline{w}_{s,\tau}(t), \overline{\mathbf{u}}_\tau(t), \overline{\chi}_\tau(t)) + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)} \int_{\Gamma_C} k(\underline{\chi}_\tau)(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))^2 \,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)} \int_{\Gamma_C} (\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))\mathfrak{c}'(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau \mathbf{n})| \left|\partial_t \mathbf{u}_{\tau,\mathrm{T}}\right| \,\mathrm{d}x\,\mathrm{d}r \\
&\le \mathscr{E}_\tau(\overline{w}_\tau(s), \overline{w}_{s,\tau}(s), \overline{\mathbf{u}}_\tau(s), \overline{\chi}_\tau(s)) + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)} \left(\int_\Omega \overline{h}_\tau \,\mathrm{d}x + \left\langle \overline{\mathbf{F}}_\tau, \partial_t \mathbf{u}_\tau \right\rangle_{\mathbf{W}} \right) \mathrm{d}r
\end{aligned} \tag{4.25}$$

for all $0 \le s \le t \le T$.

Proof. We multiply (4.10a) and (4.10b) by τ, we test (4.10c) by $\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}$, (4.10d) by $\chi_\tau^j - \chi_\tau^{j-1}$, and add the resulting relations. The terms

$$\begin{aligned}
&-\int_\Omega \Theta(w_\tau^j)\mathrm{div}(\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}) \,\mathrm{d}x, \\
&\tau \int_\Omega \varepsilon\left(\frac{\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}}{\tau}\right) \mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}}{\tau}\right) \mathrm{d}x, \\
&\int_{\Gamma_C} \mathfrak{c}(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))|\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})| \,|\mathbf{u}_{\tau,\mathrm{T}}^j - \mathbf{u}_{\tau,\mathrm{T}}^{j-1}| \,\mathrm{d}x, \\
&\tau \int_{\Gamma_C} \left| \frac{\chi_\tau^j - \chi_\tau^{j-1}}{\tau} \right|^2 \mathrm{d}x
\end{aligned} \tag{4.26}$$

cancel out, whereas other terms from the discrete enthalpy equations (4.10a) and (4.10b) combine to give the *positive* (cf. (3.20) and (3.21)) terms

$$\begin{aligned}
&\tau \int_{\Gamma_C} k(\chi_\tau^{j-1})(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))^2 \,\mathrm{d}x, \\
&\int_{\Gamma_C} (\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))\mathfrak{c}'(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))|\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})| \,|\mathbf{u}_{\tau,\mathrm{T}}^j - \mathbf{u}_{\tau,\mathrm{T}}^{j-1}| \,\mathrm{d}x \ge 0.
\end{aligned}$$

Moreover, the elementary identity $a(\mathbf{u}_\tau^j, \mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}) = \frac{1}{2}a(\mathbf{u}_\tau^j, \mathbf{u}_\tau^j) - \frac{1}{2}a(\mathbf{u}_\tau^{j-1}, \mathbf{u}_\tau^{j-1}) + \frac{1}{2}a(\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}, \mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1})$ yields

$$\frac{1}{2}a(\mathbf{u}_\tau^j, \mathbf{u}_\tau^j) - \frac{1}{2}a(\mathbf{u}_\tau^{j-1}, \mathbf{u}_\tau^{j-1}) \le a(\mathbf{u}_\tau^j, \mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}). \tag{4.27}$$

In addition, we observe that (cf. (4.23))

$$\frac{\tau}{\gamma} \int_\Omega |\varepsilon(\mathbf{u}_\tau^j)|^\gamma \,\mathrm{d}x - \frac{\tau}{\gamma} \int_\Omega |\varepsilon(\mathbf{u}_\tau^{j-1})|^\gamma \,\mathrm{d}x \le \tau \int_\Omega |\varepsilon(\mathbf{u}_\tau^j)|^{\gamma-2}\varepsilon(\mathbf{u}_\tau^j)(\varepsilon(\mathbf{u}_\tau^j) - \varepsilon(\mathbf{u}_\tau^{j-1})) \,\mathrm{d}x, \tag{4.28}$$

and

$$\frac{1}{2}\int_{\Gamma_C} \chi_\tau^j |\mathbf{u}_\tau^j|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Gamma_C} \chi_\tau^j |\mathbf{u}_\tau^{j-1}|^2 \,\mathrm{d}x \le \int_{\Gamma_C} \chi_\tau^j \mathbf{u}_\tau^j (\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}) \,\mathrm{d}x \tag{4.29}$$

for the term deriving from the discrete momentum equation (4.10c). The corresponding term on the right-hand side of (4.10d) reads, upon changing sign,

$$\frac{1}{2}\int_{\Gamma_C} |\mathbf{u}_\tau^{j-1}|^2(\chi_\tau^j - \chi_\tau^{j-1}) \,\mathrm{d}x = \frac{1}{2}\int_{\Gamma_C} \chi_\tau^j |\mathbf{u}_\tau^{j-1}|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Gamma_C} \chi_\tau^{j-1} |\mathbf{u}_\tau^{j-1}|^2 \,\mathrm{d}x, \tag{4.30}$$

leading to a further cancelation with (4.29). Furthermore, applying to the convex function ϕ_τ (4.6) inequality (4.23), we infer

$$\int_{\Gamma_C} \phi_\tau(u^j_{\tau,N})\,dx - \int_{\Gamma_C} \phi_\tau(u^{j-1}_{\tau,N})\,dx \le \int_{\Gamma_C} \eta^j_\tau \mathbf{n}(\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau)\,dx. \tag{4.31}$$

On the other hand, using that $\boldsymbol{\mu}^j_\tau \in |\mathcal{R}^j_\tau(\eta^j_\tau\mathbf{n})|\partial j\left(\frac{\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau}{\tau}\right)$, we infer

$$0 \le \int_{\Gamma_C} \mathfrak{c}(\Theta(w^j_\tau) - \Theta(w^j_{s,\tau}))|\mathcal{R}^j_\tau(\eta^j_\tau\mathbf{n})|\,|\mathbf{u}^j_{\tau,T} - \mathbf{u}^{j-1}_{\tau,T}|\,dx = \int_{\Gamma_C} \mathfrak{c}(\Theta(w^j_\tau) - \Theta(w^j_{s,\tau}))\boldsymbol{\mu}^j_\tau(\mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau)\,dx. \tag{4.32}$$

As for the terms arising from (4.10d), relying on (4.23)–(4.24) we observe that

$$\frac{\tau}{\gamma}\int_{\Gamma_C} |\chi^j_\tau|^\gamma\,dx - \frac{\tau}{\gamma}\int_{\Gamma_C} |\chi^{j-1}_\tau|^\gamma\,dx \le \tau\int_{\Gamma_C} |\chi^j_\tau|^{\gamma-2}\chi^j_\tau(\chi^j_\tau - \chi^{j-1}_\tau)\,dx, \tag{4.33}$$

$$\begin{aligned}\int_{\Gamma_C} \gamma(\chi^j_\tau)\,dx - \int_{\Gamma_C} \gamma(\chi^{j-1}_\tau)\,dx &= \int_{\Gamma_C} \gamma_\nu(\chi^j_\tau)\,dx - \int_{\Gamma_C} \gamma_\nu(\chi^{j-1}_\tau)\,dx - \frac{\nu}{2}\int_{\Gamma_C} |\chi^j_\tau|^2\,dx + \frac{\nu}{2}\int_{\Gamma_C} |\chi^{j-1}_\tau|^2\,dx \\ &\le \int_{\Gamma_C} \gamma'_\nu(\chi^j_\tau)(\chi^j_\tau - \chi^{j-1}_\tau)\,dx - \int_{\Gamma_C} \nu\chi^{j-1}_\tau(\chi^j_\tau - \chi^{j-1}_\tau)\,dx,\end{aligned} \tag{4.34}$$

$$\begin{aligned}\int_{\Gamma_C} \Theta(w^j_{s,\tau})\left(\lambda(\chi^j_\tau) - \lambda(\chi^{j-1}_\tau)\right)\,dx &= \int_{\Gamma_C} \Theta(w^j_{s,\tau})\left(\lambda_\delta(\chi^j_\tau) - \lambda_\delta(\chi^{j-1}_\tau)\right)\,dx + \frac{\delta}{2}\int_{\Gamma_C} \Theta(w^j_{s,\tau})\left(|\chi^j_\tau|^2 - |\chi^{j-1}_\tau|^2\right)\,dx \\ &\le \int_{\Gamma_C} \Theta(w^j_{s,\tau})\lambda'_\delta(\chi^{j-1}_\tau)(\chi^j_\tau - \chi^{j-1}_\tau)\,dx + \int_{\Gamma_C} \delta\Theta(w^j_{s,\tau})\chi^j_\tau(\chi^j_\tau - \chi^{j-1}_\tau)\,dx.\end{aligned} \tag{4.35}$$

Observe that the term $\int_{\Gamma_C} \Theta(w^j_{s,\tau})(\lambda(\chi^j_\tau) - \lambda(\chi^{j-1}_\tau))\,dx$ then cancels out with the one arising from (4.10b). Furthermore, we have

$$\frac{1}{2}\int_{\Gamma_C} |\nabla\chi^j_\tau|^2\,dx - \frac{1}{2}\int_{\Gamma_C} |\nabla\chi^{j-1}_\tau|^2\,dx + \int_{\Gamma_C} \widehat{\beta}(\chi^j_\tau)\,dx - \int_{\Gamma_C} \widehat{\beta}(\chi^{j-1}_\tau)\,dx \le \int_{\Gamma_C} (A(\chi^j_\tau) + \xi^j_\tau)(\chi^j_\tau - \chi^{j-1}_\tau)\,dx. \tag{4.36}$$

Combining (4.26)–(4.36) we finally obtain

$$\begin{aligned}&\mathscr{E}_\tau(w^j_\tau, w^j_{s,\tau}, \mathbf{u}^j_\tau, \chi^j_\tau) + \tau\int_{\Gamma_C} k(\chi^{j-1}_\tau)(\Theta(w^j_\tau) - \Theta(w^j_{s,\tau}))^2\,dx \\ &\quad + \tau\int_{\Gamma_C} (\Theta(w^j_\tau) - \Theta(w^j_{s,\tau}))\mathfrak{c}'(\Theta(w^j_\tau) - \Theta(w^j_{s,\tau}))|\mathcal{R}^j_\tau(\eta^j_\tau\mathbf{n})|\frac{|\mathbf{u}^j_{\tau,T} - \mathbf{u}^{j-1}_{\tau,T}|}{\tau}\,dx \\ &\quad \le \mathscr{E}_\tau(w^{j-1}_\tau, w^{j-1}_{s,\tau}, \mathbf{u}^{j-1}_\tau, \chi^{j-1}_\tau) + \left\langle \mathbf{F}^j_\tau, \mathbf{u}^j_\tau - \mathbf{u}^{j-1}_\tau\right\rangle_{\mathbf{W}} + \tau\int_\Omega h^j_\tau\,dx.\end{aligned} \tag{4.37}$$

Then, (4.25) follows, upon summing (4.37) over any couple of indexes $1 \le l < m \le J_\tau$. □

For later convenience, we also point out a second (discrete) energy inequality, which in fact derives from the discrete equations for $\mathbf{u}$ and χ only. To state it, we introduce the (mechanical) energy functional

$$\begin{aligned}\mathscr{J}_\tau(\mathbf{u}, \chi) &:= \frac{1}{2}a(u, u) + \frac{\tau}{\gamma}\int_\Omega |\varepsilon(\mathbf{u})|^\gamma\,dx + \frac{1}{2}\int_{\Gamma_C} \chi|\mathbf{u}|^2\,dx + \int_{\Gamma_C} \phi_\tau(u_N)\,dx \\ &\quad + \int_{\Gamma_C} \left(\frac{\tau}{\gamma}|\chi|^\gamma + \frac{1}{2}|\nabla\chi|^2 + W(\chi)\right)\,dx.\end{aligned} \tag{4.38}$$

Hence, we have

$$\begin{aligned}&\mathscr{J}_\tau(\overline{\mathbf{u}}_\tau(t), \overline{\chi}_\tau(t)) + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)} b(\partial_t\mathbf{u}_\tau, \partial_t\mathbf{u}_\tau)\,dr + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\int_{\Gamma_C} \mathfrak{c}(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau\mathbf{n})|\,\left|\partial_t\mathbf{u}_{\tau,T}\right|\,dx\,dr \\ &\quad + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\int_{\Gamma_C} |\partial_t\chi_\tau|^2\,dx\,dr \\ &\quad \le \mathscr{J}_\tau(\overline{\mathbf{u}}_\tau(s), \overline{\chi}_\tau(s)) + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)} \left\langle \overline{\mathbf{F}}_\tau, \partial_t\mathbf{u}_\tau\right\rangle_{\mathbf{W}}\,dr - \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\int_\Omega \Theta(\overline{w}_\tau)\mathrm{div}(\partial_t\mathbf{u}_\tau)\,dx\,dr \\ &\quad - \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\int_{\Gamma_C} (\lambda'_\delta(\underline{\chi}_\tau) + \delta\overline{\chi}_\tau)\Theta(\overline{w}_{s,\tau})\partial_t\chi_\tau\,dx\,dr\end{aligned} \tag{4.39}$$

for all $0 \le s \le t \le T$. In fact, (4.39) follows from subtracting from the discrete total energy inequality (4.25), the discrete enthalpy equations (4.21a) and (4.21b), integrated over the interval $(\bar{\mathsf{t}}_\tau(s), \bar{\mathsf{t}}_\tau(t))$.

We are now in the position to derive the a priori estimates on the approximate solutions from (4.18). In the proof of Proposition 4.6, we will deduce the first set of a priori estimates from the total energy inequality (4.25). To carry out the related calculations, we will resort to the *discrete by-part integration* formula

$$\sum_{j=1}^{J_\tau} \left\langle w_\tau^j - w_\tau^{j-1}, v_\tau^j \right\rangle_{\mathbf{W}} = \left\langle w_\tau^{J_\tau}, v_\tau^{J_\tau} \right\rangle_{\mathbf{W}} - \left\langle w_\tau^0, v_\tau^1 \right\rangle_{\mathbf{W}} - \sum_{j=2}^{J_\tau} \left\langle w_\tau^{j-1}, v_\tau^j - v_\tau^{j-1} \right\rangle_{\mathbf{W}} \tag{4.40}$$

for all $(w_\tau^j)_{j=1}^{J_\tau} \subset \mathbf{W}^*$, $(v_\tau^j)_{j=1}^{J_\tau} \subset \mathbf{W}$.

Next, we shall proceed with the estimate of the discrete enthalpy variables and deduce (4.43f) and (4.43h) through a series of careful calculations, based on an intermediate energy estimate drawn from the (mechanical) energy inequality (4.39), and on a clever adaptation of Boccardo–Gallouët estimates from [26], see also [34]. Once done this, we will obtain the *dissipativity* estimates (4.43b) and (4.43e).

Estimate (4.43j) shall be deduced by comparison in the discrete momentum equation. Let us observe that it is with values in the space

$$\mathbf{Y}_\gamma^* \quad \text{with } \mathbf{Y}_\gamma = W^{1-1/\gamma,\gamma}(\Gamma_{\mathrm{C}};\mathbb{R}^3) \tag{4.41}$$

because, due to the higher order regularizing term $-\tau\,\mathrm{div}(|\varepsilon(\mathbf{u})|^{\gamma-2}\varepsilon(\mathbf{u}))$ on the left-hand side, only test functions $\mathbf{v} \in W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^3)$ are admissible for (4.21c). In fact, without the term $-\tau\,\mathrm{div}(|\varepsilon(\mathbf{u})|^{\gamma-2}\varepsilon(\mathbf{u}))$, the family $(\overline{\boldsymbol{\eta}}_\tau)_\tau$ would be estimated in $L^2(0,T;\mathbf{Y}^*)$, and the worse character of estimate (4.43j) is only due to technical reasons.

Combining (4.43j) with the boundedness properties of the regularizing operator $\mathcal{R}$ we will obtain an estimate for $(\overline{\boldsymbol{\mu}}_\tau)_\tau$. For this, we will have to resort to a strengthened version of hypothesis (3.19), namely

Hypothesis 4.1. The operator $\mathcal{R} : L^2(0,T;\mathbf{Y}^*) \to L^\infty(0,T;L^{16/5+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3))$ extends to an operator (denoted by the same symbol)

$$\mathcal{R} : L^2(0,T;\mathbf{Y}_\gamma^*) \to L^\infty(0,T;L^{16/5+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3)) \text{ bounded in the sense of (3.18) and weakly–strongly continuous.} \tag{4.42}$$

As mentioned in the above lines, the reason why we need Hypothesis 4.1 is purely technical and can be bypassed upon devising more carefully the approximation of system (3.9): we will explore this in Remark 5.1 at the end of the proof of Theorem 1. That is why, we have stayed with the weaker conditions (3.18)–(3.19) on $\mathcal{R}$ in Section 3.1 and we will use Hypothesis 4.1 only in the proof of Proposition 4.6, and in Section 5. More precisely, we will rely on a consequence of Hypothesis 4.1, cf. Lemma 4.7, whose statement is postponed after the proof of Proposition 4.6.

Proposition 4.6 (*A Priori Estimates*)**.** *Under Hypotheses (I)–(VI), conditions (3.25) on the data $(h,\mathbf{f},\mathbf{g})$, the third of (3.27), and (4.4) on the initial data $(w_\tau^0, w_{\mathrm{s},\tau}^0, \mathbf{u}_\tau^0, \chi_0)$, there exists a constant $C>0$ such that for every $\tau>0$*

$$\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;\mathbf{W})} \le C, \tag{4.43a}$$

$$\|\mathbf{u}_\tau\|_{H^1(0,T;\mathbf{W})} \le C, \tag{4.43b}$$

$$\tau^{1/\gamma}\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^3))} \le C, \tag{4.43c}$$

$$\|\overline{\chi}_\tau\|_{L^\infty(0,T;V_{\Gamma_{\mathrm{C}}})\cap L^2(0,T;H^2(\Gamma_{\mathrm{C}}))} + \|\chi_\tau\|_{L^\infty(0,T;V_{\Gamma_{\mathrm{C}}})\cap L^2(0,T;H^2(\Gamma_{\mathrm{C}}))} \le C, \tag{4.43d}$$

$$\|\chi_\tau\|_{H^1(0,T;H_{\Gamma_{\mathrm{C}}})} \le C, \tag{4.43e}$$

$$\|\overline{w}_\tau\|_{L^r(0,T;W^{1,r}(\Omega))\cap L^\infty(0,T;L^1(\Omega))} \le C \quad \textit{for all } 1 \le r < \frac{5}{4}, \tag{4.43f}$$

$$\|\partial_t w_\tau\|_{L^1(0,T;W^{1,r'}(\Omega)^*)} \le C \quad \textit{for all } 1 \le r < \frac{5}{4}, \tag{4.43g}$$

$$\|\overline{w}_{\mathrm{s},\tau}\|_{L^\rho(0,T;W^{1,\rho}(\Gamma_{\mathrm{C}}))\cap L^\infty(0,T;L^1(\Gamma_{\mathrm{C}}))} \le C \quad \textit{for all } 1 \le \rho < \frac{4}{3}, \tag{4.43h}$$

$$\|\partial_t w_{\mathrm{s},\tau}\|_{L^1(0,T;W^{1,\rho'}(\Gamma_{\mathrm{C}})^*)} \le C \quad \textit{for all } 1 \le \rho < \frac{4}{3}, \tag{4.43i}$$

$$\|\overline{\boldsymbol{\eta}}_\tau\|_{L^2(0,T;\mathbf{Y}_\gamma^*)} \le C, \tag{4.43j}$$

$$\|\overline{\boldsymbol{\mu}}_\tau\|_{L^\infty(0,T;L^{16/5+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3))} \le C, \tag{4.43k}$$

$$\|\overline{\xi}_\tau\|_{L^2(0,T;H_{\Gamma_{\mathrm{C}}})} \le C. \tag{4.43l}$$

Proof. First a priori estimate (energy estimate). We start from the discrete energy inequality (4.25), written for $s = 0$. Preliminarily, we observe that, due to (3.5) for the bilinear form a, Korn's inequality, the positivity of ϕ_τ, and the third of (3.23), there exist positive constants C, C' such that

$$\int_\Omega w \,\mathrm{d}x + \int_{\Gamma_C} w_s \,\mathrm{d}x + \|\mathbf{u}\|_{\mathbf{W}}^2 + \tau\|\mathbf{u}\|_{W^{1,\gamma}(\Omega;\mathbb{R}^3)}^\gamma + \|\chi\|_{L^1(\Gamma_C)} + \|\nabla\chi\|_{L^2(\Gamma_C)}^2 \leq C\mathscr{E}_\tau(w, w_s, \mathbf{u}, \chi) + C'. \tag{4.44}$$

As for the right-hand side of (4.25), it follows from assumptions (3.27) and (4.4) on the initial data that the term $\mathscr{E}_\tau(\overline{w}_\tau(0), \overline{w}_{s,\tau}(0), \overline{\mathbf{u}}_\tau(0), \overline{\chi}_\tau(0))$ is bounded uniformly with respect to τ. We use (4.16) to estimate the second integral term. For the third one, we resort to the discrete by-part integration formula (4.40), yielding

$$\int_0^{\bar{\mathrm{t}}_\tau(t)} \langle \overline{\mathbf{F}}_\tau, \partial_t \mathbf{u}_\tau \rangle_{\mathbf{W}} \,\mathrm{d}r = \langle \overline{\mathbf{F}}_\tau(t), \overline{\mathbf{u}}_\tau(t) \rangle_{\mathbf{W}} - \langle \overline{\mathbf{F}}_\tau(\tau), \mathbf{u}_\tau^0 \rangle_{\mathbf{W}} - \int_\tau^{\bar{\mathrm{t}}_\tau(t)} \langle \partial_t \mathbf{F}_\tau, \underline{\mathbf{u}}_\tau \rangle_{\mathbf{W}} \,\mathrm{d}r \doteq I_1 + I_2 + I_3.$$

With the Young inequality we have

$$|I_1| \leq \frac{1}{C_a}\|\overline{\mathbf{F}}_\tau\|_{L^\infty(0,T;\mathbf{W})}^2 + \frac{C_a}{4}\|\overline{\mathbf{u}}_\tau(t)\|_{\mathbf{W}}^2 \leq C + \frac{C_a}{4}\|\overline{\mathbf{u}}_\tau(t)\|_{\mathbf{W}}^2$$

where C_a is from (3.5) and we have used (4.17). With analogous calculations, also taking into account (4.4), we find that $|I_2| \leq C$. Moreover,

$$|I_3| \leq \int_0^{\bar{\mathrm{t}}_\tau(t)} \|\partial_t \mathbf{F}_\tau\|_{\mathbf{W}^*}\|\underline{\mathbf{u}}_\tau\|_{\mathbf{W}} \,\mathrm{d}r.$$

Therefore, in view of (4.44) we deduce from (4.25) that

$$\frac{C_a}{4}\|\overline{\mathbf{u}}_\tau(t)\|_{\mathbf{W}}^2 \leq C + \int_0^{\bar{\mathrm{t}}_\tau(t)} \|\partial_t \mathbf{F}_\tau\|_{\mathbf{W}^*}\|\underline{\mathbf{u}}_\tau\|_{\mathbf{W}} \,\mathrm{d}r.$$

Again taking into account (4.17), we conclude estimate (4.43a) via the Gronwall Lemma. Then, we infer

$$\exists\, C > 0\, \forall\, t \in (0, T) : \ \mathscr{E}_\tau(\overline{w}_\tau(t), \overline{w}_{s,\tau}(t), \overline{\mathbf{u}}_\tau(t), \overline{\chi}_\tau(t)) \leq C.$$

In view of (4.44), we then conclude

$$\|\overline{w}_\tau\|_{L^\infty(0,T;L^1(\Omega))} \leq C \quad \text{and} \quad \|\overline{w}_{s,\tau}\|_{L^\infty(0,T;L^1(\Gamma_C))} \leq C, \tag{4.45}$$

due to the positivity properties (4.12), estimate (4.43c), as well as

$$\|\overline{\chi}_\tau\|_{L^\infty(0,T;V_{\Gamma_C})} \leq C \tag{4.46}$$

by the Poincaré inequality.

From inequality (4.25) we also deduce

$$\int_0^T \int_{\Gamma_C} k(\underline{\chi}_\tau)(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))^2 \,\mathrm{d}x\,\mathrm{d}t \leq C, \tag{4.47}$$

$$\int_0^T \int_{\Gamma_C} (\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))\mathfrak{c}'(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))|\mathcal{R}_\tau(\overline{\eta}_\tau\mathbf{n})|\, \left|\partial_t \mathbf{u}_{\tau,\mathrm{T}}\right| \,\mathrm{d}x\,\mathrm{d}t \leq C. \tag{4.48}$$

Intermediate energy estimate. We write the (mechanical) energy inequality (4.39) for $s = 0$ and $0 \leq t \leq T$. Clearly, the energy functional $\mathscr{J}_\tau$ partially inherits the coercivity properties of $\mathscr{E}_\tau$. In particular, it is bounded from below. Since the frictional term on the left-hand side of (4.39) is non-negative by (3.21), also taking into account (3.5) for b we arrive at

$$\begin{aligned} &C_b\int_0^{\bar{\mathrm{t}}_\tau(t)} \|\partial_t \mathbf{u}_\tau\|_{\mathbf{W}}^2 \,\mathrm{d}r + \int_0^{\bar{\mathrm{t}}_\tau(t)}\int_{\Gamma_C} |\partial_t\chi_\tau|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_0^{\bar{\mathrm{t}}_\tau(t)}\int_{\Gamma_C} \mathfrak{c}(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))|\mathcal{R}_\tau(\overline{\eta}_\tau\mathbf{n})|\,\left|\partial_t\mathbf{u}_{\tau,\mathrm{T}}\right| \,\mathrm{d}x\,\mathrm{d}r \\ &\leq C + \int_0^{\bar{\mathrm{t}}_\tau(t)} \langle \overline{\mathbf{F}}_\tau, \partial_t \mathbf{u}_\tau \rangle_{\mathbf{W}} \,\mathrm{d}r - \int_0^{\bar{\mathrm{t}}_\tau(t)}\int_\Omega \Theta(\overline{w}_\tau)\mathrm{div}(\partial_t\mathbf{u}_\tau)\,\mathrm{d}x\,\mathrm{d}r - \int_0^{\bar{\mathrm{t}}_\tau(t)}\int_{\Gamma_C} (\lambda_\delta'(\underline{\chi}_\tau) + \delta\overline{\chi}_\tau)\Theta(\overline{w}_{s,\tau})\partial_t\chi_\tau \,\mathrm{d}x\,\mathrm{d}r \\ &\doteq C + I_1 + I_2 + I_3, \end{aligned}$$

where we have again used the third of (3.27), and (4.4) to infer that $\mathscr{J}_\tau(\mathbf{u}_\tau^0, \chi_0) \leq C$. Now,

$$|I_1| \leq C\int_0^T \|\overline{\mathbf{F}}_\tau\|_{\mathbf{W}}^2 \,\mathrm{d}r + \frac{C_b}{8}\int_0^{\bar{\mathrm{t}}_\tau(t)} \|\partial_t\mathbf{u}_\tau\|_{\mathbf{W}}^2 \,\mathrm{d}r,$$

$$|I_2| \le C \int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_\tau)\|_H^2 \, dr + \frac{C_b}{8} \int_0^{\bar{t}_\tau(t)} \|\partial_t \mathbf{u}_\tau\|_{\mathbf{W}}^2 \, dr,$$

$$\begin{aligned} |I_3| \ &\le \ \int_0^{\bar{t}_\tau(t)} \|\lambda_\delta'(\underline{\chi}_\tau) + \delta \overline{\chi}_\tau\|_{L^{\bar{p}}(\Gamma_C)} \|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\epsilon}(\Gamma_C)} \|\partial_t \chi_\tau\|_{L^2(\Gamma_C)} \, dr \\ &\le \ C \int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\epsilon}(\Gamma_C)}^2 \, dr + \frac{1}{4} \int_0^{\bar{t}_\tau(t)} \|\partial_t \chi_\tau\|_{L^2(\Gamma_C)}^2 \, dr \end{aligned}$$

where $\epsilon > 0$ is sufficiently small, and $\bar{p} \in (1, \infty)$ fulfills $\frac{1}{\bar{p}} + \frac{1}{2+\epsilon} + \frac{1}{2} = 1$. We have then used that

$$\|\lambda_\delta'(\underline{\chi}_\tau) + \delta \overline{\chi}_\tau\|_{L^\infty(0,T;L^{\bar{p}}(\Gamma_C))} \le C, \tag{4.49}$$

as a consequence of (4.46) and of the continuous embedding $H^1(\Gamma_C) \subset L^p(\Gamma_C)$ for all $1 \le p < \infty$. All in all, also using (4.17) we conclude

$$\begin{aligned} &\int_0^{\bar{t}_\tau(t)} \|\partial_t \mathbf{u}_\tau\|_{\mathbf{W}}^2 \, dr + \int_0^{\bar{t}_\tau(t)} \int_{\Gamma_C} |\partial_t \chi_\tau|^2 \, dx \, dr + \int_0^{\bar{t}_\tau(t)} \int_{\Gamma_C} \mathfrak{c}(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau})) |\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau \mathbf{n})| \, \left|\partial_t \mathbf{u}_{\tau,T}\right| \, dx \, dr \\ &\quad \le C \left(1 + \int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_\tau)\|_H^2 \, dr + \int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\epsilon}(\Gamma_C)}^2 \, dr \right). \end{aligned} \tag{4.50}$$

Second a priori estimate. As in the proof of [26, Prop. 4.2], we introduce the function

$$\Pi : [0, +\infty) \to [0, 1] \text{ defined by } \Pi(w) = 1 - \frac{1}{(1+w)^s} \quad \text{for some } 0 < s < 1, \tag{4.51}$$

s to be chosen later. Next, we test (4.10a) by $\tau \Pi(w_\tau^j)$ and (4.10b) by $\tau \Pi(w_{s,\tau}^j)$, and we add the resulting relations. Note that $\Pi(w_\tau^j)$ ($\Pi(w_{s,\tau}^j)$, resp.) is well-defined, since $w_\tau^j \ge 0$ a.e. in Ω ($w_{s,\tau}^j \ge 0$ a.e. in Γ_C, resp.) and that it is an admissible test function since Π is Lipschitz continuous and $w_\tau^j \in H^1(\Omega)$ ($w_{s,\tau}^j \in H^1(\Gamma_C)$, resp.). Furthermore, $\Pi(w_\tau^j) \ge 0$ a.e. in Ω and $\Pi(w_{s,\tau}^j) \ge 0$ a.e. in Γ_C. By convexity, we have

$$\int_\Omega \Pi(w_\tau^j)(w_\tau^j - w_\tau^{j-1}) \, dx \ge \int_\Omega \left(\widehat{\Pi}(w_\tau^j) - \widehat{\Pi}(w_\tau^{j-1}) \right) \, dx \tag{4.52}$$

where we have denoted by $\widehat{\Pi}$ the primitive of Π such that $\widehat{\Pi}(0) = 0$ (hence $\widehat{\Pi}(w) \ge 0$ for $w \ge 0$). Moreover, due to (3.15), it holds

$$\tau \int_\Omega K(w_\tau^{j-1}) \nabla w_\tau^j \nabla \Pi(w_\tau^j) \, dx \ge c\tau \int_\Omega \frac{|\nabla w_\tau^j|^2}{(1 + w_\tau^j)^{s+1}} \, dx. \tag{4.53}$$

Of course, the same estimates hold for the analogous terms deriving from (4.10b). Further, we observe that

$$\begin{aligned} &\tau \int_{\Gamma_C} k(\chi_\tau^{j-1})(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))\Theta(w_\tau^j)\Pi(w_\tau^j) \, dx - \tau \int_{\Gamma_C} k(\chi_\tau^{j-1})(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))\Theta(w_{s,\tau}^j)\Pi(w_{s,\tau}^j) \, dx \\ &\quad = \tau \int_{\Gamma_C} k(\chi_\tau^{j-1})(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))^2 \Pi(w_\tau^j) \, dx \\ &\qquad + \tau \int_{\Gamma_C} k(\chi_\tau^{j-1})(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))\Theta(w_{s,\tau}^j)(\Pi(w_\tau^j) - \Pi(w_{s,\tau}^j)) \, dx \ge 0, \end{aligned} \tag{4.54}$$

the last inequality thanks the monotonicity of Θ and Π. Analogously,

$$\begin{aligned} &\tau \int_{\Gamma_C} \mathfrak{c}'(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j)) |\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})| \left| \frac{\mathbf{u}_{\tau,T}^j - \mathbf{u}_{\tau,T}^{j-1}}{\tau} \right| \left(\Theta(w_\tau^j)\Pi(w_\tau^j) - \Theta(w_{s,\tau}^j)\Pi(w_{s,\tau}^j) \right) \, dx \\ &\quad = \tau \int_{\Gamma_C} \mathfrak{c}'(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j)) |\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})| \left| \frac{\mathbf{u}_{\tau,T}^j - \mathbf{u}_{\tau,T}^{j-1}}{\tau} \right| (\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j))\Pi(w_\tau^j) \, dx \\ &\qquad + \tau \int_{\Gamma_C} \mathfrak{c}'(\Theta(w_\tau^j) - \Theta(w_{s,\tau}^j)) |\mathcal{R}_\tau^j(\eta_\tau^j \mathbf{n})| \left| \frac{\mathbf{u}_{\tau,T}^j - \mathbf{u}_{\tau,T}^{j-1}}{\tau} \right| \Theta(w_{s,\tau}^j) \left(\Pi(w_\tau^j) - \Pi(w_{s,\tau}^j) \right) \, dx \ge 0, \end{aligned} \tag{4.55}$$

again thanks the monotonicity of Θ and Π and the last of (3.21). Integrating over the interval $(0, \bar{t}_\tau(t))$, we thus conclude

$$
\begin{aligned}
&\int_\Omega \widehat{\Pi}(\overline{w}_\tau(t))\,\mathrm{d}x + c\int_0^{\bar{t}_\tau(t)}\int_\Omega \frac{|\nabla\overline{w}_\tau|^2}{(1+\overline{w}_\tau)^{s+1}}\,\mathrm{d}x\,\mathrm{d}r + \int_{\Gamma_C}\widehat{\Pi}(\overline{w}_{s,\tau}(t))\,\mathrm{d}x + c\int_0^{\bar{t}_\tau(t)}\int_{\Gamma_C}\frac{|\nabla\overline{w}_{s,\tau}|^2}{(1+\overline{w}_{s,\tau})^{s+1}}\,\mathrm{d}x\,\mathrm{d}r\\
&\le C + \int_0^{\bar{t}_\tau(t)}\int_\Omega \overline{h}_\tau \Pi(\overline{w}_\tau)\,\mathrm{d}x\,\mathrm{d}r + \int_0^{\bar{t}_\tau(t)} \Theta(\overline{w}_\tau)\mathrm{div}(\partial_t\mathbf{u}_\tau)\Pi(\overline{w}_\tau)\,\mathrm{d}x\,\mathrm{d}r\\
&\quad + \int_0^{\bar{t}_\tau(t)}\int_\Omega \varepsilon(\partial_t\mathbf{u}_\tau)\mathbb{V}\varepsilon(\partial_t\mathbf{u}_\tau)\Pi(\overline{w}_\tau)\,\mathrm{d}x\,\mathrm{d}r\\
&\quad + \int_0^{\bar{t}_\tau(t)}\int_{\Gamma_C}\Theta(\overline{w}_{s,\tau})\frac{\lambda(\overline{\chi}_\tau)-\lambda(\underline{\chi}_\tau)}{\tau}\Pi(\overline{w}_{s,\tau})\,\mathrm{d}x\,\mathrm{d}r + \int_0^{\bar{t}_\tau(t)}\int_{\Gamma_C}|\partial_t\chi_\tau|^2\,\Pi(\overline{w}_{s,\tau})\,\mathrm{d}x\,\mathrm{d}r\\
&\quad + \int_0^{\bar{t}_\tau(t)}\int_{\Gamma_C}\mathfrak{c}(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau\mathbf{n})|\,\left|\partial_t\mathbf{u}_{\tau,\mathrm{T}}\right|\,\Pi(\overline{w}_{s,\tau})\,\mathrm{d}x\,\mathrm{d}r\\
&\doteq I_1+I_2+I_3+I_4+I_5+I_6,
\end{aligned}
\tag{4.56}
$$

where we have exploited that (cf. the first of (3.27))

$$
\int_\Omega \widehat{\Pi}(w_0)\,\mathrm{d}x + \int_{\Gamma_C}\widehat{\Pi}(w_{s,0})\,\mathrm{d}x \le c(\|w_0\|_{L^1(\Omega)} + \|w_s^0\|_{L^1(\Gamma_C)}) \le C.
$$

Now, taking into account that Π takes values in the interval $[0, 1]$, we have

$$
|I_1| \le \int_0^T \|\overline{h}_\tau\|_{L^1(\Omega)}\,\mathrm{d}r \le C,
$$

the latter inequality by (4.16). We then observe that

$$
\begin{aligned}
|I_2| &\le C\int_0^{\bar{t}_\tau(t)}\|\Theta(\overline{w}_\tau)\|_H^2\,\mathrm{d}r + C\int_0^{\bar{t}_\tau(t)}\|\partial_t\mathbf{u}_\tau\|_{\mathbf{W}}^2\,\mathrm{d}r,\\
|I_3| &\le C\int_0^{\bar{t}_\tau(t)}\|\partial_t\mathbf{u}_\tau\|_{\mathbf{W}}^2\,\mathrm{d}r,\\
|I_4| &\le C\int_0^{\bar{t}_\tau(t)}\|\Theta(\overline{w}_{s,\tau})\|_{L^2(\Gamma_C)}^2\,\mathrm{d}r + C\int_0^{\bar{t}_\tau(t)}\|\partial_t\chi_\tau\|_{L^2(\Gamma_C)}^2\,\mathrm{d}r,\\
|I_5| &\le \int_0^{\bar{t}_\tau(t)}\|\partial_t\chi_\tau\|_{L^2(\Gamma_C)}^2\,\mathrm{d}r,\\
|I_6| &\le \int_0^{\bar{t}_\tau(t)}\int_{\Gamma_C}\mathfrak{c}(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau\mathbf{n})|\,\left|\partial_t\mathbf{u}_{\tau,\mathrm{T}}\right|\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
$$

where in the estimate of I_4 we have also used that λ is Lipschitz continuous. Combining these estimates with (4.50) to bound the above integral, and using that $\int_\Omega \widehat{\Pi}(\overline{w}_\tau(t))\,\mathrm{d}x \ge 0$, $\int_{\Gamma_C}\widehat{\Pi}(\overline{w}_{s,\tau}(t))\,\mathrm{d}x \ge 0$, we ultimately deduce from (4.56) that

$$
\begin{aligned}
&\int_0^{\bar{t}_\tau(t)}\int_\Omega \frac{|\nabla\overline{w}_\tau|^2}{(1+\overline{w}_\tau)^{s+1}}\,\mathrm{d}x\,\mathrm{d}r + \int_0^{\bar{t}_\tau(t)}\int_{\Gamma_C}\frac{|\nabla\overline{w}_{s,\tau}|^2}{(1+\overline{w}_{s,\tau})^{s+1}}\,\mathrm{d}x\,\mathrm{d}r\\
&\le C\left(1+\int_0^{\bar{t}_\tau(t)}\|\Theta(\overline{w}_\tau)\|_H^2\,\mathrm{d}r + \int_0^{\bar{t}_\tau(t)}\|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\epsilon}(\Gamma_C)}^2\,\mathrm{d}r\right).
\end{aligned}
\tag{4.57}
$$

With the auxiliary functions

$$
\overline{z}_\tau := (1+\overline{w}_\tau)^{(1-s)/2} \qquad \overline{z}_{s,\tau} := (1+\overline{w}_{s,\tau})^{(1-s)/2},
\tag{4.58}
$$

estimate (4.57) rewrites as

$$
\begin{aligned}
&\int_0^{\bar{t}_\tau(t)}\int_\Omega |\nabla\overline{z}_\tau|^2\,\mathrm{d}x\,\mathrm{d}r + \int_0^{\bar{t}_\tau(t)}\int_{\Gamma_C}|\nabla\overline{z}_{s,\tau}|^2\,\mathrm{d}x\,\mathrm{d}r\\
&\le C\left(1+\int_0^{\bar{t}_\tau(t)}\|\Theta(\overline{w}_\tau)\|_H^2\,\mathrm{d}r + \int_0^{\bar{t}_\tau(t)}\|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\epsilon}(\Gamma_C)}^2\,\mathrm{d}r\right).
\end{aligned}
\tag{4.59}
$$

Let us now detail the estimate for the term $\int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_\tau)\|_H^2 \, \mathrm{d}r$. In view of (3.14), we have

$$\int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_\tau)\|_H^2 \, \mathrm{d}r \le c \int_0^{\bar{t}_\tau(t)} \int_\Omega (\overline{w}_\tau + 1)^{2/\sigma} \, \mathrm{d}x \, \mathrm{d}r = \int_0^{\bar{t}_\tau(t)} \|\overline{z}_\tau\|_{L^\zeta(\Omega)}^\zeta \, \mathrm{d}r, \tag{4.60}$$

where we have used the notation $\zeta := \frac{4}{\sigma(1-s)}$. In the case when $\sigma \ge 2$, the integral term in (4.60) is readily estimated by the $L^1(0,T;L^1(\Omega))$-norm of $\overline{w}_\tau$, and by (4.45) we conclude that $\int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_\tau)\|_H^2 \, \mathrm{d}r \le C$. Otherwise (that is if $\frac{6}{5} < \sigma < 2$, in view of (3.14)), we control the $L^\zeta(\Omega)$-norm of $\overline{z}_\tau$ by applying the Gagliardo–Nirenberg inequality (cf. (3.6)). We find

$$\|\overline{z}_\tau\|_{L^\zeta(\Omega)} \le C_{GN} \left(\|\overline{z}_\tau\|_{L^1(\Omega)} + \|\nabla \overline{z}_\tau\|_H \right)^\theta \|\overline{z}_\tau\|_H^{1-\theta}, \tag{4.61}$$

where $\theta = \frac{3(\zeta-2)}{2\zeta}$. Now, the uniform bound (4.45) implies $\|\overline{z}_\tau\|_{L^\infty(0,T;H)} \le C$. Then (4.61) yields

$$\int_0^{\bar{t}_\tau(t)} \|\overline{z}_\tau\|_{L^\zeta(\Omega)} \, \mathrm{d}r \le C \int_0^{\bar{t}_\tau(t)} \left(C' + \|\nabla \overline{z}_\tau\|_H \right)^\theta \, \mathrm{d}r. \tag{4.62}$$

Combining (4.60) and (4.62) and plugging the resulting estimate into (4.59) we arrive at

$$\begin{aligned} &\int_0^{\bar{t}_\tau(t)} \int_\Omega |\nabla \overline{z}_\tau|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_0^{\bar{t}_\tau(t)} \int_{\Gamma_C} |\nabla \overline{z}_{s,\tau}|^2 \, \mathrm{d}x \, \mathrm{d}r \\ &\quad \le C \left(1 + \int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\epsilon}(\Gamma_C)}^2 \, \mathrm{d}r \right) + C \int_0^{\bar{t}_\tau(t)} \left(C' + \|\nabla \overline{z}_\tau\|_H \right)^{\theta\zeta} \, \mathrm{d}r. \end{aligned} \tag{4.63}$$

Now, we choose in (4.51) s satisfying

$$0 < s < \frac{5\sigma - 6}{5\sigma}. \tag{4.64}$$

In this way, $2 < \zeta < \frac{10}{3}$, the constraint $0 < \theta < 1$ is satisfied, and $\zeta\,\theta < 2$. Applying the Young inequality we can absorb the last term on the right-hand side of (4.63) into the first term on the left-hand side.

For $\epsilon > 0$ small enough, the term $\int_0^{\bar{t}_\tau(t)} \|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\epsilon}(\Gamma_C)}^2 \, \mathrm{d}r$ on the right-hand side of (4.63) can be absorbed into $\int_0^{\bar{t}_\tau(t)} \int_{\Gamma_C} |\nabla \overline{z}_{s,\tau}|^2 \, \mathrm{d}x \, \mathrm{d}r$. All in all, we conclude that

$$\int_0^{\bar{t}_\tau(t)} \int_\Omega |\nabla \overline{z}_\tau|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_0^{\bar{t}_\tau(t)} \int_{\Gamma_C} |\nabla \overline{z}_{s,\tau}|^2 \, \mathrm{d}x \, \mathrm{d}r \le C$$

whence, by the estimate for $(\overline{z}_\tau)_\tau$ in $L^\infty(0,T;H)$ and the analogue bound for $\|\overline{z}_{s,\tau}\|_{L^\infty(0,T;H_{\Gamma_C})}$, we conclude

$$\|\overline{z}_\tau\|_{L^2(0,T;V)} \le C, \qquad \|\overline{z}_{s,\tau}\|_{L^2(0,T;V_{\Gamma_C})} \le C. \tag{4.65}$$

As a by-product of (4.60) and (4.62) and their analogues for $\Theta(\overline{w}_{s,\tau})$ we also infer

$$\|\Theta(\overline{w}_\tau)\|_{L^2(0,T;H)} \le C, \qquad \|\Theta(\overline{w}_{s,\tau})\|_{L^2(0,T;L^{2+\epsilon}(\Gamma_C))} \le C. \tag{4.66}$$

We observe in advance that the latter bounds will be improved, in view of the enhanced regularity for $\overline{w}_\tau$ and $\overline{w}_{s,\tau}$ derived in the next estimate (cf. (4.73)–(4.74)).

Third a priori estimate. In order to prove the first bound in (4.43f), we argue by interpolation, relying on (4.65) and on the second bound in (4.43f). Indeed, applying the Hölder inequality we have

$$\begin{aligned} \int_0^T \int_\Omega |\nabla \overline{w}_\tau|^r \, \mathrm{d}x \, \mathrm{d}r &= \int_0^T \int_\Omega \left(\frac{|\nabla \overline{w}_\tau|}{(1+\overline{w}_\tau)^{(s+1)/2}} \right)^r (1+\overline{w}_\tau)^{(s+1)r/2} \, \mathrm{d}x \, \mathrm{d}r \\ &= \int_0^T \left(\frac{2}{1-s} \right)^r \int_\Omega |\nabla \overline{z}_\tau|^r |1+\overline{w}_\tau|^{(s+1)r/2} \, \mathrm{d}x \, \mathrm{d}r \\ &\le C \int_0^T \| \, |\nabla \overline{z}_\tau|^r \|_{L^{2/r}(\Omega)} \, \| \, |1+\overline{w}_\tau|^{(s+1)r/2} \|_{L^{2/(2-r)}(\Omega)} \, \mathrm{d}r \\ &= C \int_0^T \left(\int_\Omega |\nabla \overline{z}_\tau|^2 \, \mathrm{d}x \right)^{r/2} \cdot \left(\int_\Omega |\overline{z}_\tau|^\alpha \, \mathrm{d}x \right)^{(2-r)/2} \, \mathrm{d}r, \end{aligned}$$

$$\left(\text{where we have used definition (4.58) and the place-holder } \alpha := \frac{2r(1+s)}{(2-r)(1-s)}\right),$$

$$\leq \left(\int_0^T \int_\Omega |\nabla \bar{z}_\tau|^2 \,dx\,dr\right) \left(\int_0^T \int_\Omega |\bar{z}_\tau|^\alpha \,dx\,dr\right),$$

the latter estimate by the Hölder inequality, again.

Observe that $\int_0^T \int_\Omega |\nabla \bar{z}_\tau|^2 \,dx\,dr \leq C$ thanks to (4.65) (once s is chosen as in (4.64)). We estimate $\int_0^T \int_\Omega |\bar{z}_\tau|^\alpha \,dx\,dr$ via the Gagliardo–Nirenberg inequality, proceeding as in (4.61). In this way (cf. (4.62)), we arrive at

$$\int_0^T \|\bar{z}_\tau\|^\alpha_{L^\alpha(\Omega)} \,dr \leq C \int_0^T (1 + \|\nabla \bar{z}_\tau\|_H)^{\alpha\theta} \,dr, \tag{4.67}$$

where (cf. (3.6)) $\theta = \frac{3(\alpha-2)}{2\alpha}$. Next, choosing s in (4.51) satisfying (4.64) and in addition

$$0 < s \leq \frac{5-4r}{5-3r}, \tag{4.68}$$

we have $2 < \alpha \leq \frac{10}{3}, 0 < \theta < 1$, and $\alpha\theta \leq 2$. Combining the latter information with (4.67) we thus infer $\int_0^T \|\bar{z}_\tau\|^\alpha_{L^\alpha(\Omega)} \,dr \leq C(1 + \int_0^T \int_\Omega |\nabla \bar{z}_\tau|^2 \,dx\,dr) \leq C$. All in all, we conclude the estimate $\int_0^T \int_\Omega |\nabla \overline{w}_\tau|^r \,dx\,dr \leq C$, whence

$$\|\overline{w}_\tau\|_{L^r(0,T;W^{1,r}(\Omega))} \leq C \tag{4.69}$$

via the second of (4.43f) and the Poincaré inequality.

Arguing in a very similar way, the first bound in (4.43h) can be proved. We only note that the lower constraint on ϱ $(1 \leq \varrho < \frac{4}{3})$ arises from the use of the Gagliardo–Nirenberg inequality in the lower spatial dimension $d = 2$.

Fourth a priori estimate. Combining (4.50) with estimates (4.66) we conclude

$$\|\partial_t \mathbf{u}_\tau\|_{L^2(0,T;\mathbf{W})} \leq C, \tag{4.70}$$

$$\|\partial_t \chi_\tau\|_{L^2(0,T;H_{\Gamma_C})} \leq C, \tag{4.71}$$

as well as

$$\int_0^T \int_{\Gamma_C} \mathfrak{c}(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\eta}_\tau \mathbf{n})| \left|\partial_t \mathbf{u}_{\tau,T}\right| \,dx\,dr \leq C. \tag{4.72}$$

Therefore, (4.43b) and (4.43e) ensue.

Fifth a priori estimate. Via Sobolev embeddings theorems, condition (3.14) and estimates (4.43f) and (4.43h) imply

$$\|\Theta(\overline{w}_\tau)\|_{L^\infty(0,T;L^\sigma(\Omega)) \cap L^{r\sigma}(0,T;L^{p\sigma}(\Omega))} \leq C \quad \text{for all } 1 \leq r < \frac{5}{4} \text{ and for all } 1 \leq p \leq \frac{3r}{3-r}, \tag{4.73}$$

$$\|\Theta(\overline{w}_{s,\tau})\|_{L^\infty(0,T;L^\sigma(\Gamma_C)) \cap L^{\varrho\sigma}(0,T;L^{q\sigma}(\Gamma_C))} \leq C \quad \text{for all } 1 \leq \varrho < \frac{4}{3} \text{ and for all } 1 \leq q \leq \frac{2\varrho}{2-\varrho}. \tag{4.74}$$

Relying on the fact that, indeed, $\sigma = 8/5 + \epsilon$ for some $\epsilon > 0$, with an interpolation argument we derive from estimates (4.73)–(4.74) that

$$\exists\, \bar{\epsilon} > 0 \quad \text{s.t. } \|\Theta(\overline{w}_\tau)\|_{L^{2+\bar{\epsilon}}(0,T;L^{3+\bar{\epsilon}}(\Omega))} \leq C. \tag{4.75}$$

$$\exists\, \tilde{\epsilon} > 0 \quad \text{s.t. } \|\Theta(\overline{w}_{s,\tau})\|_{L^{2+\tilde{\epsilon}}(0,T;L^{6+\tilde{\epsilon}}(\Gamma_C))} \leq C. \tag{4.76}$$

For later convenience, we also remark that the first bound of (4.43f) implies

$$\|\Theta(\overline{w}_\tau)\|_{L^{r\sigma}(0,T;L^{p\sigma}(\Gamma_C))} \leq C \quad \text{for all } 1 \leq r < \frac{5}{4} \text{ and for all } 1 \leq p \leq \frac{2r}{3-r}, \tag{4.77}$$

thanks to the continuous embedding $W^{1,r}(\Omega) \subset L^p(\Gamma_C)$, with $1 \leq p \leq \frac{2r}{3-r}$, as well as the growth condition (3.14) on Θ.

Sixth a priori estimate. We argue by comparison in (4.21e). Recalling the continuous embedding $V_{\Gamma_C} \subset L^p(\Gamma_C)$, for all $p \in [1, +\infty)$, the Lipschitz continuity of γ' (cf. (3.23)), we infer from (4.46) that

$$\|\tau|\overline{\chi}_\tau|^{\gamma-2}\,\overline{\chi}_\tau + \gamma_\nu'(\overline{\chi}_\tau) - \nu\underline{\chi}_\tau\|_{L^\infty(0,T;H_{\Gamma_C})} \leq C. \tag{4.78}$$

Again, the continuous embeddings $V_{\Gamma_C} \subset L^p(\Gamma_C)$, for all $p \in [1,+\infty)$ and $\mathbf{W} \subset L^4(\Gamma_C;\mathbb{R}^3)$ (in the sense of traces), the Lipschitz continuity of λ (cf. (3.22)), and the second of estimates (4.66) ensure that the right-hand side of (4.21e) is bounded in $L^2(0,T;H_{\Gamma_C})$. Thus, a comparison in (4.21e) gives

$$\|A\overline{\chi}_\tau + \overline{\xi}_\tau\|_{L^2(0,T;L^2(\Gamma_C))} \leq C \tag{4.79}$$

whence (4.43l) and the second of (4.43d), by monotonicity and standard elliptic regularity results.

Seventh a priori estimate. We develop a comparison argument in (4.21c). Relying on the previously obtained estimates (4.43a)–(4.43d), (4.66), and on (4.17) for $\overline{\mathbf{F}}_\tau$, we conclude that

$$\exists\, C>0\ \forall\,\tau>0:\ \|\overline{\eta}_\tau\mathbf{n} + \mathfrak{c}(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))\overline{\boldsymbol{\mu}}_\tau\|_{L^2(0,T;\mathbf{Y}^*_\gamma)} \leq C. \tag{4.80}$$

Exploiting the fact that $\overline{\eta}_\tau\mathbf{n}$ and $\mathfrak{c}(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))\overline{\boldsymbol{\mu}}_\tau$ are orthogonal, and arguing in the very same way as in [19, Sec. 4], we conclude that

$$\|\overline{\eta}_\tau\mathbf{n}\|_{L^2(0,T;\mathbf{Y}^*_\gamma)} + \|\mathfrak{c}(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))\overline{\boldsymbol{\mu}}_\tau\|_{L^2(0,T;\mathbf{Y}^*_\gamma)} \leq C.$$

Hence, estimate (4.43j) follows.

Then, thanks to (4.87) (cf. also Lemma 4.7), we find that $(\overline{\mathfrak{R}}_\tau(\overline{\eta}_\tau\mathbf{n}))_\tau$ is bounded in $L^\infty(0,T;L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3))$. Now, $\overline{\boldsymbol{\mu}}_\tau = |\overline{\mathfrak{R}}_\tau(\overline{\eta}_\tau\mathbf{n})|\overline{\mathbf{z}}_\tau$. Since $\overline{\mathbf{z}}_\tau \in \partial j(\partial_t\mathbf{u}_\tau)$, we have

$$|\overline{\mathbf{z}}_\tau| \leq 1 \quad \text{a.e. in } \Gamma_C\times(0,T). \tag{4.81}$$

Therefore, estimate (4.43k) ensues.

Eighth a priori estimate. To prove (4.43g), we test (4.21a) by $v \in W^{1,r'}(\Omega)$ where r' denotes the conjugate exponent of r, $1 \leq r < \frac{5}{4}$. Observe that $v \in L^\infty(\Omega)$ and also its trace is in $L^\infty(\Gamma_C)$, since $r' > 3$. Due to (4.43b), (4.66), and (4.16), we have

$$\begin{aligned}&\left|\int_\Omega \varepsilon(\partial_t\mathbf{u}_\tau(t))\,\mathbb{V}\varepsilon(\partial_t\mathbf{u}_\tau(t))\,v\,\mathrm{d}x + \int_\Omega \Theta(\overline{w}_\tau(t))\mathrm{div}\,(\partial_t\mathbf{u}_\tau(t))\,v\,\mathrm{d}x + \int_\Omega \overline{h}_\tau(t)\,v\,\mathrm{d}x\right| \\ &\quad\leq c\left(\|\partial_t\mathbf{u}_\tau(t)\|^2_{\mathbf{W}} + \|\Theta(\overline{w}_\tau(t))\|^2_H + \|\overline{h}_\tau(t)\|_{L^1(\Omega)}\right)\|v\|_{L^\infty(\Omega)} \doteq f_1(t) \in L^1(0,T).\end{aligned} \tag{4.82}$$

Moreover, taking into account (3.15) and (4.43f), we infer

$$\left|\int_\Omega K(\underline{w}_\tau(t))\nabla\overline{w}_\tau(t)\nabla v\,\mathrm{d}x\right| \leq c_4\,\|\overline{w}_\tau(t)\|_{L^r(\Omega)}\|v\|_{W^{1,r'}(\Omega)} \doteq f_2(t) \in L^1(0,T). \tag{4.83}$$

Next, using (3.20), (4.77), (4.74), and (4.43d), we deduce

$$\begin{aligned}&\left|\int_{\Gamma_C} k(\underline{\chi}_\tau(t))\Theta(\overline{w}_\tau(t))(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))v\,\mathrm{d}x\right| \\ &\quad\leq c\left(\|\underline{\chi}_\tau(t)\|_{L^s(\Gamma_C)} + 1\right)\|\Theta(\overline{w}_\tau(t))\|_{L^{p\sigma}(\Gamma_C)}\left(\|\Theta(\overline{w}_\tau(t))\|_{L^{p\sigma}(\Gamma_C)} + \|\Theta(\overline{w}_{s,\tau}(t))\|_{L^{q\sigma}(\Gamma_C)}\right)\|v\|_{L^\infty(\Gamma_C)} \\ &\quad\doteq f_3(t) \in L^1(0,T),\end{aligned} \tag{4.84}$$

where $1 \leq p \leq \frac{2r}{3-r}$, $1 \leq q \leq \frac{2\varrho}{2-\varrho}$ and s (deriving from the continuous embedding $V_{\Gamma_C} \subset L^s(\Gamma_C)$ for all $s \in [1,+\infty)$) is such that $\frac{1}{s} + \frac{2}{p\sigma} \leq 1$ and $\frac{1}{s} + \frac{1}{p\sigma} + \frac{1}{q\sigma} \leq 1$ with q and p from (4.74) and (4.77). Observe that there exists $s > 1$ complying with both conditions because $\sigma > 8/5$. This also guarantees that $\frac{2}{r\sigma} \leq 1$ and $\frac{1}{r\sigma} + \frac{1}{\varrho\sigma} \leq 1$ for all $1 \leq r < \frac{5}{4}$ and $1 \leq \varrho < \frac{4}{3}$. Hence $f_3 \in L^1(0,T)$. Finally,

$$\begin{aligned}&\left|\int_{\Gamma_C} \Theta(\overline{w}_\tau(t))\mathfrak{c}'(\Theta(\overline{w}_\tau(t)) - \Theta(\overline{w}_{s,\tau}(t)))\|\overline{\mathfrak{R}}_\tau(\overline{\eta}_\tau(t)\mathbf{n})\|\ \left|\partial_t\mathbf{u}_{\tau,\mathrm{T}}(t)\right| v\,\mathrm{d}x\right| \\ &\quad\leq c\|\Theta(\overline{w}_\tau(t))\|_{L^{p\sigma}(\Gamma_C)}\|\overline{\mathfrak{R}}_\tau(\overline{\eta}_\tau(t)\mathbf{n})\|_{L^{16/5+\nu}(\Gamma_C)}\|\partial_t\mathbf{u}(t)\|_{L^4(\Gamma_C)}\|v\|_{L^\infty(\Gamma_C)} \\ &\quad\leq c\|\Theta(\overline{w}_\tau(t))\|_{L^{p\sigma}(\Gamma_C)}\|\overline{\mathfrak{R}}_\tau(\overline{\eta}_\tau(t)\mathbf{n})\|_{L^{16/5+\nu}(\Gamma_C)}\|\partial_t\mathbf{u}(t)\|_{\mathbf{W}}\|v\|_{L^\infty(\Gamma_C)} \doteq f_4(t) \in L^1(0,T),\end{aligned} \tag{4.85}$$

owing to (3.21), (4.43k), and (4.43b). Then, we arrive at

$$\exists\,\mathcal{F} \in L^1(0,T) \quad \text{for a.a. } t\in(0,T)\ \forall\, v \in W^{1,r'}(\Omega):\ \left|\int_\Omega \partial_t w_\tau\, v\,\mathrm{d}x\right| \leq \mathcal{F}(t)\|v\|_{W^{1,r'}(\Omega)} \tag{4.86}$$

whence (4.43g).

Estimate (4.43i) can be proved in a very similar way, testing (4.21b) by $v \in W^{1,\varrho'}(\Gamma_C)$ and arguing as in the previous lines. □

We conclude with the following result, collecting some crucial properties of the discrete operators $(\mathcal{R}_\tau^j)_{j=1}^{J_\tau}$ from (4.7), and $(\overline{\mathcal{R}}_\tau)_\tau$ from (4.20), under Hypothesis 4.1.

Lemma 4.7. *Assume Hypothesis 4.1. Then,*

(1) *the operators $\mathcal{R}_\tau^j : \mathbf{Y}_\gamma^* \to L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)$ are bounded, uniformly with respect to $\tau > 0$ and $j = 1, \dots, J_\tau$, namely*

$$\exists \bar{S} > 0 \,\forall \tau > 0 \,\forall j = 1, \dots, J_\tau \,\forall \boldsymbol{\eta} \in \mathbf{Y}_\gamma^* : \|\mathcal{R}_\tau^j(\boldsymbol{\eta})\|_{L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)} \le \bar{S}\|\boldsymbol{\eta}\|_{\mathbf{Y}_\gamma^*}; \tag{4.87}$$

(2) *the family $(\overline{\mathcal{R}}_\tau)_\tau$ approximates $\mathcal{R}$ in the sense that,*

$$\overline{\boldsymbol{\eta}}_\tau \rightharpoonup \boldsymbol{\eta} \text{ in } L^2(0,T;\mathbf{Y}_\gamma^*) \Rightarrow \overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau) \to \mathcal{R}(\boldsymbol{\eta}) \text{ in } L^p(0,T;L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)) \quad \text{for all } 1 \le p < \infty \tag{4.88}$$

cf. (4.20) for the definition of $\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau)$.

Proof. Property (4.87) is a straightforward consequence of (4.42). In order to check (4.88), observe that, by definition, $\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau)(t) = \frac{1}{\tau}\int_{\underline{t}_\tau(t)}^{\bar{t}_\tau(t)} \mathcal{R}(\overline{\boldsymbol{\eta}}_\tau)(s)\,\mathrm{d}s$ for $t \in [0,T]$. Therefore,

$$\begin{aligned}\|\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau)(t) - \mathcal{R}(\boldsymbol{\eta})(t)\|_{L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)} \le{}& \left\| \frac{1}{\tau}\int_{\underline{t}_\tau(t)}^{\bar{t}_\tau(t)} (\mathcal{R}(\overline{\boldsymbol{\eta}}_\tau)(s) - \mathcal{R}(\boldsymbol{\eta})(s))\,\mathrm{d}s \right\|_{L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)} \\ &+ \frac{1}{\tau}\int_{\underline{t}_\tau(t)}^{\bar{t}_\tau(t)} \|\mathcal{R}(\boldsymbol{\eta})(s) - \mathcal{R}(\boldsymbol{\eta})(t)\|_{L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)}\,\mathrm{d}s \doteq I_1 + I_2.\end{aligned}$$

Now,

$$I_1 \le \frac{1}{\tau}\int_{\underline{t}_\tau(t)}^{\bar{t}_\tau(t)} \|\mathcal{R}(\overline{\boldsymbol{\eta}}_\tau) - \mathcal{R}(\boldsymbol{\eta})\|_{L^\infty(0,T;L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3))}\,\mathrm{d}s \to 0 \tag{4.89}$$

as $\tau \downarrow 0$ thanks to (4.42); observe that the above convergence is uniform in $t \in (0,T)$. As for I_2, a property of the Bochner integral guarantees that $I_2 \to 0$ for almost all $t \in (0,T)$ and that, since $\mathcal{R}(\boldsymbol{\eta}) \in L^p(0,T;L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3))$ for all $1 \le p \le \infty$,

$$\int_0^T \left(\frac{1}{\tau}\int_{\underline{t}_\tau(t)}^{\bar{t}_\tau(t)} \|\mathcal{R}(\boldsymbol{\eta})(s) - \mathcal{R}(\boldsymbol{\eta})(t)\|_{L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)}\,\mathrm{d}s \right)^p \mathrm{d}t \to 0 \quad \text{as } \tau \downarrow 0 \text{ for all } 1 \le p < \infty. \tag{4.90}$$

Combining (4.89) and (4.90) concludes the proof. □

5. Proof of Theorem 1

In this section we complete the proof of Theorem 1 by passing to the limit in the discrete system (4.21) as $\tau \downarrow 0$. We split the proof in several steps.

Step 0: **compactness**. To deduce from the a priori estimates of Proposition 4.6 the convergence (along subsequences, in suitable topologies) of the family of discrete solutions, we shall resort to appropriate weak (weak-star) and strong compactness results. In what follows, for the sake of simplicity, we shall not relabel subsequences and in fact state all convergences for the full families of approximate solutions, as $\tau \downarrow 0$.

We first note that estimate (4.43b) implies

$$\|\overline{\mathbf{u}}_\tau - \mathbf{u}_\tau\|_{L^\infty(0,T;\mathbf{W})} \le C\tau^{1/2}, \tag{5.1}$$

(and the analogous bound holds for $\underline{\mathbf{u}}_\tau$). In addition, estimates (4.43a)–(4.43b) combined with well-known weak and weak-star compactness results and with (5.1) ensure that there exists $u \in H^1(0,T;\mathbf{W})$ such that (along a not relabeled subsequence)

$$\mathbf{u}_\tau \rightharpoonup \mathbf{u} \quad \text{in } H^1(0,T;\mathbf{W}), \qquad \underline{\mathbf{u}}_\tau, \overline{\mathbf{u}}_\tau, \mathbf{u}_\tau \rightharpoonup^* \mathbf{u} \quad \text{in } L^\infty(0,T;\mathbf{W}). \tag{5.2}$$

Then, applying strong compactness (see e.g. [35, Thm. 5, Cor. 4]) and trace theorems we can deduce, in particular, that

$$\underline{\mathbf{u}}_\tau, \overline{\mathbf{u}}_\tau, \mathbf{u}_\tau \to \mathbf{u} \quad \text{in } L^2(0,T;L^q(\Gamma_C)) \text{ for all } 1 \le q < 4. \tag{5.3}$$

After noting that the analogue of (5.1) holds for the functions $\overline{\chi}_\tau$, χ_τ (and $\underline{\chi}_\tau$), with the same arguments we show that estimates (4.43d)–(4.43e) imply the following convergences (still along not relabeled subsequences)

$$\chi_\tau \rightharpoonup \chi \quad \text{in } H^1(0,T;V_{\Gamma_C}), \tag{5.4}$$

$$\underline{\chi}_\tau, \overline{\chi}_\tau, \chi_\tau \rightharpoonup^* \chi \quad \text{in } L^\infty(0,T;V_{\Gamma_C}) \cap L^2(0,T;H^2(\Gamma_C)), \tag{5.5}$$

$$\underline{\chi}_\tau, \overline{\chi}_\tau, \chi_\tau \to \chi \quad \text{in } L^2(0,T;V_{\Gamma_C}) \cap L^p(\Gamma_C \times (0,T)) \cap C^0([0;T];H_{\Gamma_C}) \text{ for all } 1 \le p < \infty. \tag{5.6}$$

As for the convergence of $(\overline{w}_\tau)_\tau$, $(\overline{w}_{s,\tau})_\tau$ and $(w_\tau)_\tau$, $(w_{s,\tau})_\tau$, observe that in this case we only dispose of a L^1-in-time bound for the time derivatives of $(w_\tau)_\tau$, $(w_{s,\tau})_\tau$, cf. (4.43g) and (4.43i). Relying on estimates (4.43f)–(4.43g) for $(w_\tau)_\tau$ and the aforementioned [35, Cor. 4], we deduce that there exists $w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^p(0,T;L^1(\Omega))$ for all $1 \le p < \infty$ (recall that $1 \le r < 5/4$), such that

$$\begin{aligned} &w_\tau \rightharpoonup w \quad \text{in } L^r(0,T;W^{1,r}(\Omega)),\\ &w_\tau \to w \quad \text{in } L^r(0,T;W^{1-\epsilon,r}(\Omega)) \text{ for all } 0<\epsilon<1, \qquad w_\tau \to w \quad \text{in } L^p(0,T;L^1(\Omega)) \text{ for all } 1 \le p < \infty. \end{aligned} \tag{5.7}$$

Furthermore, the Helly-type compactness result [27, Thm. 6.1] ensures that $w \in \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$ and

$$w_\tau(t) \rightharpoonup w(t) \quad \text{in } W^{1,r'}(\Omega)^* \text{ for all } t \in [0,T]. \tag{5.8}$$

Repeating the same compactness arguments for the sequences $(\overline{w}_\tau)_\tau$ and $(\underline{w}_\tau)_\tau$, we find functions $\overline{w}$, $\underline{w} \in L^r(0,T;W^{1,r}(\Omega)) \cap L^p(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$ for which convergences (5.7) and (5.8) hold. Observe that we cannot deduce that $w(t) = \overline{w}(t) = \underline{w}(t)$ for all $t \in [0,T]$, as the analogue of estimate (5.1) is not valid now. Nonetheless, let $J \subset [0,T]$ be given by the union of the jump sets of the functions $w, \overline{w}, \underline{w}$. Clearly, J is countable and has therefore null Lebesgue measure. With a standard argument (developed in detail, for instance, in the proof of [36, Thm. 4.1]), it is possible to show that $w(t) = \overline{w}(t) = \underline{w}(t)$ for all $t \in [0,T] \setminus J$. Hence convergences (5.7) hold along (suitable subsequences of) $(\overline{w}_\tau)_\tau$ and $(\overline{w}_{s,\tau})_\tau$, as well.

Estimates (4.43h)–(4.43i) and the very same compactness tokens as in the above lines also yield that there exists $w_s \in L^\rho(0,T;W^{1,\rho}(\Gamma_C)) \cap L^p(0,T;L^1(\Gamma_C)) \cap \mathrm{BV}(0,T;W^{1,\rho'}(\Gamma_C)^*)$ for all $1 \le p < \infty$, with $1 \le \rho < 4/3$, such that

$$\begin{aligned} &\overline{w}_{s,\tau}, w_{s,\tau} \rightharpoonup w_s \quad \text{in } L^\rho(0,T;W^{1,\rho}(\Gamma_C)),\\ &\underline{w}_{s,\tau}, \overline{w}_{s,\tau}, w_{s,\tau} \to w_s \quad \text{in } L^\rho(0,T;H_{\Gamma_C}),\\ &\underline{w}_{s,\tau}, \overline{w}_{s,\tau}, w_{s,\tau} \to w_s \quad \text{in } L^p(0,T;L^1(\Gamma_C)) \text{ for all } 1 \le p < \infty, \end{aligned} \tag{5.9}$$

$$w_{s,\tau}(t) \rightharpoonup w_s(t) \quad \text{in } W^{1,\rho'}(\Omega)^* \text{ for all } t \in [0,T]. \tag{5.10}$$

As a consequence of (5.7) and (5.9), we also have the pointwise a.e. convergence results

$$\underline{w}_\tau, \overline{w}_\tau, w_\tau \to w \quad \text{a.e. in } \Omega \times (0,T), \tag{5.11}$$

$$\underline{w}_{s,\tau}, \overline{w}_{s,\tau}, w_{s,\tau} \to w_s \quad \text{a.e. in } \Gamma_C \times (0,T). \tag{5.12}$$

Since Θ is continuous, we deduce from (5.11)–(5.12) that

$$\Theta(\overline{w}_\tau) \to \Theta(w) \quad \text{a.e. in } \Omega \times (0,T), \qquad \Theta(\overline{w}_{s,\tau}) \to \Theta(w_s) \quad \text{a.e. in } \Gamma_C \times (0,T). \tag{5.13}$$

Combining convergences (5.13) with estimates (4.73)–(4.74) for $(\Theta(\overline{w}_\tau))_\tau$ and $(\Theta(\overline{w}_{s,\tau}))_\tau$, we infer that

$$\Theta(\overline{w}_\tau) \rightharpoonup^* \Theta(w) \quad \text{in } L^{r\sigma}(0,T;L^{p\sigma}(\Omega)) \cap L^\infty(0,T;L^\sigma(\Omega)), \tag{5.14}$$

$$\Theta(\overline{w}_{s,\tau}) \rightharpoonup \Theta(w_s) \quad \text{in } L^{\rho\sigma}(0,T;L^{q\sigma}(\Gamma_C)) \cap L^\infty(0,T;L^\sigma(\Gamma_C)), \tag{5.15}$$

where the indexes r, p, q, ρ are as in (4.43f), (4.43h), (4.73)–(4.74). Moreover, in view of (4.75)–(4.76) we get (at least) the following strong convergences

$$\Theta(\overline{w}_\tau) \to \Theta(w) \quad \text{in } L^2(0,T;L^2(\Omega)), \qquad \Theta(\overline{w}_{s,\tau}) \to \Theta(w_s) \quad \text{in } L^{2+\epsilon}(0,T;L^{2+\epsilon}(\Gamma_C)) \text{ for some } \epsilon > 0. \tag{5.16}$$

With analogous arguments, from (4.77) and (5.7) via standard trace theorems we deduce that

$$\Theta(\overline{w}_\tau) \to \Theta(w) \quad \text{in } L^{r\sigma}(0,T;L^{p\sigma}(\Gamma_C)) \text{ for all } 1 \le r < \frac{5}{4},\ 1 \le p < \frac{2r}{3-r}. \tag{5.17}$$

Finally, (4.43j)–(4.43l) and estimate (4.81) for $(\overline{\mathbf{z}}_\tau)_\tau$ yield that there exist functions $\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}^*_\gamma)$, $\boldsymbol{\mu} \in L^\infty(0,T;L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3))$, $\mathbf{z} \in L^\infty(\Gamma_C \times (0,T);\mathbb{R}^3)$, and $\xi \in L^2(0,T;H_{\Gamma_C})$ such that

$$\overline{\boldsymbol{\eta}}_\tau \rightharpoonup \boldsymbol{\eta} \quad \text{in } L^2(0,T;\mathbf{Y}^*_\gamma), \tag{5.18}$$

$$\overline{\boldsymbol{\mu}}_\tau \rightharpoonup^* \boldsymbol{\mu} \quad \text{in } L^\infty(0,T;L^{16/5+\nu}(\Gamma_C;\mathbb{R}^3)), \tag{5.19}$$

$$\overline{\mathbf{z}}_\tau \rightharpoonup^* \mathbf{z} \quad \text{in } L^\infty(\Gamma_C \times (0,T);\mathbb{R}^3), \tag{5.20}$$

$$\overline{\xi}_\tau \rightharpoonup \xi \quad \text{in } L^2(0,T;H_{\Gamma_C}). \tag{5.21}$$

With the very same argument as in [19, proof of Thm. 2.1, Sec. 5.1], one sees that also $\underline{\boldsymbol{\eta}}_\tau \rightharpoonup \boldsymbol{\eta}$ in $L^2(0,T;\mathbf{Y}^*_\gamma)$. Furthermore, in view of the strong–weak closedness of the graph of (the operator induced by) β (on $L^2(0,T;H_{\Gamma_C})$), combining (5.21) with the strong convergence (5.6) of $(\overline{\chi}_\tau)_\tau$ allows us to deduce that $\xi \in \beta(\chi)$ a.e. in $\Gamma_C \times (0,T)$.

Step 1: **passage to the limit in the discrete** χ**-equation** (4.21e). We shall exploit convergences (5.2)–(5.6), (5.15), and (5.21). First let us point out that, due to estimate (4.43d) the term $\tau|\overline{\chi}_\tau|^{\gamma-2}\overline{\chi}_\tau \to 0$ in $L^p(\Gamma_C \times (0,T))$ for all $1 \le p < \infty$. We now pass to the limit in the nonlinear terms involving γ_ν' and λ_δ'. We first observe that they are continuous functions (cf. (3.22)–(3.23) and (4.8)) and

$$|\gamma_\nu'(x)|,\ |\lambda_\delta'(x)| \le c(1+|x|). \tag{5.22}$$

Thus, relying on the strong convergence (5.6) and on a generalized version of the Lebesgue theorem we deduce

$$\lambda_\delta'(\underline{\chi}_\tau),\ \gamma_\nu'(\overline{\chi}_\tau) \rightharpoonup \lambda_\delta'(\chi),\ \gamma_\nu'(\chi) \quad \text{in } L^p(\Gamma_C \times (0,T)) \text{ for all } 1 \le p < \infty. \tag{5.23}$$

Note that a comparison in (4.21e) ensures that the term $-(\lambda_\delta'(\underline{\chi}_\tau) + \delta\overline{\chi}_\tau)\Theta(\overline{w}_{s,\tau})$ is bounded, e.g., in $L^2(0,T;H_{\Gamma_C})$. Since it converges pointwise a.e. in $\Gamma_C \times (0,T)$ (see (5.6) and (5.12)) we can deduce that

$$-(\lambda_\delta'(\underline{\chi}_\tau) + \delta\overline{\chi}_\tau)\Theta(\overline{w}_{s,\tau}) \rightharpoonup -(\lambda_\delta'(\chi) + \delta\chi)\Theta(w_s) \quad \text{in } L^2(0,T;H_{\Gamma_C}). \tag{5.24}$$

Finally, let us consider the last term on the right-hand side of (4.21e), i.e. the quadratic term $-\frac{1}{2}|\underline{\mathbf{u}}_\tau|^2$. Due to (4.43a) it is bounded in $L^2(0,T;H_{\Gamma_C})$. Moreover, strong convergence (5.3) yields that it converges almost everywhere. Thus, we can identify its weak limit in $L^2(0,T;H_{\Gamma_C})$ as

$$-\frac{1}{2}|\underline{\mathbf{u}}_\tau|^2 \rightharpoonup -\frac{1}{2}|\mathbf{u}|^2 \quad \text{in } L^2(0,T;H_{\Gamma_C}). \tag{5.25}$$

This concludes the limit passage in (4.21e).

For later convenience, we also show that the weak convergence (5.4) improves to a strong one, namely that

$$\partial_t\chi_\tau \to \partial_t\chi \quad \text{in } L^2(0,T;H). \tag{5.26}$$

To this aim we proceed by semicontinuity arguments, testing (4.21e) by $\partial_t\chi_\tau$, integrating over $(0,t)$ and using (5.2)–(5.6), the second of (5.16), (5.23)–(5.25) to show that

$$\limsup_{\tau\searrow 0} \int_0^t \int_\Omega |\partial_t\chi_\tau|^2 \,\mathrm{d}x\,\mathrm{d}r \le \int_0^t \int_\Omega |\partial_t\chi|^2 \,\mathrm{d}x\,\mathrm{d}r. \tag{5.27}$$

Hence (5.26) follows.

Step 2: **passage to the limit in the discrete momentum equation** (4.21c). First of all, it follows from estimate (4.43c) that $\tau|\varepsilon(\overline{\mathbf{u}}_\tau)|^{\gamma-2}\varepsilon(\overline{\mathbf{u}}_\tau) \to 0$ in $L^\gamma(\Omega;\mathbb{R}^{3\times3})$. Concerning the frictional contribution (i.e. the seventh integral term on the left-hand side of (4.21c)), combining the pointwise convergences (5.13) with the condition that $\mathfrak{c} \in L^\infty(\mathbb{R})$ (cf. (3.21)) and using the generalized Lebesgue theorem we gather that

$$\mathfrak{c}(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau})) \to \mathfrak{c}(\Theta(w) - \Theta(w_s)) \quad \text{in } L^p(\Gamma_C \times (0,T)) \text{ for all } 1 \le p < \infty. \tag{5.28}$$

Hence, exploiting convergences (4.17) for $(\overline{\mathbf{F}}_\tau)_\tau$, (5.2)–(5.3), (5.6), (5.14), (5.18), (5.19), and (5.28) we pass to the limit in (4.21c), with test functions $v \in W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^3)$ and eventually get

$$\begin{aligned} &b(\partial_t\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_\Omega \Theta(w)\mathrm{div}\,\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_C} \chi\mathbf{u}\mathbf{v}\,\mathrm{d}x + \langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}_\gamma} + \int_{\Gamma_C} \mathfrak{c}(\Theta(w) - \Theta(w_s))\boldsymbol{\mu}\mathbf{v}\,\mathrm{d}x \\ &\quad = \langle\mathbf{F},\mathbf{v}\rangle \quad \text{for all } v \in W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^3) \end{aligned} \tag{5.29}$$

a.e. in $(0,T)$. In order to conclude the passage to the limit in (4.21c), it remains to prove that $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ fulfill (3.29d) and (3.29e).

As for $\boldsymbol{\eta}$, it follows from (5.29) that

$$\langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}_\gamma} = -b(\partial_t\mathbf{u},\mathbf{v}) - a(\mathbf{u},\mathbf{v}) - \int_\Omega \Theta(w)\mathrm{div}\,\mathbf{v} - \int_{\Gamma_C} \chi\mathbf{u}\mathbf{v} - \int_{\Gamma_C} \mathfrak{c}(\Theta(w) - \Theta(w_s))\boldsymbol{\mu}\mathbf{v}\,\mathrm{d}x + \langle\mathbf{F},\mathbf{v}\rangle. \tag{5.30}$$

Observe that, due the regularity of the limit quadruplet $(w, w_s, \mathbf{u}, \chi)$, all the terms on the right-hand side of (5.30) make sense with test functions $\mathbf{v} \in \mathbf{W}$. That is why, in the proof of (3.29d) we will consider $\boldsymbol{\eta}$ as an element in $L^2(0,T;\mathbf{Y}^*)$, instead of $L^2(0,T;\mathbf{Y}_\gamma^*)$, write $\langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}}$ in place of $\langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}_\gamma}$, and work with the formulation of (5.29) against test functions in $\mathbf{W}$, even though for the time being we have just obtained it with test functions in $W^{1,\gamma}_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^3)$. This "shortcut" can be rigorously justified through a double approximation procedure, cf. Remark 5.1 later on.

To show that $\boldsymbol{\eta}(t) \in \partial\boldsymbol{\varphi}(\mathbf{u}(t))$ in $\mathbf{Y}^*$ for almost all $t \in (0,T)$, we test (4.21c) by $\overline{\mathbf{u}}_\tau$. For every $t \in [0,T]$ we have

$$\begin{aligned}
\limsup_{\tau\to 0}\int_0^t\int_{\Gamma_C}\overline{\boldsymbol{\eta}}_\tau\overline{\mathbf{u}}_\tau\,dx\,ds &\overset{(1)}{\leq} -\liminf_{\tau\to 0}\Big(\int_0^t\Big(b(\partial_t\mathbf{u},\overline{\mathbf{u}}_\tau)+a(\overline{\mathbf{u}}_\tau,\overline{\mathbf{u}}_\tau)+\int_\Omega\Theta(\overline{w}_\tau)\mathrm{div}\,\overline{\mathbf{u}}_\tau\,dx\Big)\,ds\\
&\quad+\int_0^t\int_{\Gamma_C}\big(\overline{\chi}_\tau|\overline{\mathbf{u}}_\tau|^2+\mathfrak{c}(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))\overline{\boldsymbol{\mu}}_\tau\overline{\mathbf{u}}_\tau\big)\,dx\,ds-\int_0^t\big\langle\overline{\mathbf{F}}_\tau,\overline{\mathbf{u}}_\tau\big\rangle_{\mathbf{W}}\,dr\Big)\\
&\overset{(2)}{\leq}-\int_0^t\Big(b(\partial_t\mathbf{u},\mathbf{u})+a(\mathbf{u},\mathbf{u})+\int_\Omega\Theta(w)\mathrm{div}(\mathbf{u})\,dx\\
&\quad+\int_{\Gamma_C}\chi|\mathbf{u}|^2\,dx+\int_{\Gamma_C}\mathfrak{c}(\Theta(w)-\Theta(w_s))\boldsymbol{\mu}\mathbf{u}-\langle\mathbf{F},\mathbf{u}\rangle_{\mathbf{W}}\Big)\,ds\\
&\overset{(3)}{=}\int_0^t\langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}}\,ds. \qquad (5.31)
\end{aligned}$$

Here, (1) follows from (4.21c) where we have neglected the term $\tau|\varepsilon(\overline{\mathbf{u}}_\tau)|^{\gamma-2}\varepsilon(\overline{\mathbf{u}}_\tau)$ for consistency with the simplified procedure outlined in the above lines. Inequality (2) follows from convergences (4.17), (5.2)–(5.3), (5.6), (5.16), (5.19), and (5.28) via lower semicontinuity arguments, also using that

$$\begin{aligned}
-\liminf_{\tau\to 0}\int_0^t b(\partial_t\mathbf{u},\overline{\mathbf{u}}_\tau)\,ds &= -\liminf_{\tau\to 0}\int_0^{\bar{t}_\tau(t)} b(\partial_t\mathbf{u},\overline{\mathbf{u}}_\tau)\,ds\\
&\leq -\liminf_{\tau\to 0}\frac{1}{2}b(\overline{\mathbf{u}}_\tau(t),\overline{\mathbf{u}}_\tau(t))-\frac{1}{2}b(u_\tau^0,u_\tau^0)\\
&\leq \frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0)-\frac{1}{2}b(\mathbf{u}(t),\mathbf{u}(t))=-\int_0^t b(\partial_t\mathbf{u},\mathbf{u})\,ds \qquad (5.32)
\end{aligned}$$

in view of (4.4). Finally, (3) ensues from (5.29). Then, we conclude

$$\begin{aligned}
\int_0^t\langle\boldsymbol{\eta},\mathbf{v}-\mathbf{u}\rangle_{\mathbf{Y}}\,ds &\overset{(1)}{\leq}\liminf_{\tau\to 0}\int_0^t\int_{\Gamma_C}\overline{\boldsymbol{\eta}}_\tau(\mathbf{v}-\overline{\mathbf{u}}_\tau)\,dx\,ds\overset{(2)}{\leq}\liminf_{\tau\to 0}\int_0^t\int_{\Gamma_C}\big(\phi_\tau(v_N)-\phi_\tau(\overline{u}_{\tau,N})\big)\,dx\\
&\overset{(3)}{\leq}\int_0^t(\boldsymbol{\varphi}(\mathbf{v})-\boldsymbol{\varphi}(\mathbf{u}))\,ds,
\end{aligned}$$

for all $\mathbf{v}\in\mathbf{Y}$ and $t\in[0,T]$, where $\overline{u}_{\tau,N}$ denotes the normal component of $\overline{\mathbf{u}}_\tau$. Here, (1) follows from (5.18) and the previously proved inequality (5.31), (2) from the fact that $\overline{\boldsymbol{\eta}}_\tau=\overline{\eta}_\tau\mathbf{n}$, with $\overline{\eta}_\tau=\phi'_\tau(\overline{u}_{\tau,N})$, and (3) from the fact that ϕ_τ Mosco-converges as $\tau\downarrow 0$ to ϕ, and from the definition (3.13) of $\boldsymbol{\varphi}$. All in all, we conclude (3.29d).

Let us now prove (3.29e): in the ensuing calculations, we will go back to dealing with $\boldsymbol{\eta}\in\mathbf{Y}^*_\gamma$ and in fact resort to the enhanced Hypothesis 4.1 on the operator $\mathcal{R}$. Preliminarily, for every fixed $w\in L^r(0,T;W^{1,r}(\Omega))$, $w_s\in L^\rho(0,T;W^{1,\rho}(\Gamma_C))$, and $\boldsymbol{\eta}\in L^2(0,T;\mathbf{Y}^*_\gamma)$, we introduce the functional $\mathcal{J}_{(w,w_s,\boldsymbol{\eta})}:L^2(0,T;L^4(\Gamma_C;\mathbb{R}^3))\to[0,+\infty)$ by

$$\begin{aligned}
\mathcal{J}_{(w,w_s,\boldsymbol{\eta})}(\mathbf{v}):&=\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(w(x,t))-\Theta(w_s(x,t)))|\mathcal{R}(\boldsymbol{\eta})(x,t)|j(\mathbf{v}(x,t))\,dx\,dt\\
&=\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(w(x,t))-\Theta(w_s(x,t)))|\mathcal{R}(\boldsymbol{\eta})(x,t)|\,|\mathbf{v}_T(x,t)|\,dx\,dt.
\end{aligned}$$

Clearly, $\mathcal{J}_{(w,w_s,\boldsymbol{\eta})}$ is a convex and lower semicontinuous functional on $L^2(0,T;L^4(\Gamma_C;\mathbb{R}^3))$. It can be easily verified that the subdifferential $\partial\mathcal{J}_{(w,w_s,\boldsymbol{\eta})}:L^2(0,T;L^4(\Gamma_C;\mathbb{R}^3))\rightrightarrows L^2(0,T;L^{4/3}(\Gamma_C;\mathbb{R}^3))$ of $\mathcal{J}_{(w,w_s,\boldsymbol{\eta})}$ is given at every $\mathbf{v}\in L^2(0,T;L^4(\Gamma_C;\mathbb{R}^3))$ by

$$\mathbf{h}\in\partial\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{v})\ \Leftrightarrow\ \begin{cases}\mathbf{h}\in L^2(0,T;L^{4/3}(\Gamma_C;\mathbb{R}^3)),\\ \mathbf{h}(x,t)\in\mathfrak{c}(\vartheta(x,t)-\vartheta_s(x,t))|\mathcal{R}(\boldsymbol{\eta})(x,t)|\partial j(\mathbf{v}(x,t))\end{cases} \qquad (5.33)$$

for almost all $(x,t)\in\Gamma_C\times(0,T)$, where ∂j is given by (1.3). We shall prove that

$$\mathcal{J}_{(w,w_s,\boldsymbol{\eta})}(\mathbf{w})-\mathcal{J}_{(w,w_s,\boldsymbol{\eta})}(\partial_t\mathbf{u})\geq\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(w)-\Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}(\mathbf{w}-\partial_t\mathbf{u})\,dx\,dt \qquad (5.34)$$

for all $\mathbf{w}\in L^2(0,T;L^4(\Gamma_C;\mathbb{R}^3))$. From (5.34) we will conclude that $\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\in\partial\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\partial_t\mathbf{u})$, hence the desired (3.29e) by (5.33) and the strict positivity (3.21) of $\mathfrak{c}$. We first observe that

$$\limsup_{\tau\to 0}\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau)|\overline{\mathbf{z}}_\tau\partial_t\mathbf{u}_\tau\,dx\,dt\leq\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(w)-\Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\partial_t\mathbf{u}\,dx\,dt. \qquad (5.35)$$

This can be checked by testing (4.21c) by $\partial_t \mathbf{u}_\tau$ and passing to the limit in (4.21c) via convergences (4.17) for $(\overline{\mathbf{F}}_\tau)_\tau$, (5.2)–(5.3), (5.6), (5.14), and (5.18), also applying convexity inequalities as in (4.23). It also follows from convergence (4.88) for $(\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau))_\tau$ that

$$\lim_{\tau\to 0}\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau)|\overline{\mathbf{z}}_\tau\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(w)-\Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t \tag{5.36}$$

for all $\mathbf{w}\in L^2(0,T;L^4(\Gamma_C;\mathbb{R}^3))$. Therefore, we have

$$\begin{aligned}&\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(w)-\Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}(\mathbf{w}-\partial_t\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\liminf_{\tau\downarrow 0}\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau)|\overline{\mathbf{z}}_\tau(\mathbf{w}-\partial_t\mathbf{u}_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\liminf_{\tau\downarrow 0}\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))|\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau)|(|\mathbf{w}_T|-|(\partial_t\mathbf{u}_\tau)_T|)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\int_0^T\int_{\Gamma_C}\mathfrak{c}(\Theta(w)-\Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|(|\mathbf{w}_T|-|(\partial_t\mathbf{u})_T|)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{5.37}$$

Then, (5.34) ensues. Hence we conclude that $(w,\mathbf{u},\chi)$ comply with the weak formulation (3.29c) of the momentum equation.

Last but not least, note that, testing (4.21c) by $\partial_t\mathbf{u}_\tau$ and integrating in time we can actually prove that

$$\limsup_{\tau\downarrow 0}\int_0^T b(\partial_t\mathbf{u}_\tau,\partial_t\mathbf{u}_\tau)\,\mathrm{d}t\le\int_0^T b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,\mathrm{d}t \tag{5.38}$$

from which we also deduce the strong convergence

$$\partial_t\mathbf{u}_\tau\to\partial_t\mathbf{u}\quad\text{in } L^2(0,T;\mathbf{W}). \tag{5.39}$$

Step 3: **passage to the limit in the discrete enthalpy equations** (4.21a) **and** (4.21b). Let us first tackle the limit passage in (4.21a), with test functions in $v\in W^{1,r'}(\Omega)$. We integrate it in time, over the interval (s,t) with $0\le s\le t\le T$, and exploit convergences (4.16) for $(\overline{h}_\tau)_\tau$, (5.2), (5.6)–(5.8), (5.15), and (5.17), to pass to the limit in the (integrated version of the) first four integral terms on the left-hand side of (4.21a). In particular, the fact that k is Lipschitz combined with (5.6) ensures that $k(\underline{\chi}_\tau)\to k(\chi)$ in $L^p(\Gamma_C\times(0,T))$ for all $1\le p<\infty$. Besides, since K is continuous the pointwise convergence (5.11) yields that $K(\underline{w}_\tau)\to K(w)$ a.e. in $\Omega\times(0,T)$. Using that K is bounded by (3.17), with the Lebesgue theorem we conclude that for every $v\in W^{1,r'}(\Omega)$ the convergence $K(\underline{w}_\tau)\nabla v\to K(w)\nabla v$ in $L^{r'}(\Omega\times(0,T))$. Hence

$$\int_s^t\int_\Omega K(\underline{w}_\tau)\nabla\overline{w}_\tau\nabla v\,\mathrm{d}x\,\mathrm{d}r\to\int_s^t\int_\Omega K(w)\nabla w\nabla v\,\mathrm{d}x\,\mathrm{d}r\quad\text{for all } v\in W^{1,r'}(\Omega).$$

For the limit passage in the fourth term on the left-hand side of (4.21a), we combine (5.6), yielding that $k(\underline{\chi}_\tau)\to k(\chi)$ in $L^q(\Gamma_C\times(0,T))$ for all $1\le q<\infty$ by the Lipschitz continuity of k with the strong convergences (5.16) and (5.17) for $\Theta(\overline{w}_{s,\tau})$ and $\Theta(\overline{w}_\tau)$. All in all, we conclude that

$$\begin{aligned}&\int_s^t\int_{\Gamma_C}k(\underline{\chi}_\tau)\Theta(\overline{w}_\tau)(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))v\,\mathrm{d}x\,\mathrm{d}r\\
&\quad\to\int_s^t\int_{\Gamma_C}k(\chi)\Theta(w)(\Theta(w)-\Theta(w_s))v\,\mathrm{d}x\,\mathrm{d}r\qquad\text{for all } v\in W^{1,r'}(\Omega).\end{aligned}\tag{5.40}$$

As for the fifth term we observe that

$$\mathfrak{c}'(\Theta(\overline{w}_\tau)-\Theta(\overline{w}_{s,\tau}))\to\mathfrak{c}'(\Theta(w)-\Theta(w_s))\quad\text{in } L^q(\Gamma_C\times(0,T))\text{ for all } 1\le q<\infty \tag{5.41}$$

(cf. (5.28)), which we combine with convergence (4.88) for $(\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau))_\tau$, with the strong convergences (5.39) for $\partial_t\mathbf{u}_\tau$, yielding that $\partial_t\mathbf{u}_{\tau,T}\to\partial_t\mathbf{u}_T$ in $L^2(0,T;L^4(\Gamma_C))$, and with (5.17) for $\Theta(\overline{w}_{s,\tau})$. Therefore

$$\begin{aligned}&\int_s^t\int_{\Gamma_C}\Theta(\overline{w}_{s,\tau}(t))\mathfrak{c}'(\Theta(\overline{w}_\tau(t))-\Theta(\overline{w}_{s,\tau}(t)))|\overline{\mathcal{R}}_\tau(\overline{\boldsymbol{\eta}}_\tau(t))|\,|\partial_t\mathbf{u}_\tau(t)|v\,\mathrm{d}x\,\mathrm{d}r\\
&\quad\to\int_s^t\int_{\Gamma_C}\Theta(w_s(t))\mathfrak{c}'(\Theta(w(t))-\Theta(w_s(t)))|\mathcal{R}(\boldsymbol{\eta}(t))|\,|\partial_t\mathbf{u}(t)|v\,\mathrm{d}x\,\mathrm{d}r\quad\text{for all } v\in W^{1,r'}(\Omega).\end{aligned}\tag{5.42}$$

The limit passage on the right-hand side of (4.21a) again ensues from (5.39). In this way, we conclude that the limit quadruplet $(w, w_s, \mathbf{u}, \chi)$ fulfills

$$
\begin{aligned}
&\langle w(t) - w(s), v\rangle_{W^{1,r'}(\Omega)} - \int_s^t \int_\Omega \Theta(w)\mathrm{div}\,(\partial_t \mathbf{u})\, v \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_\Omega K(w)\nabla w \nabla v \,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_s^t \int_{\Gamma_\mathrm{C}} \Big(k(\chi)\Theta(w)(\Theta(w) - \Theta(w_s)) + \Theta(w)\mathfrak{c}'(\Theta(w) - \Theta(w_s))|\mathcal{R}(\boldsymbol{\eta})|\; |\partial_t \mathbf{u}| \Big) v \,\mathrm{d}x\,\mathrm{d}r \\
&= \int_s^t \int_\Omega \varepsilon\,(\partial_t \mathbf{u})\, \mathbb{V}\varepsilon\,(\partial_t \mathbf{u})\, v \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_\Omega h v \,\mathrm{d}x\,\mathrm{d}r \quad \text{for all } v \in W^{1,r'}(\Omega), \text{ for all } 0 \le s \le t \le T.
\end{aligned} \tag{5.43}
$$

Since (5.43) holds for any $0 \le s \le t \le T$ and any test function in $W^{1,r'}(\Omega)$, we deduce from it that w is absolutely continuous with values in $W^{1,r'}(\Omega)^*$. Thus, we recover the improved regularity (3.37), and the improved formulation (3.38) of the enthalpy equation.

Analogously proceeding, we integrate equation (4.21b) over the time interval (s, t). Hence we take the limit as $\tau \downarrow 0$. The passage to the limit in the (integrated version of the) first and third term on the left-hand side of (4.21b) follows from convergences (5.9)–(5.12), arguing as in the above lines. For the second term, we use that $\tau^{-1}(\lambda(\overline{\chi}_\tau) - \lambda(\underline{\chi}_\tau))$ is the (a.e. defined) derivative of the piecewise linear interpolant Λ_τ of the values $(\lambda(\chi_\tau^j))_{j=1}^{J_\tau}$. Since λ is a Lipschitz continuous function, estimate (4.43e) yields that the sequence $(\Lambda_\tau)_\tau$ is bounded in $L^\infty(0, T; V_{\Gamma_\mathrm{C}}) \cap H^1(0, T; H_{\Gamma_\mathrm{C}})$. Therefore, taking into account convergences (5.4)–(5.6) we easily deduce that $\partial_t \Lambda_\tau = \tau^{-1}(\lambda(\overline{\chi}_\tau) - \lambda(\underline{\chi}_\tau))$ weakly converges to $\partial_t \lambda(\chi)$ in $L^2(0, T; H_{\Gamma_\mathrm{C}})$. We combine this with the strong convergence (5.16) for $\Theta(\overline{w}_{s,\tau})$ in $L^2(0, T; H_{\Gamma_\mathrm{C}})$ to pass to the limit. As for the right-hand side of (4.21b), for the limit passage in the first and the third term with test functions in $W^{1,\rho'}(\Gamma_\mathrm{C})$, we argue in the very same way as in (5.40)–(5.41). For the second term, we use that $\mathfrak{c}(\Theta(\overline{w}_\tau) - \Theta(\overline{w}_{s,\tau})) \to \mathfrak{c}(\Theta(w) - \Theta(w_s))$ in $L^q(\Gamma_\mathrm{C} \times (0, T))$ for all $1 \le q < \infty$ (cf. (5.41)), and again convergences (4.88) for $(\mathcal{R}_\tau(\overline{\boldsymbol{\eta}}_\tau))_\tau$ and (5.39) for $\partial_t \mathbf{u}_\tau$. The limit passage in the last term results from the previously proved strong convergence (5.26). In this way we pass to the limit and obtain the integrated version of (3.39). From this we infer that w_s is absolutely continuous with values in $W^{1,\rho'}(\Gamma_\mathrm{C})^*$ (cf. the second of (3.37), and that (3.39) holds pointwise).

Step 4: **positivity**. Property (3.41) ensues from the positivity (4.12) of the discrete enthalpies, and from convergences (5.11) and (5.12).

This concludes the proof of Theorem 1. ∎

Remark 5.1. As already mentioned, the enhanced Hypothesis 4.1 on $\mathcal{R}$ is motivated by the fact that the family $(\overline{\boldsymbol{\eta}}_\tau = \phi'_\tau(\overline{u}_{\tau,\mathrm{N}})\mathbf{n})_\tau$ is estimated, by comparison in the discrete momentum equation (4.21c), in $L^2(0, T; \mathbf{Y}^*_\gamma)$, in place of $L^2(0, T; \mathbf{Y}^*)$. This is due to the presence of the higher order regularizing term $-\tau\, \mathrm{div}(|\varepsilon(\overline{\mathbf{u}}_\tau)|^{\gamma-2}\varepsilon(\overline{\mathbf{u}}_\tau))$ on the left-hand side of (4.21c).

A possible way to avoid this stronger condition on $\mathcal{R}$ would be to resort to a double approximation procedure, keeping the parameter of the Moreau–Yosida approximation of ϕ, say ϵ, distinct from the time step. Hence, one would have to first pass to the limit with the time step, exploiting the fact that the term $\phi'_\epsilon(\overline{u}_{\tau,\mathrm{N}})\mathbf{n}$ inherits the estimates of $\overline{\mathbf{u}}_\tau$ by the Lipschitz continuity of ϕ'_ϵ. Subsequently, one would perform the passage to the limit with respect to ϵ.

Acknowledgments

The authors are grateful to Christian Heinemann for suggesting the splittings (4.11).

E.B. and R.R. acknowledge support from the MIUR-PRIN'10-11 grant for the project "Calculus of Variations". The authors have also been partially supported by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni), of INdAM (Istituto Nazionale di Alta Matematica).

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