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LONG-TIME BEHAVIOUR OF A THERMOMECHANICAL MODEL FOR ADHESIVE CONTACT

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Abstract. This paper deals with the large-time analysis of a PDE system modelling contact with adhesion, in the case when thermal effects are taken into account. The phenomenon of adhesive contact is described in terms of phase transitions for a surface damage model proposed by M. Frémond. Thermal effects are governed by entropy balance laws. The resulting system is highly nonlinear, mainly due to the presence of internal constraints on the physical variables and the coupling of equations written in a domain and on a contact surface. We prove existence of solutions on the whole time interval $(0, +\infty)$ by a double approximation procedure. Hence, we are able to show that solution trajectories admit cluster points which fulfil the stationary problem associated with the evolutionary system, and that in the large-time limit dissipation vanishes.

1. **Introduction.** This paper is concerned with the large-time analysis of a PDE system describing adhesive contact between a thermo-viscoelastic body and a rigid support. The model has been recently introduced, and global-in-time existence results have been proved on finite-time intervals, both in isothermal cases (see [2] in the case of an irreversible damage evolution on the contact surface, and [3] for the reversible case), and for PDE systems including thermal effects (see [4]). The modelling approach for contact with adhesion which we apply refers to a damage theory described by phase transitions, and it is due to M. Frémond (see [13]). The idea consists in describing the adhesion between viscoelastic bodies in terms of a surface damage theory, in which the damage parameter is related to the active bonds which are responsible for the adhesion between the bodies. Hence, the equations of the evolutionary system are recovered from thermomechanical laws, and they are written in the domain of the viscoelastic body and on the contact surface.

It turns out to be interesting, both from a theoretical point of view and in view of applications, to investigate how the thermomechanical system (i.e., the body and the rigid support it is in contact with) behaves for large times. More

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precisely, we shall investigate if the trajectories of the solutions to the resulting PDE system present some cluster point, in the limit as time goes to $+\infty$. Then, we shall look for a relation between these limit states and the stationary system associated with our evolution problem. In particular, we aim to prove that, in the limit, solution trajectories reach a thermomechanical equilibrium state in which dissipation vanishes. This kind of large-time analysis was performed in [3] for the reversible model in the isothermal case.

Before introducing the long-time behaviour analysis of the problem, we shall briefly recall the model and make some comments on the existence of solutions on the whole time interval $(0, +\infty)$. Moreover, we shall point out that this paper also presents a novelty in the formulation of the model itself, as we generalize the convex potential usually ensuring internal constraints on the damage parameter.

The model. We mainly refer to the recent contribution [4], in which (a slightly different version of) the thermomechanical model has been introduced. The state variables, in terms of which the equilibrium of the system is established, are defined in the domain $\Omega \subset \mathbb{R}^3$ (where the body is located), and on the contact surface Γ_c. Namely, we shall take Ω to be a sufficiently smooth bounded domain in $\mathbb{R}^3$, with boundary $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_c$. The sets Γ_i are open subsets in the relative topology of $\partial\Omega$, with smooth boundary and disjoint one from each other. In particular, Γ_c is the contact surface. We suppose that Γ_c and Γ_1 have strictly positive measures and, for the sake of simplicity, we identify Γ_c with a subset of $\mathbb{R}^2$. Thus, we shall treat Γ_c as a flat surface.

The state variables we shall consider in Ω are the absolute temperature ϑ of the body and the macroscopic deformations, given in terms of the linearized strain tensor $\varepsilon(\mathbf{u})$ ($\mathbf{u}$ represents the vector of small displacements). On the contact surface, we introduce the surface absolute temperature (the reader may think of the temperature of the adhesive glue) ϑ_s and a damage parameter χ, related to the active bonds in the glue ensuring adhesion. For the moment, we do not require any constraints on the values assumed by χ. Taking into account local interactions (in the glue and between the glue and the body), we include the gradient $\nabla\chi$ and the displacement trace $\mathbf{u}_{|\Gamma_c}$ among the state variables on the contact surface. The free energy in Ω is written as follows

$$\Psi_\Omega = \vartheta(1 - \ln(\vartheta)) + \vartheta \mathrm{tr}\, \varepsilon(\mathbf{u}) + \frac{1}{2}\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}), \tag{1}$$

where K is the elasticity tensor and the coefficient ϑ multiplying $\mathrm{tr}\varepsilon(\mathbf{u})$ accounts for the thermal expansion energy.

Remark 1. Notice that here we have taken the term $\vartheta(1 - \ln(\vartheta))$ for the purely thermal contribution in the free energy Ψ_Ω (and, similarly, for the free energy Ψ_{Γ_c} below), while in [4] we have considered a more general concave function. The particular choice in this paper is very frequent in the literature, as it has some analytical and modelling advantages. From the latter viewpoint, the presence of the logarithm in (1) yields an internal constraint on the temperature: indeed, the domain of Ψ_Ω is well-defined for $\vartheta > 0$, which is in agreement with thermodynamical consistency. On the analytical level, this form of the thermal contribution shall allow us to simplify the procedure exploited in [4] to prove existence of solutions (see Remark 13).

Next, we specify the free energy in Γ_c, which presents some novelty with respect to the model introduced in [4] (cf. also [2] and [3]). In fact, we shall consider

$$\begin{aligned}\Psi_{\Gamma_c} = \vartheta(1-\ln(\vartheta_s)) + \lambda(\chi)(\vartheta_s - \vartheta_{eq}) + \widehat{\beta}(\chi) + \sigma(\chi) \\ + \frac{1}{2}|\nabla\chi|^2 + \frac{1}{2}\chi^+|\mathbf{u}_{|\Gamma_c}|^2 + I_-(\mathbf{u}_{|\Gamma_c}\cdot\mathbf{n}),\end{aligned} \tag{2}$$

where ϑ_{eq} is a critical temperature, and $\widehat{\beta}$ is a convex, lower-semicontinuous (proper) function. The indicator function I_- forces the scalar product $\mathbf{u}_{|\Gamma_c}\cdot\mathbf{n}$ to be non-positive, as it is defined on $\mathbb{R}$ by $I_-(y)=0$ if $y\leq 0$ and $I_-(y)=+\infty$ for $y>0$. This renders the impenetrability condition between the body and the support. In the same way, the term $\widehat{\beta}$ may yield a constraint on the values assumed by χ. For example, a proper choice of $\widehat{\beta}$ may enforce positivity of χ (see [2, 3, 4]). In particular, this occurs when, classically, $\widehat{\beta}=I_{[0,1]}$, forcing $\chi\in[0,1]$. Then, the coefficient of $|\mathbf{u}_{|\Gamma_c}|^2$ remains non-negative, in accord with physical consistency. However, in the present paper we shall allow the potential $\widehat{\beta}$ to be more general and we do not impose any a priori restriction on its domain. Hence, to ensure physical consistency, we take the deformation coefficient to be χ^+ (using the notation $r^+=\max(r,0)$ for every $r\in\mathbb{R}$). Finally, the function λ is related to the latent heat, while σ takes into account possibly non-convex contributions in the free energy. In particular, we include in σ cohesive effects in the glue, which are represented by a non-increasing function in χ (a simple choice is $\sigma(\chi)=w(1-\chi)$, with a positive parameter w).

Then, the evolution of the system is governed by two convex potentials (non-negative and assuming their minimum 0 if there is no dissipation), namely the pseudo-potentials of dissipation written in Ω and in Γ_c. We have (here K_v is a viscosity matrix)

$$\Phi_\Omega = \frac{1}{2}|\nabla\vartheta|^2 + \frac{1}{2}\varepsilon(\mathbf{u}_t)K_v\varepsilon(\mathbf{u}_t), \tag{3}$$

and

$$\Phi_{\Gamma_c} = \frac{1}{2}|\nabla\vartheta_s|^2 + \frac{1}{2}|\chi_t|^2 + \frac{1}{2}k(\chi)(\vartheta_{|\Gamma_c}-\vartheta_s)^2. \tag{4}$$

Notice that the dissipation in Ω depends on $\varepsilon(\mathbf{u}_t)$ and on $\nabla\vartheta$, while the dissipation in Γ_c depends on $\nabla\vartheta_s$, χ_t, and on $(\vartheta_{|\Gamma_c}-\vartheta_s)$. The function k, which accounts for the heat exchange between the body and the adhesive material, shall be taken non-negative and smooth enough. Actually, to characterize the large-time behaviour of the system, we need to assume that k is bounded from below by some positive constant (see Remark 3).

The PDE system. Proceeding as in [4], we refer to thermomechanical laws and, after specifying the constitutive equations in terms of the above potentials, we arrive at the following PDE system (T is a fixed final time)

$$\partial_t(\ln(\vartheta)) - \mathrm{div}(\mathbf{u}_t) - \Delta\vartheta = h \qquad \text{in } \Omega \times (0,T), \tag{5}$$

$$\partial_n \vartheta = \begin{cases} 0 & \text{in } (\partial\Omega \setminus \Gamma_c) \times (0,T), \\ -k(\chi)(\vartheta - \vartheta_s) & \text{in } \Gamma_c \times (0,T), \end{cases} \tag{6}$$

$$\partial_t(\ln(\vartheta_s)) - \partial_t(\lambda(\chi)) - \Delta\vartheta_s = k(\chi)(\vartheta - \vartheta_s) \qquad \text{in } \Gamma_c \times (0,T), \tag{7}$$

$$\partial_n \vartheta_s = 0 \qquad \text{in } \partial\Gamma_c \times (0,T), \tag{8}$$

$$-\,\mathrm{div}\,(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \vartheta\mathbf{1}) = \mathbf{f} \qquad \text{in } \Omega \times (0,T), \tag{9}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_1 \times (0,T), \quad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \vartheta\mathbf{1})\mathbf{n} = \mathbf{g} \qquad \text{in } \Gamma_2 \times (0,T), \tag{10}$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t) + \vartheta\mathbf{1})\mathbf{n} + \chi^+\mathbf{u} + \partial I_-(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \qquad \text{in } \Gamma_c \times (0,T), \tag{11}$$

$$\begin{aligned} \chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\vartheta_{\mathrm{eq}} + \mathrm{H}(\chi)\frac{1}{2}|\mathbf{u}|^2 & \\ \ni -\lambda'(\chi)\vartheta_s \qquad \text{in } \Gamma_c \times (0,T), & \end{aligned} \tag{12}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \qquad \text{in } \partial\Gamma_c \times (0,T) \tag{13}$$

where h is an external entropy source, $\mathbf{f}$ a volume force, and g a traction. Moreover, $\beta = \partial\widehat{\beta}$ and $\mathrm{H} = \partial\mathrm{p}$, where $\mathrm{p}(r) = r^+$ for all $r \in \mathbb{R}$. Hence, the Heaviside maximal monotone operator $\mathrm{H} : \mathbb{R} \to 2^{\mathbb{R}}$ is defined by $\mathrm{H}(r) = 0$ if $r < 0$, $\mathrm{H}(0) = [0,1]$, and $\mathrm{H}(r) = 1$ if $r > 0$. We warn that, here and in what follows, we shall omit for simplicity the index $v_{|\Gamma_c}$ to denote the trace on Γ_c of a function v, defined in Ω.

Remark 2. Let us comment on the above equations, while referring the to [4] for their rigorous derivation. First of all, we point out that (5) and (7) are entropy equations. The possibility of describing thermal effects in phase transitions by the use of an entropy equation, in place of the more standard energy balance, has only recently been introduced. In particular, let us point out that the entropy in Ω is defined as $\ln(\vartheta) - \mathrm{div}(\mathbf{u})$, and in Γ_c as $\ln(\vartheta_s) - \lambda(\chi)$. Using entropy equations brings to some advantages both for the analytical treatment and the modelling of the phenomenon. In particular, from (5) and (7) one directly recovers the positivity of the temperature, which is necessary for thermodynamical consistency, avoiding the application of any maximum principle argument. We do not enter the details of this theory and refer, among the others, to the papers [7] and [8]. Then, (9) is derived from the momentum balance, in which accelerations are not taken into account. Equation (12) is recovered as a balance equation for micro-movements related to the evolution of the phase parameter (see [13] for the theory of the generalized principle of virtual power including micro-movements and micro-forces responsible for the phase transition).

Remark 3. We emphasize that the structure of (12) is more complicated than the analogous equation in the model studied in [4]. Indeed, the maximal monotone operator β is more general than the one considered in [4], for we do not impose any restriction on its domain. Moreover, the presence of the operator H in (12) introduces a new nonlinearity in the equation. From the physical viewpoint, since $\mathrm{H}(0) = [0,1]$, a residual influence of macroscopic displacements on the mechanical behaviour of the glue may persist also when the glue is damaged, e.g. if $\chi = 0$ (the parameter χ representing here the proportion of active bonds). This corresponds to assuming that a local interaction between the body and the support is preserved even when the bonds in the glue are completely damaged. Notice that this is reasonable, if one takes into account distance forces. An analogous argument justifies the assumption that k in (7) is bounded from below by some positive constant, see (H5) later on. This ensures that the thermal local interaction between the body and the support is conserved even when the adhesion is not active.

Our first main result (see Theorem 2.1 later on) states that for every $T > 0$ the Cauchy problem for system (5)–(13) admits at least one solution. In this way, we parallel the global existence result of [4]: therein, as we mentioned before, we considered a slightly different free energy on the contact surface, which resulted in an equation governing the evolution of the parameter χ simpler than (12). Nonetheless, the proof of Theorem 2.1 (which shall be developed in Section 4), closely follows the argument developed in [4]. It hinges upon a double approximation procedure (depending on two approximating parameters), and a subsequent passage to the limit argument with respect to the mentioned parameters. One of them is used to regularize the nonlinearities in the equations by means of Yosida approximations. Furthermore, some viscosity terms in ϑ and ϑ_s are added in (5) and (7), depending on the second parameter. The *local* existence of a solution for the approximate system (supplemented with suitable regularized initial data for ϑ and ϑ_s, due to the presence of viscosity), is obtained with the Schauder theorem, while uniqueness for the approximate problem follows by contraction arguments. Hence, we conclude the existence of global-in-time solutions by proving suitable a priori estimates (which in fact directly hold in the time interval $(0, +\infty)$, as they do not depend on the final time horizon T), independent of the approximating parameters. The very same estimates allow us to pass to the limit in the approximate problem, firstly as the viscosity parameter, and secondly as the parameter of the Yosida regularizations vanish. Finally, we point out that, due to the strongly nonlinear character of system (5)–(13), we do not expect uniqueness of solutions for the related Cauchy problem.

Large-time analysis. As previously mentioned, the ultimate aim of this paper is investigating the large-time behaviour of system (5)–(13) (supplemented with suitable initial conditions). More precisely, we are interested in finding cluster points of solution trajectories and characterizing a sort of thermomechanical equilibrium of the system in the limit, in which there is no dissipation. This corresponds to proving that solution trajectories converge to solutions of the stationary problem associated with our system, in which dissipation is zero.

Now, some results in this direction have been obtained in the literature concerning the long-time behaviour of phase-field systems with non-convex potentials (see, for example, [11, 12, 14, 16, 17]). Typically, these results apply to binary systems (i.e., macroscopic deformations are not included), see among the others [8] dealing with a singular entropy equation.

The main difficulties related to our analysis are due to the singular character of the entropy equations, to the presence of general multivalued operators on the state variables, and to the nonlinear coupling between the equations written in the domain Ω and the ones set in Γ_c.

Remark 4. On the other hand, the analysis of the large-time behaviour in terms of the global attractor for the dynamical system generated by (5)–(13) might also be addressed. Indeed, the existence of the global attractor would signify that the system dissipation is controlled in the evolution. However, for the moment being, proving the existence of the attractor seems out of our reach. In fact, the strongly nonlinear character of the equations essentially prevents us from obtaining those estimates on the solutions which would guarantee the existence of a compact and absorbing set, for the dynamical system, in the phase space dictated by the choice of the initial data.

Prior to addressing the large-time analysis of system (5)–(13), we specify that we consider a quadruple $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ to be a solution of (5)–(13) in $(0, +\infty)$, if $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ fulfils (5)–(13) in the finite-time interval $(0, T)$, for every $T > 0$. Hence, to perform the asymptotic analysis on the solutions of (5)–(13) as time goes to $+\infty$, we shall rely on some further estimates improving the solution regularity of the existence Theorem 2.1. Only in this enhanced setting, shall we obtain (see Proposition 1) the bounds on the solutions (in suitable functional spaces, on the whole half-line $(0, +\infty)$), necessary to prove that, for every solution trajectory, its ω-limit set (i.e., the set of its cluster points) is non-empty. These further estimates shall be first formally derived in Section 3, and then made rigorous in Section 4 by performing all the related calculations on the approximate system used for proving Theorem 2.1. That is why, our asymptotic analysis solely applies to the solutions of (5)–(13) originating from the aforementioned double approximation procedure. Once proven that the ω-limit is non-empty, we shall show that its elements solve the stationary system associated with the evolutionary problem (5)–(13) (see Theorem 2.2). As a by-product of this procedure, we shall see that in the limit as $t \to \infty$ the dissipation vanishes (cf., in particular, Remark 9).

Plan of the paper. In Section 2 we enlist all of the assumptions on the problem data and state our results. A (partially formal) proof of our Theorem 2.2 on the long-time behaviour of the PDE system (5)–(13) is developed in Section 3 and rigorously justified in Section 4, which also contains the proof of the global existence Theorem 2.1.

2. Main results.

2.1. Preliminaries.

Notation. Throughout the paper, given a Banach space X, we shall denote by ${}_{X'}\langle \cdot, \cdot \rangle_X$ the duality pairing between X' and X itself, and by $\| \cdot \|_X$ both the norm in X and in any power of X; $\mathrm{C}^0_{\mathrm{w}}([0, T]; X)$ shall be the space of the weakly continuous X-valued functions on $[0, T]$.

Henceforth, we shall suppose that Ω is a bounded smooth set of $\mathbb{R}^3$, such that Γ_c is a smooth bounded domain of $\mathbb{R}^2$, and use the notation

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and}$$
$$\mathbf{W} := \left\{ \mathbf{v} \in V^3 \ : \ \mathbf{v} = \mathbf{0} \ \text{ a.e. on } \Gamma_1 \right\} ,$$

the latter space endowed with the norm induced by V^3. We shall denote by $\mathcal{R}$ the standard Riesz operator

$$\mathcal{R} : V \to V' \ \text{ given by } \ {}_{V'}\langle \mathcal{R}(u), v \rangle_V := \int_\Omega uv + \int_\Omega \nabla u \nabla v \quad \text{for all } u, \, v \in V, \quad (14)$$

and by $\mathcal{R}_{\Gamma_c}$ the analogously defined Riesz operator, mapping $H^1(\Gamma_c)$ into $(H^1(\Gamma_c))'$. We shall extensively use that

$$V \subset L^p(\Gamma_c) \text{ with a continuous (compact) embedding for } 1 \leq p \leq 4 \ \ (1 \leq p < 4, \text{ resp.}), \tag{15}$$

$$H^1(\Gamma_c) \subset L^p(\Gamma_c) \text{ with a compact embedding for } 1 \leq p < \infty. \tag{16}$$

For notational simplicity, we shall write $\int_{\Gamma_c} \mathbf{uv}$ ($\int_{\Gamma_2} \mathbf{uv}$, respectively) for the duality pairing ${}_{(H^{-1/2}(\Gamma_c))^3}\langle \mathbf{u}, \mathbf{v}\rangle_{(H^{1/2}(\Gamma_c))^3}$ between $(H^{-1/2}(\Gamma_c))^3$ and $(H^{1/2}(\Gamma_c))^3$ (between $(H^{-1/2}(\Gamma_2))^3$ and $(H^{1/2}(\Gamma_2))^3$, resp.). Finally, given a subset $\mathcal{O} \subset \mathbb{R}^N$, $N = 1, 2, 3$, we shall denote by $|\mathcal{O}|$ its Lebesgue measure and, for a given $v \in V'$, the symbol $m(v)$ shall signify its mean value $1/|\Omega|\ {}_{V'}\langle v, 1\rangle_V$.

Variational formulation of the elasticity equation. We introduce the standard bilinear forms which allow to give a variational formulation of (the boundary value problem for) equation (9). As usual in elasticity theory, we may assume that the material is isotropic and hence suppose that the rigidity matrix K in (9)–(11) can be represented as

$$K\varepsilon(\mathbf{u}) = \lambda \mathrm{tr}\ (\varepsilon(\mathbf{u}))\, \mathbf{1} + 2\mu\varepsilon(\mathbf{u}),$$

where $\lambda, \mu > 0$ are the so-called Lamé constants and $\mathbf{1}$ is the identity matrix. Also, for the sake of simplicity but without loss of generality, we set $K_v = \mathbf{1}$ in (9)–(11). Therefore, (9) may be formulated by means of the following bilinear symmetric forms $a, b : \mathbf{W} \times \mathbf{W} \to \mathbb{R}$, defined by

$$a(\mathbf{u}, \mathbf{v}) := \lambda \int_\Omega \mathrm{div}(\mathbf{u})\mathrm{div}(\mathbf{v}) + 2\mu \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \text{for all } \mathbf{u},\ \mathbf{v} \in \mathbf{W},$$

$$b(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \text{for all } \mathbf{u},\ \mathbf{v} \in \mathbf{W}.$$

Note that the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous and, since Γ_1 has positive measure, by Korn's inequality they are $\mathbf{W}$-elliptic as well, so that

$$\exists\, C_a,\ K_a > 0 \ \ \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{W} : \ a(\mathbf{u}, \mathbf{u}) \geq C_a \|\mathbf{u}\|^2_{\mathbf{W}} \quad |a(\mathbf{u}, \mathbf{v})| \leq K_a \|\mathbf{u}\|_{\mathbf{W}} \|\mathbf{v}\|_{\mathbf{W}}\,, \tag{17}$$

$$\exists\, C_b,\ K_b > 0 \ \ \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{W} : \ b(\mathbf{u}, \mathbf{u}) \geq C_b \|\mathbf{u}\|^2_{\mathbf{W}} \quad |b(\mathbf{u}, \mathbf{v})| \leq K_b \|\mathbf{u}\|_{\mathbf{W}} \|\mathbf{v}\|_{\mathbf{W}}\,. \tag{18}$$

2.2. **A global existence result.**

Statement of the assumptions. In equation (12) we consider

$$\text{a maximal monotone operator } \ \beta : \mathbb{R} \to 2^{\mathbb{R}}, \tag{H1}$$

and denote by $\widehat{\beta} : \overline{\mathrm{D}(\beta)} \to (-\infty, +\infty]$ a proper, l.s.c. and convex function such that $\beta = \partial\widehat{\beta}$. Instead of dealing with the pointwise operator $\partial I_- : \mathbb{R} \to 2^{\mathbb{R}}$ in (11), we shall work with a suitable generalization, defined in the duality relation between $H^{-1/2}(\Gamma_c)^3$ and $H^{1/2}(\Gamma_c)^3$. To this aim, we introduce

$$\widehat{\alpha} : (H^{1/2}(\Gamma_c))^3 \to [0, +\infty] \text{ a proper, convex and l.s.c. functional, with } \widehat{\alpha}(\mathbf{0}) = 0 = \min \widehat{\alpha}, \tag{H2}$$

and set
$$\alpha := \partial\widehat{\alpha}:\ (H^{1/2}(\Gamma_c))^3 \to 2^{(H^{-1/2}(\Gamma_c))^3}.$$

Remark 5. In order to render the impenetrability constraint mentioned in the Introduction by means of the operator α, we may proceed as follows. We consider $j(\mathbf{u}) = I_-(\mathbf{u}\cdot\mathbf{n})$ and associate with j the following functional

$$\widehat{\alpha}(\mathbf{v}) = \int_{\Gamma_c} j(\mathbf{v}) \quad \text{if} \quad \mathbf{v} \in (H^{1/2}(\Gamma_c))^3 \quad \text{and} \quad j(\mathbf{v}) \in L^1(\Gamma_c), \tag{19}$$

$$\widehat{\alpha}(\mathbf{v}) = +\infty \quad \text{otherwise}. \tag{20}$$

Since $\widehat{\alpha}$ is a proper, convex and lower semicontinuous functional on $(H^{1/2}(\Gamma_c))^3$, its subdifferential (cf. [1, Cap. II, p. 52])

$$\alpha := \partial\widehat{\alpha}:\ (H^{1/2}(\Gamma_c))^3 \to 2^{(H^{-1/2}(\Gamma_c))^3} \tag{21}$$

is a maximal monotone operator. Notice that, in this case, (19) implies that, if $\eta \in \alpha(\mathbf{v})$, then $\mathbf{v}$ belongs to the domain of j and thus fulfils $\mathbf{v}\cdot\mathbf{n} \le 0$, which corresponds to the impenetrability condition.

We assume that the nonlinearities σ and λ comply with

$$\sigma \in \mathrm{C}^{1,1}(\mathbb{R})\,, \tag{H3}$$

$$\lambda \in \mathrm{C}^{1,1}(\mathbb{R})\,, \tag{H4}$$

(and denote by L_σ and L_λ the Lipschitz constants of the functions $\sigma' : \mathbb{R} \to \mathbb{R}$ and $\lambda' : \mathbb{R} \to \mathbb{R}$, respectively), and that (cf. Remark 3)

$$\begin{gathered} k:\mathbb{R} \to (0,+\infty) \ \text{ is Lipschitz continuous , with Lipschitz constant } L_k, \text{ and} \\ \exists\, c_k > 0 \ \ \forall\, x \in \mathbb{R}:\ \ k(x) \ge c_k\,. \end{gathered} \tag{H5}$$

Remark 6. We point out that (H3), (H4), and (H5) respectively entail that

$$\exists\, C_\sigma > 0\ \forall\, x \in \mathbb{R} \ :\ \ |\sigma(x)| \le C_\sigma(x^2+1)\,, \tag{22a}$$

$$\exists\, C_\lambda > 0\ \forall\, x \in \mathbb{R} \ :\ \ |\lambda'(x)| \le C_\lambda(|x|+1)\,, \tag{22b}$$

$$\exists\, C_k > 0\ \forall\, x \in \mathbb{R} \ :\ \ |k(x)| \le C_k(|x|+1)\,. \tag{22c}$$

As far as the problem data are concerned, we suppose

$$h \in L^2(0,T;V') \cap L^1(0,T;H)\,, \tag{H6}$$

$$\mathbf{f} \in L^2(0,T;\mathbf{W}')\,, \tag{H7}$$

$$\mathbf{g} \in L^2(0,T;(H^{-1/2}(\Gamma_2))^3)\,. \tag{H8}$$

It follows from (H7)–(H8) that, defining $\mathbf{F} : (0,T) \to \mathbf{W}'$ via

$$_{\mathbf{W}'}\langle \mathbf{F}(t), \mathbf{v}\rangle_{\mathbf{W}} := {}_{\mathbf{W}'}\langle \mathbf{f}(t), \mathbf{v}\rangle_{\mathbf{W}} + \int_{\Gamma_2} \mathbf{g}(t)\cdot\mathbf{v} \quad \forall\, \mathbf{v} \in W \quad \text{for a.e.}\, t \in (0,T),$$

there holds

$$\mathbf{F} \in L^2(0,T;\mathbf{W}')\,. \tag{23}$$

Finally, we require that the initial data fulfil

$$\vartheta_0 \in L^{\bar p}(\Omega)\,, \ \text{with } \bar p \ge \frac{6}{5}, \ \text{ and } \ \ln(\vartheta_0) \in H\,, \tag{24}$$

$$\vartheta_s^0 \in L^{\bar q}(\Gamma_c)\,, \ \text{with } \bar q > 1, \ \text{ and } \ \ln(\vartheta_s^0) \in L^2(\Gamma_c)\,, \tag{25}$$

$$\mathbf{u}_0 \in \mathbf{W} \ \text{ and } \ \mathbf{u}_0 \in \mathrm{D}(\widehat{\alpha})\,, \tag{26}$$

$$\chi_0 \in H^1(\Gamma_c) \ \text{ and } \ \widehat{\beta}(\chi_0) \in L^1(\Gamma_c)\,. \tag{27}$$

Note that the first of (24) and of (25) respectively yield

$$\vartheta_0 \in V', \qquad \vartheta_s^0 \in H^1(\Gamma_c)'\,. \tag{28}$$

Variational formulation and existence theorem. The variational formulation of the initial-boundary value problem for system (5)–(13) reads as follows.
Problem (P). Under the standing assumptions (H1)–(H8), given a quadruple of initial data $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ fulfilling (24)–(27), find functions $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$, with the regularity

$$\begin{aligned} &\vartheta \in L^2(0,T;V) \cap L^\infty(0,T;L^1(\Omega))\,, \\ &\ln(\vartheta) \in L^\infty(0,T;H) \cap H^1(0,T;V')\,, \end{aligned} \tag{29}$$

$$\begin{aligned} &\vartheta_s \in L^2(0,T;H^1(\Gamma_c)) \cap L^\infty(0,T;L^1(\Gamma_c))\,, \\ &\ln(\vartheta_s) \in L^\infty(0,T;L^2(\Gamma_c)) \cap H^1(0,T;H^1(\Gamma_c)')\,, \end{aligned} \tag{30}$$

$$\mathbf{u} \in H^1(0,T;\mathbf{W})\,, \tag{31}$$

$$\boldsymbol{\eta} \in L^2(0,T;(H^{-1/2}(\Gamma_c))^3)\,, \tag{32}$$

$$\chi \in L^2(0,T;H^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c))\,, \tag{33}$$

$$\xi \in L^2(0,T;L^2(\Gamma_c))\,, \tag{34}$$

$$\zeta \in L^\infty(\Gamma_c \times (0,T))\,, \tag{35}$$

satisfying the initial conditions

$$\vartheta(0) = \vartheta_0 \quad \text{a.e. in } \Omega\,, \tag{36}$$

$$\vartheta_s(0) = \vartheta_s^0 \quad \text{a.e. in } \Gamma_c\,, \tag{37}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega\,, \tag{38}$$

$$\chi(0) = \chi_0 \quad \text{a.e. in } \Gamma_c\,, \tag{39}$$

and

$$\begin{aligned} {}_{V'}\langle \partial_t \ln(\vartheta), v\rangle_V - \int_\Omega \operatorname{div}(\mathbf{u}_t)\, v + \int_\Omega \nabla\vartheta\, \nabla v + \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s) v \\ = {}_{V'}\langle h, v\rangle_V \quad \forall\, v \in V \ \text{ a.e. in } (0,T)\,, \end{aligned} \tag{40}$$

$$\begin{aligned} &{}_{H^1(\Gamma_c)'}\langle \partial_t \ln(\vartheta_s), v\rangle_{H^1(\Gamma_c)} - \int_{\Gamma_c} \partial_t \lambda(\chi)\, v + \int_{\Gamma_c} \nabla\vartheta_s\, \nabla v \\ &= \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s) v \quad \forall\, v \in H^1(\Gamma_c) \ \text{ a.e. in } (0,T)\,, \end{aligned} \tag{41}$$

$$\begin{aligned} b(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_\Omega \vartheta \operatorname{div}(\mathbf{v}) + \int_{\Gamma_c} (\chi^+ \mathbf{u} + \boldsymbol{\eta}) \cdot \mathbf{v} \\ = {}_{\mathbf{W}'}\langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \forall \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0,T)\,, \end{aligned} \tag{42}$$

$$\boldsymbol{\eta} \in \alpha(\mathbf{u}) \ \text{ in } (H^{-1/2}(\Gamma_c))^3 \quad \text{a.e. in } (0,T), \tag{43}$$

$$\chi_t - \Delta\chi + \xi + \sigma'(\chi) = -\lambda'(\chi)\vartheta_s - \frac{1}{2}\zeta|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_c \times (0,T), \tag{44}$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } \Gamma_c \times (0,T), \tag{45}$$

$$\zeta \in \mathrm{H}(\chi) \quad \text{a.e. in } \Gamma_c \times (0,T), \tag{46}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \ \text{a.e. in } \partial\Gamma_c \times (0,T)\,. \tag{47}$$

Notice that the contribution $-\lambda'(\chi)\vartheta_{\mathrm{eq}}$ occurring in (12) has been incorporated into the term $\sigma'(\chi)$ in (44).

Theorem 2.1. *Assume (H1)–(H8).*

1. *Then, Problem* $(\mathbf{P})$ *admits a global solution* $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$ *on the interval* $(0, T)$.
2. *If, in addition,*

$$\mathbf{f} \in W^{1,1}(0,T;\mathbf{W}'), \qquad g \in W^{1,1}(0,T;(H^{-1/2}(\Gamma_2))^3), \qquad h \in W^{1,1}(0,T;V'), \tag{48}$$

then there exists a solution $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$ *having for all* $\delta > 0$ *the further regularity*

$$\vartheta \in L^\infty(\delta,T;V) \cap H^1(\delta,T;L^{12/7}(\Omega))\,, \tag{49a}$$

$$\vartheta_s \in L^\infty(\delta,T;H^1(\Gamma_c)) \cap H^1(\delta,T;L^{2-\rho}(\Gamma_c)) \quad \text{for all } \rho \in (0,2), \tag{49b}$$

$$\chi \in L^\infty(\delta,T;H^2(\Gamma_c)) \cap H^1(\delta,T;H^1(\Gamma_c)) \cap W^{1,\infty}(\delta,T;L^2(\Gamma_c))\,, \tag{49c}$$

$$\xi \in L^\infty(\delta,T;L^2(\Gamma_c))\,, \tag{49d}$$

$$\mathbf{u} \in W^{1,\infty}(\delta,T;\mathbf{W})\,. \tag{49e}$$

From (49a)–(49c) it follows in particular that for all $\delta > 0$

$$\vartheta \in \mathrm{C}^0_{\mathrm{w}}([\delta,T];V)\,, \qquad \vartheta_s \in \mathrm{C}^0_{\mathrm{w}}([\delta,T];H^1(\Gamma_c))\,, \qquad \chi \in \mathrm{C}^0_{\mathrm{w}}([\delta,T];H^2(\Gamma_c))\,. \tag{50}$$

We refer to Remark 16 for some further comments concerning the above statement.

Remark 7. The regularity of ϑ_0 and ϑ_s^0 required in (24)–(25) turns out to be necessary in the proof of our existence result Theorem 2.1 for (the Cauchy problem for) system (5)–(13). Indeed, since we are going to prove existence of solutions by passing to the limit in a viscosity approximation of equations (5) and (7), we shall need to dispose of more regular approximate initial data. Our construction of such data (see Lemma 4.2) apparently hinges upon the regularity (24)–(25) of ϑ_0 and ϑ_s^0. However, we are not able to recover for ϑ and ϑ_s the regularity corresponding to assumptions (24) and (25), namely $\vartheta \in \mathrm{C}^0_{\mathrm{w}}([0,T];L^{\bar p}(\Omega))$ and $\vartheta_s \in \mathrm{C}^0_{\mathrm{w}}([0,T];L^{\bar q}(\Omega))$, with $\bar p$ and $\bar q$ as in (24)–(25). This is mainly due to the highly nonlinear character of PDE system and, in some sense, to the fact that the natural initial conditions for (40) and (41) are written for $\ln(\vartheta)$ and $\ln(\vartheta_s)$. Nonetheless, notice that the regularity required for the initial data is preserved (see (50)) for $t \geq \delta > 0$, for every $\delta > 0$, in the more regular framework of (48).

The proof of the above result is based on a double approximation procedure which we shall detail in Section 4. The related passages to the limit rely on suitable a priori estimates on the approximate solutions, which we shall formally perform on the (un-approximated) system (40)–(47), and directly on the time-interval $(0, +\infty)$, within the (formal) proof of Proposition 1. Such estimates shall be rendered rigorous in Sec. 4.3. Finally, in Sec. 4.4 we shall conclude the proof of Theorem 2.1.

2.3. Results on the long-time behaviour of Problem (P).

Within the scope of the present section, we shall say that

a quadruple $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$, with the regularity (29)–(31) and (33),
is a solution to Problem $(\mathbf{P})$ if it fulfils (40)–(41) and
there exists a triple $(\boldsymbol{\eta}, \xi, \zeta)$ for which (32), (34), (35) and (42)–(47) hold.

In view of the long-time analysis of the solutions to Problem (**P**), we shall hereafter suppose that

$$\forall\, R>0 \;\; \exists\, C_R>0 \;\; \forall\, x \in \mathrm{dom}(\widehat{\beta}) : \quad \widehat{\beta}(x) + C_R \geq R x^2 \,. \tag{H9}$$

Notice that (H9) is trivially fulfilled in the case $\mathrm{dom}(\widehat{\beta})$ is a bounded interval, whereas, if $\mathrm{dom}(\widehat{\beta})$ is unbounded, it is implied by a super-quadratic growth of $\widehat{\beta}$ at infinity. We shall also require some summability on $(0,+\infty)$ for the problem data:

$$\mathbf{f} \in L^\infty(0,+\infty;\mathbf{W}') \;\; \text{and} \;\; \mathbf{f}_t \in L^1(0,+\infty;\mathbf{W}')\,, \tag{H10}$$

$$\mathbf{g} \in L^\infty(0,+\infty;(H^{-1/2}(\Gamma_2))^3) \;\; \text{and} \;\; \mathbf{g}_t \in L^1(0,+\infty;(H^{-1/2}(\Gamma_2))^3)\,, \tag{H11}$$

$$h \in L^\infty(0,+\infty;V') \cap L^1(0,+\infty;H) \;\; \text{and} \;\; h_t \in L^1(0,+\infty;V')\,. \tag{H12}$$

The above assumptions yield in particular

$$\mathbf{F} \in L^\infty(0,+\infty;\mathbf{W}') \;\; \text{and} \;\; \mathbf{F}_t \in L^1(0,+\infty;\mathbf{W}')\,, \tag{51}$$

$$h \in L^2(0,+\infty;V')\,. \tag{52}$$

Furthermore, using (51), it is not difficult to prove (see [3, Remark 2.3] for all details) that

$$\exists\, \mathbf{F}_\infty \in \mathbf{W}' : \quad \mathbf{F}(t) \to \mathbf{F}_\infty \;\; \text{in } \mathbf{W}' \text{ as } t \to +\infty. \tag{53}$$

Hence, Theorem 2.1 ensures that for every quadruple of initial data $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ fulfilling (24)–(27) there exists (at least) one solution trajectory

$$(\vartheta, \vartheta_s, \mathbf{u}, \chi) : (0,+\infty) \to V \times H^1(\Gamma_c) \times \mathbf{W} \times H^1(\Gamma_c) \;\; \text{originating from } (\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0).$$

The ensuing Proposition 1 contains some suitable large-time a priori estimates for such trajectories. Such bounds shall enable us to conclude that the associated ω-limit set (57) is non-empty, and that its elements solve the stationary system associated with Problem (**P**) (see Theorem 2.2).

As we shall see, these results in fact hold for a class of solutions of Problem (**P**), namely *approximable solutions* which, in order not to overburden the paper, we shall precisely define in Section 4 only (cf. Definition 4.4). Here, we may just mention that the notion of *approximable solution* is tightly linked to the approximation procedure developed in Sec. 4 to prove the global existence of solutions to Problem (**P**) (see Theorem 2.1). Such a solution notion allows us to perform rigourously on system (40)–(47) some of the a priori estimates on which our large-time analysis relies (cf. Remark 8).

Long-time a priori estimates.

Proposition 1. *Assume* (H1)–(H5) *and* (H9)–(H12). *Let* $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ *be a quadruple of initial data complying with* (24)–(27). *Then, there exists a constant* $K_1 > 0$*, only depending on the functions* λ*,* k*,* σ*, and on the quantity*

$$\begin{aligned} M := {} & \|\vartheta_0\|_{L^1(\Omega)} + \|\vartheta_s^0\|_{L^1(\Gamma_c)} + \|\mathbf{u}_0\|_{\mathbf{W}} + \widehat{\alpha}(\mathbf{u}_0) + \|\chi_0\|_{H^1(\Gamma_c)} \\ & + \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_c)} + \|\mathbf{F}\|_{L^\infty(0,+\infty;\mathbf{W}')} + \|\mathbf{F}_t\|_{L^1(0,+\infty;\mathbf{W}')} \\ & + \|h\|_{L^\infty(0,+\infty;V')\cap L^1(0,+\infty;H)} + \|h_t\|_{L^1(0,+\infty;V')}\,, \end{aligned} \tag{54}$$

such that for every approximable solution *(in the sense of Definition 4.4)* $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ *to Problem* $(\mathbf{P})$*, fulfilling initial conditions* (36)–(39)*, there holds*

$$\|\nabla\vartheta\|_{L^2(0,+\infty;H)} + \|\nabla\vartheta_s\|_{L^2(0,+\infty;L^2(\Gamma_c))} \leq K_1\,, \tag{55a}$$

$$\|\vartheta\|_{L^\infty(0,+\infty;L^1(\Omega))} + \|\vartheta_s\|_{L^\infty(0,+\infty;L^1(\Gamma_c))} \leq K_1\,, \tag{55b}$$

$$\|\vartheta - \vartheta_s\|_{L^2(0,+\infty;L^2(\Gamma_c))} \leq K_1\,, \tag{55c}$$

$$\begin{aligned}\|\mathbf{u}_t\|_{L^2(0,+\infty;\mathbf{W})} + \|\mathbf{u}\|_{L^\infty(0,+\infty;\mathbf{W})}&\\ + \|\widehat{\alpha}(\mathbf{u})\|_{L^\infty(0,+\infty)} + \|\chi_t\|_{L^2(0,+\infty;L^2(\Gamma_c))} &\leq K_1\,,\end{aligned} \tag{55d}$$

$$\|\chi\|_{L^\infty(0,+\infty;H^1(\Gamma_c))} + \|\widehat{\beta}(\chi)\|_{L^\infty(0,+\infty;L^1(\Gamma_c))} \leq K_1\,, \tag{55e}$$

$$\|\partial_t \ln(\vartheta)\|_{L^2(0,+\infty;V')} + \|\partial_t \ln(\vartheta_s)\|_{L^2(0,+\infty;H^1(\Gamma_c)')} \leq K_1\,. \tag{55f}$$

Furthermore, for all $\delta > 0$ *there exist constants* $K_2(\delta)$, $K_3(\delta,\rho) > 0$*, depending on* δ*, on the functions* λ*,* k*,* σ*, and on the quantity* M (54) *(*$K_3(\delta,\rho)$ *on* $\rho \in (0,2)$ *as well), but independent of* $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$*, such that the following estimates hold*

$$\begin{aligned}&\|\vartheta\|_{L^\infty(\delta,+\infty;V)} + \|\vartheta_s\|_{L^\infty(\delta,+\infty;H^1(\Gamma_c))} + \|\chi\|_{L^\infty(\delta,+\infty;H^2(\Gamma_c))}\\ &\quad + \|\chi_t\|_{L^2(\delta,+\infty;H^1(\Gamma_c))\cap L^\infty(\delta,+\infty;L^2(\Gamma_c))} + \|\mathbf{u}\|_{W^{1,\infty}(\delta,+\infty;\mathbf{W})} \leq K_2(\delta)\,,\end{aligned} \tag{56a}$$

$$\|\vartheta_t\|_{L^2(\delta,+\infty;L^{12/7}(\Omega))} + \|\partial_t\vartheta_s\|_{L^2(\delta,+\infty;L^{2-\rho}(\Gamma_c))} \leq K_3(\delta,\rho)\,. \tag{56b}$$

Remark 8. In Section 3.1 we shall give a formal proof of the above result, in which all the estimates leading to (55) and (56) shall be performed on the PDE system (40)–(47) directly. In particular, this shall involve the formal differentiation of equations (42) and (44), as well as formally testing (40), (41) by ϑ_t and $\partial_t\vartheta_s$, respectively. All of these calculations shall be rigorously justified, by working on a suitable approximation of Problem $(\mathbf{P})$, in Section 4.3.

Results on the ω-limit of solution trajectories. Now, for a given quadruple of initial data $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$, complying with (24)–(27), let $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ be an *approximable solution* (in the sense of Definition 4.4), starting from $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$, (its existence is ensured by Theorem 2.1). We aim to investigate the cluster points for large times of the trajectory of $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$, in the topology of the space $H^{1-\epsilon}(\Omega)\times H^{1-\epsilon}(\Gamma_c)\times(H^{1-\epsilon}(\Omega))^3\times H^{2-\epsilon}(\Gamma_c)$, with an arbitrary $\epsilon > 0$. To this aim, we define the ω-limit set $\omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ of the trajectory $(\vartheta(t), \vartheta_s(t), \mathbf{u}(t), \chi(t))_{t\geq 0}$ as

$$\begin{aligned}\omega(\vartheta, \vartheta_s, \mathbf{u}, \chi) :=\big\{&(\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty) \in V \times H^1(\Gamma_c) \times \mathbf{W} \times H^1(\Gamma_c)\ :\\ &\exists\{t_n\} \subset [0,+\infty),\ t_n \nearrow +\infty \text{ as } n \uparrow \infty,\\ &\text{with } (\vartheta(t_n), \vartheta_s(t_n), \mathbf{u}(t_n), \chi(t_n)) \to (\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty)\\ &\text{in } H^{1-\epsilon}(\Omega) \times H^{1-\epsilon}(\Gamma_c) \times \big(H^{1-\epsilon}(\Omega)\big)^3 \times H^{2-\epsilon}(\Gamma_c)\big\}.\end{aligned} \tag{57}$$

For simplicity, we choose to omit the dependence on the initial data $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ (and on the parameter ϵ), in the notation $\omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$. Notice that the latter would be replaced by the more customary $\omega(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ if we additionally disposed of a uniqueness result for the *approximable solutions* to Problem $(\mathbf{P})$.

The following theorem shall be proved in Section 3.2.

Theorem 2.2. *Assume* (H1)–(H5) *and* (H9)–(H12)*.*

Then, for every quadruple of initial data $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ *complying with* (24)–(27) *and for every approximable solution* $(\vartheta(t), \vartheta_s(t), \mathbf{u}(t), \chi(t))_{t\geq 0}$ *originating from*

$(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$, *for all* $\epsilon > 0$ *the associated* ω*-limit set* $\omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ *is a non-empty, compact and connected subset of* $H^{1-\epsilon}(\Omega) \times H^{1-\epsilon}(\Gamma_c) \times \left(H^{1-\epsilon}(\Omega)\right)^3 \times H^{2-\epsilon}(\Gamma_c)$.

Furthermore, every quadruple $(\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty) \in \omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ *is a solution of the stationary system associated with Problem* (**P**), *namely*

$$\exists\, \overline{\vartheta}_\infty \geq 0 : \quad \vartheta_\infty(x) \equiv \overline{\vartheta}_\infty \quad \text{for a.e. } x \in \Omega, \qquad \vartheta_{s,\infty}(x) \equiv \overline{\vartheta}_\infty \quad \text{for a.e. } x \in \Gamma_c, \tag{58a}$$

$$a(\mathbf{u}_\infty, \mathbf{v}) + \overline{\vartheta}_\infty \int_\Omega \operatorname{div}(\mathbf{v}) + \int_{\Gamma_c} (\chi_\infty^+ \mathbf{u}_\infty + \boldsymbol{\eta}_\infty) \cdot \mathbf{v} = \mathbf{F}_\infty \quad \forall \mathbf{v} \in \mathbf{W}, \qquad \boldsymbol{\eta}_\infty \in \alpha(\mathbf{u}_\infty) \quad \text{in } (H^{-1/2}(\Gamma_c))^3, \tag{58b}$$

$\chi_\infty \in H^2(\Gamma_c)$ *and*

$$\begin{cases} -\Delta \chi_\infty + \xi_\infty + \sigma'(\chi_\infty) = -\lambda'(\chi_\infty)\overline{\vartheta}_\infty - \frac{1}{2}\zeta_\infty |\mathbf{u}_\infty|^2 \quad \text{a.e. in } \Gamma_c, \\ \xi_\infty \in L^2(\Gamma_c), \quad \xi_\infty \in \beta(\chi_\infty) \text{ a.e. in } \Gamma_c, \\ \zeta_\infty \in L^\infty(\Gamma_c), \quad \zeta_\infty \in H(\chi_\infty) \text{ a.e. in } \Gamma_c, \\ \partial_{\mathbf{n}_s} \chi_\infty = 0 \quad \text{a.e. in } \partial\Gamma_c. \end{cases} \tag{58c}$$

Remark 9. Let us emphasize that, in the limit as $t \to \infty$, the system is in a state of thermomechanical equilibrium. Indeed, the dissipation, described by the pseudo-potentials (3) and (4), has vanished in (58a)–(58c).

No uniqueness result is available on the stationary system (58a)–(58c). Hence, one cannot deduce directly from Theorem 2.2 that $\omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ is a singleton and, thus, that the *whole* solution trajectory $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ converges, as $t \to +\infty$, to a unique equilibrium. However, the next result (whose proof is postponed to Section 3.2) shows that, under more specific assumptions on the operator β and on the nonlinearities λ and σ, it is possible to uniquely determine the χ-component of the elements in $\omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$.

Corollary 1. *Under assumptions* (H1)–(H5) *and* (H10)–(H12), *suppose further that*

$$\beta = \partial I_{[m_*, m^*]}, \quad \text{for some } -\infty < m_* < m^* < +\infty, \tag{59a}$$

$$\lambda \quad \text{is non-decreasing on } [m_*, m^*], \tag{59b}$$

$$\sigma'(x) > 0 \quad \text{for all } x \in [m_*, m^*]. \tag{59c}$$

Then, for any quadruple $(\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty) \in \omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ *there holds*

$$\chi_\infty(x) \equiv m_* \quad \forall\, x \in \Gamma_c\,, \tag{60}$$

and we have as $t \to +\infty$

$$\chi(t) \to \chi_\infty \quad \text{in } H^{2-\epsilon}(\Gamma_c) \text{ for all } \epsilon > 0.$$

Remark 10. Corollary 1 ensures that, in the case when the latent heat is positive, β has a bounded domain (which is the interesting case from a physical point of view), and cohesion in the material (which is included in the decreasing part of σ, see (2)) is not too large with respect to the remaining part of the potential σ, the glue tends to be completely damaged in the large-time limit. We remark that this is the result one would expect from experience.

3. Proofs.

Notation. Henceforth, for the sake of notational simplicity, we shall write $\langle\cdot,\cdot\rangle$ for all the duality pairings ${}_{\mathbf{W}'}\langle\cdot,\cdot\rangle_{\mathbf{W}}$, ${}_{V'}\langle\cdot,\cdot\rangle_{V}$, and ${}_{H^1(\Gamma_c)'}\langle\cdot,\cdot\rangle_{H^1(\Gamma_c)}$, and, further, denote by the symbols c, c' C, C' most of the (positive) constants occurring in calculations and estimates. Notice that the above constants shall not depend on the final time T.

3.1. Proof of Proposition 1.

The following is a *formal* proof, based on two main a priori estimates on system (40)–(47), which shall be revisited in Section 4.3. Therein, we shall also rigorously prove the further solution regularity (49a)–(49e).

First (formal) estimate. We test (40) by ϑ, (41) by ϑ_s, (42) by $\mathbf{u}_t$ and (44) by χ_t, add the resulting relations and integrate them on the interval $(0,t)$, with $t\in(0,+\infty)$. Now, we take into account the *formal* identities

$$\begin{aligned}&\int_0^t \langle \partial_t \ln(\vartheta),\vartheta\rangle = \|\vartheta(t)\|_{L^1(\Omega)} - \|\vartheta_0\|_{L^1(\Omega)}\,,\\ &\int_0^t \langle \partial_t \ln(\vartheta_s),\vartheta_s\rangle = \|\vartheta_s(t)\|_{L^1(\Gamma_c)} - \|\vartheta_s^0\|_{L^1(\Gamma_c)}\,,\end{aligned}\tag{61}$$

and the chain rule for the convex functionals $\widehat{\alpha}$, and $\widehat{\beta}$ (cf. with [10, Lemma 4.1] and [9, Lemma 3.3], respectively), and for the smooth function σ, yielding

$$\begin{aligned}&\int_0^t\int_{\Gamma_c} \boldsymbol{\eta}\cdot\mathbf{u}_t = \widehat{\alpha}(\mathbf{u}(t)) - \widehat{\alpha}(\mathbf{u}_0)\,, \qquad \int_0^t\int_{\Gamma_c}\xi\chi_t = \int_{\Gamma_c}\widehat{\beta}(\chi(t)) - \int_{\Gamma_c}\widehat{\beta}(\chi_0)\,,\\ &\qquad\qquad \int_0^t\int_{\Gamma_c}\sigma'(\chi)\chi_t = \int_{\Gamma_c}\sigma(\chi(t)) - \int_{\Gamma_c}\sigma(\chi_0)\,.\end{aligned}\tag{62}$$

In the same way, an integration by parts and the chain rule for $\mathrm{p}(\cdot) = (\cdot)^+$ lead to

$$\begin{aligned}\int_0^t\int_{\Gamma_c}\chi^+\mathbf{u}\mathbf{u}_t &= \frac{1}{2}\int_0^t\int_{\Gamma_c}\chi^+\partial_t|\mathbf{u}|^2\\ &= -\frac{1}{2}\int_0^t\int_{\Gamma_c}\zeta\chi_t|\mathbf{u}|^2 + \frac{1}{2}\int_{\Gamma_c}\chi^+(t)|\mathbf{u}(t)|^2 - \frac{1}{2}\int_{\Gamma_c}\chi_0^+|\mathbf{u}_0|^2\,,\end{aligned}\tag{63}$$

where we recall that $\zeta\in\mathrm{H}(\chi) = \partial\mathrm{p}(\chi)$. Finally, we observe that, by the properties (17) and (18) of the forms a and b, respectively, we have

$$\int_0^t a(\mathbf{u},\mathbf{u}_t) \geq \frac{C_a}{2}\|\mathbf{u}(t)\|^2_{\mathbf{W}} - \frac{1}{2}K_a\|\mathbf{u}_0\|^2_{\mathbf{W}}\,, \qquad \int_0^t b(\mathbf{u}_t,\mathbf{u}_t) \geq C_b\int_0^t\|\mathbf{u}_t\|^2_{\mathbf{W}}\,. \tag{64}$$

Collecting (61)–(64) and observing that some terms cancel out, we get

$$
\begin{aligned}
&\|\vartheta(t)\|_{L^1(\Omega)} + \int_0^t \|\nabla\vartheta\|_H^2 + \int_0^t \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s)^2 + \|\vartheta_s(t)\|_{L^1(\Gamma_c)} \\
&\qquad + \int_0^t \|\nabla\vartheta_s\|_{L^2(\Gamma_c)}^2 + C_b \int_0^t \|\mathbf{u}_t\|_{\mathbf{W}}^2 + \frac{C_a}{2}\|\mathbf{u}(t)\|_{\mathbf{W}}^2 + \widehat{\alpha}(\mathbf{u}(t)) \\
&\qquad + \frac{1}{2}\int_{\Gamma_c} \chi^+(t)|\mathbf{u}(t)|^2 + \int_0^t \|\chi_t\|_{L^2(\Gamma_c)}^2 + \frac{1}{2}\|\nabla\chi(t)\|_{L^2(\Gamma_c)}^2 \\
&\qquad + \int_{\Gamma_c} \Big(\widehat{\beta}(\chi(t)) + \sigma(\chi(t))\Big) \\
&\qquad \le \|\vartheta_0\|_{L^1(\Omega)} + \|\vartheta_s^0\|_{L^1(\Gamma_c)} + \frac{K_a}{2}\|\mathbf{u}_0\|_{\mathbf{W}}^2 + \widehat{\alpha}(\mathbf{u}_0) + \frac{1}{2}\|\nabla\chi_0\|_{L^2(\Gamma_c)}^2 \\
&\qquad + c\|\chi_0\|_{L^2(\Gamma_c)}\|\mathbf{u}_0\|_{\mathbf{W}}^2 + \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_c)} + \|\sigma(\chi_0)\|_{L^1(\Gamma_c)} + I_1 + I_2\,,
\end{aligned}
\tag{65}
$$

where have also used the continuous embedding (15) for the third term on the right-hand side of (63). Then, we estimate

$$
\begin{aligned}
I_1 = \int_0^t \langle h,\vartheta\rangle &\le \int_0^t \|h\|_{V'}\|\vartheta - m(\vartheta)\|_V + \int_0^t \|h\|_{V'}\|m(\vartheta)\|_V \\
&\le C\int_0^t \|h\|_{V'}^2 + \frac{1}{2}\int_0^t \|\nabla\vartheta\|_H^2 + \frac{1}{|\Omega|^{1/2}}\int_0^t \|h\|_{V'}\|\vartheta\|_{L^1((\Omega)}\,,
\end{aligned}
\tag{66}
$$

the second inequality due to the Poincaré inequality for functions with zero mean value, and, with an integration by parts, we get

$$
\begin{aligned}
I_2 = \int_0^t \langle \mathbf{F},\mathbf{u}_t\rangle &= -\int_0^t \langle \mathbf{F}_t,\mathbf{u}\rangle + \langle \mathbf{F}(t),\mathbf{u}(t)\rangle - \langle \mathbf{F}(0),\mathbf{u}_0\rangle \\
&\le \int_0^t \|\mathbf{F}_t\|_{\mathbf{W}'}\|\mathbf{u}\|_{\mathbf{W}} + \frac{C_a}{4}\|\mathbf{u}(t)\|_{\mathbf{W}}^2 \\
&\quad + \frac{1}{2}\|\mathbf{u}_0\|_{\mathbf{W}}^2 + C\|\mathbf{F}\|_{L^\infty(0,+\infty;\mathbf{W}')}^2\,.
\end{aligned}
\tag{67}
$$

Hence, taking into account our assumptions on the data (H10)–(H12) (which yield (51) and (52)), as well as (24)–(27), we may apply a variant of the Gronwall Lemma (cf. [9, Lemma A.5]) and conclude estimates (55a), (55b), and (55d), while (55c) follows from the bound for $(k(\chi))^{1/2}(\vartheta-\vartheta_s)$ in $L^2(0,+\infty;L^2(\Gamma_c))$ and the fact that k is bounded from below by a strictly positive constant, see (H5). Finally, thanks to (H9) and (22a), we deduce from (65) that

$$
\begin{aligned}
&\exists\, C,\, c,\, c' > 0 \quad \forall\, t \in (0,+\infty)\ : \\
&\qquad c\|\chi(t)\|_{L^2(\Gamma_c)}^2 - c' \le \int_{\Gamma_c} \Big(\widehat{\beta}(\chi(t)) + \sigma(\chi(t))\Big) \le C\,.
\end{aligned}
\tag{68}
$$

Joint with the bound for $\nabla\chi$ in $L^\infty(0,+\infty;L^2(\Gamma_c))$, this yields the estimate for $\|\chi\|_{L^\infty(0,+\infty;H^1(\Gamma_c))}$. A fortiori, in view of (H9) we also recover the bound for $\widehat{\beta}(\chi)$, and (55e) ensues. In the end, estimate (55f) for $\partial_t \ln(\vartheta)$ and for $\partial_t \ln(\vartheta_s)$ follows from a comparison in equations (40) and (41), respectively. For example, using (H5) and (55c), (55e) one observes that

$$
\|k(\chi)(\vartheta-\vartheta_s)\|_{L^2(0,+\infty;V')} \le C(\|\chi\|_{L^\infty(0,+\infty;L^4(\Gamma_c))}+1)\|\vartheta-\vartheta_s\|_{L^2(0,+\infty;L^2(\Gamma_c))} \le C'\,.
$$

Hence, in view of the bound (55a) for $\nabla\vartheta$ in $L^2(0,+\infty;H)$, of (55d) for $\mathrm{div}(\mathbf{u}_t)$ in $L^2(0,+\infty;H^3)$, and of (52), one concludes the estimate for $\partial_t \ln(\vartheta)$ in $L^2(0,+\infty;V')$.

Second (formal) estimate. In what follows, we shall formally treat the maximal monotone operators α, H, and β as nondecreasing and *Lipschitz* functions. Indeed, the following estimates can be rigorously justified as in Section 4 regularizing these nonlinearities by their Yosida approximations (which are in fact Lipschitz functions). Further, we shall work with a solution $(\vartheta,\vartheta_s,\mathbf{u},\chi)$ enjoying the further regularity (49) (see also (50)). Formally proceeding, for all $\delta>0$ we let a constant K_δ, only depending on λ, k, σ, on the quantity M (54), and possibly on δ, such that

$$\|\vartheta(\delta)\|_V^2+\|\vartheta_s(\delta)\|_{H^1(\Gamma_c)}^2+\|\mathbf{u}_t(\delta)\|_{\mathbf{W}}^2+\|\chi_t(\delta)\|_{L^2(\Gamma_c)}^2\le K_\delta\,. \tag{69}$$

This formal assumption shall be discarded once we put forth the rigorous arguments of Section 4, see Remark 11 for further comments. Hence, we are in the position of performing the following calculations. We test (40) by ϑ_t and (41) by $\partial_t\vartheta_s$, differentiate (42) w.r.t. time and test it by $\mathbf{u}_t$, and differentiate (44) w.r.t. time and multiply it by χ_t. We add the resulting relations and integrate them on the interval (δ,t), with $t\in(\delta,+\infty)$, also adding $1/2(\|\vartheta(t)\|_{L^1(\Omega)}^2+\|\vartheta_s(t)\|_{L^1(\Gamma_c)}^2)$ to both sides. Indeed, the Poincaré inequality for the zero mean value functions yields

$$\begin{aligned}&C_P\|\vartheta(t)\|_V^2\le\frac12\left(\|\nabla\vartheta(t)\|_H^2+\|\vartheta(t)\|_{L^1(\Omega)}^2\right),\\&C_P\|\vartheta_s(t)\|_{H^1(\Gamma_c)}^2\le\frac12\left(\|\nabla\vartheta_s(t)\|_{L^2(\Gamma_c)}^2+\|\vartheta_s(t)\|_{L^1(\Gamma_c)}^2\right)\end{aligned}\tag{70}$$

for some positive constant C_P independent of $t\in(0,+\infty)$. We also notice that, for some other constant c also depending on the embeddings (15)–(16), there holds

$$c\|\vartheta(t)-\vartheta_s(t)\|_{L^4(\Gamma_c)}^2\le\frac{C_P}{2}\left(\|\vartheta(t)\|_V^2+\|\vartheta_s(t)\|_{H^1(\Gamma_c)}^2\right)\quad\text{for all }t\in(0,+\infty).\tag{71}$$

Further, we remark that

$$\int_\delta^t\int_{\Gamma_c}\partial_t(\chi^+\mathbf{u})\mathbf{u}_t=\int_\delta^t\int_{\Gamma_c}\chi^+|\mathbf{u}_t|^2+\int_\delta^t\int_{\Gamma_c}\mathrm{H}(\chi)\chi_t\mathbf{u}\mathbf{u}_t\,.\tag{72}$$

Taking into account the cancellation of some terms and the coercivity and continuity of the forms a and b (17)–(18), using the *formal* identities

$$\int_\delta^t\langle\partial_t\ln(\vartheta),\vartheta_t\rangle=\int_\delta^t\int_\Omega\frac{|\vartheta_t|^2}{\vartheta}\,,\qquad\int_\delta^t\langle\partial_t\ln(\vartheta_s),\partial_t\vartheta_s\rangle=\int_\delta^t\int_{\Gamma_c}\frac{|\partial_t\vartheta_s|^2}{\vartheta_s}\,,\tag{73}$$

as well as (70)–(72), we end up with

$$\begin{aligned}&\int_\delta^t\int_\Omega\frac{|\vartheta_t|^2}{\vartheta}+\frac{C_P}{2}\|\vartheta(t)\|_V^2+\int_\delta^t\int_{\Gamma_c}k(\chi)(\vartheta-\vartheta_s)\partial_t(\vartheta-\vartheta_s)\\&+\int_\delta^t\int_{\Gamma_c}\frac{|\partial_t\vartheta_s|^2}{\vartheta_s}+\frac{C_P}{2}\|\vartheta_s(t)\|_{H^1(\Gamma_c)}^2+c\|\vartheta(t)-\vartheta_s(t)\|_{L^4(\Gamma_c)}^2\\&+\frac{C_b}{2}\|\mathbf{u}_t(t)\|_{\mathbf{W}}^2+C_a\int_\delta^t\|\mathbf{u}_t\|_{\mathbf{W}}^2+\int_\delta^t\int_{\Gamma_c}\chi^+|\mathbf{u}_t|^2+\int_\delta^t\langle\alpha'(\mathbf{u})\mathbf{u}_t,\mathbf{u}_t\rangle\\&+\frac12\|\chi_t(t)\|_{L^2(\Gamma_c)}^2+\int_\delta^t\int_{\Gamma_c}|\nabla\chi_t|^2+\int_\delta^t\int_{\Gamma_c}\beta'(\chi)|\chi_t|^2\\&\le CK_\delta+K_1^2+I_3+I_4+I_5+I_6+I_7+I_8\,,\end{aligned}\tag{74}$$

in which we have controlled the term $1/2(\|\vartheta(t)\|^2_{L^1(\Omega)} + \|\vartheta_s(t)\|^2_{L^1(\Gamma_c)})$ on the right-hand side by (55b). Integrating by parts, we have

$$
\begin{aligned}
I_3 &= \int_\delta^t \langle h, \vartheta_t\rangle \\
&= -\int_\delta^t \langle h_t, \vartheta\rangle + \langle h(t), \vartheta(t)\rangle - \langle h(\delta), \vartheta(\delta)\rangle \\
&\leq \int_\delta^t \|h_t\|_{V'}\|\vartheta\|_V + C\|h\|^2_{L^\infty(0,+\infty;V')} + \frac{C_P}{4}\|\vartheta(t)\|^2_V + K_\delta\,,
\end{aligned}
\tag{75}
$$

while we estimate

$$
I_4 = \int_\delta^t \langle \mathbf{F}_t, \mathbf{u}_t\rangle \leq \int_\delta^t \|\mathbf{F}_t\|_{\mathbf{W}'}\|\mathbf{u}_t\|_{\mathbf{W}}\,,
\tag{76}
$$

$$
\begin{aligned}
I_5 &= -2\int_\delta^t \int_{\Gamma_c} \mathrm{H}(\chi)\chi_t \mathbf{u}\mathbf{u}_t \\
&\leq 2\int_\delta^t \|\chi_t\|_{L^2(\Gamma_c)}\|\mathbf{u}\|_{L^4(\Gamma_c)}\|\mathbf{u}_t\|_{L^4(\Gamma_c)} \\
&\leq C\|\mathbf{u}\|_{L^\infty(0,+\infty;\mathbf{W})}\left(\int_\delta^t \int_{\Gamma_c} \|\chi_t\|^2_{L^2(\Gamma_c)} + \int_\delta^t \int_{\Gamma_c} \|\mathbf{u}_t\|^2_{\mathbf{W}}\right) \leq C\,,
\end{aligned}
\tag{77}
$$

$$
I_6 = -\frac{1}{2}\int_\delta^t \int_{\Gamma_c} \mathrm{H}'(\chi)|\chi_t|^2|\mathbf{u}|^2 \leq 0\,,
\tag{78}
$$

(77) due to the continuous embedding (15) and to estimate (55d), and (78) following by monotonicity. Moreover, recalling (H3), (H4), and (55d), and applying the Hölder inequality

$$
I_7 = -\int_\delta^t \int_{\Gamma_c} \sigma''(\chi)|\chi_t|^2 \leq L_\sigma K_1^2\,,
\tag{79}
$$

$$
\begin{aligned}
I_8 &= -\int_\delta^t \int_{\Gamma_c} \lambda''(\chi)|\chi_t|^2\vartheta_s \\
&\leq L_\lambda \int_\delta^t \|\chi_t\|_{L^2(\Gamma_c)}\|\chi_t\|_{L^4(\Gamma_c)}\|\vartheta_s\|_{L^4(\Gamma_c)} \\
&\leq \frac{1}{4}\int_\delta^t \|\nabla\chi_t\|^2_{L^2(\Gamma_c)} + \frac{K_1^2}{4} + C\int_\delta^t \|\chi_t\|^2_{L^2(\Gamma_c)}\|\vartheta_s\|^2_{H^1(\Gamma_c)}\,,
\end{aligned}
\tag{80}
$$

C also depending on the constant of the continuous embedding (16). Finally, we integrate by parts the third term on the left-hand side of (74), so that

$$
\begin{aligned}
&\int_\delta^t \int_{\Gamma_c} k(\chi)(\vartheta - \vartheta_s)\partial_t(\vartheta - \vartheta_s) \\
&\quad = \frac{1}{2}\int_{\Gamma_c} k(\chi(t))|\vartheta(t) - \vartheta_s(t)|^2 - \frac{1}{2}\int_{\Gamma_c} k(\chi(\delta))|\vartheta(\delta) - \vartheta_s(\delta)|^2 - I_9 \\
&\quad \geq \frac{c_k}{2}\|\vartheta(t) - \vartheta_s(t)\|^2_{L^2(\Gamma_c)} \\
&\qquad\quad - c\left(\|\chi(\delta)\|_{L^2(\Gamma_c)} + 1\right)\left(\|\vartheta(\delta)\|^2_V + \|\vartheta_s(\delta)\|^2_{H^1(\Gamma_c)}\right) - I_9 \\
&\quad \geq \frac{c_k}{2}\|\vartheta(t) - \vartheta_s(t)\|^2_{L^2(\Gamma_c)} - C\left(K_\delta + 1\right) - I_9\,,
\end{aligned}
\tag{81}
$$

the latter inequality due (H5), to (22c), (55e), and (69), while, using (H5) and (55d), we estimate

$$\begin{aligned} I_9 &= \frac{1}{2}\int_\delta^t \int_{\Gamma_c} k'(\chi)\chi_t|\vartheta-\vartheta_s|^2 \\ &\le \frac{L_k}{2}\int_\delta^t \|\chi_t\|_{L^4(\Gamma_c)}\|\vartheta-\vartheta_s\|_{L^2(\Gamma_c)}\|\vartheta-\vartheta_s\|_{L^4(\Gamma_c)} \\ &\le \frac{1}{2}\int_\delta^t \|\nabla\chi_t\|^2_{L^2(\Gamma_c)} + \frac{K_1^2}{2} + C\int_\delta^t \|\vartheta-\vartheta_s\|^2_{L^2(\Gamma_c)}\|\vartheta-\vartheta_s\|^2_{L^4(\Gamma_c)} \end{aligned} \tag{82}$$

Next, we collect (74)–(82), observing that the ninth term on the left-hand side of (74) is non-negative, and so are the tenth and the thirteenth terms, by monotonicity of α and β, respectively. Then, taking into account the summability properties (51) and (52), as well as estimate (55d) for $\|\chi_t\|_{L^2(0,+\infty;L^2(\Gamma_c))}$ and (55c) for $\|\vartheta-\vartheta_s\|_{L^2(0,+\infty;L^2(\Gamma_c))}$, we apply the standard Gronwall lemma and its aforementioned variant (cf. [9, Lemmas A.3, A.5]) and conclude estimate (56a) for ϑ, ϑ_s, $\mathbf{u}$, and χ_t. A comparison in (44) yields (possibly for a larger $K_2(\delta)$)

$$\|-\Delta\chi+\xi\|^2_{L^\infty(\delta,+\infty;L^2(\Gamma_c))} \le K_2(\delta)\,,$$

whence an estimate both for $\|\xi\|_{L^\infty(\delta,+\infty;L^2(\Gamma_c))}$ and for $\|\Delta\chi\|_{L^\infty(\delta,+\infty;L^2(\Gamma_c))}$ by the monotonicity of β. Then, the bound for $\|\chi\|_{L^\infty(\delta,+\infty;H^2(\Gamma_c))}$ follows from elliptic regularity. Furthermore, the estimate on the first and on the fourth integral terms on the left-hand side of (74) gives (notice that $\vartheta>0$ a.e. in $\Omega\times(0,+\infty)$ and $\vartheta_s>0$ a.e. in $\Gamma_c\times(0,+\infty)$)

$$\|\partial_t\vartheta^{1/2}\|_{L^2(\delta,+\infty;H)} + \|\partial_t\vartheta_s^{1/2}\|_{L^2(\delta,+\infty;L^2(\Gamma_c))} \le C(\delta)\,. \tag{83}$$

On the other hand, due to the continuous embeddings $V\subset L^p(\Omega)$ for all $1\le p\le 6$ and $H^1(\Gamma_c)\subset L^q(\Gamma_c)$ for all $1\le q<\infty$, (56a) in particular yields

$$\begin{aligned} &\forall\, 1\le q<\infty \quad \exists\, C_q(\delta)>0\,: \\ &\qquad \|\vartheta^{1/2}\|_{L^\infty(\delta,+\infty;L^{12}(\Omega))} + \|\vartheta_s^{1/2}\|_{L^\infty(\delta,+\infty;L^q(\Gamma_c))} \le C_q(\delta)\,. \end{aligned} \tag{84}$$

Hence, (56b) follows from (56a), (83), (84), and the Hölder inequality, giving

$$\begin{aligned} &\|\vartheta_t\|_{L^{12/7}(\Omega)} \le \|\partial_t\vartheta^{1/2}\|_H\|\vartheta^{1/2}\|_{L^{12}(\Omega)} \\ &\|\partial_t\vartheta_s\|_{L^{2-\rho}(\Gamma_c)} \le \|\partial_t\vartheta_s^{1/2}\|_{L^2(\Gamma_c)}\|\vartheta_s^{1/2}\|_{L^{(4-2\rho)/\rho}(\Gamma_c)} \quad \text{for all } 0<\rho<2. \end{aligned} \tag{85}$$

This concludes the proof. □

Remark 11. Following [6], we point out that a possible way to perform estimate (74) more rigorously, without assuming (69), would be to fix a smooth "cut-off" function $\varsigma:[0,+\infty)\to[0,+\infty)$, with $\varsigma,\varsigma'\in L^\infty(0,+\infty)$ (for example, $\varsigma(t)=\tanh(t)$ for all $t\ge 0$), and test (40) by $\varsigma\vartheta_t$, (41) by $\varsigma\partial_t\vartheta_s$, the time derivative of (42) by $\varsigma\mathbf{u}_t$, and the time derivative of (44) by $\varsigma\chi_t$. However, to keep calculations simpler we have postponed this procedure to the rigorous proof of Proposition 1 in Sec. 4.

3.2. **Proof of Theorem 2.2.** In view of estimates (55d) and (56a), for all $\delta>0$ the trajectory $\{(\vartheta(t),\vartheta_s(t),\mathbf{u}(t),\chi(t)),\, t\ge\delta\}$ of every approximable solution $(\vartheta,\vartheta_s,\mathbf{u},\chi)$, starting from a quadruple of initial data $(\vartheta_0,\vartheta_s^0,\mathbf{u}_0,\chi_0)$ as in (24)–(27), is bounded in $V\times H^1(\Gamma_c)\times\mathbf{W}\times H^2(\Gamma_c)$. Hence, it is relatively compact in $H^{1-\epsilon}(\Omega)\times H^{1-\epsilon}(\Gamma_c)\times(H^{1-\epsilon}(\Omega))^3\times H^{2-\epsilon}(\Gamma_c)$ for all $\epsilon>0$, and with a standard argument one finds that the set $\omega(\vartheta,\vartheta_s,\mathbf{u},\chi)$ is a non-empty and compact subset of the latter product space. Thanks to (50), $\omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ is connected in $H^{1-\epsilon}(\Omega) \times H^{1-\epsilon}(\Gamma_c) \times (H^{1-\epsilon}(\Omega))^3 \times H^{2-\epsilon}(\Gamma_c)$ for all $\epsilon > 0$ as well, by a well-known result in the theory of dynamical systems, see e.g. [15].

Now, let us fix $(\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty) \in \omega(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ and an increasing sequence $\{t_n\} \subset (0, +\infty)$ such that $t_n \nearrow +\infty$ as $n \uparrow \infty$ and

$$\begin{aligned} (\vartheta(t_n), \vartheta_s(t_n), &\mathbf{u}(t_n), \chi(t_n)) \to (\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty) \\ &\text{in } H^{1-\epsilon}(\Omega) \times H^{1-\epsilon}(\Gamma_c) \times \left(H^{1-\epsilon}(\Omega)\right)^3 \times H^{2-\epsilon}(\Gamma_c). \end{aligned} \tag{86}$$

Following a well-established procedure, for all $n \in \mathbb{N}$ and $T > 0$ we introduce the translated functions

$$\begin{aligned} &\vartheta_n(t) := \vartheta(t + t_n), \quad \vartheta_{s,n}(t) := \vartheta_s(t + t_n), \\ &\mathbf{u}_n(t) := \mathbf{u}(t + t_n), \quad \chi_n(t) := \chi(t + t_n)\,, \end{aligned}$$

for $t \in [0, T]$. We also set for a.e. $t \in (0, T)$

$$\begin{aligned} &\boldsymbol{\eta}_n(t) := \boldsymbol{\eta}(t + t_n), \quad \xi_n(t) := \xi(t + t_n), \quad \zeta_n(t) := \zeta(t + t_n), \\ &\mathbf{F}_n(t) := \mathbf{F}(t + t_n), \quad h_n(t) := h(t + t_n). \end{aligned}$$

Clearly, $\boldsymbol{\eta}_n(t) \in \alpha(\mathbf{u}_n(t))$ for a.e. $t \in (0, T)$, and $\xi_n(x, t) \in \beta(\chi_n(x, t))$, $\zeta_n(x, t) \in \mathrm{H}(\chi_n(x, t))$ for a.e. $(x, t) \in \Gamma_c \times (0, T)$. For later convenience, we point out that, in view of (51) and (H12), for all $n \in \mathbb{N}$ and $T > 0$

$$\|\mathbf{F}_n\|_{L^\infty(0,T;W')} \le \|\mathbf{F}\|_{L^\infty(0,+\infty;W')}, \tag{87}$$

$$\|h_n\|_{L^\infty(0,T;V')} \le \|h\|_{L^\infty(0,+\infty;V')}, \qquad \|h_n\|_{L^1(0,T;H)} \le \|h\|_{L^1(0,+\infty;H)}\,. \tag{88}$$

Furthermore, (52) leads to

$$\int_0^T \|h_n\|_{V'}^2 = \int_{t_n}^{t_n+T} \|h\|_{V'}^2 \le \int_{t_n}^{+\infty} \|h\|_{V'}^2 \to 0 \quad \text{as } n \to \infty \tag{89}$$

whereby

$$h_n \to 0 \;\text{ in } L^2(0, T; V'), \quad h_n \rightharpoonup^* 0 \qquad \text{in } L^\infty(0, T; V'), \tag{90}$$

whereas, using (53), we verify that

$$\exists\, \mathbf{F}_\infty \in \mathbf{W}' : \quad \mathbf{F}_n(t) \to \mathbf{F}_\infty \;\text{ in } L^p(0, T; \mathbf{W}') \text{ for all } 1 \le p < \infty\,. \tag{91}$$

Now, $(\vartheta_n, \vartheta_{s,n}, \mathbf{u}_n, \chi_n, \boldsymbol{\eta}_n, \xi_n, \zeta_n)$ is a solution to Problem $(\mathbf{P})$ (with data h_n and $\mathbf{F}_n$ in place of h and $\mathbf{F}$) on the interval $(0, T)$, and, by construction, the functions ϑ_n, $\vartheta_{s,n}$, $\mathbf{u}_n$ and χ_n comply the initial conditions

$$\begin{aligned} &\vartheta_n(0) = \vartheta(t_n), \quad \mathbf{u}_n(0) = \mathbf{u}(t_n) \quad \text{in } \Omega, \\ &\vartheta_{s,n}(0) = \vartheta_s(t_n), \quad \chi_n(0) = \chi(t_n) \quad \text{in } \Gamma_c. \end{aligned} \tag{92}$$

Exploiting (86), we shall prove that the quadruple $(\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty)$ fulfils the stationary system (58) by passing to the limit as $n \to \infty$ in the abovementioned initial-boundary problem.

To this aim, in view of estimates (55) and (56) (indeed, we may suppose without loss of generality that, e.g., $t_n \ge 1$ and choose $\rho = \frac{12}{7}$ in (56b)), we remark that there exists a constant $K_4 > 0$, independent of $n \in \mathbb{N}$ and $T > 0$, such that

$$\begin{aligned}
&\|\vartheta_n\|_{L^\infty(0,T;V)} + \|\nabla\vartheta_n\|_{L^2(0,T;H)} \leq K_4\,,\\
&\|\partial_t \ln(\vartheta_n)\|_{L^2(0,T;V')} + \|\partial_t\vartheta_n\|_{L^2(0,T;L^{12/7}(\Omega))} \leq K_4\,,\\
&\|\vartheta_{s,n}\|_{L^\infty(0,T;H^1(\Gamma_c))} + \|\nabla\vartheta_{s,n}\|_{L^2(0,T;L^2(\Gamma_c))} \leq K_4\,,\\
&\|\partial_t \ln(\vartheta_{s,n})\|_{L^2(0,T;H^1(\Gamma_c)')} + \|\partial_t\vartheta_{s,n}\|_{L^2(0,T;L^{12/7}(\Gamma_c))} \leq K_4\,,\\
&\|\vartheta_n - \vartheta_{s,n}\|_{L^2(0,T;L^2(\Gamma_c))} \leq K_4\,,\\
&\|\mathbf{u}_n\|_{W^{1,\infty}(0,T;\mathbf{W})} + \|\partial_t\mathbf{u}_n\|_{L^2(0,T;\mathbf{W})} + \|\widehat{\alpha}(\mathbf{u}_n)\|_{L^\infty(0,T)} \leq K_4\,,\\
&\|\chi_n\|_{L^\infty(0,T;H^2(\Gamma_c))\cap W^{1,\infty}(0,T;L^2(\Gamma_c)} + \|\partial_t\chi_n\|_{L^2(0,T;H^1(\Gamma_c))}\\
&\qquad + \|\widehat{\beta}(\chi_n)\|_{L^\infty(0,T;L^1(\Gamma_c))} + \|\xi_n\|_{L^\infty(0,T;L^2(\Gamma_c))} \leq K_4\,.
\end{aligned} \tag{93}$$

Moreover, since $0 \leq \zeta_n \leq 1$ a.e. in $\Gamma_c \times (0,T)$ there holds

$$\|\zeta_n\|_{L^\infty(\Gamma_c\times(0,T))} \leq 1\,. \tag{94}$$

Finally, let us prove a further estimate independent of $n \in \mathbb{N}$ but depending on $T > 0$. Exploiting (93) (which also yields a bound, independent of $n \in \mathbb{N}$ and $T > 0$, for $\|\chi_n^+\mathbf{u}_n\|_{L^\infty(0,T;H^{-1/2}(\Gamma_c))}$) and (87), we argue by comparison in (42) and conclude that

$$\|\boldsymbol{\eta}_n\|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq K_5(T) \quad \text{for all } n \in \mathbb{N}. \tag{95}$$

Estimates (93)– (95), joint with standard weak-compactness arguments, the compactness results [18, Thm. 4, Cor. 5], and the Ascoli-Arzelà theorem in the framework of the weak topologies of $\mathbf{W}$ and $H^1(\Gamma_c)$, yield that there exist a subsequence of $\{(\vartheta_n, \vartheta_{s,n}, \mathbf{u}_n, \chi_n, \boldsymbol{\eta}_n, \xi_n, \zeta_n)\}$ (which we do not relabel), and functions $(\bar\vartheta, \bar\vartheta_s, \bar{\mathbf{u}}, \bar\chi, \bar{\boldsymbol{\eta}}, \bar\xi, \bar\zeta)$ for which the following convergences hold as $n \uparrow \infty$

$$\begin{aligned}
&\vartheta_n \rightharpoonup^* \bar\vartheta \quad \text{in } L^\infty(0,T;V) \cap H^1(0,T;L^{12/7}(\Omega)),\\
&\vartheta_n \to \bar\vartheta \quad \text{in } \mathrm{C}^0([0,T];H^{1-\rho}(\Omega)) \text{ for all } \rho \in (0,1),
\end{aligned} \tag{96}$$

$$\begin{aligned}
&\vartheta_{s,n} \rightharpoonup^* \bar\vartheta_s \quad \text{in } L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^{12/7}(\Gamma_c)),\\
&\vartheta_{s,n} \to \bar\vartheta_s \quad \text{in } \mathrm{C}^0([0,T];H^{1-\rho}(\Gamma_c)) \text{ for all } \rho \in (0,1),
\end{aligned} \tag{97}$$

$$\begin{aligned}
&\mathbf{u}_n \rightharpoonup \bar{\mathbf{u}} \quad \text{in } H^1(0,T;\mathbf{W})\\
&\mathbf{u}_n \to \bar{\mathbf{u}} \quad \text{in } \mathrm{C}^0([0,T];H^{1-\rho}(\Omega)^3) \quad \text{for all } \rho \in (0,1)\,,
\end{aligned} \tag{98}$$

$$\boldsymbol{\eta}_n \rightharpoonup \bar{\boldsymbol{\eta}} \quad \text{in } L^2(0,T;(H^{-1/2}(\Gamma_c))^3)\,, \tag{99}$$

$$\begin{aligned}
&\chi_n \rightharpoonup^* \bar\chi \quad \text{in } L^\infty(0,T;H^2(\Gamma_c)) \cap H^1(0,T;H^1(\Gamma_c)) \cap W^{1,\infty}(0,T;L^2(\Gamma_c)),\\
&\chi_n \to \bar\chi \quad \text{in } \mathrm{C}^0([0,T];H^{2-\rho}(\Gamma_c)) \quad \text{for all } \rho \in (0,2),
\end{aligned} \tag{100}$$

$$\xi_n \rightharpoonup^* \bar\xi \quad \text{in } L^\infty(0,T;L^2(\Gamma_c))\,, \tag{101}$$

$$\zeta_n \rightharpoonup^* \bar\zeta \quad \text{in } L^\infty(0,T;L^p(\Gamma_c)) \quad \text{for all } 1 \leq p < \infty\,. \tag{102}$$

Clearly, from the positivity of the sequences $\{\vartheta_n\}$ and $\{\vartheta_{s,n}\}$ we deduce that

$$\begin{aligned}
&\bar\vartheta(x,t) \geq 0 \qquad \text{for a.e. } (x,t) \in \Omega \times (0,T),\\
&\bar\vartheta_s(x,t) \geq 0 \qquad \text{for a.e. } (x,t) \in \Gamma_c \times (0,T)\,.
\end{aligned}$$

Furthermore, arguing in the same way as for (89)–(90), one sees that estimate (55f) for $\|\partial_t \ln(\vartheta)\|_{L^2(0,+\infty;V')}$ and $\|\partial_t \ln(\vartheta_{s,n})\|_{L^2(0,+\infty;H^1(\Gamma_c)')}$ entails that, as $n \to \infty$,

$$\partial_t \ln(\vartheta_n) \to 0 \quad \text{in } L^2(0,T;V'), \qquad \partial_t \ln(\vartheta_{s,n}) \to 0 \quad \text{in } L^2(0,T;H^1(\Gamma_c)'). \tag{103}$$

Analogously, (55d) implies that

$$\partial_t \mathbf{u}_n \to 0 \quad \text{in } L^2(0,T;\mathbf{W}), \tag{104}$$

and

$$\partial_t \chi_n \to 0 \quad \text{in } L^2(0,T;L^2(\Gamma_c)). \tag{105}$$

In the same way, in view of (56b) for ϑ_t and $\partial_t \vartheta_s$, we conclude that

$$\partial_t \vartheta_n \to 0 \quad \text{in } L^2(0,T;L^{12/7}(\Omega)), \qquad \partial_t \vartheta_{s,n} \to 0 \quad \text{in } L^2(0,T;L^{12/7}(\Gamma_c)). \tag{106}$$

Finally, from (55a) and (55c), we infer that

$$\begin{aligned} &(\vartheta_n - \vartheta_{s,n}) \to 0 \quad \text{in } L^2(0,T;L^2(\Gamma_c)), \\ &\nabla \vartheta_n \to 0 \quad \text{in } L^2(0,T;H), \quad \nabla \vartheta_{s,n} \to 0 \quad \text{in } L^2(0,T;L^2(\Gamma_c)). \end{aligned} \tag{107}$$

With (104) we get $\bar{\mathbf{u}}_t = \mathbf{0}$ a.e. in Ω, so that $\bar{\mathbf{u}}$ is constant in time in Ω. Hence,

$$\bar{\mathbf{u}}(x,t) = \bar{\mathbf{u}}(x,0) = \lim_{n\uparrow\infty} \mathbf{u}(x,t_n) = \mathbf{u}_\infty(x) \quad \text{for a.e. } x \in \Omega, \quad \text{for all } t \in [0,T],$$

and, analogously, from (105) and (106) we deduce for all $t \in [0,T]$,

$$\bar{\vartheta}(x,t) = \vartheta_\infty(x) \ \text{ for a.e. } x \in \Omega\,, \qquad \bar{\vartheta}_s(x,t) = \vartheta_{s,\infty}(x)\,,$$
$$\bar{\chi}(x,t) = \chi_\infty(x) \ \text{ for a.e. } x \in \Gamma_c\,.$$

Finally, owing to (107) we get that ϑ_∞ and $\vartheta_{s,\infty}$ are constant (as $\nabla\vartheta_\infty = \mathbf{0}$ and $\nabla\vartheta_{s,\infty} = \mathbf{0}$) and

$$\vartheta_\infty|_{\Gamma_c} = \vartheta_{s,\infty} \qquad \text{a.e. in } \Gamma_c\,. \tag{108}$$

We also point out that (98) and (102) yield, as $n \to \infty$,

$$\zeta_n |\mathbf{u}_n|^2 \rightharpoonup^* \bar{\zeta}|\bar{\mathbf{u}}|^2 \quad \text{in } L^\infty(0,T;L^{2-\rho}(\Gamma_c)) \text{ for all } 0<\rho<2. \tag{109}$$

Further, combining (106) with (93) and (H4), we easily conclude that $\partial_t\lambda(\chi_n) = \lambda'(\chi_n)\partial_t\chi_n \to 0$ in $L^2(0,T;L^{2-\rho}(\Gamma_c))$ for all $\rho \in (0,2)$. Arguing in the same way as in the proof of [4, Prop. 4.4], we use (96)–(100) and (103)–(109) to pass to the limit in Problem (**P**) and, taking into account (90)–(91), we conclude that, almost everywhere in $(0,T)$, the functions $(\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty, \bar{\boldsymbol{\eta}}, \bar{\xi}, \bar{\zeta})$ fulfil (58a) and

$$a(\mathbf{u}_\infty, \mathbf{v}) + \int_\Omega \vartheta_\infty \mathrm{div}(\mathbf{v}) + \int_{\Gamma_c} (\chi_\infty^+ \mathbf{u}_\infty + \bar{\boldsymbol{\eta}}) \cdot \mathbf{v} = \mathbf{F}_\infty \quad \forall \mathbf{v} \in \mathbf{W}\,, \tag{110}$$

$$-\Delta\chi_\infty + \bar{\xi} + \sigma'(\chi_\infty) = -\lambda'(\chi_\infty)\vartheta_{s,\infty} - \frac{1}{2}\bar{\zeta}|\mathbf{u}_\infty|^2 \quad \text{a.e. in } \Gamma_c\,, \tag{111}$$

joint with the no-flux boundary conditions (47). Combining convergences (100), (101), and (102) with the strong-weak closedness of the graphs β and H, we find that

$$\bar{\xi} \in \beta(\chi_\infty), \quad \bar{\zeta} \in \mathrm{H}(\chi_\infty) \qquad \text{a.e. in}\, \Gamma_c \times (0,T)\,, \tag{112}$$

while, in view of the maximal monotonicity of the operator (induced by) α on $L^2(0,T;(H^{-1/2}(\Gamma_c))^3)$ and of [1, Lemma 1.3, p. 42], we deduce that

$$\bar{\boldsymbol{\eta}} \in \alpha(\mathbf{u}_\infty) \quad \text{in } (H^{-1/2}(\Gamma_c))^3, \ \text{ a.e. in}\,(0,T) \tag{113}$$

from the inequality

$$\limsup_{n\uparrow\infty} \int_0^t \int_{\Gamma_c} \boldsymbol{\eta}_n \cdot \mathbf{u}_n \le \int_0^t \int_{\Gamma_c} \bar{\boldsymbol{\eta}} \cdot \mathbf{u}_\infty \quad \text{for all } t \in (0,T)\,,$$

which can be verified in the same way as in the proof of [4, Prop. 4.4].

Remark 12. Notice that $(\vartheta_\infty, \vartheta_{s,\infty}, \mathbf{u}_\infty, \chi_\infty)$ solve the stationary problem (58a)-(58c) associated with the evolution system (40)-(47). Moreover, we observe that (104)–(107) entail that dissipation vanishes in the limit as $t \to \infty$ (see (3) and (4)).

□

Proof of Corollary 1. For the sake of completeness, here we repeat the same argument developed in the proof of [3, Prop. 2.5]. We test the first of (58c) by $(\chi_\infty - m_*)$ and integrate on Γ_c. We obtain

$$\int_{\Gamma_c} |\nabla(\chi_\infty - m_*)|^2 + \int_{\Gamma_c} \xi_\infty(\chi_\infty - m_*)$$
$$= -\int_{\Gamma_c} \left(\sigma'(\chi_\infty) + \lambda'(\chi_\infty)\vartheta_{s,\infty} + \frac{1}{2}\zeta_\infty|\mathbf{u}_\infty|^2\right)(\chi_\infty - m_*) \leq 0\,,$$

the latter inequality due to (59b)–(59c) and the fact that $m_* \leq \chi_\infty \leq m^*$ a.e. in Γ_c. On the other hand, the second term on the left-hand side of the above inequality is non-negative by monotonicity, so that we deduce that $\nabla(\chi_\infty - m_*) \equiv 0$ a.e. in Γ_c. Thus, there exists some constant $\varrho \geq m_*$ such that $\chi_\infty \equiv \varrho$ a.e. in Γ_c. Now, integrating (58c) and again recalling (59b)–(59c), we find that

$$\int_{\Gamma_c} \xi_\infty < 0\,.$$

Hence, necessarily $\varrho = m_*$, and (60) ensues. □

4. Rigorous estimates.

Outlook: This section is devoted to the rigorous proof of Theorem 2.1 and of Proposition 1. Thus, in Sec. 4.2 we shall specify the variational problem (depending on two parameters $\varepsilon > 0$ and $\mu > 0$) approximating Problem (**P**), and outline the main steps of the proof of its global well-posedness. Hence, in Sec. 4.3 we shall prove some estimates on the approximate solutions, which are independent of the parameters ε and μ and in fact hold on $(0, +\infty)$. This shall enable us to pass to the limit in the approximate problem first as $\varepsilon \searrow 0$, and secondly as $\mu \searrow 0$, and to conclude the proof of Theorem 2.1 in Sec. 4.4, and of Proposition 1 in Sec. 4.5.

Since the (double) approximation procedure for Problem (**P**) strongly relies on the usage of Yosida regularizations of the nonlinear operators ln, α , β, and of the Heaviside operator H, in the following section we recapitulate some related preparatory results.

4.1. Recaps on Yosida regularizations.

Regularization of ln**:** For fixed $\mu > 0$, we denote by

$$\mathsf{r}_\mu := (\mathrm{Id} + \mu \ln)^{-1} : \mathbb{R} \to \mathbb{R} \tag{114}$$

the resolvent operator associated with the logarithm ln (where $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ is the identity function), and recall that $\mathsf{r}_\mu : \mathbb{R} \to (0, +\infty)$ is a contraction. With easy calculations, one sees that $\mathsf{r}_\mu(1) = 1$, so that, by contractivity, there holds

$$\mathsf{r}_\mu(x) \leq |x| + 2 \qquad \text{for all } x \in \mathbb{R}. \tag{115}$$

The Yosida regularization of ln is then defined by

$$\ln_\mu := \frac{\mathrm{Id} - \mathsf{r}_\mu}{\mu} : \mathbb{R} \to \mathbb{R}\,. \tag{116}$$

It follows from [9, Prop. 2.6] that for all $\mu > 0$ the function $\ln_\mu : \mathbb{R} \to \mathbb{R}$ is non-decreasing and Lipschitz continuous, with Lipschitz constant $1/\mu$.

For later convenience, as in [5] we also introduce the following function

$$\mathscr{I}_\mu(x) := \int_0^x s \ln'_\mu(s)\, \mathrm{d}s\,. \tag{117}$$

We point out that, since $\mathscr{I}_\mu$ is decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$, there holds for all $\mu > 0$

$$\mathscr{I}_\mu(x) \geq \mathscr{I}_\mu(0) = 0 \qquad \text{for all } x \in \mathbb{R}. \tag{118}$$

The following result collects some properties of $\ln_\mu$ and $\mathscr{I}_\mu$ which shall play a crucial role in the proof of the forthcoming Proposition 2.

Lemma 4.1. *The following inequalities hold:*

$$\exists\, \mu_* > 0 : \quad \forall\, \mu \in (0, \mu_*) \quad \forall\, x > 0 \quad \ln'_\mu(x) \leq \frac{2}{x}\,, \tag{119a}$$

$$\forall\, \mu > 0 \quad \forall\, x \in \mathbb{R} \quad \ln'_\mu(x) \geq \frac{1}{|x| + 2 + \mu}\,. \tag{119b}$$

As a consequence, $\mathscr{I}_\mu$ *satisfies*

$$\exists\, \mu_* > 0 : \quad \forall\, \mu \in (0, \mu_*) \quad \forall\, x \geq 0 \quad \mathscr{I}_\mu(x) \leq 2x\,, \tag{120a}$$

$$\exists\, C_1, C_2 > 0 : \quad \forall\, \mu > 0 \quad \forall\, x \in \mathbb{R} \quad \mathscr{I}_\mu(x) \geq C_1 |x| - C_2\,. \tag{120b}$$

Proof. **Ad (119).** Using the definitions (114) and (116) of r_μ and $\ln_\mu$ and repeating the calculations in the proof of [5, Lemma 4.2], it is possible to show that

$$\ln'_\mu(x) = \frac{1}{\mathsf{r}_\mu(x) + \mu} \qquad \text{for all } x \in \mathbb{R},$$

which, combined with (115), yields (119b). For the proof of (119a), which follows the very same lines, we directly refer to [5, Lemma 4.2].

Ad (120). Estimate (120a) is an immediate consequence of (119a) and of the definition of $\mathscr{I}_\mu$. We shall now prove (120b) for $x \geq 0$ (the inequality in the case $x < 0$ being completely analogous). Indeed, from the inequality

$$s \ln'_\mu(s) \geq 1 - \frac{\mu + 2}{s + \mu + 2} \qquad \text{for all } s \geq 0$$

(which is an immediate consequence of (119b)), we deduce that

$$\begin{aligned}\mathscr{I}_\mu(x) &\geq x - \int_0^x \frac{\mu + 2}{s + \mu + 2}\, \mathrm{d}s \\ &= x - (\mu + 2) \ln(x + \mu + 2) + (\mu + 2) \ln(\mu + 2) \geq C_1 x - C_2\,,\end{aligned}$$

for some suitable positive constants C_1 and C_2. □

Regularization of α: For fixed $\mu > 0$, we shall denote by

$$\alpha_\mu : (H^{1/2}(\Gamma_c))^3 \to (H^{-1/2}(\Gamma_c))^3 \quad \text{the } \mu\text{-Yosida regularization of } \alpha,$$

and recall that, by [1, Prop. II.1.1], the operator α_μ is single-valued, monotone, bounded and demi-continuous. Being $\widehat{\alpha}$ a non-negative functional by (H2), its primitive

$$\widehat{\alpha}_\mu(\mathbf{u}) := \min_{\mathbf{v} \in D(\alpha)} \left(\frac{\|\mathbf{u} - \mathbf{v}\|^2_{H^{1/2}(\Gamma_c)}}{2\mu} + \widehat{\alpha}(\mathbf{v}) \right) \qquad \text{for all } \mathbf{u} \in (H^{1/2}(\Gamma_c))^3$$

(which is Fréchet differentiable on $(H^{1/2}(\Gamma_c))^3$), fulfils

$$0 \le \widehat{\alpha}_\mu(\mathbf{u}) \le \widehat{\alpha}(\mathbf{u}) \qquad \text{for all } \mathbf{u} \in \overline{D(\alpha)}. \tag{121}$$

Regularization of β: We shall also use

$$\beta_\mu : \mathbb{R} \to \mathbb{R} \ \text{ the } \mu\text{-Yosida regularization of } \beta.$$

With straightforward calculations one verifies that, thanks to (H9), the Yosida approximation

$$\widehat{\beta}_\mu(x) := \min_{y \in \overline{D(\beta)}} \left(\frac{|y-x|^2}{2\mu} + \widehat{\beta}(y) \right) \qquad \text{for all } x \in \mathbb{R}$$

of $\widehat{\beta}$ verifies

$$\begin{cases} \forall\, R > 0 \quad \exists\, C_R > 0 \ \ \forall\, \mu > 0\ \forall\, x \ge 0 : \ \ \widehat{\beta}_\mu(x) \ge \frac{R}{2\mu R+1} x^2 - C_R\,, \\ \qquad\qquad\ \exists\, \overline{C} > 0 \ \ \forall\, \mu > 0\ \forall\, x < 0 : \ \ \widehat{\beta}_\mu(x) \ge \frac{x^2}{2\mu} - \overline{C}\,. \end{cases} \tag{122}$$

Regularization of H: For fixed $\mu > 0$, we shall denote by

$$\begin{aligned} &\mathrm{H}_\mu = \mathrm{p}_\mu' : \mathbb{R} \to \mathbb{R} \ \text{ the } \mu\text{-Yosida regularization of H}, \\ &\mathrm{p}_\mu : \mathbb{R} \to \mathbb{R} \ \text{ the } \mu\text{-Yosida approximation of } (\cdot)^+. \end{aligned}$$

In particular, it can be checked that

$$\mathrm{p}_\mu(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{x^2}{2\mu} & \text{if } 0 < x < \mu, \\ x - \dfrac{\mu}{2} & \text{if } x \ge \mu. \end{cases} \tag{123}$$

Using the definition of H_μ and p_μ, it is straightforward to verify that

$$0 \le \mathrm{H}_\mu(x) \le 1 \qquad \text{for all } x \in \mathbb{R}, \tag{124}$$

$$0 \le \mathrm{p}_\mu(x) \le (x)^+ \qquad \text{for all } x \in \mathbb{R}. \tag{125}$$

4.2. **Approximation of Problem (P).**

A double approximation procedure. Let ε, $\mu > 0$ be two strictly positive parameters. We consider the approximation of Problem (**P**) obtained in the following way:

1. we add to (40) the regularizing viscosity term $\varepsilon\mathcal{R}(\vartheta_t)$ and to (41) the viscosity term $\varepsilon\mathcal{R}_{\Gamma_c}(\partial_t\vartheta_s)$ ($\mathcal{R}$ and $\mathcal{R}_{\Gamma_c}$ being the Riesz operators introduced in Notation 2.1);
2. we replace the operators α in (42), β and H in (44) with their Yosida regularization $\alpha_\mu : \left(H^{1/2}(\Gamma_c)\right)^3 \to \left(H^{-1/2}(\Gamma_c)\right)^3$, $\beta_\mu : \mathbb{R} \to \mathbb{R}$, and $\mathrm{H}_\mu : \mathbb{R} \to \mathbb{R}$; accordingly, we replace the term $\chi^+\mathbf{u}$ in equation (42) by $\mathrm{p}_\mu(\chi)\mathbf{u}$;
3. both in (40) and in (41) we replace the logarithm ln with its Yosida regularization $\ln_\mu$.

Approximate initial data. In order to properly state our approximate problem, depending on the parameters $\varepsilon > 0$ and $\mu > 0$, we shall need some enhanced regularity on the initial data for ϑ and ϑ_s. The following result concerns the construction of sequences of suitable approximate initial data $\{\vartheta_\varepsilon^0\}$ and $\{\vartheta_{s,\varepsilon}^0\}$, which in fact depend on the parameter $\varepsilon > 0$ only.

Lemma 4.2. *Assume that the initial data ϑ_0 and ϑ_s^0 respectively comply with (24) and (25). Then,*

1. *there exists a sequence $\{\vartheta_\varepsilon^0\} \subset V$ such that for all $\varepsilon > 0$ and $\mu \in (0, \mu_*)$ (μ_* being as in Lemma 4.1) there hold*

$$\varepsilon^{1/2} \|\vartheta_\varepsilon^0\|_V \le \|\vartheta_0\|_{V'}\,, \tag{126a}$$

$$\int_\Omega \mathscr{I}_\mu(\vartheta_\varepsilon^0) \le 2\|\vartheta_0\|_{L^1(\Omega)}\,, \tag{126b}$$

$$\vartheta_\varepsilon^0 \to \vartheta_0 \quad \text{in } L^{\bar p}(\Omega) \text{ as } \varepsilon \searrow 0\,, \tag{126c}$$

$$\ln_\mu(\vartheta_\varepsilon^0) \to \ln_\mu(\vartheta_0) \quad \text{in } L^{\bar p}(\Omega) \text{ as } \varepsilon \searrow 0\,; \tag{126d}$$

2. *there exists $\{\vartheta_{s,\varepsilon}^0\} \subset H^1(\Gamma_c)$ such that for all $\varepsilon > 0$ and $\mu \in (0, \mu_*)$*

$$\varepsilon^{1/2} \|\vartheta_{s,\varepsilon}^0\|_{H^1(\Gamma_c)} \le \|\vartheta_s^0\|_{H^1(\Gamma_c)'}\,, \tag{127a}$$

$$\int_{\Gamma_c} \mathscr{I}_\mu(\vartheta_{s,\varepsilon}^0) \le 2\|\vartheta_s^0\|_{L^1(\Gamma_c)}\,, \tag{127b}$$

$$\vartheta_{s,\varepsilon}^0 \to \vartheta_s^0 \quad \text{in } L^{\bar q}(\Gamma_c) \text{ as } \varepsilon \searrow 0\,, \tag{127c}$$

$$\ln_\mu(\vartheta_{s,\varepsilon}^0) \to \ln_\mu(\vartheta_s^0) \text{ in } L^{\bar q}(\Gamma_c) \text{ as } \varepsilon \searrow 0\,. \tag{127d}$$

Proof. We shall carry out the construction of the sequence $\{\vartheta_\varepsilon^0\}$ only, the argument for $\{\vartheta_{s,\varepsilon}^0\}$ being completely analogous. For all ε, $\mu > 0$, we let $\vartheta_\varepsilon^0 \in V$ be the solution of the variational equation

$$\int_\Omega \vartheta_\varepsilon^0 v + \varepsilon^{1/2} \int_\Omega \nabla\vartheta_\varepsilon^0 \nabla v = \int_\Omega \vartheta_0 v \qquad \text{for all } v \in V \tag{128}$$

(notice that $\vartheta_\varepsilon^0 \in V$ is well-defined, thanks to (28)). Then, (126a) can be straightforwardly proved, together with

$$\vartheta_\varepsilon^0 \to \vartheta_0 \qquad \text{in } V' \text{ as } \varepsilon \searrow 0. \tag{129}$$

Furthermore, the maximum principle shows that, being $\vartheta_0 > 0$ a.e. in Ω thanks to the second of (24),

$$\vartheta_\varepsilon^0 \ge 0 \qquad \text{a.e. in } \Omega.$$

Then, testing (128) by 1, one obtains

$$\|\vartheta_\varepsilon^0\|_{L^1(\Omega)} = \|\vartheta_0\|_{L^1(\Omega)} \quad \text{for all } \varepsilon > 0. \tag{130}$$

Therefore, (126b) follows from combining (130) with (120a). Finally, to check (126c) we test (128) by $(\vartheta_\varepsilon^0)^{\bar p - 1}$ and with easy calculations deduce that $\|\vartheta_\varepsilon^0\|_{L^{\bar p}(\Omega)} \le \|\vartheta_0\|_{L^{\bar p}(\Omega)}$. Combining this estimate with (129) using the uniform convexity of $L^{\bar p}(\Omega)$, we conclude the strong convergence of $\{\vartheta_\varepsilon^0\}$ in $L^{\bar p}(\Omega)$. Hence, (126d) is a direct consequence of (126c) and the Lipschitz continuity of $\ln_\mu$. □

Variational formulation of the approximate problem. We thus obtain the following boundary value problem, which we directly state on the half-line $(0,+\infty)$ in view of the long-time a priori estimates of Proposition 2.

Problem $(\mathbf{P}_\varepsilon^\mu)$. Given a quadruple of initial data $(\vartheta_\varepsilon^0, \vartheta_{s,\varepsilon}^0, \mathbf{u}_0, \chi_0)$, $\mathbf{u}_0$ and χ_0 being as in (26)–(27), and ϑ_ε^0 and $\vartheta_{s,\varepsilon}^0$ fulfilling (126) and (127) respectively, find functions $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$, with the regularity

$$\vartheta \in H^1(0,T;V), \quad \vartheta_s \in H^1(0,T;H^1(\Gamma_c)), \quad \mathbf{u} \in H^1(0,T;\mathbf{W}),$$
$$\chi \in L^2(0,T;H^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c))$$

for all $T>0$, fulfilling the initial conditions

$$\vartheta(0) = \vartheta_\varepsilon^0 \quad \text{a.e. in}\,\Omega, \qquad \vartheta_s(0) = \vartheta_{s,\varepsilon}^0 \quad \text{a.e. in}\,\Gamma_c \tag{131}$$

and (38)–(39), and the equations

$$\begin{aligned} &\varepsilon\int_\Omega \vartheta_t\, v + \int_\Omega \partial_t \ln_\mu(\vartheta) v - \int_\Omega \operatorname{div}(\mathbf{u}_t)\, v + \varepsilon\int_\Omega \nabla\vartheta_t\,\nabla v + \int_\Omega \nabla\vartheta\,\nabla v \\ &\qquad + \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s)v = \langle h, v\rangle \quad \forall\, v\in V \ \text{ a.e. in } (0,+\infty), \end{aligned} \tag{132}$$

$$\begin{aligned} &\varepsilon\int_{\Gamma_c} \partial_t\vartheta_s\, v + \int_{\Gamma_c} \partial_t \ln_\mu(\vartheta_s) v + \varepsilon\int_{\Gamma_c} \nabla\partial_t\vartheta_s \nabla v \\ &-\int_{\Gamma_c} \partial_t\lambda(\chi)\, v + \int_{\Gamma_c} \nabla\vartheta_s\,\nabla v = \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s)v \\ &\qquad\qquad \forall\, v\in H^1(\Gamma_c) \ \text{ a.e. in } (0,+\infty), \end{aligned} \tag{133}$$

$$\begin{aligned} &b(\mathbf{u}_t,\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_\Omega \vartheta \operatorname{div}(\mathbf{v}) + \int_{\Gamma_c} (\mathrm{p}_\mu(\chi)\mathbf{u} + \alpha_\mu(\mathbf{u}))\cdot\mathbf{v} \\ &\qquad\qquad = {}_{\mathbf{W}'}\langle \mathbf{F},\mathbf{v}\rangle_{\mathbf{W}} \quad \forall \mathbf{v}\in\mathbf{W} \ \text{ a.e. in } (0,+\infty), \end{aligned} \tag{134}$$

$$\begin{aligned} &\chi_t - \Delta\chi + \beta_\mu(\chi) + \sigma'(\chi) = \\ &\qquad -\lambda'(\chi)\vartheta_s - \frac{1}{2}|\mathbf{u}|^2 \mathrm{H}_\mu(\chi) \quad \text{a.e. in } \Gamma_c\times(0,+\infty), \\ &\partial_{\mathbf{n}_s}\chi = 0 \ \text{a.e. in } \partial\Gamma_c\times(0,+\infty). \end{aligned} \tag{135}$$

Remark 13. Note that the approximate system (132)-(135) presents fewer technical difficulties than the analogous approximate version introduced in [4]. This is mainly due to the fact that, as we deal with the specific choice of $\ln(\vartheta)$ and $\ln(\vartheta_s)$ in the entropy (cf. Remark 1), we are allowed to directly introduce the Yosida regularization of the logarithm, instead of the more intricate approximating procedure exploited in [4].

4.2.1. *Outline of the proof of global well-posedness for Problem* $(\mathbf{P}_\varepsilon^\mu)$. Since Problem $(\mathbf{P}_\varepsilon^\mu)$ only slightly differs from the approximate problem considered in [4], the global well-posedness for Problem $(\mathbf{P}_\varepsilon^\mu)$, on any interval $(0,T)$, can be obtained arguing in the very same way as in [4, Sec. 3], to which we refer the reader for all details. Here, we shall just sketch the main steps of the proof.

Step 1.: First of all, one proves the existence of a local (in time) solution to (the Cauchy problem for) Problem $(\mathbf{P}_\varepsilon^\mu)$ with the use of a Schauder fixed point argument. This involves establishing intermediate well-posedness results for each of the approximate equations, for which we refer to the calculations developed in [4, Sec. 3.2], and defining a solution operator which complies with the conditions of the Schauder fixed point theorem (cf. [4, Sec. 3.3]).

Step 2.: Next, to extend the local solution to the whole interval $(0,T)$, one needs global (in time) a priori estimates. The latter substantially coincide with the ones formally performed in the proof of Proposition 1 and shall be repeated on the approximate system (132)–(135) within the proof of Proposition 2.

Step 3.: Finally, uniqueness of solutions to Problem ($\mathbf{P}_\varepsilon^\mu$) follows from the very same contraction estimates performed in [4, Sec. 3.5].

Remark 14. We briefly justify our construction of the approximate Problem ($\mathbf{P}_\varepsilon^\mu$), referring to [4, Rems. 3.2, 4.2] for more comments.

The viscosity terms $\varepsilon\mathcal{R}(\vartheta_t)$ and $\varepsilon\mathcal{R}_{|_{\Gamma_c}}(\partial_t\vartheta_s)$ have been inserted in (132) and (133), respectively, for technical reasons, related to the fixed point construction of a local solution for Problem ($\mathbf{P}_\varepsilon^\mu$). In particular, the contributions $\varepsilon\mathcal{R}(\vartheta_t)$ and $\varepsilon\mathcal{R}_{|_{\Gamma_c}}(\partial_t\vartheta_s)$ are essential to prove uniqueness of solutions to (the Cauchy problems for) approximate equations (132) and (133). Furthermore, they also play a crucial role to make the estimate leading to the further regularity (56a)–(56b) rigorous, see the ensuing proof of Proposition 2. For the same reason, we have replaced the maximal monotone operators ln, α and β by their Yosida regularizations, and correspondingly substituted the coupling terms $\chi^+\mathbf{u}$ and $-1/2\zeta|\mathbf{u}|^2$ in equations (42) and (44), with $\mathrm{p}_\mu(\chi)\mathbf{u}$ and $-1/2\mathrm{H}_\mu(\chi)|\mathbf{u}|^2$, respectively.

Now, as in the approximation of the system considered in [4], we shall keep the viscosity parameter ε distinct from the Yosida parameter μ in both approximate equations (132) and (133). Thus, we shall prove the existence of solutions to Problem ($\mathbf{P}$) by passing to the limit in Problem ($\mathbf{P}_\varepsilon^\mu$) first as $\varepsilon \searrow 0$ for $\mu > 0$ fixed, and then as $\mu \searrow 0$. This procedure shall enable us to recover on any interval $(0,T)$ the $L^\infty(0,T;H)$-regularity for $\ln(\vartheta)$ (the $L^\infty(0,T;L^2(\Gamma_c))$-regularity for $\ln(\vartheta_s)$, respectively) by testing the μ-approximation of (40) (of (41), respectively), by the term $\ln_\mu(\vartheta)$ ($\ln_\mu(\vartheta_s)$, resp.), and obtaining some bound independent of the approximation parameter μ. In fact, such an estimate may be performed on equation (132) (on (133), resp.) only when $\varepsilon = 0$. For, if one kept $\varepsilon > 0$, one would not obtain estimates on $\ln_\mu(\vartheta)$ independent of the parameters ε and μ, essentially because the term $\langle\varepsilon\mathcal{R}(\vartheta_t), \ln_\mu(\vartheta)\rangle$ ($\langle\varepsilon\mathcal{R}_{|_{\Gamma_c}}(\partial_t\vartheta_s), \ln_\mu(\vartheta_s)\rangle$, resp.) cannot be dealt with by monotonicity arguments.

4.3. Estimates on solutions to Problem ($\mathbf{P}_\varepsilon^\mu$).

Proposition 2. *Assume (H1)–(H5) and (H9)–(H12). Then,*

1. *there exists a constant $K_6 > 0$, only depending on the functions λ, k, σ, and on the quantity M (54), such that for all ε, $\mu > 0$ the following estimates hold*

for the family of solutions $\{(\vartheta_{\varepsilon\mu}, \vartheta_{s,\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \chi_{\varepsilon\mu})\}_{\varepsilon,\mu}$ *to Problem* $(\mathbf{P}_{\varepsilon}^{\mu})$*:*

$$\|\nabla\vartheta_{\varepsilon\mu}\|_{L^2(0,+\infty;H)} + \|\nabla\vartheta_{s,\varepsilon\mu}\|_{L^2(0,+\infty;L^2(\Gamma_c))} \leq K_6\,, \tag{136a}$$

$$\varepsilon^{1/2}\|\vartheta_{\varepsilon\mu}\|_{L^\infty(0,+\infty;V)} + \varepsilon^{1/2}\|\vartheta_{s,\varepsilon\mu}\|_{L^\infty(0,+\infty;H^1(\Gamma_c))} \leq K_6\,, \tag{136b}$$

$$\|\vartheta_{\varepsilon\mu}\|_{L^\infty(0,+\infty;L^1(\Omega))} + \|\vartheta_{s,\varepsilon\mu}\|_{L^\infty(0,+\infty;L^1(\Gamma_c))} \leq K_6\,, \tag{136c}$$

$$\|\vartheta_{\varepsilon\mu} - \vartheta_{s,\varepsilon\mu}\|_{L^2(0,+\infty;L^2(\Gamma_c))} \leq K_6\,, \tag{136d}$$

$$\|\partial_t\mathbf{u}_{\varepsilon\mu}\|_{L^2(0,+\infty;\mathbf{W})} + \|\mathbf{u}_{\varepsilon\mu}\|_{L^\infty(0,+\infty;\mathbf{W})} + \|\widehat{\alpha}_\mu(\mathbf{u}_{\varepsilon\mu})\|_{L^\infty(0,+\infty)} \leq K_6\,, \tag{136e}$$

$$\begin{aligned}&\|\chi_{\varepsilon\mu}\|_{L^\infty(0,+\infty;H^1(\Gamma_c))} + \|\partial_t\chi_{\varepsilon\mu}\|_{L^2(0,+\infty;L^2(\Gamma_c))}\\ &+ \|\widehat{\beta}_\mu(\chi_{\varepsilon\mu})\|_{L^\infty(0,+\infty;L^1(\Gamma_c))} + \|\mathrm{H}_\mu(\chi_{\varepsilon\mu})\|_{L^\infty(\Gamma_c\times(0,+\infty))} \leq K_6\,,\end{aligned} \tag{136f}$$

$$\|\partial_t \ln_\mu(\vartheta_{\varepsilon\mu})\|_{L^2(0,+\infty;V')} + \|\partial_t \ln_\mu(\vartheta_{s,\varepsilon\mu})\|_{L^2(0,+\infty;H^1(\Gamma_c)')} \leq K_6\,. \tag{136g}$$

2. *Furthermore, for all* $\delta > 0$ *there exist constants* $K_7(\delta)$, $K_8(\delta,\rho) > 0$*, depending on* δ*, on the functions* λ*,* k*,* σ*, and on the quantity* M (54) ($K_8(\delta,\rho)$ *on* $\rho \in (0,2)$ *as well), but independent of* $\varepsilon > 0$ *and* $\mu > 0$*, such that the following estimates hold*

$$\begin{aligned}&\|\vartheta_{\varepsilon\mu}\|_{L^\infty(\delta,+\infty;V)} + \|\vartheta_{s,\varepsilon\mu}\|_{L^\infty(\delta,+\infty;H^1(\Gamma_c))}\\ &+ \|\chi_{\varepsilon\mu}\|_{L^\infty(\delta,+\infty;H^2(\Gamma_c))} + \|\partial_t\chi_{\varepsilon\mu}\|_{L^2(\delta,+\infty;H^1(\Gamma_c))\cap L^\infty(\delta,+\infty;L^2(\Gamma_c))}\\ &+ \|\beta_\mu(\chi_{\varepsilon\mu})\|_{L^\infty(\delta,+\infty;L^2(\Gamma_c))} + \|\mathbf{u}_{\varepsilon\mu}\|_{W^{1,\infty}(\delta,+\infty;\mathbf{W})} \leq K_7(\delta)\,,\end{aligned} \tag{137a}$$

$$\|\partial_t\vartheta_{\varepsilon\mu}\|_{L^2(\delta,+\infty;L^{12/7}(\Omega))} + \|\partial_t\vartheta_{s,\varepsilon\mu}\|_{L^2(\delta,+\infty;L^{2-\rho}(\Gamma_c))} \leq K_8(\delta,\rho)\,, \tag{137b}$$

Proof. We shall prove (136) and (137) by performing on Problem $(\mathbf{P}_{\varepsilon}^{\mu})$ the same a priori estimates as in the proof of Proposition 1 and obtaining bounds independent of ε, $\mu > 0$ and of $t \in (0,+\infty)$.

First estimate. We test (132) by $\vartheta_{\varepsilon\mu}$, (133) by $\vartheta_{s,\varepsilon\mu}$, (134) by $\partial_t\mathbf{u}_{\varepsilon\mu}$, and (135) by $\partial_t\chi_{\varepsilon\mu}$, add the resulting relations and integrate them on the interval $(0,t)$, with $t \in (0,+\infty)$. Basically, we put forth the same calculations as throughout (61)–(67), up to the following changes. Instead of the formal identities (61), using the definition (117) of $\mathscr{I}_\mu$ we find

$$\begin{aligned}\int_0^t\int_\Omega \partial_t \ln_\mu(\vartheta_{\varepsilon\mu})\vartheta_{\varepsilon\mu} &= \int_0^t\int_\Omega \mathscr{I}_\mu'(\vartheta_{\varepsilon\mu})\partial_t\vartheta_{\varepsilon\mu}\\ &= \int_\Omega \mathscr{I}_\mu\left(\vartheta_{\varepsilon\mu}(t)\right) - \int_\Omega \mathscr{I}_\mu\left(\vartheta_{\varepsilon}^0\right)\\ &\geq C_1\|\vartheta_{\varepsilon\mu}(t)\|_{L^1(\Omega)} - C_2 - 2\|\vartheta_0\|_{L^1(\Omega)}\,,\end{aligned}$$

the last inequality ensuing from (120b) and (126b). In the same way, we have

$$\int_0^t\int_{\Gamma_c} \partial_t \ln_\mu(\vartheta_{s,\varepsilon\mu})\vartheta_{s,\varepsilon\mu} \geq C_1\|\vartheta_{s,\varepsilon\mu}(t)\|_{L^1(\Gamma_c)} - C_2 - 2\|\vartheta_s^0\|_{L^1(\Gamma_c)}\,,$$

whereas we estimate

$$\int_0^t \langle\varepsilon\mathcal{R}(\partial_t\vartheta_{\varepsilon\mu}), \vartheta_{\varepsilon\mu}\rangle = \frac{\varepsilon}{2}\|\vartheta_{\varepsilon\mu}(t)\|_V^2 - \frac{\varepsilon}{2}\|\vartheta_\varepsilon^0\|_V^2 \geq \frac{\varepsilon}{2}\|\vartheta_{\varepsilon\mu}(t)\|_V^2 - \frac{1}{2}\|\vartheta_0\|_{V'}^2$$

in view of (126a), and analogously for $\vartheta_{s,\varepsilon\mu}$. Then, we perform the same passages as in (62)–(63), up to replacing $\widehat{\alpha}$, $(\cdot)^+$, and $\widehat{\beta}$ with their approximations $\widehat{\alpha}_\mu$, p_μ, and $\widehat{\beta}_\mu$. Repeating the very same calculations as in (66)–(67) and taking into account some cancellations, as well as (126a) and (127a), we arrive at

$$
\begin{aligned}
&\frac{\varepsilon}{2}\|\vartheta_{\varepsilon\mu}(t)\|_V^2 + \|\vartheta_{\varepsilon\mu}(t)\|_{L^1(\Omega)} + \frac{1}{2}\int_0^t \|\nabla\vartheta_{\varepsilon\mu}\|_H^2 + \int_0^t\int_{\Gamma_c} k(\chi_{\varepsilon\mu})(\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu})^2 \\
&\frac{\varepsilon}{2}\|\vartheta_{s,\varepsilon\mu}(t)\|_{H^1(\Gamma_c)}^2 + \|\vartheta_s(t)\|_{L^1(\Gamma_c)} + \int_0^t \|\nabla\vartheta_s\|_{L^2(\Gamma_c)}^2 \\
&+ C_b\int_0^t \|\partial_t \mathbf{u}_{\varepsilon\mu}\|_{\mathbf{W}}^2 + \frac{C_a}{4}\|\mathbf{u}_{\varepsilon\mu}(t)\|_{\mathbf{W}}^2 + \widehat{\alpha}_\mu(\mathbf{u}_{\varepsilon\mu}(t)) + \frac{1}{2}\int_{\Gamma_c} \mathrm{p}_\mu(\chi_{\varepsilon\mu}(t))|\mathbf{u}_{\varepsilon\mu}(t)|^2 \\
&+ \int_0^t \|\partial_t\chi_{\varepsilon\mu}\|_{L^2(\Gamma_c)}^2 + \frac{1}{2}\|\nabla\chi_{\varepsilon\mu}(t)\|_{L^2(\Gamma_c)}^2 + \int_{\Gamma_c}\Big(\widehat{\beta}_\mu(\chi_{\varepsilon\mu}(t)) + \sigma(\chi_{\varepsilon\mu}(t))\Big) \\
&\leq C\Big(1 + \|\vartheta_0\|_{L^1(\Omega)} + \|\vartheta_0\|_{V'} + \|\vartheta_s^0\|_{L^1(\Gamma_c)} + \|\vartheta_s^0\|_{H^1(\Gamma_c)'} \\
&\qquad + \|\mathbf{u}_0\|_{\mathbf{W}}^2 + \widehat{\alpha}(\mathbf{u}_0) + \|\nabla\chi_0\|_{L^2(\Gamma_c)}^2 \\
&\qquad + \|\chi_0\|_{L^2(\Gamma_c)}\|\mathbf{u}_0\|_{\mathbf{W}}^2 + \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_c)} + \|\sigma(\chi_0)\|_{L^1(\Gamma_c)} + \int_0^t \|h\|_{V'}^2 \\
&\qquad + \|\mathbf{F}\|_{L^\infty(0,+\infty;\mathbf{W}')}^2\Big) + \frac{1}{|\Omega|^{1/2}}\int_0^t \|h\|_{V'}\|\vartheta\|_{L^1((\Omega)} + \int_0^t \|\mathbf{F}_t\|_{\mathbf{W}'}\|\mathbf{u}_{\varepsilon\mu}\|_{\mathbf{W}}\,.
\end{aligned}
$$

As in the proof of Proposition 1, from this inequality we deduce the bounds (136a)–(136c) and (136e)–(136f) (note that estimate (68) holds on this approximate level, too, thanks to (122), and the estimate for $\mathrm{H}_\mu(\chi_{\varepsilon\mu})$ trivially follows from (124)). Further, (136d) ensues from the bound

$$\|(k(\chi_{\varepsilon\mu}))^{1/2}(\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu})\|_{L^2(0,+\infty;L^2(\Gamma_c))} \leq C \tag{138}$$

and from (H5). Again, the bound (136g) for $\partial_t \ln_\mu(\vartheta_{\varepsilon\mu})$ and $\partial_t \ln_\mu(\vartheta_{s,\varepsilon\mu})$ is a consequence of the previous estimates and a comparison in equations (132) and (133).

Second estimate. We test (132) by $\tanh(\cdot)\partial_t\vartheta_{\varepsilon\mu}$, (133) by $\tanh(\cdot)\partial_t\vartheta_{s,\varepsilon\mu}$, differentiate (134) w.r.t. time and test it by $\tanh(\cdot)\partial_t\mathbf{u}_{\varepsilon\mu}$, and differentiate (135) and multiply it by $\tanh(\cdot)\partial_t\chi_{\varepsilon\mu}$. Notice that such an estimate is rigorous on this level, for $\partial_t\vartheta_{\varepsilon\mu} \in V$ and $\partial_t\vartheta_{s,\varepsilon\mu} \in H^1(\Gamma_c)$. Further, we sum the resulting relations and integrate in time. In place of using the formal identities (73), we have the following inequalities, which are direct consequences of (119b):

$$
\begin{aligned}
&\int_0^t\int_\Omega \partial_t \ln_\mu(\vartheta_{\varepsilon\mu}(\cdot,r))\tanh(r)\partial_t\vartheta_{\varepsilon\mu}(\cdot,r)\mathrm{d}r \\
&\quad = \int_0^t\int_\Omega \tanh(r)|\partial_t\vartheta_{\varepsilon\mu}(\cdot,r)|^2 \ln_\mu'(\vartheta_{\varepsilon\mu}(\cdot,r))\mathrm{d}r \\
&\quad \geq \int_0^t\int_\Omega \tanh(r)\frac{|\partial_t\vartheta_{\varepsilon\mu}(\cdot,r)|^2}{|\vartheta_{\varepsilon\mu}(\cdot,r)|+\mu+2}\mathrm{d}r \\
&\quad = 4\int_0^t\int_\Omega \tanh(r)|\partial_t\Theta_{\varepsilon\mu}^{1/2}(\cdot,r)|^2\mathrm{d}r\,,
\end{aligned}
\tag{139}
$$

and, analogously,

$$
\begin{aligned}
&\int_0^t\int_{\Gamma_c} \partial_t \ln_\mu(\vartheta_{s,\varepsilon\mu}(\cdot,r))\tanh(r)\partial_t\vartheta_{s,\varepsilon\mu}(\cdot,r)\mathrm{d}r \\
&\quad \geq 4\int_0^t\int_{\Gamma_c} \tanh(r)|\partial_t\Theta_{s,\varepsilon\mu}^{1/2}(\cdot,r)|^2\mathrm{d}r\,,
\end{aligned}
\tag{140}
$$

where we have set for all $\varepsilon,\ \mu>0$

$$\Theta_{\varepsilon\mu} := |\vartheta_{\varepsilon\mu}| + \mu + 2\,, \qquad \Theta_{s,\varepsilon\mu} := |\vartheta_{s,\varepsilon\mu}| + \mu + 2\,. \tag{141}$$

Then, we observe that the remaining computations are not affected by the factor $\tanh(\cdot)$ in a substantial way. In fact, the same calculations as in the proof of the second (formal) estimate in Proposition 1 go through, up to dealing with the integration by parts more carefully due to the presence of $\tanh(\cdot)$. In particular, we have

$$\begin{aligned}
&\int_0^t \int_\Omega \nabla\vartheta_{\varepsilon\mu}(\cdot,r)\nabla\left(\tanh(r)\partial_t\vartheta_{\varepsilon\mu}(\cdot,r)\right)\mathrm{d}r \\
&\qquad = \frac{1}{2}\int_\Omega \tanh(t)|\nabla\vartheta_{\varepsilon\mu}(t)|^2 - \frac{1}{2}\int_0^t \tanh'(r)\int_\Omega |\nabla\vartheta_{\varepsilon\mu}(\cdot,r)|^2\mathrm{d}r \\
&\qquad \geq \frac{\tanh(t)}{2}\int_\Omega |\nabla\vartheta_{\varepsilon\mu}(t)|^2 - \frac{K_6^2}{2}\,,
\end{aligned} \tag{142}$$

in which we have used the fact that $\tanh(0)=0$, that $0<\tanh'(r)\leq 1$ for all $r\in\mathbb{R}$, and estimate (136a). We handle the corresponding term for $\vartheta_{s,\varepsilon\mu}$ in the same way. Next, we estimate

$$\begin{aligned}
&\int_0^t \int_{\Gamma_c} k(\chi_{\varepsilon\mu}(\cdot,r))\tanh(r)(\vartheta_{\varepsilon\mu}(\cdot,r)-\vartheta_{s,\varepsilon\mu}(\cdot,r))\partial_t\left(\vartheta_{\varepsilon\mu}(\cdot,r)-\vartheta_{s,\varepsilon\mu}(\cdot,r)\right)\mathrm{d}r \\
&\qquad = \frac{\tanh(t)}{2}\int_{\Gamma_c} k(\chi_{\varepsilon\mu}(t))\left(\vartheta_{\varepsilon\mu}(t)-\vartheta_{s,\varepsilon\mu}(t)\right)^2 + I_{10} + I_{11}\,,
\end{aligned} \tag{143}$$

where, integrating by parts

$$I_{10} = -\frac{1}{2}\int_0^t\int_{\Gamma_c} \tanh'(r)k(\chi_{\varepsilon\mu}(\cdot,r))\left(\vartheta_{\varepsilon\mu}(\cdot,r)-\vartheta_{s,\varepsilon\mu}(\cdot,r)\right)^2 \geq -C \tag{144}$$

$$\begin{aligned}
I_{11} &= -\frac{1}{2}\int_0^t\int_{\Gamma_c} \tanh(r)k'(\chi_{\varepsilon\mu}(\cdot,r))\partial_t\chi_{\varepsilon\mu}(\cdot,r)\left(\vartheta_{\varepsilon\mu}(\cdot,r)-\vartheta_{s,\varepsilon\mu}(\cdot,r)\right)^2\mathrm{d}r \\
&\geq -\frac{L_k}{2}\int_0^t \|\partial_t\chi_{\varepsilon\mu}\|_{L^4(\Gamma_c)}\|\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu}\|_{L^4(\Gamma_c)}\|\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu}\|_{L^2(\Gamma_c)} \\
&\geq -\nu\int_0^t \|\partial_t\chi_{\varepsilon\mu}\|^2_{L^4(\Gamma_c)} - C_\nu\int_0^t \|\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu}\|^2_{L^4(\Gamma_c)}\|\vartheta_{\varepsilon\mu}-\vartheta_{s,\varepsilon\mu}\|^2_{L^2(\Gamma_c)}
\end{aligned} \tag{145}$$

the inequality in (144) due to (138), while the first passage in (145) ensues from (H5) and the second one from the Hölder and Young inequalities, ν being a positive constant to be chosen small enough. Furthermore, also taking into account (18), we have

$$\begin{aligned}
&\int_0^t \tanh(r)b\left(\partial^2_{tt}\mathbf{u}_{\varepsilon\mu}(\cdot,r),\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r)\right)\mathrm{d}r \\
&\qquad \geq \frac{C_b}{2}\tanh(t)\|\partial_t\mathbf{u}_{\varepsilon\mu}(t)\|^2_{\mathbf{W}} - \frac{1}{2}\int_0^t \tanh'(r)b\left(\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r),\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r)\right)\mathrm{d}r \\
&\qquad \geq \frac{C_b}{2}\tanh(t)\|\partial_t\mathbf{u}_{\varepsilon\mu}(t)\|^2_{\mathbf{W}} - \frac{K_b}{2}K_6^2\,,
\end{aligned} \tag{146}$$

the last passage due to (136e) and (18). In the same way, we find

$$\int_0^t \tanh(r)\partial_{tt}^2\chi_{\varepsilon\mu}(\cdot,r)\partial_t\chi_{\varepsilon\mu}(\cdot,r)\mathrm{d}r \geq \frac{\tanh(t)}{2}\|\partial_t\chi_{\varepsilon\mu}(t)\|^2_{L^2(\Gamma_c)} - \frac{K_6^2}{2}\,, \tag{147}$$

$$\begin{aligned}&\int_0^t\int_{\Gamma_c}\tanh(r)\nabla\left(\partial_t\chi_{\varepsilon\mu}(\cdot,r)\right)\nabla\left(\partial_t\chi_{\varepsilon\mu}(\cdot,r)\right)\mathrm{d}r\\ &\qquad = \int_0^t\tanh(r)\int_{\Gamma_c}|\partial_t(\nabla\chi_{\varepsilon\mu}(\cdot,r))|^2\mathrm{d}r\,,\end{aligned} \tag{148}$$

and we remark that

$$\begin{aligned}&\int_0^t\tanh(r)\langle\alpha'_\mu(\mathbf{u}_{\varepsilon\mu}(\cdot,r))\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r),\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r)\rangle\mathrm{d}r\geq 0\,,\\ &\int_0^t\tanh(r)\int_{\Gamma_c}\beta'_\mu(\chi_{\varepsilon\mu}(\cdot,r))|\partial_t\chi_{\varepsilon\mu}(\cdot,r)|^2\mathrm{d}r\geq 0\,.\end{aligned} \tag{149}$$

Finally, we perform the same computations as in (79)–(80), and observe that

$$\int_0^t\int_{\Gamma_c}\tanh(r)\partial_t\left(\mathrm{p}_\mu(\chi_{\varepsilon\mu}(\cdot,r))\mathbf{u}_{\varepsilon\mu}(\cdot,r)\right)\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r)\mathrm{d}r = I_{12}+I_{13}\,, \tag{150}$$

$$I_{12} = \int_0^t\int_{\Gamma_c}\tanh(r)\mathrm{p}_\mu(\chi_{\varepsilon\mu}(\cdot,r))|\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r)|^2\mathrm{d}r\geq 0\,, \tag{151}$$

$$\begin{aligned}I_{13} &= \int_0^t\tanh(r)\int_{\Gamma_c}\mathrm{H}_\mu(\chi_{\varepsilon\mu}(\cdot,r))\partial_t\chi_{\varepsilon\mu}(\cdot,r)\mathbf{u}_{\varepsilon\mu}(\cdot,r)\partial_t\mathbf{u}_{\varepsilon\mu}(\cdot,r)\mathrm{d}r\\ &\geq -C\int_0^t\|\partial_t\chi_{\varepsilon\mu}\|_{L^2(\Gamma_c)}\|\mathbf{u}_{\varepsilon\mu}\|_{\mathbf{W}}\|\partial_t\mathbf{u}_{\varepsilon\mu}\|_{\mathbf{W}}\geq -CK_6^3\,,\end{aligned} \tag{152}$$

the passages in (152) due to the Hölder inequality, the continuous embedding (15), and estimates (136e)–(136f). In the end, we point out that the analogues of (70)–(71) and (75)–(76) go through. Then, collecting (139)–(152) and putting forth the same arguments as in the proof of Proposition 1, we conclude estimate (137a) for all $\delta>0$. We also find (recall notation (141))

$$\|\partial_t\Theta_{\varepsilon\mu}^{1/2}\|_{L^2(\delta,+\infty;H)} + \|\partial_t\Theta_{s,\varepsilon\mu}^{1/2}\|_{L^2(\delta,+\infty;L^2(\Gamma_c))}\leq C_\delta\,,$$

and, arguing in the same way as in (84)–(85), we infer that for all $\rho\in(0,2)$

$$\|\partial_t\Theta_{\varepsilon\mu}\|_{L^2(\delta,+\infty;L^{12/7}(\Omega))} + \|\partial_t\Theta_{s,\varepsilon\mu}\|_{L^2(\delta,+\infty;L^{2-\rho}(\Gamma_c))}\leq C_{\delta,\rho}\,,$$

whence the bound (137b) ensues. □

4.4. **Proof of Theorem 2.1.**

4.4.1. *Passage to the limit in Problem* $(\mathbf{P}_\varepsilon^\mu)$ *as* $\varepsilon\searrow 0$. First, we introduce the boundary value problem obtained by taking $\varepsilon=0$ in Problem $(\mathbf{P}_\varepsilon^\mu)$, which we shall supplement with the initial data (24)–(27).

Problem $(\mathbf{P}^\mu)$. Given a quadruple of initial data $(\vartheta_0,\vartheta_s^0,\mathbf{u}_0,\chi_0)$ complying with (24)–(27), find functions $(\vartheta,\vartheta_s,\mathbf{u},\chi)$, $\mathbf{u}$ with the regularity (31), χ with (33), and, for all $T>0$,

$$\vartheta\in L^2(0,T;V)\cap L^\infty(0,T;L^1(\Omega)),\quad \ln_\mu(\vartheta)\in H^1(0,T;V')\,, \tag{153}$$

$$\vartheta_s\in L^2(0,T;H^1(\Gamma_c))\cap L^\infty(0,T;L^1(\Gamma_c)),\quad \ln_\mu(\vartheta_s)\in H^1(0,T;H^1(\Gamma_c)')\,, \tag{154}$$

fulfilling the initial conditions

$$
\begin{aligned}
&\ln_\mu(\vartheta(0)) = \ln_\mu(\vartheta_0) \ \text{ a.e. in}\,\Omega\,, \qquad \ln_\mu(\vartheta_s(0)) = \ln_\mu(\vartheta_s^0) \ \text{ a.e. in}\,\Gamma_c\,,\\
&\mathbf{u}(0)=\mathbf{u}_0 \ \text{ a.e. in}\,\Omega\,, \qquad \chi(0)=\chi_0 \ \text{ a.e. in}\,\Gamma_c\,,
\end{aligned}
\tag{155}
$$

the equations

$$
\begin{aligned}
\langle \partial_t \ln_\mu(\vartheta), v\rangle - \int_\Omega \operatorname{div}(\mathbf{u}_t)\, v + \int_\Omega \nabla\vartheta\,\nabla v + \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s)v&\\
= \langle h, v\rangle \quad \forall\, v\in V \ \text{ a.e. in } (0,+\infty)\,,&
\end{aligned}
\tag{156}
$$

$$
\begin{aligned}
\langle \partial_t \ln_\mu(\vartheta_s), v\rangle - \int_{\Gamma_c} \partial_t\lambda(\chi)\, v + \int_{\Gamma_c} \nabla\vartheta_s\,\nabla v&\\
= \int_{\Gamma_c} k(\chi)(\vartheta-\vartheta_s)v \quad \forall\, v\in H^1(\Gamma_c) \ \text{ a.e. in } (0,+\infty)\,,&
\end{aligned}
\tag{157}
$$

and such that $(\mathbf{u},\chi)$ comply with (134)–(135).

Notice that no uniqueness result is available for (the Cauchy problem for) Problem $(\mathbf{P}^\mu)$.

The following result is a straightforward consequence of Proposition 2.

Proposition 3. *Assume (H1)–(H5) and (H9)–(H12). Let $\mu>0$ be fixed. Then, there exists a (not relabeled) subsequence of $\{(\vartheta_{\varepsilon\mu},\vartheta_{s,\varepsilon\mu},\mathbf{u}_{\varepsilon\mu},\chi_{\varepsilon\mu})\}_\varepsilon$ and a quadruple $(\vartheta,\vartheta_s,\mathbf{u},\chi)$ such that for all $T>0$ as $\varepsilon\searrow 0$ the following convergences hold*

$$
\vartheta_{\varepsilon\mu} \rightharpoonup \vartheta \quad \text{in } L^2(0,T;V), \qquad \varepsilon\mathcal{R}(\vartheta_{\varepsilon\mu})\to 0 \quad \text{in } L^\infty(0,T;V'),
\tag{158a}
$$

$$
\begin{aligned}
&\vartheta_{s,\varepsilon\mu} \rightharpoonup \vartheta_s \quad \text{in } L^2(0,T;H^1(\Gamma_c)),\\
&\varepsilon\mathcal{R}_{\Gamma_c}(\vartheta_{s,\varepsilon\mu})\to 0 \quad \text{in } L^\infty(0,T;H^1(\Gamma_c)'),
\end{aligned}
\tag{158b}
$$

$$
\ln_\mu(\vartheta_{\varepsilon\mu}) \rightharpoonup \ln_\mu(\vartheta) \quad \text{in } L^2(0,T;V)\,,
\tag{158c}
$$

$$
\begin{aligned}
&\ln_\mu(\vartheta_{\varepsilon\mu}) \to \ln_\mu(\vartheta) \quad \text{in } \mathrm{C}^0([0,T];X)\\
&\qquad\qquad \text{for every Banach space } X \text{ with } V' \Subset X\,,
\end{aligned}
\tag{158d}
$$

$$
\varepsilon\mathcal{R}(\vartheta_{\varepsilon\mu}) + \ln_\mu(\vartheta_{\varepsilon\mu}) \rightharpoonup \ln_\mu(\vartheta) \quad \text{in } H^1(0,T;V')\,,
\tag{158e}
$$

$$
\ln_\mu(\vartheta_{s,\varepsilon\mu}) \rightharpoonup \ln_\mu(\vartheta_s) \quad \text{in } L^2(0,T;H^1(\Gamma_c))\,,
\tag{158f}
$$

$$
\begin{aligned}
&\ln_\mu(\vartheta_{s,\varepsilon\mu}) \to \ln_\mu(\vartheta_s) \quad \text{in } \mathrm{C}^0([0,T];Y)\\
&\qquad\qquad \text{for every Banach space } Y \text{ with } H^1(\Gamma_c)' \Subset Y\,,
\end{aligned}
\tag{158g}
$$

$$
\varepsilon\mathcal{R}_{\Gamma_c}(\vartheta_{s,\varepsilon\mu}) + \ln_\mu(\vartheta_{s,\varepsilon\mu}) \rightharpoonup \ln_\mu(\vartheta_s) \quad \text{in } H^1(0,T;H^1(\Gamma_c)')\,,
\tag{158h}
$$

$$
\begin{aligned}
&\chi_{\varepsilon\mu} \rightharpoonup^* \chi \quad \text{in } L^2(0,T;H^2(\Gamma_c))\cap L^\infty(0,T;H^1(\Gamma_c))\cap H^1(0,T;L^2(\Gamma_c)),\\
&\chi_{\varepsilon\mu}\to\chi \quad \text{in } L^2(0,T;H^{2-\rho}(\Gamma_c))\cap \mathrm{C}^0([0,T];H^{1-\rho}(\Gamma_c)) \quad \text{for all } \rho\in(0,1),
\end{aligned}
\tag{158i}
$$

$$
\begin{aligned}
&\mathrm{p}_\mu(\chi_{\varepsilon\mu}) \rightharpoonup^* \mathrm{p}_\mu(\chi) \quad \text{in } L^\infty(0,T;H^1(\Gamma_c))\cap H^1(0,T;L^2(\Gamma_c)),\\
&\mathrm{p}_\mu(\chi_{\varepsilon\mu}) \to \mathrm{p}_\mu(\chi) \quad \text{in } \mathrm{C}^0([0,T];H^{1-\rho}(\Gamma_c)) \quad \text{for all } \rho\in(0,1),
\end{aligned}
\tag{158j}
$$

$$
\beta_\mu(\chi_{\varepsilon\mu}) \rightharpoonup \beta_\mu(\chi) \quad \text{in } L^2(0,T;L^2(\Gamma_c)),
\tag{158k}
$$

$$
\mathrm{H}_\mu(\chi_{\varepsilon\mu}) \rightharpoonup^* \mathrm{H}_\mu(\chi) \quad \text{in } L^\infty(0,T;L^p(\Gamma_c)) \quad \text{for all } 1\le p<\infty,
\tag{158l}
$$

$$
\alpha_\mu(\mathbf{u}_{\varepsilon\mu}) \rightharpoonup \alpha_\mu(\mathbf{u}) \quad \text{in } L^2(0,T;\mathbf{W}'),
\tag{158m}
$$

$$
\begin{aligned}
&\mathbf{u}_{\varepsilon\mu} \rightharpoonup \mathbf{u} \quad \text{in } H^1(0,T;\mathbf{W})\\
&\mathbf{u}_{\varepsilon\mu} \to \mathbf{u} \quad \text{in } \mathrm{C}^0([0,T];H^{1-\rho}(\Omega)^3) \quad \text{for all } \rho\in(0,1)\,,
\end{aligned}
\tag{158n}
$$

(the symbol $\Subset$ in (158d) and (158g) signifying compact inclusion), and $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ is a solution to Problem $(\mathbf{P}^\mu)$.

Furthermore, $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ has the additional regularity (49a)–(49c) and (49e) on every interval (δ, T), for all $0 < \delta < T$ and, up to the extraction of a further subsequence, for $\{(\vartheta_{\varepsilon\mu}, \vartheta_{s,\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \chi_{\varepsilon\mu})\}_\varepsilon$ the enhanced convergences hold as $\varepsilon \searrow 0$ for all $0 < \delta < T$

$$\begin{aligned} &\vartheta_{\varepsilon\mu} \rightharpoonup^* \vartheta \quad \text{in } L^\infty(\delta, T; V) \cap H^1(\delta, T; L^{12/7}(\Omega)), \\ &\vartheta_{\varepsilon\mu} \to \vartheta \quad \text{in } \mathrm{C}^0([\delta, T]; H^{1-\rho}(\Omega)) \text{ for all } \rho \in (0,1), \end{aligned} \tag{159a}$$

$$\begin{aligned} &\vartheta_{s,\varepsilon\mu} \rightharpoonup^* \vartheta_s \quad \text{in } L^\infty(\delta, T; H^1(\Gamma_c)) \cap H^1(\delta, T; L^{2-\rho}(\Gamma_c)) \text{ for all } \rho \in (0,2) \\ &\vartheta_{s,\varepsilon\mu} \to \vartheta_s \quad \text{in } \mathrm{C}^0([\delta, T]; H^{1-\rho}(\Gamma_c)) \text{ for all } \rho \in (0,1), \end{aligned} \tag{159b}$$

$$\mathbf{u}_{\varepsilon\mu} \rightharpoonup^* \mathbf{u} \quad \text{in } W^{1,\infty}(\delta, T; \mathbf{W}), \tag{159c}$$

$$\begin{aligned} &\chi_{\varepsilon\mu} \rightharpoonup^* \chi \quad \text{in } L^\infty(\delta, T; H^2(\Gamma_c)) \cap H^1(\delta, T; H^1(\Gamma_c)) \cap W^{1,\infty}(\delta, T; L^2(\Gamma_c)), \\ &\chi_{\varepsilon\mu} \to \chi \quad \text{in } \mathrm{C}^0([\delta, T]; H^{2-\rho}(\Gamma_c)) \quad \text{for all } \rho \in (0,2), \end{aligned} \tag{159d}$$

$$\beta_\mu(\chi_{\varepsilon\mu}) \rightharpoonup^* \beta_\mu(\chi) \quad \text{in } L^\infty(\delta, T; L^2(\Gamma_c)). \tag{159e}$$

Sketch of the proof. Since the proof of this result substantially goes along the same lines as the proof of [4, Prop. 4.4], we shall just outline its main steps, referring to [4] for all details.

- First of all, using estimates (136) and standard compactness results (cf. [18]), with a diagonalization procedure we extract from $\{(\vartheta_{\varepsilon\mu}, \vartheta_{s,\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \chi_{\varepsilon\mu})\}_\varepsilon$ a subsequence for which convergences (158a)–(158n) hold as $\varepsilon \searrow 0$. In order to show that $\vartheta \in L^\infty(0, T; L^1(\Omega))$, as in the proof of [4, Thm. 1] we exploit a *Lebesgue point argument*. Indeed, using the first of (158a) and the convexity of $|\cdot|$, we find that for all $t_0 \in (0, T)$ and $r > 0$ such that $(t_0 - r, t_0 + r) \subset (0, T)$
$$\begin{aligned} \int_{t_0 - r}^{t_0 + r} \int_\Omega |\vartheta| &\leq \liminf_{\varepsilon \searrow 0} \left(\int_{t_0 - r}^{t_0 + r} \int_\Omega |\vartheta_{\varepsilon\mu}| \right) \\ &\leq 2r \sup_{\varepsilon > 0} \|\vartheta_{\varepsilon\mu}\|_{L^\infty(0,T;L^1(\Omega))} \leq 2r K_6 , \end{aligned} \tag{160}$$
the latter inequality due to (136c). We now divide the above relation by r and let $r \downarrow 0$. Using that the Lebesgue point property holds at almost every $t_0 \in (0, T)$, we obtain for all $T > 0$ the estimate
$$\|\vartheta\|_{L^\infty(0,T;L^1(\Omega))} \leq K_6 . \tag{161}$$
A completely analogous argument may be developed for proving that $\vartheta_s \in L^\infty(0, T; L^1(\Gamma_c))$.
- The identification of the limits of the sequences $\{\beta_\mu(\chi_{\varepsilon\mu})\}$ and $\{\mathrm{H}_\mu(\chi_{\varepsilon\mu})\}$ follows from the strong-weak closedness of the graphs of the operators β_μ, and H_μ. As for the limits of $\{\alpha_\mu(\mathbf{u}_{\varepsilon\mu})\}$, $\{\ln_\mu(\vartheta_{\varepsilon\mu})\}$ and $\{\ln_\mu(\vartheta_{s,\varepsilon\mu})\}$, thanks to [1, Lemma 1.3, p. 42], it is sufficient to prove that for all $t > 0$
$$\begin{aligned} &\limsup_{\varepsilon \searrow 0} \int_0^t \int_\Omega \ln_\mu(\vartheta_{\varepsilon\mu}) \vartheta_{\varepsilon\mu} \leq \int_0^t \int_\Omega \ln_\mu(\vartheta) \vartheta , \\ &\limsup_{\varepsilon \searrow 0} \int_0^t \int_{\Gamma_c} \ln_\mu(\vartheta_{s,\varepsilon\mu}) \vartheta_{s,\varepsilon\mu} \leq \int_0^t \int_\Omega \ln_\mu(\vartheta_s) \vartheta_s , \\ &\limsup_{\varepsilon \searrow 0} \int_0^t \int_{\Gamma_c} \alpha_\mu(\mathbf{u}_{\varepsilon\mu}) \cdot \mathbf{u}_{\varepsilon\mu} \leq \int_0^t \int_{\Gamma_c} \alpha_\mu(\mathbf{u}) \cdot \mathbf{u} . \end{aligned}$$

Convergences (158j) for $\{\mathrm{p}_\mu(\chi_{\varepsilon\mu})\}$ ensue from the corresponding (158i) for $\{\chi_{\varepsilon\mu}\}$ and from the fact that $0 \le \mathrm{p}'_\mu(r) = \mathrm{H}_\mu(r) \le 1$ for all $r \in \mathbb{R}$, cf. (124). Notice that the pair $(\mathbf{u}, \chi)$ complies with the initial conditions (38)–(39) in view of convergences (158n) and (158i). In the same way, combining convergences (126d) and (127d) of the sequences $\{\ln_\mu(\vartheta^0_\varepsilon)\}$, $\{\ln_\mu(\vartheta^0_{s,\varepsilon})\}$, with convergences (158d) and (158g), respectively, we conclude the initial conditions (155) for $\ln_\mu(\vartheta)$ and $\ln_\mu(\vartheta_s)$. Finally, we refer to the proof of [4, Prop. 4.4] for the conclusion of the argument showing that $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ is a solution to Problem $(\mathbf{P}^\mu)$.

- The further regularity (49a)–(49c) and (49e) for $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ and the enhanced convergences (159) ensue from estimates (137), see also the proof of Theorem 2.2.

□

In view of the definition of *approximable solution* to Problem $(\mathbf{P})$ which we shall give in Section 4.5, we select, among solutions of Problem $(\mathbf{P}^\mu)$, only the ones just constructed by passing to the limit in Problem $(\mathbf{P}^\mu_\varepsilon)$ as the viscosity parameter ε vanishes. Albeit improperly, within the scope of this section we shall refer to them as *viscosity solutions*.

Definition 4.3 (Viscosity solutions of Problem $(\mathbf{P}^\mu)$)**.** Let $\mu > 0$ be fixed. We say that a quadruple $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ is a *viscosity solution* of Problem $(\mathbf{P}^\mu)$ if

1. it is a solution to Problem $(\mathbf{P}^\mu)$;
2. there exist a sequence $\varepsilon_k \searrow 0$ and a family $\{(\vartheta_{\varepsilon_k\mu}, \vartheta_{s,\varepsilon_k\mu}, \mathbf{u}_{\varepsilon_k\mu}, \chi_{\varepsilon_k\mu})\}_k$ of solutions of Problem $(\mathbf{P}^\mu_{\varepsilon_k})$ converging as $\varepsilon_k \searrow 0$ to $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ in the sense specified by (158a)–(158n), on every interval $(0, T)$.

In particular, it follows from the last part of Proposition 3 that every *viscosity solution* to Problem $(\mathbf{P}^\mu)$ has the further regularity (49a)–(49c) and (49e).

4.4.2. *Passage to the limit as* $\mu \searrow 0$.

Proposition 4 (Estimates on viscosity solutions of Problem $(\mathbf{P}^\mu)$)**.** *Assume* (H1)–(H5) *and* (H9)–(H12)*. Then,*

1. *estimates* (136) *on the half-line* $(0, +\infty)$*,* (137a) *on* $(\delta, +\infty)$ *for all* $\delta > 0$*, and* (137b) *on* $(\delta, +\infty)$ *for all* $\delta > 0$ *and* $\rho \in (0, 2)$*, hold, with the same constants* K_6*,* $K_7(\delta)$*, and* $K_8(\delta, \rho)$*, for all* $\mu > 0$ *and for every viscosity solution* $\{(\vartheta_\mu, \vartheta_{s,\mu}, \mathbf{u}_\mu, \chi_\mu)\}$ *to Problem* $(\mathbf{P}^\mu)$*.*
2. *Furthermore, for all* $T > 0$ *there exists a constant* $K_9(T)$*, only depending on* T*, on the quantity* M (54)*, on the functions* λ*,* k*, and* σ*, as well as on* $\|\ln(\vartheta_0)\|_H$ *and* $\|\ln(\vartheta^0_s)\|_{L^2(\Gamma_c)}$*, such that for all* $\mu > 0$

$$\|\ln_\mu(\vartheta_\mu)\|_{L^\infty(0,T;H)} + \|\ln_\mu(\vartheta_{s,\mu})\|_{L^\infty(0,T;L^2(\Gamma_c))} \le K_9(T)\,. \tag{162}$$

As already mentioned in Remark 14, the calculations leading to (162) cannot be performed on the solutions of Problem $(\mathbf{P}^\mu_\varepsilon)$, with $\varepsilon > 0$.

Sketch of the proof. The first part of the statement is an obvious consequence of the Definition (4.3) of viscosity solution, of estimates (136) and (137), and of a trivial lower-semicontinuity argument. The enhanced estimate (162) for $\ln_\mu(\vartheta_\mu)$ and $\ln_\mu(\vartheta_{s,\mu})$ is obtained by testing (156) by $\ln_\mu(\vartheta_\mu)$, (157) by $\ln_\mu(\vartheta_{s,\mu})$, adding the resulting relations and integrating on time. We refer to [4, Sec. 4.2] for all details.

Here, we just point out that, being $\ln(\vartheta_0) \in H$, thanks to (155) there holds

$$\| \ln_\mu(\vartheta_\mu(0))\|_H = \| \ln_\mu(\vartheta_0)\|_H \leq \| \ln(\vartheta_0)\|_H \,,$$

and the analogous bound for $\| \ln_\mu(\vartheta_{s,\mu}(0))\|_{L^2(\Gamma_c)}$ ensues from the fact that $\ln(\vartheta_s^0) \in L^2(\Gamma_c)$. Such estimates are then used in the calculations yielding (162). □

Thus, we have the following proposition, which concludes the proof of Theorem 2.1.

Proposition 5. *Assume (H1)–(H5) and (H9)–(H12). Let $\{(\vartheta_\mu, \vartheta_{s,\mu}, \mathbf{u}_\mu, \chi_\mu)\}_\mu$ be a sequence of* viscosity solutions *to Problem $(\mathbf{P}^\mu)$. Then, there exists a (not relabeled) subsequence of $\{(\vartheta_\mu, \vartheta_{s,\mu}, \mathbf{u}_\mu, \chi_\mu)\}_\mu$ and functions $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$ such that for all $T > 0$ the following convergences hold as $\mu \searrow 0$*

$$\vartheta_\mu \rightharpoonup \vartheta \quad \text{in } L^2(0,T;V), \tag{163a}$$

$$\vartheta_{s,\mu} \rightharpoonup \vartheta_s \quad \text{in } L^2(0,T;H^1(\Gamma_c)), \tag{163b}$$

$$\ln_\mu(\vartheta_\mu) \rightharpoonup^* \ln(\vartheta) \quad \text{in } L^\infty(0,T;H) \cap H^1(0,T;V')\,, \tag{163c}$$

$$\begin{aligned} &\ln_\mu(\vartheta_\mu) \to \ln(\vartheta) \quad \text{in } \mathrm{C}^0([0,T];X) \\ &\qquad\qquad \text{for every Banach space } X \text{ with } H \Subset X\,, \end{aligned} \tag{163d}$$

$$\ln_\mu(\vartheta_{s,\mu}) \rightharpoonup^* \ln(\vartheta_s) \quad \text{in } L^\infty(0,T;L^2(\Gamma_c)) \cap H^1(0,T;H^1(\Gamma_c)')\,, \tag{163e}$$

$$\begin{aligned} &\ln_\mu(\vartheta_{s,\mu}) \to \ln(\vartheta_s) \quad \text{in } \mathrm{C}^0([0,T];Y) \\ &\qquad\qquad \text{for every Banach space } Y \text{ with } L^2(\Gamma_c) \Subset Y\,, \end{aligned} \tag{163f}$$

$$\begin{aligned} &\chi_\mu \rightharpoonup^* \chi \quad \text{in } L^2(0,T;H^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c)), \\ &\chi_\mu \to \chi \quad \text{in } L^2(0,T;H^{2-\rho}(\Gamma_c)) \cap \mathrm{C}^0([0,T];H^{1-\rho}(\Gamma_c)) \quad \text{for all } \rho \in (0,1), \end{aligned} \tag{163g}$$

$$\begin{aligned} &\mathrm{p}_\mu(\chi_\mu) \rightharpoonup^* \chi^+ \quad \text{in } L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c)), \\ &\mathrm{p}_\mu(\chi_\mu) \to \chi^+ \quad \text{in } \mathrm{C}^0([0,T];H^{1-\rho}(\Gamma_c)) \quad \text{for all } \rho \in (0,1), \end{aligned} \tag{163h}$$

$$\beta_\mu(\chi_\mu) \rightharpoonup \xi \quad \text{in } L^2(0,T;L^2(\Gamma_c)), \qquad \xi \in \beta(\chi) \quad \text{a.e. in } \Gamma_c \times (0,T), \tag{163i}$$

$$\begin{aligned} &\mathrm{H}_\mu(\chi_\mu) \rightharpoonup^* \zeta \quad \text{in } L^\infty(0,T;L^p(\Gamma_c)) \quad \text{for all } 1 \leq p < \infty, \\ &\zeta \in \mathrm{H}(\chi) \quad \text{a.e. in } \Gamma_c \times (0,T), \end{aligned} \tag{163j}$$

$$\alpha_\mu(\mathbf{u}_\mu) \rightharpoonup \boldsymbol{\eta} \quad \text{in } L^2(0,T;\mathbf{W}'), \qquad \boldsymbol{\eta} \in \alpha(\mathbf{u}) \ \text{a.e. in } (0,T), \tag{163k}$$

$$\begin{aligned} &\mathbf{u}_\mu \rightharpoonup \mathbf{u} \quad \text{in } H^1(0,T;\mathbf{W}) \\ &\mathbf{u}_\mu \to \mathbf{u} \quad \text{in } \mathrm{C}^0([0,T];H^{1-\rho}(\Omega)^3) \quad \text{for all } \rho \in (0,1)\,, \end{aligned} \tag{163l}$$

and $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$ is a solution to Problem $(\mathbf{P})$.

Furthermore, the functions ϑ, ϑ_s, $\mathbf{u}$, χ, and ξ have the additional regularity (49) on every interval (δ, T), for all $0 < \delta < T$ and, up to the extraction of another subsequence, for $\{(\vartheta_\mu, \vartheta_{s,\mu}, \mathbf{u}_\mu, \chi_\mu)\}_\mu$ the enhanced convergences (159) hold for all $0 < \delta < T$ as $\mu \searrow 0$.

Sketch of the proof. The proof follows the very same lines as the argument for Proposition 3 (cf. also the proof of [4, Thm. 1]). We just point out that convergences (163h) ensue from the pointwise convergence $\mathrm{p}_\mu(\chi_\mu) \to \chi^+$ a.e. in $\Gamma_c \times (0,T)$ (which can be deduced from (163g), exploiting formula (123)), and again from the bound (124) on p_μ'. Furthermore, (163c)–(163f) are a consequence of the additional estimate (162). Finally, as in the proof of Proposition 3, the strong convergences (163d) and (163f), as well as the Yosida convergences $\ln_\mu(\vartheta_0) \rightharpoonup \ln(\vartheta_0)$ in H

and $\ln_\mu(\vartheta_s^0) \rightharpoonup \ln(\vartheta_s^0)$ in $L^2(\Gamma_c)$ as $\mu \searrow 0$, enable us to pass to the limit in initial conditions (155) and deduce that

$$\ln(\vartheta(0)) = \ln(\vartheta_0) \text{ a.e. in } \Omega, \qquad \ln(\vartheta_s(0)) = \ln(\vartheta_s^0) \text{ a.e. in } \Gamma_c\,,$$

whence initial conditions (36) and (37). □

Remark 15. Notice that, in the statement of Proposition (5), assumptions (H10)–(H12) on the data $\mathbf{f}$, g, and h are stronger than the data requirements in Theorem 2.1, and we assume the additional condition (H9) on $\widehat{\beta}$. This is due to the fact that, to avoid unnecessary repetitions, we have chosen to unify in the present section the proof of global existence with the proof of the long-time estimates.

In fact, a closer perusal of the proof of Theorem 2.1 and a comparison with the argument for [4, Thm. 1] show that the sole (H1) on β and (H6)–(H8) on the data h, $\mathbf{f}$, and g are sufficient for the global existence of solutions to Problem (**P**) on the finite-time interval $(0, T)$. Further, under assumptions (48) (which are the finite-time versions of (H10)–(H12)) one proves the further regularity (49) on the interval $(0, T)$. In particular, condition (H9) on $\widehat{\beta}$ is not necessary for the aforementioned finite-time results.

4.5. **Rigorous proof of Proposition 1.** We are now in the position of specifying approximable solutions to Problem (**P**) as the ones arising as limits of viscosity solutions to Problem ($\mathbf{P}^\mu$).

Definition 4.4 (Approximable solutions of Problem (**P**))**.** We say that a seven-tuple $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$ is an *approximable solution* of Problem (**P**) if

1. it is a solution to Problem (**P**),
2. there exist a sequence $\mu_k \searrow 0$ and a family $\{(\vartheta_{\mu_k}, \vartheta_{s,\mu_k}, \mathbf{u}_{\mu_k}, \chi_{\mu_k})\}_k$ of viscosity solutions of Problem ($\mathbf{P}^{\mu_k}$) converging, as $\mu_k \searrow 0$, to $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$, in the sense specified by (163a)–(163l), on every interval $(0, T)$.

Remark 16. Taking into account Remark 15, we might state Theorem 2.1 in the following more precise way: under assumptions (H1)–(H8), on every interval $(0, T)$ Problem (**P**) admits at least an approximable solution $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$. If, in addition, (48) holds, then every approximable solution has the further regularity (49).

Rigorous proof of Proposition 1. It follows from the Definition 4.4 of approximable solution, from the estimates on viscosity solutions to Problem ($\mathbf{P}^\mu$) specified in Proposition 4, and from elementary lower-semicontinuity arguments. □

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