A vanishing viscosity approach to a rate-independent damage model

Dorothee Knees [*] Riccarda Rossi[†] Chiara Zanini[‡]

Abstract

We analyze a rate-independent model for damage evolution in elastic bodies. The central quantities are a stored energy functional and a dissipation functional, which is assumed to be positively homogeneous of degree one. Since the energy is not simultaneously (strictly) convex in the damage variable and the displacements, solutions may have jumps as a function of time. The latter circumstance makes it necessary to recur to suitable notions of weak solution. However, the by-now classical concept of global energetic solution fails to describe accurately the behavior of the system at jumps.

Hence, we consider rate-independent damage models as limits of systems driven by viscous, rate-dependent dissipation. We use a technique for taking the vanishing viscosity limit, which is based on arc-length reparameterization. In this way, in the limit we obtain a novel formulation for the rate-independent damage model, which highlights the interplay of viscous and rate-independent effects in the jump regime, and provides a better description of the energetic behavior of the system at jumps.

1 Introduction

In this paper, we focus on the modeling of damage in an elastic body $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, during a time interval $[0, T]$, as a *rate-independent, activated* process. The phenomenon is described in terms of a *damage* parameter $z : \Omega \times [0, T] \to \mathbb{R}$, assessing the soundness of the material: usually, z takes values in the interval $[0, 1]$, and one has $z(x, t) = 0$ ($z(x, t) = 1$, respectively), when the system at the process time $t \in [0, T]$ is fully damaged (completely sound), "locally" around $x \in \Omega$. The driving energy is a function of time (through the external loading), of the damage parameter z, and of the displacement variable u. We consider small strains and assume that the elastic energy is quadratic. The external loading encompasses time-dependent displacement boundary conditions, as well as volume and surface loading. All in all, the stored energy functional $\mathcal{E} : [0, T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ (with the state space $\mathcal{U} = \left\{ v \in H^1(\Omega, \mathbb{R}^d) \, ; \, v|_{\Gamma_D} = 0 \right\}$ for u, and $\mathcal{Z}$ the Sobolev-Slobodeckij space $H^s(\Omega)$, $s \in \{1, \frac{3}{2}\}$, for z), is

$$\mathcal{E}(t, u, z) := \frac{1}{2} a_s(z, z) + \int_\Omega f(z) \, \mathrm{d}x + \frac{1}{2} \int_\Omega g(z) \mathbb{C} \varepsilon(u + u_D(t)) : \varepsilon(u + u_D(t)) \, \mathrm{d}x - \langle \ell(t), u \rangle_{\mathcal{U}}, \tag{1.1}$$

where a_s is the bilinear form associated with the H^s semi-norm on $\mathcal{Z} = H^s(\Omega)$, $s \in \{1, \frac{3}{2}\}$, and $\varepsilon(u)$ is the symmetrized strain tensor. For the precise assumptions on the nonlinearities $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

[*]Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany, E-Mail: dorothee.knees@wias-berlin.de

[†]Department of Mathematics, University of Brescia, Via Valotti 9, 25133 Brescia, Italy, E-Mail: riccarda.rossi@ing.unibs.it

[‡]Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy, E-Mail: chiara.zanini@polito.it

the elasticity tensor $\mathbb{C}$, the external loading $\ell = \ell(t)$, and the Dirichlet datum $u_D = u_D(t)$, we refer the reader to Section 2. We impose that at each time $t \in [0, T]$ the displacement $u(t)$ minimizes the energy $\mathcal{E}(t, \cdot, z(t))$, namely

$$u(t) \in \operatorname{argmin}_{v \in \mathcal{U}} \mathcal{E}(t, v, z(t)). \tag{1.2a}$$

Dissipation occurs through the internal, *fast* variable z. As in [MR06, BMR09, TM10], we stay in the rate-independent framework, which characterizes phenomena where the external loading is much slower than the internal relaxation times. Hence, the evolution of z is described by the doubly nonlinear equation

$$\partial\mathcal{R}_1(z'(t)) + \mathrm{D}_z\mathcal{E}(t, u(t), z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T). \tag{1.2b}$$

The above differential inclusion features the 1-positively homogeneous, *unidirectional* dissipation functional $\mathcal{R}_1 : \mathcal{Z} \to [0, \infty]$ defined, for a given fracture toughness $\kappa > 0$ and $\eta \in \mathcal{Z}$, by

$$\mathcal{R}_1(\eta) = \begin{cases} \int_\Omega \kappa\,|\eta(x)|\,\mathrm{d}x & \text{if } \eta \leq 0 \text{ a.e. in } \Omega, \\ \infty & \text{else.} \end{cases} \tag{1.3}$$

In (1.2b), $\mathrm{D}_z\mathcal{E}$ is the Gâteaux derivative of $\mathcal{E}$ w.r. to z, whereas $\partial\mathcal{R}_1 : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ is the (convex analysis) subdifferential of $\mathcal{R}_1$ in the frame of the duality between $\mathcal{Z}^*$ and $\mathcal{Z}$, i.e., for a given $\eta \in \mathrm{dom}(\mathcal{R}_1)$

$$\zeta \in \partial\mathcal{R}_1(\eta) \ \text{ if and only if } \ \mathcal{R}_1(w) - \mathcal{R}_1(\eta) \geq \langle \zeta, w - \eta \rangle_{\mathcal{Z}} \ \text{ for all } w \in \mathcal{Z}. \tag{1.4}$$

We observe that the dissipation potential $\mathcal{R}_1$ only depends on z, and this causes a lack of time compactness for u, but introducing the *reduced* energy functional $\mathcal{I}$ allows us to deal with this degeneracy with respect to u, and to formulate the problem only in the variable z. With

$$\mathcal{I} : [0, T] \times \mathcal{Z} \to \mathbb{R}, \quad \text{defined by } \mathcal{I}(t, z) = \inf_{v \in \mathcal{U}} \mathcal{E}(t, v, z), \tag{1.5}$$

equations (1.2a) and (1.2b) are combined into

$$\partial\mathcal{R}_1(z'(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T). \tag{1.6}$$

It will be shown in Lemma 2.7 that the Gâteaux derivative $\mathrm{D}_z\mathcal{I}$ is well defined on $[0, T] \times \mathcal{Z}$ and, taking into account the concrete structure of $\mathrm{D}_z\mathcal{I}$, (1.6) can be rewritten as

$$\partial\mathcal{R}_1(z'(t)) + A_s z(t) + f'(z(t)) + \frac{1}{2} g'(z(t))\mathbb{C}\varepsilon(u(t) + u_D(t)) : \varepsilon(u(t) + u_D(t)) \ni 0 \quad \text{for a.a. } t \in (0, T), \tag{1.7}$$

where A_s is the operator associated with the bilinear form $a_s(\cdot, \cdot)$, and u fulfills (1.2a). The model studied here falls into the class of damage models introduced in [FN96, Kac86].

Notice that the range of $\partial\mathcal{R}_1$ is $\partial\mathcal{R}_1(0)$, which, due to (1.3), is an unbounded subset of $\mathcal{Z}^*$. This will cause difficulties when deriving uniform a-priori bounds. Since the *reduced* energy $\mathcal{I}$ arises as the *minimum* of a family of not necessarily convex functionals, one cannot rely on standard convexity arguments to study (1.6). Moreover, because of rate-independence we can only hope for a BV-control in time for z, which does not prevent jumps. This calls for weak solvability notions for (1.6).

A well-established framework to describe rate-independent processes is the *global energetic* formulation developed by Mielke & Theil, see [MT99, MT04, Mie05], and used, in the context of damage modeling, in [MR06, BMR09, TM10, FKS12]. There, the evolution is characterized via a global stability criterion and an energy balance, which must be satisfied during the whole evolution. Due to the *global* stability condition, the prediction of the jumps of the solutions turns out to be not entirely satisfactory. Indeed, global energetic solutions may change instantaneously in a very drastic way, jumping into very far-apart energetic configurations (see, for instance, [Mie03, Ex. 6.1], [KMZ08, Ex. 6.3], and [MRS09, Ex. 1]), while a *local* force balance criterion would predict a slow evolution.

In this paper, we discuss the vanishing viscosity approach as an alternative for the derivation of a *local* rate-independent damage model. The philosophy that rate-independence should be considered as limit of systems with smaller and smaller viscosity has by now been widely adopted in the applications, see, e.g., [TZ09, DDMM08, Cag08]. In the mainstream of the papers [EM06, MRS09, MRS12a, MZ12] on general rate-independent systems, [KMZ08, KZM10, LT11] for rate-independent models of crack propagation, and [BFM12, DDS11, DDS12] for rate-independent models in plasticity, we exploit this vanishing viscosity approach to obtain a more precise description of the system behavior at jumps. Hence, we approximate (1.6) with the doubly nonlinear equation

$$\partial\mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0,T), \tag{1.8}$$

where the dissipation functional $\mathcal{R}_\epsilon$ features an additional L^2-viscosity term, viz.

$$\mathcal{R}_\epsilon(\eta) = \mathcal{R}_1(\eta) + \mathcal{R}_{2,\epsilon}(\eta) \quad \text{ with } \mathcal{R}_{2,\epsilon}(\eta) = \tfrac{\epsilon}{2}\,\|\eta\|^2_{L^2(\Omega)} \doteq \epsilon\mathcal{R}_2(\eta) \text{ for } \eta \in \mathcal{Z}. \tag{1.9}$$

Let us mention that damage models with *viscous* dissipation (possibly with viscosity and inertia in the displacement equation, and coupled with thermal effects) have been analyzed in [BS04, BSS05, BB08, FK06], as well as in [HK10], where damage is coupled with phase separation processes. Bridging a connection between the *rate-dependent* and *rate-independent* modeling approaches, in this paper we aim to study the limit of (1.8), as the viscosity parameter ϵ tends to zero. Our vanishing viscosity results hinge on a preliminary analysis of the Cauchy problem for (1.8), for which we establish an existence result, cf. Theorem 3.3.

As it was shown in [EM06, MRS09, MRS12a, MZ12] for general rate-independent systems, passing to the limit as $\epsilon \searrow 0$ in (1.8) leads to an alternative weak formulation of (1.6), featuring a finer description of the solution jumps, which anyway occur later than for global energetic solutions. The key idea from [EM06] is that, at jumps the vanishing viscosity solutions to (1.6) follow a path which is reminiscent of the viscous approximation. To reveal this, one has to go over to an extended state space and study the limiting behavior of the sequence $(\hat{t}_\epsilon, \hat{z}_\epsilon)_\epsilon$ as $\epsilon \downarrow 0$, for a suitable reparameterization $\hat{z}_\epsilon = z_\epsilon \circ \hat{t}_\epsilon$ of a family $(z_\epsilon)_\epsilon$ of *viscous* solutions to (1.8), see (1.14) and (1.15) below. After establishing the relevant a-priori estimates in Section 4, in Section 5 we will prove that, up to a subsequence, the functions $(\hat{t}_\epsilon, \hat{z}_\epsilon)_\epsilon$ converge to a so-called *$\mathcal{Z}$-parameterized solution* of (1.6). While referring to Definition 5.2 for the precise assessment of $\mathcal{Z}$-parameterized solutions, here we just mention that the limit pair $(\hat{t}, \hat{z})$ is a Lipschitz continuous curve $(\hat{t}, \hat{z}) : [0, S] \to [0, T] \times \mathcal{Z}$, fulfilling a *parameterized* doubly nonlinear evolution equation, viz.

$$\partial\mathcal{R}_1(\hat{z}'(s)) + \lambda(s)\hat{z}'(s) + \mathrm{D}_z\mathcal{I}(\hat{t}(s), \hat{z}(s)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } s \in (0,S), \tag{1.10}$$

where $\lambda : (0, S) \to (0, +\infty)$ is a Borel function such that $\hat{t}'(s)\lambda(s) = 0$ a.e. in $(0, S)$. Notice that (1.10) encompasses both rate-independent evolution and, when the system jumps, the influence of rate-dependent dissipation. To reveal this, we observe that the time function $\hat{t} : [0, S] \to [0, T]$ encodes the (slow) external time scale. When $\hat{t}' > 0$ on some interval (s_1, s_2), we have $\lambda = 0$ on (s_1, s_2), hence (1.10) is simply a *parameterized* version of (1.6): the system dissipation is only due to rate-independent, dry friction. When $\hat{t}' = 0$ on some interval (s_1, s_2), the external time is frozen. Indeed, the system has switched to a different regime, which is seen as a *jump* in the slow external time scale. If $\lambda > 0$ in (1.10), also *viscous* dissipation is active. This is in accordance with the following interpretation: jumps are fast (with respect to the slow external time scale) transitions between two metastable states, during which the system may switch to a viscous regime. We refer to [EM06, MRS09, MRS12a], and to Section 5, for further observations on (1.10).

Let us shortly compare our model and results with the results for the damage model developed in [GL09, FG06]. In these papers, the influence of the damage state on the elastic properties is not postulated as in our case (where the effective tensor is defined by $g(z)\mathbb{C}$), but it is the outcome of a certain homogenization procedure that takes place during the evolution process. It is shown in [GL09] that the solutions are (possibly discontinuous) *threshold solutions*. Roughly speaking, this means in particular that solutions do not jump before the forces reach a certain critical value. In the one-dimensional setting, the model from [GL09] can be reformulated in terms of a convex, but not strictly convex, reduced energy $\mathcal{I}$ (cf. Remark 6 in [FG06]) and the dissipation potential $\mathcal{R}_1$ from (1.3). In this one-dimensional case, due to the convexity of $\mathcal{I}$, the threshold solutions from [GL09] coincide with solutions of the corresponding global energetic model, as well as with the vanishing viscosity solutions considered here.

The main difficulties for the existence and vanishing viscosity analysis of (1.8) are of course related to its *doubly nonlinear* character. In particular, let us note in (1.7) the simultaneous presence of a quadratic term in $\varepsilon(u)$ (featured from the derivative $\mathrm{D}_z\mathcal{I}$ of the *nonconvex* energy $\mathcal{I}$), and of the multivalued operator $\partial\mathcal{R}_1$. Indeed, differently from [EM06, MRS09, MRS12a, MZ12], here we are enforcing irreversibility, hence the operator $\partial\mathcal{R}_1 : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ is *unbounded*. This makes it difficult to derive suitable bounds for the thermodynamically conjugated force, i.e. the derivative $\mathrm{D}_z\mathcal{I}$, and it motivates the presence of the regularizing term a_s in the energy functional, hence our choice of the $\mathcal{Z}$-topology, which is stronger than the natural one associated with $\mathcal{R}_1$. Indeed, one technical difficulty is that on the one hand it is possible to derive an estimate for z' in $L^2(0,T;L^2(\Omega))$ and for the term $\mathrm{D}_z\mathcal{I}(t,z(t))$ in the space $L^\infty(0,T;\mathcal{Z}^*)$, and a comparison argument in (1.8) will not give additional information on $\mathrm{D}_z\mathcal{I}$, due to the unboundedness of the term $\partial\mathcal{R}_1(z'(t))$. On the other hand, it is crucial both, for the existence and for the vanishing viscosity analysis of (1.8), that the terms $\mathrm{D}_z\mathcal{I}$ and z' are in *duality*.

In fact, the key step for the proof of *existence* of viscous solutions to (1.8) (cf. Theorem 3.3), is to obtain for viscous solutions $(z_\epsilon)_{\epsilon>0} \subset H^1(0,T;L^2(\Omega))$, the improved bound

$$\|z_\epsilon'\|_{L^2(0,T;\mathcal{Z})} \leq C_\epsilon, \tag{1.11}$$

where C_ϵ is a positive constant which depends on the viscosity parameter ϵ and explodes as $\epsilon \searrow 0$. We will prove (1.11) by means of careful estimates carried out for related time-incremental problems. Here we rely on a refined elliptic regularity result for the Euler-Lagrange equation for the minimum problem (1.2a), from the recent [HMW11]. We highlight that this regularity result does not hinge on smoothness of the boundary $\partial\Omega$, and thus it allows us to deal with a broad class of domains, as well as with mixed boundary conditions, which is crucial for real-world applications.

Next, we develop enhanced estimates, relying on the *parabolic* character of (1.7) and partially drawn from [MZ12]. In this way, we obtain for viscous solutions the further BV-bound

$$\|z_\epsilon'\|_{L^1(0,T;\mathcal{Z})} \leq C, \tag{1.12}$$

for a constant C which is now *independent* of $\epsilon > 0$. Indeed, estimate (1.12) is the starting point for the vanishing viscosity analysis developed Theorem 5.3. Thanks to (1.12) it is possible to reparameterize viscous solutions $(z_\epsilon)_{\epsilon>0}$ by the $\mathcal{Z}$-arclength of their graph, which leads in the limit $\epsilon \searrow 0$ to the aforementioned $\mathcal{Z}$-parameterized solutions.

In Section 4 we shall prove the estimates (1.11) and (1.12) by working on the *time-discretization* scheme associated with (1.8): in particular, we shall suitably adapt to the time-discrete setting the arguments from [MZ12, Lemma 3.4]. Note that this is not trivial and will involve an extension of the sharp, time-discrete Gronwall-type Lemma in [NSV00], cf. Lemma 4.9 later on. Moving from (1.12), we shall perform the vanishing viscosity analysis as $\epsilon \searrow 0$ in Section 5.

For a different approach to the proof of estimates (1.11) and (1.12) the reader is referred to the Preprint version of this article [KRZ11], where we regularize (1.8) by adding a $\mathcal{Z}$-viscosity term, modulated by a "small" parameter $\delta > 0$. We obtain both estimate (1.11) (with a constant depending on ϵ but independent of δ) and estimate (1.12) (with a constant independent of ϵ and δ), for the solutions of the δ-regularized viscous problem. Hence, we pass to the limit as $\delta \searrow 0$ and conclude the existence of solutions to the Cauchy problem for (1.8).

We emphasize that the time-discrete approach is clearly interesting in view of numerical analysis. Indeed, it has been proved in [MRS12a] (cf. also the forthcoming [MRS12c]) for general rate-independent systems, and in [KMZ08] for a crack propagation model that, passing to the limit in the time-discretization scheme for (1.8) as *both* the viscosity parameter *and* the time-step tend to zero, leads to the so-called BV solutions to (1.6). Loosely speaking, the latter concept is the "non-parameterized" version of the notion of parameterized solution. We plan to address within our damage model this simultaneous passage to the limit, as well as the analysis of BV solutions. Taking into account also spatial discretizations, the task is to derive relations between the discretization parameters (time-step size τ_n, mesh size h_n, viscosity ν_n) such that in the limit $(\tau_n, h_n, \nu_n) \to 0$ the approximate solutions converge to the vanishing viscosity solution (and not, for instance, to global energetic solutions), cf. [KS12], where a similar question for a crack propagation model is discussed. The arguments developed in Section 4 on the time-discrete problems will be fundamental for these studies.

A second issue we are going to address in the future, is to replace the *linear* s-Laplacian in (1.7) with the *nonlinear* p-Laplacian operator, which is usually found in models for damage, cf. [MR06, BMR09, TM10]. The key step for doing so will be to obtain, via regularity arguments, enhanced estimates for the term $\mathrm{D}_z\mathcal{I}(t, z(t))$ in (1.8). We are convinced that the careful study of the properties of the reduced functional $\mathcal{I}$ performed in Section 2 as well as the discretization tools and estimates that have been established in Section 4, will be of some interest also in other contexts.

Our main result. For the reader's convenience, we collect here the main results of this paper, referring to the statements throughout the paper for the precise assumptions.

First, in Section 2, relying on the recent regularity results from [HMW11] we derive basic properties of the marginal energy functional $\mathcal{I}$. These include the Gâteaux-differentiability as well as continuity properties of $\mathrm{D}_z\mathcal{I}$ and $\partial_t\mathcal{I}$ with respect to weak and strong convergence.

Estimates (1.11) and (1.12) are derived based on a time-discretization scheme associated with (1.8). We prove that for given $\epsilon > 0$, $z_0 \in \mathcal{Z}$, and a partition $\{0 = t_0^\tau < \ldots < t_N^\tau = T\}$ of the time interval $[0, T]$ with time step $\tau = t_{k+1}^\tau - t_k^\tau$, $k = 0, \ldots, N-1$, there exist elements $(z_k^\tau)_{0 \le k \le N}$ such that $z_0^\tau = z_0$ and

$$\partial\mathcal{R}_1\left(\frac{z_{k+1}^\tau - z_k^\tau}{\tau}\right) + \epsilon\frac{z_{k+1}^\tau - z_k^\tau}{\tau} + \mathrm{D}_z\mathcal{I}(t_{k+1}^\tau, z_{k+1}^\tau) \ni 0. \tag{1.13}$$

Hence we consider the piecewise linear interpolants $(\hat{z}_\tau)_\tau$ of the discrete solutions to (1.13) and, under suitable assumptions, in Theorem 4.3 show that for every sequence $\tau^j \searrow 0$ as $j \to \infty$ there exists a (not relabeled) subsequence of $(\hat{z}_{\tau^j})$ and $z_\epsilon \in H^1(0, T; \mathcal{Z})$ such that $\hat{z}_{\tau^j} \rightharpoonup z_\epsilon$ weakly in $H^1(0, T; \mathcal{Z})$ and z_ϵ is a solution to the Cauchy problem for (1.8). Under additional regularity assumptions, in Theorem 3.7 we prove that the latter Cauchy problem in fact admits a *unique* solution $z_\epsilon \in H^1(0, T; \mathcal{Z})$. Further (cf. Proposition 4.5), in the case of dimension $d = 2$, if the initial datum z_0 fulfills $z_0(x) \in [0, 1]$ for a.a. $x \in \Omega$, then every viscous solution z_ϵ constructed via time-discretization also fulfills

$$z_\epsilon(t,x) \in [0,1] \quad \text{for a.a. } x \in \Omega, \text{ for all } t \in [0,T].$$

The starting point for the vanishing viscosity analysis as $\epsilon \to 0$ is the crucial BV-estimate

$$\sup_{\epsilon>0} \int_0^T \|z_\epsilon'(t)\|_{\mathcal{Z}} \, \mathrm{d}t \le C_0$$

for viscous solutions to (1.8), see Theorem 3.3 and Corollary 4.8. Based on this, we are entitled to consider the graph $\mathrm{Graph}(z_\epsilon) := \{ (t, z_\epsilon(t)) \,;\, t \in [0,T] \} \subset [0,T] \times \mathcal{Z}$ and its $\mathcal{Z}$-arclength parameterization

$$s_\epsilon(t) = t + \int_0^t \|z_\epsilon'(r)\|_{\mathcal{Z}} \, \mathrm{d}r. \tag{1.14}$$

For $S_\epsilon = s_\epsilon(T)$ we introduce the functions $\hat{t}_\epsilon : [0, S_\epsilon] \to [0,T]$ and $\hat{z}_\epsilon : [0, S_\epsilon] \to \mathcal{Z}$

$$\hat{t}_\epsilon(s) := s_\epsilon^{-1}(s), \quad \hat{z}_\epsilon(s) := z_\epsilon(\hat{t}_\epsilon(s)) \tag{1.15}$$

and study the limiting behavior as $\epsilon \to 0$ of the parameterized viscous trajectories $\{ (\hat{t}_\epsilon(s), \hat{z}_\epsilon(s)) \,;\, s \in [0, S_\epsilon] \}$.

In Theorem 5.3 and Proposition 5.7 we prove that there exist $S > 0$ such that for every sequence $\epsilon_n \searrow 0$, up to a not relabeled subsequence there exists a pair $(\hat{t}, \hat{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S]; [0,T] \times \mathcal{Z})$ with

$$(\hat{t}_{\epsilon_n}, \hat{z}_{\epsilon_n}) \overset{*}{\rightharpoonup} (\hat{t}, \hat{z}) \quad \text{in } W^{1,\infty}(0,S; [0,T] \times \mathcal{Z}),$$

and $(\hat{t}, \hat{z})$ is a $\mathcal{Z}$-parameterized solution in the sense of Def. 5.2, fulfilling the parameterized doubly nonlinear evolution equation (1.10).

Plan of the paper. In Section 2 we set up the model and thoroughly analyze the properties of the reduced energy $\mathcal{I}$. Next, in Section 3 we state Theorem 3.3 (=existence of solutions and a priori estimates uniform w.r. to the viscosity parameter ϵ) for the Cauchy problem associated with (1.8). In Sec. 3 we also discuss uniqueness of viscous solutions under special assumptions. We prove Thm. 3.3 via a time-discretization in Section 4. Finally, in Section 5 we develop the vanishing viscosity analysis of (1.8).

2 The energy functional and its properties

2.1 Set-up

Hereafter $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded domain with Lipschitz boundary, and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, with the open Dirichlet boundary Γ_D such that $\mathcal{H}^{d-1}(\Gamma_D) > 0$, and the Neumann boundary Γ_N. We shall assume that

$$\Gamma_D \text{ and } \Gamma_N \text{ are regular in the sense of Gröger, [Grö89],} \tag{2.1}$$

viz., loosely speaking, that the hypersurface separating Γ_D and Γ_N is Lipschitz.

Notation. For a given Banach space X, we shall denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X^* and X, and, if X is a Hilbert space, we shall use the symbol $(\cdot, \cdot)_X$ for its scalar product. For matrices $A, B \in \mathbb{R}^{m \times d}$ the inner product is defined by $A : B = \mathrm{tr}(B^\top A) = \sum_{i=1}^m \sum_{j=1}^d a_{ij} b_{ij}$.

The letter Q shall stand for the space-time cylinder $\Omega \times (0,T)$. The following function spaces and notation shall be used for $\sigma \ge 0$, $p \in [1, \infty]$:

- $W^{\sigma,p}(\Omega)$ Sobolev-Slobodeckij spaces, $H^\sigma(\Omega) := W^{\sigma,2}(\Omega)$,
- $W^{1,p}_{\Gamma_D}(\Omega) := \left\{ u \in W^{1,p}(\Omega)\,;\, u\big|_{\Gamma_D} = 0 \right\}$ and $W^{-1,p}_{\Gamma_D}(\Omega) := \big(W^{1,p'}_{\Gamma_D}(\Omega)\big)^*$ the dual space, $\frac{1}{p} + \frac{1}{p'} = 1$.

We shall denote by $u : \Omega \to \mathbb{R}^d$ the displacement, and by $z : \Omega \to \mathbb{R}$ the (scalar) damage variable. The corresponding state spaces are

$$\mathcal{U} = \left\{ v \in H^1(\Omega, \mathbb{R}^d)\,;\, v\big|_{\Gamma_D} = 0 \right\} = W^{1,2}_{\Gamma_D}(\Omega, \mathbb{R}^d) \tag{2.2}$$

$$\mathcal{Z} = H^s(\Omega), \quad \text{with } s \geq \frac{d}{2}. \tag{2.3}$$

In fact, we restrict to the case $s < 2$, so that the associated bilinear form on $\mathcal{Z}$ is:

$$a_s(z_1, z_2) = \int_\Omega \nabla z_1 \cdot \nabla z_2 \,\mathrm{d}x \quad \text{if } s = 1, \tag{2.4a}$$

$$a_s(z_1, z_2) = \int_\Omega \int_\Omega \frac{\big(\nabla z_1(x) - \nabla z_1(y)\big) \cdot \big(\nabla z_2(x) - \nabla z_2(y)\big)}{|x-y|^{d+2(s-1)}} \,\mathrm{d}x\,\mathrm{d}y \quad \text{if } s \in (1,2). \tag{2.4b}$$

Recall that $\mathcal{Z}$ is a Hilbert space, with the inner product $(z_1, z_2)_{\mathcal{Z}} = (z_1, z_2)_{L^2(\Omega)} + a_s(z_1, z_2)$. We denote by $A_s : \mathcal{Z} \to \mathcal{Z}^*$ the associated operator, viz.

$$\langle A_s(z), w \rangle_{\mathcal{Z}} := a_s(z, w) \quad \text{for every } z,\, w \in \mathcal{Z}. \tag{2.5}$$

Notice that

$$\mathcal{Z} \Subset L^r(\Omega) \quad \text{for every } r \in [1, \infty). \tag{2.6}$$

Furthermore, since $\mathcal{Z}$ is dense in $L^2(\Omega)$, we have that $(\mathcal{Z}, L^2(\Omega) \cong L^2(\Omega)^*, \mathcal{Z}^*)$ is a *Hilbert triple*. In particular, every element of $L^2(\Omega)$ is identified with an element in $\mathcal{Z}^*$, and we thus have

$$\mathcal{Z} \subset L^2(\Omega) \subset \mathcal{Z}^*. \tag{2.7}$$

2.2 The energy functional

The energy is given by the sum of the elastic energy and an energy only depending on the damage variable. As for the latter contribution, we consider a function

$$f \in C^2(\mathbb{R}), \quad \text{such that} \quad \exists\, K_1,\, K_2,\, K_3 > 0 \quad \forall\, x \in \mathbb{R} : \quad |f''(x)| \leq K_1 \quad \text{and} \quad f(x) \geq K_2\, |x|^2 - K_3. \tag{2.8}$$

A typical choice for f is $f(z) = (1-z)^2$, see [Gia05]. We then have the functional

$$\mathcal{I}_1 : \mathcal{Z} \to \mathbb{R} \ \text{ defined by } \ \mathcal{I}_1(z) := \frac{1}{2} a_s(z,z) + \int_\Omega f(z) \,\mathrm{d}x\,.$$

Linearly elastic materials are considered with an elastic energy density

$$W(x, \eta) = \frac{1}{2}\mathbb{C}(x)\eta : \eta, \quad \text{for } \eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ and almost every } x \in \Omega.$$

Hereafter, we shall suppose for the elasticity tensor that

$$\mathbb{C} \in L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}_{\mathrm{sym}})) \tag{2.9a}$$

$$\exists\, \gamma_0 > 0 \quad \text{for all } \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ and almost all } x \in \Omega : \quad \mathbb{C}(x)\xi : \xi \geq \gamma_0\, |\xi|^2\,. \tag{2.9b}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a further constitutive function such that

$$g \in \mathrm{C}^2(\mathbb{R}), \text{ with } g',\, g'' \in L^\infty(\mathbb{R}), \text{ and } \exists\, \gamma_1,\, \gamma_2 > 0 \,:\, \forall\, z \in \mathbb{R} \,:\, \gamma_1 \leq g(z) \leq \gamma_2. \tag{2.10}$$

Given an external loading $\ell \in C^0([0,T],\mathcal{U}^*)$ and a Dirichlet datum $u_D \in C^0([0,T];H^1(\Omega,\mathbb{R}^d))$, we take the elastic energy

$$\mathcal{E}_2 : [0,T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R} \;\text{ defined by }\; \mathcal{E}_2(t,u,z) := \int_\Omega g(z) W(\varepsilon(u+u_D(t)))\,\mathrm{d}x - \langle \ell(t), u\rangle_{\mathcal{U}} \tag{2.11}$$

where $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized strain tensor. For $u \in \mathcal{U}$ and $z \in \mathcal{Z}$ the stored energy is then defined as

$$\mathcal{E}(t,u,z) = \mathcal{I}_1(z) + \mathcal{E}_2(t,u,z).$$

Minimizing the stored energy with respect to the displacements we obtain the reduced energy

$$\mathcal{I} : [0,T] \times \mathcal{Z} \to \mathbb{R} \;\text{ given by }\; \mathcal{I}(t,z) = \mathcal{I}_1(z) + \mathcal{I}_2(t,z) \text{ with } \mathcal{I}_2(t,z) = \inf\{\, \mathcal{E}_2(t,v,z)\,;\, v \in \mathcal{U}\,\}\,. \tag{2.12}$$

Remark 2.1. For our main results on the vanishing viscosity analysis of (1.6) (cf. Theorems 3.3 and 5.3 later on), it will be sufficient to suppose that the index s in (2.3) fulfills $s = \frac{d}{2}$. In particular, let us highlight that, in the bi-dimensional case $d = 2$, we have $s = 1$, hence the operator A_s reduces to the usual Laplacian operator.

Remark 2.2. As we have already pointed out, the irreversibility of the damage process is enforced in our model through the choice of the dissipation functional (1.3). Instead, so far we have not included in our model the constraint that the damage variable z only take values in $[0,1]$: indeed, the term $I_{[0,1]}(z)$ does not contribute to the energy $\mathcal{I}$. However, in Section 4 we shall prove via a time-discretization procedure that, in the bi-dimensional case $d = 2$ and under suitable assumptions on the nonlinearities g and f, if the initial datum z_0 satisfies $z_0(x) \in [0,1]$ for almost every $x \in \Omega$, then there exists a *viscous* solution $z \in H^1(0,T;\mathcal{Z})$ with $z(0) = z_0$ and $z(x) \in [0,1]$ for almost every $x \in \Omega$. Ultimately, with the vanishing viscosity analysis developed in Section 5, we shall obtain *parameterized* solutions to the rate-independent system for damage, which only take values in $[0,1]$. The proof relies on a comparison principle argument that cannot be adapted to the case with H^s-regularizations, $s \geq \frac{3}{2}$. However, using a p-Laplace-like regularization instead of a H^s-regularization would allow us to prove the non-negativity of z also in higher space dimension.

Notation 2.3. Hereafter, throughout the paper we shall use the symbols c, c', C, and C' for various positive constants which only depend on known quantities, and whose meaning may vary even in the same line.

2.3 Properties of the energy functional

A regularity result from [HMW11]. The following result has been recently proved in [HMW11] (cf. Thm. 1.1 therein): For $\mathbb{C}$ as in (2.9a), g as in (2.10), and $z \in \mathcal{Z}$, let L_z be the linear elliptic operator defined by

$$\langle L_z(v), w\rangle_{\mathcal{U}} := \int_\Omega g(z(x))\mathbb{C}(x)\varepsilon(v(x)) : \varepsilon(w(x))\,\mathrm{d}x \quad \text{for all } v,\, w \in \mathcal{U}. \tag{2.13}$$

Then,

$$\text{there exists } p > 2 \text{ s.t. for all } \widetilde{p} \in [2,p] \quad L_z : W^{1,\widetilde{p}}_{\Gamma_D}(\Omega;\mathbb{R}^d) \to W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega;\mathbb{R}^d) \text{ is an isomorphism,} \tag{2.14}$$

and there exists some constant $c_0 > 0$, only depending on $\|\mathbb{C}\|_{L^\infty(\Omega)}$ and $\|g\|_{L^\infty(\mathbb{R})}$, such that

$$\|L_z^{-1} h\|_{W^{1,\widetilde{p}}_{\Gamma_D}(\Omega)} \leq c_0 \|h\|_{W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega)} \quad \text{for all } h \in W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega) \text{ and } \widetilde{p} \in [2,p]. \tag{2.15}$$

Notice that, in particular, the integrability exponent p and the constant c_0 are *independent* of $z \in \mathcal{Z}$. Relying on this regularity result, in the next lemmatas we prove some crucial properties of the reduced energy (2.12).

Assumptions on the initial data. Hereafter, we shall require that

$$\ell \in C^{1,1}([0,T]; W^{-1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d)), \quad u_D \in C^{1,1}([0,T]; W^{1,p}(\Omega;\mathbb{R}^d)) \text{ with } p > 2 \text{ from (2.14)}. \tag{2.16}$$

Coercivity of the reduced energy and properties of minimizers.

Lemma 2.4 (Existence of minimizers and their regularity).
Let $s = d/2$. Under assumptions (2.1), (2.8)–(2.10), *and* (2.16), *for every $(t,z) \in [0,T] \times \mathcal{Z}$ there exists a unique $u_{\min}(t,z) \in \mathcal{U}$, which minimizes $\mathcal{E}(t,z,\cdot)$. Moreover, there exists $p > 2$ such that for all $\widetilde{p} \in [2,p]$ and $(t,z) \in [0,T] \times \mathcal{Z}$ it holds that $u_{\min}(t,z) \in W^{1,\widetilde{p}}_{\Gamma_D}(\Omega)$, and*

$$\forall\,(t,z) \in [0,T] \times \mathcal{Z} : \quad \|u_{\min}(t,z)\|_{W^{1,\widetilde{p}}(\Omega)} \le c_0 \big(\|\ell(t)\|_{W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega)} + \|u_D(t)\|_{W^{1,\widetilde{p}}(\Omega)} \big), \tag{2.17}$$

where c_0 is the constant from (2.15)*. Furthermore, the following coercivity inequality for $\mathcal{I}$ is valid: There exist constants $c_1, c_2 > 0$ such that for all $(t,z) \in [0,T] \times \mathcal{Z}$ it holds*

$$\mathcal{I}(t,z) \ge c_1 \big(\|z\|^2_{H^s(\Omega)} + \|u_{\min}(t,z)\|^2_{H^1(\Omega)} \big) - c_2. \tag{2.18}$$

Proof. Taking into account (2.10), (2.16), and employing Korn's inequality, it is immediate to see that for every $(t,z) \in [0,T] \times \mathcal{Z}$ the functional $\mathcal{E}_2(t,z,\cdot)$ is uniformly convex on $\mathcal{U}$. Therefore, $\mathcal{E}_2(t,z,\cdot)$ (and, hence, $\mathcal{E}(t,z,\cdot)$), has a unique minimizer $u_{\min}(t,z)$, satisfying the Euler equation

$$L_z(u_{\min}(t,z) + u_D(t)) = \ell(t) \quad \text{for every } t \in [0,T]. \tag{2.19}$$

Since $\ell(t) \in W^{-1,p}_{\Gamma_D}(\Omega)$ for all $t \in [0,T]$ by (2.16), from (2.14) we deduce that $u_{\min}(t,z) \in W^{1,\widetilde{p}}_{\Gamma_D}(\Omega)$ for all $\widetilde{p} \in [2,p]$. Then, (2.17) follows from (2.15) and assumption (2.16).

Finally, estimate (2.18) follows from combining

$$\begin{aligned} \mathcal{I}_2(t,z) = \mathcal{E}_2(t, u_{\min}(t,z), z) &\ge \frac{\gamma_1\gamma_0}{2} \int_\Omega |\varepsilon(u_{\min}(t,z) + u_D(t))|^2 \,\mathrm{d}x - \|\ell(t)\|_{H^{-1}(\Omega)} \|u_{\min}(t,z)\|_{\mathcal{U}} \\ &\ge C\,\|u_{\min}(t,z)\|^2_{\mathcal{U}} - C' \left(\|u_D\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\ell\|^2_{L^\infty(0,T;H^{-1}(\Omega))} \right), \end{aligned}$$

(where we have again used (2.10), Korn's inequality, (2.16) and (2.17)), with

$$\mathcal{I}_1(t,z) \ge \frac{1}{2} a_s(z,z) + C\|z\|^2_{L^2(\Omega)} - C' \ge C\|z\|^2_{H^s(\Omega)} - C',$$

where the first inequality follows from (2.8), and the second one from a Poincaré-type inequality. □

Lemma 2.5 (Continuous dependence on the data).
Let $s = d/2$. Under assumptions (2.1), (2.8)–(2.10), *there exists a constant $c_3 > 0$ such that for all ℓ and u_D with* (2.16), *all $z_1, z_2 \in \mathcal{Z}$, all $t_1, t_2 \in [0,T]$ and all $\widetilde{p} \in [2,p)$ it holds with $r = p\widetilde{p}(p-\widetilde{p})^{-1}$*

$$\begin{aligned} &\|u_{\min}(t_1,z_1) - u_{\min}(t_2,z_2)\|_{W^{1,\widetilde{p}}(\Omega)} \\ &\qquad \le c_3 \big(|t_1 - t_2| + \|z_1 - z_2\|_{L^r(\Omega)} \big) \big(\|\ell\|_{C^1([0,T]; W^{-1,p}_{\Gamma_D}(\Omega))} + \|u_D\|_{C^1([0,T]; W^{1,p}(\Omega))} \big). \end{aligned} \tag{2.20}$$

Proof. For $i = 1, 2$, let $u_i := u_{\min}(t_i, z_i) \in W^{1,p}(\Omega)$, with p from Lemma 2.4. From the Euler-Lagrange equation (2.19) written for u_i, $i = 1, 2$, with algebraic manipulations we obtain that $u_1 - u_2$ satisfies for all $v \in \mathcal{U}$

$$\begin{aligned}\int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1-u_2)\colon \varepsilon(v)\,\mathrm{d}x &= \int_\Omega \big(g(z_2)-g(z_1)\big)\mathbb{C}\varepsilon(u_2)\colon\varepsilon(v)\,\mathrm{d}x\\ &\quad - \int_\Omega \big(g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\big)\colon\varepsilon(v)\,\mathrm{d}x + \langle \ell(t_1)-\ell(t_2), v\rangle_{\mathcal{U}}\,.\end{aligned} \tag{2.21}$$

Hence, the function $u_1 - u_2$ fulfills

$$\int_\Omega g(z_1)\mathbb{C}\varepsilon(u_1-u_2) : \varepsilon(v)\,\mathrm{d}x = \langle \tilde{\ell}_{1,2}, v\rangle \quad \text{for all } v \in \mathcal{U},$$

where $\tilde{\ell}_{1,2} \in W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega)$ subsumes the terms on the right-hand side of (2.21). Therefore, (2.15) gives

$$\|u_1-u_2\|_{W^{1,\widetilde{p}}(\Omega)} \le c_0 \left\|\tilde{\ell}_{1,2}\right\|_{W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega)},$$

whence we deduce the estimate

$$\begin{aligned}\|u_1-u_2\|_{W^{1,\widetilde{p}}(\Omega)} \le c_0\big(&\|\ell(t_1)-\ell(t_2)\|_{W^{-1,\widetilde{p}}_{\Gamma_D}(\Omega)} + \|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^{\widetilde{p}}(\Omega)}\\ &+ \|g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\|_{L^{\widetilde{p}}(\Omega)}\big).\end{aligned} \tag{2.22}$$

Now, the Lipschitz continuity of g and Hölder's inequality imply that

$$\|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_2)\|_{L^{\widetilde{p}}(\Omega)} \le C\|z_1-z_2\|_{L^r(\Omega)}\|\varepsilon(u_2)\|_{L^p(\Omega)} \le C'\|z_1-z_2\|_{L^r(\Omega)} \tag{2.23}$$

with $r = p\widetilde{p}(p-\widetilde{p})^{-1}$, where the second inequality follows from (2.16) and from estimate (2.17). We use (2.23) to estimate the second term on the right-hand side of (2.22). As for the third summand, we have

$$\begin{aligned}&\|g(z_1)\mathbb{C}\varepsilon(u_D(t_1)) - g(z_2)\mathbb{C}\varepsilon(u_D(t_2))\|_{L^{\widetilde{p}}(\Omega)}\\ &\quad\le \|(g(z_1)-g(z_2))\mathbb{C}\varepsilon(u_D(t_1))\|_{L^{\widetilde{p}}(\Omega)} + \|g(z_2)\mathbb{C}(\varepsilon(u_D(t_1))) - \varepsilon(u_D(t_2)))\|_{L^{\widetilde{p}}(\Omega)}\\ &\qquad\le C\big(\|z_1-z_2\|_{L^r(\Omega)}\|u_D\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|u_D(t_1)-u_D(t_2)\|_{W^{1,p}(\Omega)}\big),\end{aligned} \tag{2.24}$$

where the latter inequality again follows from (2.23), and from the fact that $g \in L^\infty(\mathbb{R})$. Combining all of the above inequalities, and relying on (2.16), we finally arrive at (2.20). □

Differentiability w.r. to time.

Lemma 2.6 (Differentiability and growth w.r. to time)**.**
Let $s = d/2$. Under assumptions (2.1), (2.8)–(2.10) and (2.16), for every $z \in \mathcal{Z}$ the map $t \mapsto \mathcal{I}(t, z)$ belongs to $C^1([0, T], \mathbb{R})$ with

$$\partial_t\mathcal{I}(t,z) = \int_\Omega g(z)\mathbb{C}(\varepsilon(u_{\min}(t,z)+u_D(t)))\colon\varepsilon(\dot{u}_D(t))\,\mathrm{d}x - \langle\dot{\ell}(t), u_{\min}(t,z)\rangle. \tag{2.25}$$

Moreover, there exists a constant $c_4 > 0$ such that for all $t \in [0, T]$, $z \in \mathcal{Z}$ and u_D, ℓ with (2.16) we have

$$|\partial_t\mathcal{I}(t,z)| \le c_4\big(\|u_D\|^2_{C^1([0,T];W^{1,p}(\Omega))} + \|\ell\|^2_{C^1([0,T];W^{-1,p}_{\Gamma_D}(\Omega))}\big). \tag{2.26}$$

Finally, for all $r \in [\frac{p}{p-2}, \infty)$ there exists a constant $c_5 > 0$ depending on $\|\ell\|_{C^{1,1}([0,T];W^{-1,p}_{\Gamma_D}(\Omega))}$ and $\|u_D\|_{C^{1,1}([0,T];W^{1,p}(\Omega))}$ such that for all $t_i \in [0, T]$ and $z_i \in \mathcal{Z}$ we have

$$|\partial_t\mathcal{I}(t_1,z_1) - \partial_t\mathcal{I}(t_2,z_2)| \le c_5\big(|t_1-t_2| + \|z_1-z_2\|_{L^r(\Omega)}\big). \tag{2.27}$$

Proof. Relation (2.25) follows from direct calculations. Then,

$$|\partial_t \mathcal{I}(t,z)| \le c_3\big(\|u_D\|_{C^1([0,T];H^1(\Omega))} + \|\ell\|_{C^1([0,T];\mathcal{U}^*)} \big) \|u_{\min}(t,z)\|_{H^1(\Omega)} + c_5 \|u_D\|^2_{C^1([0,T];H^1(\Omega))} .$$

In view of (2.17) we arrive at (2.26). In order to prove (2.27), we calculate

$$\begin{aligned}
&\partial_t \mathcal{I}(t_1,z_1) - \partial_t \mathcal{I}(t_2,z_2)\\
&= \int_\Omega (g(z_1){-}g(z_2))\mathbb{C}(\varepsilon(u_{\min}(t_1,z_1)+u_D(t_1))) : \varepsilon(\dot u_D(t_1))\,\mathrm{d}x\\
&+ \int_\Omega g(z_2)\mathbb{C}(\varepsilon(u_{\min}(t_1,z_1)+u_D(t_1)){-}\varepsilon(u_{\min}(t_2,z_2)+u_D(t_2))) : \varepsilon(\dot u_D(t_1))\,\mathrm{d}x\\
&+ \int_\Omega g(z_2)\mathbb{C}(\varepsilon(u_{\min}(t_2,z_2)+u_D(t_2))) : (\varepsilon(\dot u_D(t_1)){-}\varepsilon(\dot u_D(t_2)))\,\mathrm{d}x\\
&- \langle \dot\ell(t_1){-}\dot\ell(t_2), u_{\min}(t_1,z_1)\rangle + \langle \dot\ell(t_2), u_{\min}(t_2,z_2){-}u_{\min}(t_1,z_1)\rangle \doteq I_1+I_2+I_3+I_4+I_5
\end{aligned}$$

To estimate I_1, I_2, and I_3 we rely on the fact that g, $g' \in L^\infty(\mathbb{R})$, on the previously proved (2.17) and (2.20), and on the following Hölder-estimate: For $z \in \mathcal{Z}$ and $v_i \in W^{1,q}(\Omega)$ we have

$$\|z\,|\nabla v_1|\,|\nabla v_2|\|_{L^1(\Omega)} \le \|z\|_{L^r(\Omega)} \|v_1\|_{W^{1,q}(\Omega)} \|v_2\|_{W^{1,q}(\Omega)} ,$$

with q defined by $\frac{1}{r}{+}\frac{2}{q} = 1$, i.e. $r = q/(q{-}2)$. The estimates for I_4 and I_5 ensue from (2.16) and (2.20). □

Differentiability w.r.t. to z. The differentiability of $\mathcal{I}$ with respect to z will be studied in the $\mathcal{Z} - \mathcal{Z}^*$ duality. In particular, $\mathrm{D}_z\mathcal{I}(t,\cdot) : \mathcal{Z} \to \mathcal{Z}^*$ shall denote the Gâteaux-differential of the functional $\mathcal{I}(t,\cdot)$.

Lemma 2.7 (Gâteaux-differentiability)**.**
Let $s = d/2$. Under assumptions (2.1), (2.8)–(2.10) and (2.16), for all $t \in [0,T]$ the functional $\mathcal{I}(t,\cdot) : \mathcal{Z} \to \mathbb{R}$ is Gâteaux-differentiable at all $z \in \mathcal{Z}$, and for all $\eta \in \mathcal{Z}$ we have

$$\mathrm{D}_z\mathcal{I}(t,z)[\eta] = a_s(z,\eta) + \int_\Omega f'(z)\eta\,\mathrm{d}x + \int_\Omega g'(z)\widetilde{W}(t,\nabla u_{\min}(t,z))\eta\,\mathrm{d}x, \tag{2.28}$$

where we use the abbreviation $\widetilde{W}(t,\nabla v) = W(\nabla v{+}\nabla u_D(t)) = \frac{1}{2}\mathbb{C}\varepsilon(v{+}u_D(t)){:}\varepsilon(v{+}u_D(t))$. In particular, the following estimate holds

$$\exists\, c_6 > 0\ \forall\,(t,z) \in [0,T]\times\mathcal{Z} :\quad \|\mathrm{D}_z\mathcal{I}(t,z)\|_{\mathcal{Z}^*} \le c_6\,(\|z\|_{\mathcal{Z}} + 1)\,. \tag{2.29}$$

Proof. The Gâteaux-differentiability of $\mathcal{I}_1$ follows from the definition of the bilinear form $a_s(\cdot,\cdot)$ and assumption (2.8) on f. We only have to verify the Gâteaux-differentiability of $\mathcal{I}_2(t,z)$. In this direction, let $u \in W^{1,\widetilde{p}}_{\Gamma_D}(\Omega)$ with $\widetilde{p} \in (2,p)$, p as in Lemma 2.4, and $t \in [0,T]$ be fixed. The mapping $\mathcal{E}_2(t,u,\cdot) : \mathcal{Z} \to \mathbb{R}$ is Gâteaux-differentiable, as shown by the following calculations: Let $z,\eta \in \mathcal{Z}$, $h \in \mathbb{R}\backslash\{0\}$, and set

$$b_h(z,\eta) := h^{-1}\big(\mathcal{E}_2(t,u,z+h\eta) - \mathcal{E}_2(t,u,z)\big) = \int_\Omega\int_0^1 g'(z+\sigma h\eta)\eta\widetilde{W}(t,\nabla u)\mathrm{d}\sigma\,\mathrm{d}x.$$

For $h \to 0$ the integrand pointwise converges to $g'(z)\eta\widetilde{W}(t,\nabla u)$. Moreover, since $\mathcal{Z} \subset L^r(\Omega)$ for all $r \in [1,\infty)$ and since $\widetilde{W}(t,\nabla u) \in L^{\frac{\widetilde{p}}{2}}(\Omega)$, the function $x \mapsto \|g'\|_{L^\infty(\mathbb{R})}\,|\eta(x)|\,\widetilde{W}(t,\nabla u(x))$ is an integrable majorant. Hence, with Lebesgue's theorem it follows that for $h \to 0$ the sequence $(b_h(z,\eta))_h$ converges to $b(z,\eta) := \int_\Omega g'(z)\eta\widetilde{W}(t,\nabla u)\,\mathrm{d}x$. Observe that for every z the mapping $b(z,\cdot) : \mathcal{Z} \to \mathbb{R}$ is an element of $\mathcal{Z}^*$. This proves that $\mathcal{E}_2(t,u,\cdot)$ is Gâteaux differentiable, with

$$\mathrm{D}_z\mathcal{E}_2(t,u,z)[\eta] = \int_\Omega g'(z)\eta\widetilde{W}(t,\nabla u)\,\mathrm{d}x \quad \text{for all } \eta \in \mathcal{Z}. \tag{2.30}$$

The previous calculations show that for $h \searrow 0$ we have

$$\limsup_{h\searrow 0} h^{-1}\big(\mathcal{I}_2(t,z+h\eta)-\mathcal{I}_2(t,z)\big) \le \lim_{h\searrow 0} h^{-1}\big(\mathcal{E}_2(t,u_{\min}(t,z),z+h\eta)-\mathcal{E}_2(t,u_{\min}(t,z),z)\big)$$
$$= \mathrm{D}_z\mathcal{E}_2(t,u_{\min}(t,z),z)[\eta].$$

On the other hand, for $h>0$ the following inequality is valid:

$$h^{-1}\big(\mathcal{I}_2(t,z+h\eta)-\mathcal{I}_2(t,z)\big) \ge h^{-1}\big(\mathcal{E}_2(t,u_{\min}(t,z+h\eta),z+h\eta)-\mathcal{E}_2(t,u_{\min}(t,z+h\eta),z)\big)$$
$$= \int_\Omega\int_0^1 g'(z+\sigma h\eta)\eta\widetilde{W}(t,\nabla u_{\min}(t,z+h\eta))\mathrm{d}\sigma\,\mathrm{d}x. \tag{2.31}$$

Choose $2<\widetilde{p}<p$ with p from Lemma 2.4. From (2.20) it follows that $u_{\min}(t,z+h\sigma\eta)\overset{h\to 0}{\to} u_{\min}(t,z)$ strongly in $W^{1,\widetilde{p}}(\Omega)$. Hence, $\widetilde{W}(t,\nabla u_{\min}(t,z+h\sigma\eta))$ converges strongly in $L^{\frac{\widetilde{p}}{2}}(\Omega)$ to $\widetilde{W}(t,\nabla u_{\min}(t,z))$. Moreover, $g'(z+h\sigma\eta)\eta$ converges to $g'(z)\eta$ strongly in $L^{\widetilde{p}/(\widetilde{p}-2)}(\Omega)$, since g' is continuous and bounded. Hence, the right-hand side in (2.31) converges to $\mathrm{D}_z\mathcal{E}_2(t,u_{\min}(t,z),z)[\eta]$ given by (2.30). This proves that for every $(t,z)\in[0,T]\times\mathcal{Z}$

$$\mathrm{D}_z\mathcal{I}_2(t,z)[\eta] = \int_\Omega g'(z)\eta\widetilde{W}(t,\nabla u_{\min}(t,z))\,\mathrm{d}x \quad \text{for all } \eta\in\mathcal{Z}, \tag{2.32}$$

whence (2.28). Relying on (2.8), which in particular yields that f' is Lipschitz continuous on $\mathbb{R}$, on (2.10), and on (2.16), we easily deduce the estimate

$$\|\mathrm{D}_z\mathcal{I}(t,z)\|_{\mathcal{Z}^*} \le C\left(1+\|z\|_{\mathcal{Z}}+\|u_{\min}\|^2_{H^1(\Omega)}\right).$$

Then, (2.29) ensues from estimate (2.17). □

Lemma 2.8 (Lipschitz continuity of $\mathrm{D}_z\mathcal{I}$)**.**
Let $s=d/2$. Assume (2.1)*,* (2.8)*–*(2.10) *and* (2.16)*, and set*

$$\widetilde{\mathcal{I}}(t,z) := \mathcal{I}_2(t,z)+\int_\Omega f(z)\,\mathrm{d}x \quad \textit{for all } (t,z)\in[0,T]\times\mathcal{Z}. \tag{2.33}$$

For every $r\in(\frac{3p}{p-2},+\infty)$ (where p is as in (2.14)*), there exists a constant $c_7>0$ depending on r, $\|\ell\|_{C^1([0,T];W^{-1,p}_{\Gamma_D}(\Omega))}$ and $\|u_D\|_{C^1([0,T];W^{1,p}(\Omega))}$, such that for all $t_i\in[0,T]$ and $z_i\in\mathcal{Z}$, $i=1,2$, we have*

$$\big\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1)-\mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\big\|_{L^{\sigma'}(\Omega)} \le c_7(|t_1-t_2|+\|z_1-z_2\|_{L^r(\Omega)}) \tag{2.34}$$

with $\sigma=\frac{rp}{pr-3p-2r}\in(1,+\infty)$ and σ' its conjugate exponent. In particular, there exists a constant c_8 depending on c_7 and r such that

$$\big\|\mathrm{D}_z\widetilde{\mathcal{I}}(t_1,z_1)-\mathrm{D}_z\widetilde{\mathcal{I}}(t_2,z_2)\big\|_{\mathcal{Z}^*} \le c_8(|t_1-t_2|+\|z_1-z_2\|_{L^r(\Omega)}). \tag{2.35}$$

Hence,

$$\textit{if } t_n\to t \textit{ and } z_n\rightharpoonup z \textit{ weakly in } \mathcal{Z}, \textit{ then } \mathrm{D}_z\widetilde{\mathcal{I}}(t_n,z_n)\to\mathrm{D}_z\widetilde{\mathcal{I}}(t,z) \textit{ strongly in } \mathcal{Z}^*. \tag{2.36}$$

Proof. Since f' is Lipschitz, in order to prove estimate (2.35) it remains to investigate the properties of $\mathrm{D}_z\mathcal{I}_2$, given by (2.28). For $i=1,2$, let $u_i:=u_{\min}(t_i,z_i)\in W^{1,p}(\Omega)$. For every fixed $r\in(\frac{3p}{p-2},\infty)$, set $\widetilde{p}=\frac{rp}{p+r}$, and notice that $2<\frac{3p}{p+1}<\widetilde{p}<p$, and that $r=\frac{p\widetilde{p}}{p-\widetilde{p}}>\frac{\widetilde{p}}{\widetilde{p}-2}$. Hence, the exponent σ defined by

$\sigma^{-1}+r^{-1}+2\widetilde{p}^{-1}=1$ belongs to $(1,\infty)$. For all $t_i\in[0,T]$, $z_i\in\mathcal{Z}$ and $\eta\in L^\sigma(\Omega)$ it follows with Hölder's inequality, and relying on the Lipschitz continuity of g, that

$$\begin{aligned}
&\int_\Omega \big(\mathrm{D}_z\mathcal{I}_2(t_1,z_1)-\mathrm{D}_z\mathcal{I}_2(t_2,z_2)\big)\eta\,\mathrm{d}x\\
&\leq \|\eta\|_{L^\sigma(\Omega)}\Big(\|g'(z_1)-g'(z_2)\|_{L^r(\Omega)}\Big\|\widetilde{W}(t_1,\nabla u_1)\Big\|_{L^{\frac{\widetilde{p}}{2}}(\Omega)}\\
&\qquad\qquad\qquad\qquad + \|g'(z_2)\|_{L^r(\Omega)}\Big\|\widetilde{W}(t_1,\nabla u_1)-\widetilde{W}(t_2,\nabla u_2)\Big\|_{L^{\frac{\widetilde{p}}{2}}(\Omega)}\Big)\\
&\leq C\,\|\eta\|_{L^\sigma(\Omega)}\Big(\|z_1-z_2\|_{L^r(\Omega)}\big(\|u_1\|^2_{W^{1,p}(\Omega)}+\|u_D\|^2_{L^\infty(0,T;W^{1,p}(\Omega))}\big)\\
&\qquad + \big(\|u_1+u_D(t_1)\|_{W^{1,p}(\Omega)}+\|u_2+u_D(t_2)\|_{W^{1,p}(\Omega)}\big)\big(\|u_1-u_2\|_{W^{1,\widetilde{p}}(\Omega)}+\|u_D(t_1)-u_D(t_2)\|_{W^{1,p}(\Omega)}\big)\Big)\\
&\leq C\,\|\eta\|_{L^\sigma(\Omega)}\big(\|z_1-z_2\|_{L^r(\Omega)}+|t_1-t_2|\big)\Big(\|\ell\|_{C^1([0,T];W^{-1,p}_{\Gamma_D}(\Omega))}+\|u_D\|_{C^1([0,T];W^{1,p}(\Omega))}\Big)^2. \qquad (2.37)
\end{aligned}$$

For the last estimate we have used (2.17) and (2.20), and (2.34) follows. Since for every $\sigma\in(1,\infty)$ the space $\mathcal{Z}$ is embedded in $L^\sigma(\Omega)$, hence $L^{\sigma'}(\Omega)\subset\mathcal{Z}^*$, we finally arrive at (2.35). Observe that the constant c_8 also depends on the embedding constant for $L^{\sigma'}(\Omega)\subset\mathcal{Z}^*$, and thus ultimately on r. □

Corollary 2.9 (Fréchet differentiability of $\mathcal{I}$)**.**
Let $s=d/2$. Under assumptions (2.1), (2.8)–(2.10) *and* (2.16)*, the functional $\mathcal{I}$ is Fréchet differentiable on $[0,T]\times\mathcal{Z}$ with a Lipschitz continuous derivative, i.e. $\mathcal{I}\in C^{1,1}([0,T]\times\mathcal{Z},\mathbb{R})$. Furthermore, $\widetilde{\mathcal{I}}$ (defined in* (2.33)*), $\partial_t\mathcal{I}$ and $\mathrm{D}_z\mathcal{I}$ are weakly continuous and $\mathcal{I}$ is weakly lower semicontinuous, i.e.*

$$t_n\to t \text{ and } z_n\rightharpoonup z \text{ weakly in } \mathcal{Z} \text{ implies } \begin{cases}\liminf_{n\to\infty}\mathcal{I}(t_n,z_n)\geq\mathcal{I}(t,z),\\ \widetilde{\mathcal{I}}(t_n,z_n)\to\widetilde{\mathcal{I}}(t,z),\\ \partial_t\mathcal{I}(t_n,z_n)\to\partial_t\mathcal{I}(t,z),\\ \mathrm{D}_z\mathcal{I}(t_n,z_n)\rightharpoonup\mathrm{D}_z\mathcal{I}(t,z) \text{ weakly in } \mathcal{Z}^*.\end{cases} \qquad (2.38)$$

Proof. This follows from the previous Lemmatas 2.6 and 2.8. Notice that the continuity property (2.38) of $\partial_t\mathcal{I}$ and $\mathrm{D}_z\mathcal{I}$ is an immediate consequence of estimates (2.27) and (2.35), joint with the compact embedding of $\mathcal{Z}$ in $L^r(\Omega)$. □

A further consequence of Lemma 2.8 is that $\mathrm{D}_z\mathcal{I}$ fulfills a "generalized" monotonicity property.

Corollary 2.10.
Let $s=d/2$. Under assumptions (2.1), (2.8)–(2.10) *and* (2.16)*, for every $r\in(\frac{3p}{p-2},+\infty)$ (where p is as in* (2.14)*), there exist constants c_9, $c_{10}>0$ such that for all $t\in[0,T]$ and $z_i\in\mathcal{Z}$, $i=1,2$, we have*

$$\|z_1-z_2\|^2_{L^2(\Omega)}+\langle\mathrm{D}_z\mathcal{I}(t,z_1)-\mathrm{D}_z\mathcal{I}(t,z_2),z_1-z_2\rangle_{\mathcal{Z}}\geq c_9\,\|z_1-z_2\|^2_{\mathcal{Z}}-c_{10}\,\|z_1-z_2\|^2_{L^2(\Omega)}\,. \qquad (2.39)$$

Proof. It is sufficient to observe that for any $r\in(\frac{3p}{p-2},+\infty)$ (where p is as in (2.14)) there holds

$$\begin{aligned}
&\|z_1-z_2\|^2_{L^2(\Omega)}+\langle\mathrm{D}_z\mathcal{I}(t,z_1)-\mathrm{D}_z\mathcal{I}(t,z_2),z_1-z_2\rangle_{\mathcal{Z}}\\
&=\|z_1-z_2\|^2_{L^2(\Omega)}+a_s(z_1-z_2,z_1-z_2)+\langle\mathrm{D}_z\widetilde{\mathcal{I}}(t,z_1)-\mathrm{D}_z\widetilde{\mathcal{I}}(t,z_2),z_1-z_2\rangle_{\mathcal{Z}}\\
&\geq\|z_1-z_2\|^2_{\mathcal{Z}}-c_8\,\|z_1-z_2\|_{\mathcal{Z}}\,\|z_1-z_2\|_{L^r(\Omega)}\,,
\end{aligned}$$

where $\widetilde{\mathcal{I}}$ is defined as in (2.33). Then, (2.39) follows upon using that $\mathcal{Z}\Subset L^r(\Omega)\subset L^2(\Omega)$, and the well-known fact that for every $\eta>0$ there exists $C_\eta>0$ such that for all $z\in\mathcal{Z}$ we have $\|z\|_{L^r(\Omega)}\leq\eta\,\|z\|_{\mathcal{Z}}+C_\eta\,\|z\|_{L^2(\Omega)}$. □

2.4 Improved estimates under special conditions

If the boundary of Ω is smooth and if the coefficients $g(z)\mathbb{C}$ in the elastic energy functional (2.11) are continuous on $\overline{\Omega}$, then the previous estimates (2.20), (2.27), and (2.35) can be refined. These improvements will be relevant for the uniqueness analysis of the viscous problem, see Section 3.2.

Throughout this section, in addition to (2.1) and (2.9a) we suppose that

$$\begin{gathered}\Omega \subset \mathbb{R}^d,\ d \in \{2,3\},\ \text{is a bounded domain with } \mathrm{C}^1\text{-boundary } \partial\Omega \\ \text{and Dirichlet boundary } \Gamma_D = \partial\Omega,\end{gathered} \tag{2.40}$$

$$s > d/2. \tag{2.41}$$

Observe that (2.41) implies

$$\mathcal{Z} \Subset \mathrm{C}^{0,\alpha}(\overline{\Omega}) \ \text{ for some } \alpha \in (0,1]. \tag{2.42}$$

We shall then also require that, for the same $\alpha \in (0,1]$,

$$\mathbb{C} \in \mathrm{C}^{0,\alpha}(\overline{\Omega}, \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}_{\mathrm{sym}})). \tag{2.43}$$

Under these conditions, we may apply to the linear elliptic operator $L_z(u) = -\operatorname{div}(g(z)\mathbb{C}\varepsilon(u))$ (cf. (2.13)) a $W^{1,p}$-regularity result for weak solutions of partial differential equations on smooth domains, see e.g. [Giu03, Section 10.4]. Adapted to our situation it reads:

$$\text{for every } p \in [2,\infty) \text{ the operator } L_z : W^{1,p}_0(\Omega;\mathbb{R}^d) \to W^{1,p'}_0(\Omega;\mathbb{R}^d)^* \text{ is an isomorphism,} \tag{2.44}$$

and the operator-norm of L_z^{-1} depends uniformly on the ellipticity constant γ_0, and on the Hölder-norm of $\mathbb{C}$ and of $g(z)$ (thus, ultimately, on $\|z\|_{\mathcal{Z}}$ in view of (2.42)).

In this setting, we have the following improved estimates.

Proposition 2.11. *In addition to* (2.1), (2.8)–(2.10), *assume* (2.40), (2.41), *and* (2.43). *Let $p \in (2,\infty)$ be fixed, and suppose that* (2.16) *holds for the index p. Then, the estimates in Lemmatas 2.4–2.8 are valid, with constants depending uniformly on $\|z\|_{\mathcal{Z}}$. In particular, for $p = 4$ there holds: for all $M > 0$ there exist positive constants $\tilde{c}_0 = \tilde{c}_0(M)$, $\tilde{c}_3 = \tilde{c}_3(M)$, such that*

$$\forall\,(t,z) \in [0,T]\times\mathcal{Z},\ \|z\|_{\mathcal{Z}} \le M :\quad \|u_{\min}(t,z)\|_{W^{1,4}_{\Gamma_D}(\Omega)} \le \tilde{c}_0\left(\|\ell(t)\|_{W^{-1,4}_{\Gamma_D}(\Omega)} + \|u_D(t)\|_{W^{1,4}(\Omega)}\right), \tag{2.45}$$

$$\begin{aligned}&\forall\,(t_i,z_i) \in [0,T]\times\mathcal{Z},\ \|z_i\|_{\mathcal{Z}} \le M,\ i=1,2 :\\ &\|u_{\min}(t_1,z_1) - u_{\min}(t_2,z_2)\|_{W^{1,4}_{\Gamma_D}(\Omega)}\\ &\le \tilde{c}_3\left(|t_1-t_2| + \|z_1-z_2\|_{\mathcal{Z}}\right)\left(\|\ell\|_{\mathrm{C}^1([0,T];W^{-1,4}_{\Gamma_D}(\Omega))} + \|u_D(t)\|_{\mathrm{C}^1([0,T];W^{1,4}(\Omega))}\right).\end{aligned} \tag{2.46}$$

Proof. Estimate (2.45) can be proved by the very same argument as for (2.17), relying on (2.44) for $p = 4$. Estimate (2.46) can be obtained as in the proof of Lemma 2.5, up to the following changes: One chooses $\widetilde{p} = 4$ in (2.22), and $p = \widetilde{p} = 4$ and $r = \infty$ in (2.23) and (2.24). □

3 The viscous problem

The viscous approximation. Recall that $\mathcal{R}_\epsilon = \mathcal{R}_1 + \mathcal{R}_{2,\epsilon}$, with $\mathcal{R}_{2,\epsilon}(\eta) = \frac{\epsilon}{2}\|\eta\|^2_{L^2(\Omega)}$, denotes the *viscous* dissipation functional, and $\partial\mathcal{R}_\epsilon : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ is its subdifferential (in the sense of convex analysis), in the duality between $\mathcal{Z}^*$ and $\mathcal{Z}$, cf. with (1.4). Throughout this section, we shall analyze the *viscous* doubly nonlinear evolution equation

$$\partial\mathcal{R}_\epsilon(z'(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a.}\, t \in (0, T), \tag{3.1}$$

with the initial condition, featuring $z_0 \in \mathcal{Z}$,

$$z(0) = z_0. \tag{3.2}$$

We shall denote by $\mathcal{R}_\varepsilon^*$ the convex conjugate of the functional $\mathcal{R}_\varepsilon$, taken in the $\mathcal{Z} - \mathcal{Z}^*$ duality, viz.

$$\mathcal{R}_\epsilon^*(\sigma) = \sup\left\{ \langle\sigma, \eta\rangle_{\mathcal{Z}} - \mathcal{R}_\epsilon(\eta) \ : \ \eta \in \mathcal{Z} \right\}.$$

The following lemma collects, for later use, two crucial formulae for $\partial\mathcal{R}_\epsilon$ and $\mathcal{R}_\epsilon^*$.

Lemma 3.1. *There holds*

$$\partial\mathcal{R}_\epsilon(\eta) = \partial\mathcal{R}_1(\eta) + \partial\mathcal{R}_{2,\epsilon}(\eta) = \partial\mathcal{R}_1(\eta) + \epsilon\eta \quad \textit{for all } \eta \in \mathcal{Z}; \tag{3.3}$$

$$\mathcal{R}_\epsilon^*(\sigma) = \inf_{\mu\in\partial\mathcal{R}_1(0)} \frac{1}{\epsilon}\widetilde{\mathcal{R}}_2(\sigma-\mu) = \frac{1}{\epsilon}\min_{\mu\in\partial\mathcal{R}_1(0)} \widetilde{\mathcal{R}}_2(\sigma-\mu)$$
$$\textit{with } \widetilde{\mathcal{R}}_2(\sigma) := \begin{cases} \frac{1}{2}\|\sigma\|_{L^2(\Omega)}^2 & \textit{if } \sigma \in L^2(\Omega), \\ +\infty & \textit{if } \sigma \in \mathcal{Z}^*\backslash L^2(\Omega). \end{cases} \tag{3.4}$$

Proof. The first identity in (3.3) follows from [AE84, Cor. IV.6]. Next, we observe $\partial\mathcal{R}_{2,\epsilon}(\eta) = \{\mathrm{D}\mathcal{R}_{2,\epsilon}(\eta)\}$, as $\mathcal{R}_{2,\epsilon}$ is Fréchet differentiable on $\mathcal{Z}$. On account for (2.7), $\mathrm{D}\mathcal{R}_{2,\epsilon}$ coincides with the Fréchet derivative of $\mathcal{R}_{2,\epsilon}$ in the $L^2(\Omega)$-topology, whence the second identity in (3.3).

The $\inf - \sup$ convolution formula (see, e.g., [IT79, Theorem 3.3.4.1]), for $\mathcal{R}_\epsilon^*$ yields

$$\mathcal{R}_\epsilon^*(\sigma) = \inf\left\{\mathcal{R}_{2,\epsilon}^*(\sigma-\mu) \ : \ \mu \in \partial\mathcal{R}_1(0)\right\}$$

whence the first identity in (3.4). Using that $\partial\mathcal{R}_1(0) \subset \mathcal{Z}^*$ is weakly closed, it can be easily checked that the inf is in fact a min. □

As a consequence of (3.3) and of (2.28), the doubly nonlinear evolution equation (3.1) reads

$$\partial\mathcal{R}_1(z'(t)) + \epsilon z'(t) + A_s(z(t)) + f'(z(t)) + g'(z(t))\widetilde{W}(t, \nabla u_{\min}(t, z(t))) \ni 0 \quad \text{for a.a. } t \in (0, T). \tag{3.5}$$

Finally, for later convenience we observe that, by the 1-positive homogeneity of $\mathcal{R}_1$, its convex analysis subdifferential $\partial\mathcal{R}_1$ satisfies the following relations for every $v \in \mathcal{Z}$:

$$\eta \in \partial\mathcal{R}_1(v) \ \Rightarrow \ \begin{cases} \langle\eta, v\rangle_{\mathcal{Z}} = \mathcal{R}_1(v), \\ \langle\eta, w\rangle_{\mathcal{Z}} \le \mathcal{R}_1(w) \text{ for all } w \in \mathcal{Z}. \end{cases} \tag{3.6}$$

3.1 Existence and a priori estimates for viscous solutions

The following result clarifies the properties of solutions to (3.1) (equivalently, of (3.5)), with the regularity $z \in H^1(0, T; \mathcal{Z})$.

Proposition 3.2. *Let $s = d/2$. Assume* (2.1), (2.8)–(2.10), *and* (2.16). *Then, for a curve $z \in H^1(0, T; \mathcal{Z})$ the following are equivalent:*

1. *z is a solution to* (3.1);

2. *z fulfills for all* $0 \le s \le t \le T$ *the* energy identity

$$\int_s^t \mathcal{R}_\epsilon(z'(\tau))\mathrm{d}\tau + \int_s^t \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\tau, z(\tau)))\mathrm{d}\tau + \mathcal{I}(t, z(t)) = \mathcal{I}(s, z(s)) + \int_s^t \partial_t\mathcal{I}(\tau, z(\tau))\mathrm{d}\tau; \quad (3.7)$$

3. *z fulfills for all* $0 \le t \le T$ *the* energy inequality

$$\int_0^t \mathcal{R}_\epsilon(z'(\tau))\mathrm{d}\tau + \int_0^t \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\tau, z(\tau)))\mathrm{d}\tau + \mathcal{I}(t, z(t)) \le \mathcal{I}(0, z(0)) + \int_0^t \partial_t\mathcal{I}(\tau, z(\tau))\mathrm{d}\tau. \quad (3.8)$$

Proof. We start by observing the following crucial fact: For every $z \in H^1(0, T; \mathcal{Z})$, there holds $\mathrm{D}_z\mathcal{I}(\cdot, z(\cdot)) \in L^\infty(0, T; \mathcal{Z}^*)$ (thanks to (2.29)), and Corollary 2.9 guarantees the *chain rule* identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t, z(t)) = \partial_t\mathcal{I}(t, z(t)) + \langle \mathrm{D}_z\mathcal{I}(t, z(t)), z'(t)\rangle_{\mathcal{Z}} \quad \text{for a.a.}\, t \in (0, T). \quad (3.9)$$

Clearly, (3.7) implies (3.8). Suppose now that z fulfills (3.8): applying (3.9) we have that $\mathcal{I}(0, z(0)) + \int_0^t \partial_t\mathcal{I}(\tau, z(\tau))\mathrm{d}\tau = \mathcal{I}(t, z(t)) + \int_0^t \langle -\mathrm{D}_z\mathcal{I}(\tau, z(\tau)), z'(\tau)\rangle_{\mathcal{Z}}\, \mathrm{d}\tau$, so that from (3.8) we deduce

$$\int_0^t \mathcal{R}_\epsilon(z'(\tau))\mathrm{d}\tau + \int_0^t \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\tau, z(\tau)))\mathrm{d}\tau + \mathcal{I}(t, z(t)) \le \mathcal{I}(t, z(t)) + \int_0^t \langle -\mathrm{D}_z\mathcal{I}(\tau, z(\tau)), z'(\tau)\rangle_{\mathcal{Z}}\, \mathrm{d}\tau.$$

Taking into account the elementary convex analysis inequality $\langle \zeta, v\rangle_{\mathcal{Z}} \le \mathcal{R}_\epsilon(v) + \mathcal{R}_\epsilon^*(\zeta)$ for all $z \in \mathcal{Z}$, $\zeta \in \mathcal{Z}^*$, we immediately conclude that the above integral inequality indeed holds as an equality, in fact pointwise

$$\mathcal{R}_\epsilon(z'(t)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(t, z(t))) - \langle -\mathrm{D}_z\mathcal{I}(t, z(t)), z'(t)\rangle_{\mathcal{Z}} = 0 \quad \text{for a.a.}\, t \in (0, T).$$

Again by convex analysis, from the above relation we infer that $-\mathrm{D}_z\mathcal{I}(t, z(t)) \in \partial\mathcal{R}_\epsilon(z'(t))$ for almost all $t \in (0, T)$, i.e. z is a solution to (3.1).

Suppose now that z fulfills (3.1), test it by $z'(t)$, and use for every $\xi(t) \in \partial\mathcal{R}_\epsilon(z'(t))$ the convex analysis identity $\langle \xi(t), z'(t)\rangle_{\mathcal{Z}} = \mathcal{R}_\epsilon(z'(t)) + \mathcal{R}_\epsilon^*(\xi(t))$ for a.a. $t \in (0, T)$. Then, (3.7) follows upon applying the chain rule (3.9), and integrating on (s, t) for all $0 \le s \le t$. □

We may now state our main result on the viscous problem (3.1).

Theorem 3.3. *Let* $s = d/2$. *Assume* (2.1), (2.8)–(2.10), *and* (2.16). *Suppose that the initial datum* $z_0 \in \mathcal{Z}$ *additionally fulfills*

$$\mathrm{D}_z\mathcal{I}(0, z_0) \in L^2(\Omega). \quad (3.10)$$

Then,

1. *for every* $\epsilon > 0$ *there exists a viscous solution* $z_\epsilon \in H^1(0, T; \mathcal{Z})$ *to the Cauchy problem* (3.1)–(3.2), *satisfying for all* $0 \le s \le t \le T$ *the* energy identity

$$\begin{aligned} \int_s^t \big(\mathcal{R}_1(z_\epsilon'(\tau)) + \epsilon\mathcal{R}_2(z_\epsilon'(\tau)) + \frac{1}{\epsilon}\min_{\mu \in \partial\mathcal{R}_1(0)} \widetilde{\mathcal{R}}_2(-\mathrm{D}_z\mathcal{I}(\tau, z_\epsilon(\tau)) - \mu)\big)\,\mathrm{d}\tau + \mathcal{I}(t, z_\epsilon(t)) \\ = \mathcal{I}(s, z_\epsilon(s)) + \int_s^t \partial_t\mathcal{I}(\tau, z_\epsilon(\tau))\mathrm{d}\tau. \end{aligned} \quad (3.11)$$

2. *There exists a family of viscous solutions* $(z_\epsilon)_{\epsilon>0}$ *and a constant* $C_0 > 0$ *such that*

$$\sup_{\epsilon>0} \int_0^T \|z_\epsilon'(\tau)\|_{\mathcal{Z}}\, \mathrm{d}\tau \le C_0. \quad (3.12)$$

Outlook to the proof of Theorem 3.3. The *proof* will be developed in Section 4 working on the time-discretization scheme associated with (3.1).

First, we will prove for the approximate solutions $(\hat{z}_\tau)_\tau$ (obtained through linear interpolation of the discrete solutions with time-step τ; we omit to highlight in the notation their dependence on ϵ), the estimate $\sup_\tau \|\hat{z}_\tau\|_{H^1(0,T;\mathcal{Z})} \leq C(\epsilon)$, with $C(\epsilon)$ depending on $\epsilon > 0$ and exploding as $\epsilon \to 0$. Second, for $(\hat{z}_\tau)_\tau$ we will obtain an estimate in $\mathrm{BV}([0,T];\mathcal{Z})$, for a constant *independent* of ϵ.

Furthermore, arguing on the time-discrete approximation of (3.1), we shall prove the following remarkable fact (cf. Proposition 4.5): Under special conditions, if $d = 2$ and if the initial datum z_0 fulfills $z_0(x) \in [0,1]$ for a.a. $x \in \Omega$, then there exist viscous solutions z_ϵ with $z_\epsilon(x) \in [0,1]$ for a.a. $x \in \Omega$.

Remark 3.4 (A different approach to the proof of Theorem 3.3). As an alternative to the time-discretization scheme for (3.1), one might consider the "augmented" viscous dissipation for $\eta \in \mathcal{Z}$

$$\mathcal{R}_{\epsilon,\delta}(\eta) := \mathcal{R}_\epsilon(\eta) + \mathcal{R}_{\mathcal{Z},\delta}(\eta) = \mathcal{R}_1(\eta) + \mathcal{R}_{2,\epsilon}(\eta) + \mathcal{R}_{\mathcal{Z},\delta}(\eta) \quad \text{with} \quad \mathcal{R}_{\mathcal{Z},\delta}(\eta) = \tfrac{\delta}{2}\,|\eta|^2_{\mathcal{Z}}\,,$$

where $|\eta|_{\mathcal{Z}} := \sqrt{a_s(\eta,\eta)}$ denotes the semi-norm on $\mathcal{Z}$ induced by the bilinear form a_s and $\delta > 0$. This approach was developed in detail in [KRZ11, Sec. 4]. The arguments to prove Theorem 3.3 are slightly shorter compared to the proof via time-discretization. However, in view of further developments in the direction of numerical analysis we have confined ourselves to the approach via time-discretization. Moreover, let us stress that in the special framework of Prop. 4.5, time-discretization brings about additional information. If the solutions to the Cauchy problem (3.1)–(3.2) are unique (see Section 3.2 for sufficient conditions), then the solutions arising as limits of the time-discrete scheme, and the solutions as limits of the δ-viscous problem as $\delta \to 0$, do coincide.

Remark 3.5 (Results under the sole condition $z_0 \in \mathcal{Z}$). As it will be clear from the discussion in Section 4 (cf. Remark 4.4), we are not able to prove existence of viscous solutions under the *sole* condition $z_0 \in \mathcal{Z}$. Indeed, with the latter condition, and standard energy estimates in the time-discrete approximate problem, we can prove that there exists $z_\epsilon \in L^\infty(0,T;\mathcal{Z}) \cap H^1(0,T;L^2(\Omega))$, with $z_\epsilon(0) = z_0$, fulfilling the *energy inequality* (3.8) for all $0 \leq t \leq T$. However, without the additional condition (3.10), we are not able to obtain the further regularity $z_\epsilon \in H^1(0,T;\mathcal{Z})$ (viz., the higher spatial regularity $z_\epsilon' \in L^2(0,T;\mathcal{Z})$ for z_ϵ'). On the other hand, only such a regularity ensures the validity of the chain rule (3.9), and (3.9) is the key point for deducing, from (3.8), that z_ϵ in fact fulfills (3.1), cf. the proof of Proposition 3.2.

3.2 Uniqueness for the viscous problem under special conditions

In the setting of Proposition 2.11, we have the following uniqueness result for viscous solutions.

Proposition 3.6 (Uniqueness for viscous solutions)**.** *In addition to* (2.1)*,* (2.8)*–*(2.10)*, assume* (2.40)*,* (2.41)*, and* (2.43)*. Suppose further that for $i \in \{1,2\}$ the data (u_D^i, ℓ^i) satisfy* (2.16) *for $p = 4$, and that for $z_0^i \in \mathcal{Z}$ it holds $\mathrm{D}_z\mathcal{I}(0, z_0^i) \in L^2(\Omega)$. Let z_1, $z_2 \in H^1(0,T;\mathcal{Z})$ be solutions to* (3.1)*, supplemented with the data (u_D^1, ℓ^1) and (u_D^2, ℓ^2), respectively. Set $M = \sum_{i=1}^2 \|z_i\|_{L^\infty(0,T;\mathcal{Z})}$. Then, there exists a constant $C_1 > 0$, depending on M, on T, and on γ_2 (cf.* (2.10)*), such that for almost all $t \in (0,T)$ it holds*

$$\begin{aligned} &\|z_1(t) - z_2(t)\|_{\mathcal{Z}} + \sqrt{\epsilon}\,\|z_1' - z_2'\|_{L^2(0,t;L^2(\Omega))} \\ &\qquad \leq C_1 \left(\left\|z_0^1 - z_0^2\right\|_{\mathcal{Z}} + \left\|u_D^1 - u_D^2\right\|_{L^2(0,T;W^{1,4}(\Omega))} + \left\|\ell^1 - \ell^2\right\|_{L^2(0,T;W^{-1,4}_{\Gamma_D}(\Omega))} \right). \end{aligned} \tag{3.13}$$

Proof. We subtract the differential inclusion (3.1) for z_2 from (3.1) for z_1, and we use $z_1' - z_2'$ as test function. Taking into account that, by monotonicity, $\langle \partial\mathcal{R}_1(z_1'(t)) - \partial\mathcal{R}_1(z_2'(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}} \geq 0$ for almost all $t \in (0,T)$ (where with abuse of notation we have written $\partial\mathcal{R}_1$ as single-valued), we arrive at the following inequality

$$\langle \epsilon(z_1'(t) - z_2'(t)) + \mathrm{D}_z\mathcal{I}(t, z_1(t)) - \mathrm{D}_z\mathcal{I}(t, z_2(t)), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}} \le 0 \quad \text{for a.a. } t \in (0,T)$$

We rearrange the terms, and add and subtract $\langle z_1(t) - z_2(t), z_1'(t) - z_2'(t)\rangle_{\mathcal{Z}}$. Thus,

$$\begin{aligned}\epsilon \|z_1(t) - z_2(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_1(t) - z_2(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a_s(z_1(t) - z_2(t), z_1(t) - z_2(t))\\ \le -\int_\Omega (f'(z_1(t)) - z_1(t) - f'(z_2(t)) + z_2(t))(z_1'(t) - z_2'(t))\,\mathrm{d}x \\ -\int_\Omega (G_1(t) - G_2(t))(z_1'(t) - z_2'(t))\,\mathrm{d}x \doteq S_1 + S_2,\end{aligned} \tag{3.14}$$

where we have used the short-hand notation $G_i(t) := g'(z_i(t))\widetilde{W}(t, \nabla u_{\min}(t, z_i(t)))$, and $u_{\min}(t, z_i(t))$ is the minimizer of $\mathcal{E}_2^i(t; \cdot, z_i(t))$ with the data $u_D^i(t)$, $\ell^i(t)$.

Now, for almost all $t \in (0,T)$ the following estimate holds

$$\begin{aligned}&\|G_1(t) - G_2(t)\|^2_{L^2(\Omega)}\\ &\le 2\int_\Omega |g'(z_1(t)) - g'(z_2(t))|^2 |\widetilde{W}(t, \nabla u_{\min}(t, z_1(t)))|^2\,\mathrm{d}x\\ &\quad + 2\int_\Omega |g'(z_2(t))|^2 |\widetilde{W}(t, \nabla u_{\min}(t, z_1(t))) - \widetilde{W}(t, \nabla u_{\min}(t, z_2(t)))|^2\,\mathrm{d}x\\ &\le C\|z_1(t) - z_2(t)\|^2_{L^\infty(\Omega)} (\|u_{\min}(t, z_1(t))\|^4_{W^{1,4}(\Omega)} + \|u_D^1(t)\|^4_{W^{1,4}(\Omega)})\\ &\quad + C'\|g(z_2(t))\|^2_{L^\infty(\Omega)} (\|u_{\min}(t, z_1(t)) + u_{\min}(t, z_2(t))\|^2_{W^{1,4}(\Omega)} + \|u_D^1(t) + u_D^2(t)\|^2_{W^{1,4}(\Omega)})\\ &\qquad \times (\|u_{\min}(t, z_1(t)) - u_{\min}(t, z_2(t))\|^2_{W^{1,4}(\Omega)} + \|u_D^1(t) - u_D^2(t)\|^2_{W^{1,4}(\Omega)})\\ &\le C(M, \gamma_2)\Big(1 + \sum_{i=1}^2 \|u_D^i(t)\|^4_{W^{1,4}(\Omega)} + \|\ell^i(t)\|^4_{W^{-1,4}_{\Gamma_D}(\Omega)}\Big)\\ &\qquad \times \Big(\|z_1(t) - z_2(t)\|^2_{\mathcal{Z}} + \|u_D^1(t) - u_D^2(t)\|^2_{W^{1,4}(\Omega)} + \|\ell^1(t) - \ell^2(t)\|^2_{W^{-1,4}_{\Gamma_D}(\Omega)}\Big)\end{aligned}$$

where the second inequality follows from (2.10) and Hölder's inequality, and the last one from the estimate

$$\begin{aligned}&\|u_{\min}(t, z_1(t)) - u_{\min}(t, z_2(t))\|_{W^{1,4}(\Omega)}\\ &\quad \le \|u_{\min}(t, z_1(t), \ell^1(t), u_D^1(t)) - u_{\min}(t, z_1(t), \ell^2(t), u_D^2(t))\|_{W^{1,4}(\Omega)}\\ &\qquad + \|u_{\min}(t, z_1(t), \ell^2(t), u_D^2(t)) - u_{\min}(t, z_2(t), \ell^2(t), u_D^2(t))\|_{W^{1,4}(\Omega)}\\ &\quad \le C'(M, \gamma_2)\big(\|z_1(t) - z_2(t)\|_{\mathcal{Z}} + \|\ell^1(t) - \ell^2(t)\|_{W^{-1,4}_{\Gamma_D}(\Omega)} + \|u_D^1(t) - u_D^2(t)\|_{W^{1,4}(\Omega)}\big)\end{aligned}$$

for some $C'(M, \gamma_2)$, which follows from (2.45) and (2.46). Notice that the constants $C(M, \gamma_2)$ and $C'(M, \gamma_2)$ depend on M and on γ_2.

Then, also taking into account that f' is Lipschitz continuous, the terms S_1 and S_2 on the right-hand side of (3.14) can be estimated via

$$\begin{aligned}|S_1| &\le \frac{\epsilon}{4}\|z_1'(t) - z_2'(t)\|^2_{L^2(\Omega)} + C\|z_1(t) - z_2(t)\|^2_{L^2(\Omega)},\\ |S_2| &\le \frac{\epsilon}{4}\|z_1'(t) - z_2'(t)\|^2_{L^2(\Omega)} + C\|G_1(t) - G_2(t)\|^2_{L^2(\Omega)}\\ &\le \frac{\epsilon}{4}\|z_1'(t) - z_2'(t)\|^2_{L^2(\Omega)}\\ &\qquad + C\Big(\|z_1(t) - z_2(t)\|^2_{\mathcal{Z}} + \|\ell^1(t) - \ell^2(t)\|^2_{W^{-1,4}_{\Gamma_D}(\Omega)} + \|u_D^1(t) - u_D^2(t)\|^2_{W^{1,4}(\Omega)}\Big).\end{aligned}$$

Now, we integrate (3.14) on $(0, t)$, and, taking into account all of the above calculations, conclude

$$\frac{\epsilon}{2}\left\|z_1'-z_2'\right\|^2_{L^2(0,t;L^2(\Omega))}+c\left\|z_1(t)-z_2(t)\right\|^2_{\mathcal{Z}}$$
$$\leq C\left\|z_0^1-z_0^2\right\|^2_{\mathcal{Z}}+C\int_0^t\left(\left\|z_1(\tau)-z_2(\tau)\right\|^2_{\mathcal{Z}}+\left\|\ell^1(\tau)-\ell^2(\tau)\right\|^2_{W^{-1,4}_{\Gamma_D}(\Omega)}+\left\|u_D^1(\tau)-u_D^2(\tau)\right\|^2_{W^{1,4}(\Omega)}\right)\mathrm{d}\tau.$$

Gronwall's inequality (cf. e.g. [Bre73, Lemma A.4]) finally yields (3.13). □

We conclude this section with the following corollary of Proposition 3.6.

Theorem 3.7. *In addition to* (2.1), (2.8)–(2.10), *assume* (2.40), (2.41), (2.43), *and that* (2.16) *holds for* $p = 4$. *Then, for every initial datum* $z_0 \in \mathcal{Z}$ *fulfilling* (3.10), *there exists a* unique *solution* $z \in H^1(0, T; \mathcal{Z})$ *to the Cauchy problem for* (3.1).

4 Time-discrete viscous approximation and uniform estimates

In this section, we shall prove Theorem 3.3 by passing to the limit in the time-discretization scheme which we set up below. First, in Section 4.1 we show the *existence* of viscous solutions. Next, in Section 4.2 we prove the BV-estimate (3.12).

Throughout this section, we omit the dependence of the discrete solutions on $\epsilon > 0$, and only highlight their dependence on the fineness of the non-constant time-steps.

Time-incremental problem. We consider the following time-discrete incremental minimization problem: Given $\epsilon > 0$, $z_0 \in \mathcal{Z}$ and a partition $\{0 = t_0^\tau < \ldots < t_N^\tau = T\}$ of the time interval $[0, T]$ with fineness $\tau = \sup_{0\leq k\leq N}(t_{k+1}^\tau - t_k^\tau)$, the elements $(z_k^\tau)_{0\leq k\leq N}$ are determined through $z_0^\tau = z_0$ and

$$z_{k+1}^\tau \in \operatorname{Argmin}\left\{\mathcal{I}(t_{k+1}^\tau, z)+\tau_k\mathcal{R}_\epsilon\left(\frac{z-z_k^\tau}{\tau_k}\right)\ ;\ z\in\mathcal{Z}\right\}. \tag{4.1}$$

Here, $\tau_k = t_{k+1}^\tau - t_k^\tau$ and $\mathcal{R}_\epsilon$ is defined in (1.9). The existence of minimizers follows with the direct method in the calculus of variations, thanks to the properties of the reduced energy $\mathcal{I}$ formulated in Section 2.3. Relying on Corollary 2.10, it can be easily shown that, indeed, the minimum problem 4.1 has a *unique* solution provided that τ is small enough.

We point out that any family $\{z_1^\tau, \ldots, z_N^\tau\} \subset \mathcal{Z}$ of minimizers of the incremental problem (4.1) satisfy for all $k \in \{0, \ldots, N-1\}$ the *discrete Euler-Lagrange* equation

$$\partial\mathcal{R}_1\left(\frac{z_{k+1}^\tau-z_k^\tau}{\tau_k}\right)+\epsilon\frac{z_{k+1}^\tau-z_k^\tau}{\tau_k}+\mathrm{D}_z\mathcal{I}(t_{k+1}, z_{k+1})\ni 0, \tag{4.2}$$

also taking into account (3.3).

Notation 4.1. The following piecewise constant and piecewise linear interpolation functions will be used in the sequel:

$$\begin{aligned}
&\overline{z}_\tau(t) = z_{k+1}^\tau && \text{for } t\in(t_k^\tau, t_{k+1}^\tau],\\
&\underline{z}_\tau(t) = z_k^\tau && \text{for } t\in[t_k^\tau, t_{k+1}^\tau),\\
&\hat{z}_\tau(t) = z_k^\tau+\frac{t-t_k^\tau}{\tau_k}(z_{k+1}^\tau-z_k^\tau) && \text{for } t\in[t_k^\tau, t_{k+1}^\tau].
\end{aligned}$$

Furthermore, we shall use the notation

$$\tau(r)=\tau_k \text{ for } r\in(t_k^\tau, t_{k+1}^\tau),\quad \overline{t}_\tau(r)=t_{k+1}^\tau \text{ for } r\in(t_k^\tau, t_{k+1}^\tau],\quad \underline{t}_\tau(r)=t_k^\tau \text{ for } r\in[t_k^\tau, t_{k+1}^\tau).$$

Clearly,

$$\overline{t}_\tau(t),\ \underline{t}_\tau(t) \to t \quad \text{ as } \tau \to 0 \text{ for all } t \in [0,T]. \tag{4.3}$$

Moreover, for any given function b which is piecewise constant on the intervals (t_i^τ, t_{i+1}^τ) we set

$$\triangle_{\tau(r)} b(r) = b(r) - b(s) \text{ for } r \in (t_k^\tau, t_{k+1}^\tau) \text{ and } s \in (t_{k-1}^\tau, t_k^\tau).$$

With the above notation, (4.2) can be reformulated as

$$\partial\mathcal{R}_1\left(\hat{z}_\tau'(t)\right) + \epsilon\hat{z}_\tau'(t) + \mathrm{D}_z\mathcal{I}(\overline{t}_\tau(t), \overline{z}_\tau(t)) \ni 0 \quad \text{for a.a.}\, t \in (0,T). \tag{4.4}$$

4.1 Existence of viscous solutions

The following result states the crucial a priori estimate on the approximate solutions $(\hat{z}_\tau)_\tau$.

Proposition 4.2. *Let $s = d/2$, and assume* (2.1), (2.8)–(2.10) *and* (2.16). *Suppose that $z_0 \in \mathcal{Z}$ also fulfills* (3.10), *viz.* $\mathrm{D}_z\mathcal{I}(0,z_0) \in L^2(\Omega)$.

Then, there exist constants C_{12}, $C_{13} > 0$ such that for every $\epsilon, \tau > 0$ the solutions of the time incremental problem (4.1) *satisfy*

$$\int_0^T \|\hat{z}_\tau'(t)\|_{\mathcal{Z}}^2 \,\mathrm{d}t \le \left(1 + \epsilon^{-1}\|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{L^2(\Omega)}^2\right) \exp\left(C_{12}(1+\tfrac{T}{\epsilon})\right), \tag{4.5}$$

$$\epsilon\left\|\hat{z}_\tau'\left(\frac{t_1^\tau}{2}\right)\right\|_{L^2(\Omega)} \le C_{13}\left(\sqrt{\epsilon} + \|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{L^2(\Omega)}\right)\exp\left(\frac{C_{12}}{\epsilon}t_1^\tau\right), \tag{4.6}$$

where t_1^τ is the first non-zero node of the partition of $[0,T]$.

Proof. Let $\tau > 0$ and let $\{z_1^\tau, \dots, z_N^\tau\} \subset \mathcal{Z}$ be minimizers of the incremental problem (4.1). For $t \in (t_k^\tau, t_{k+1}^\tau)$ we define $\overline{h}_\tau(t) := \epsilon\hat{z}_\tau'(t) + A_s\overline{z}_\tau(t) + \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(t), \overline{z}_\tau(t))$, where $\widetilde{\mathcal{I}}$ is as in (2.33). Hence, relation (4.4) is equivalent to $-\overline{h}_\tau(t) \in \partial\mathcal{R}_1(\hat{z}_\tau'(t))$ for $t \in (t_k^\tau, t_{k+1}^\tau)$. From the 1-homogeneity of $\mathcal{R}_1$ using (3.6) we deduce

$$\forall\, t \in (t_k^\tau, t_{k+1}^\tau) \qquad -\mathcal{R}_1(\hat{z}_\tau'(t)) = \langle \overline{h}_\tau(t), \hat{z}_\tau'(t)\rangle_{\mathcal{Z}}, \tag{4.7}$$

$$\forall\, r \in [0,T]\backslash\{t_0^\tau, \dots, t_N^\tau\} \qquad \mathcal{R}_1(\hat{z}_\tau'(t)) \ge \langle -\overline{h}_\tau(r), \hat{z}_\tau'(t)\rangle_{\mathcal{Z}}. \tag{4.8}$$

Adding both relations and choosing $\rho \in (t_i^\tau, t_{i+1}^\tau)$ and $\sigma \in (t_{i-1}^\tau, t_i^\tau)$, it follows

$$0 \ge \tau_i^{-1}\langle \overline{h}_\tau(\rho) - \overline{h}_\tau(\sigma), \hat{z}_\tau'(\rho)\rangle_{\mathcal{Z}}.$$

This relation can be rewritten as

$$\begin{aligned}\epsilon\tau_i^{-1}\langle \hat{z}_\tau'(\rho) - \hat{z}_\tau'(\sigma), \hat{z}_\tau'(\rho)\rangle_{L^2(\Omega)} &+ \tau_i^{-1}\langle A_s(\overline{z}_\tau(\rho) - \overline{z}_\tau(\sigma)), \hat{z}_\tau'(\rho)\rangle_{\mathcal{Z}} \\ &\le -\tau_i^{-1}\langle \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(\rho), \overline{z}_\tau(\rho)) - \mathrm{D}_z\widetilde{\mathcal{I}}(\overline{t}_\tau(\sigma), \overline{z}_\tau(\sigma)), \hat{z}_\tau'(\rho)\rangle_{\mathcal{Z}}.\end{aligned} \tag{4.9}$$

Observe that $\tau_i^{-1}(\overline{z}_\tau(\rho) - \overline{z}_\tau(\sigma)) = \hat{z}_\tau'(\rho)$, hence the second term on the left-hand side can be replaced with $a_s(\hat{z}_\tau'(\rho), \hat{z}_\tau'(\rho))$. Moreover, using that $2a(a-b) = a^2 - b^2 + (a-b)^2$, the first term is equal to

$$\frac{\epsilon}{\tau_i}\langle \hat{z}_\tau'(\rho) - \hat{z}_\tau'(\sigma), \hat{z}_\tau'(\rho)\rangle_{L^2(\Omega)} = \frac{\epsilon}{2\tau_i}\left(\|\hat{z}_\tau'(\rho)\|_{L^2(\Omega)}^2 - \|\hat{z}_\tau'(\sigma)\|_{L^2(\Omega)}^2 + \|\hat{z}_\tau'(\rho) - \hat{z}_\tau'(\sigma)\|_{L^2(\Omega)}^2\right).$$

Next, we "integrate" (4.9) on the time interval (τ_0, t) (that is, we multiply both sides of (4.9) by τ_i and sum for $i = 1, \ldots, k$, assuming $t \in (t_k^\tau, t_{k+1}^\tau)$). Since $\hat{z}_\tau' \equiv \frac{z_1^\tau - z_0}{\tau_0}$ on $(0, \tau_0)$, in particular $\frac{z_1^\tau - z_0}{\tau_0} = \hat{z}_\tau'\left(\frac{\tau_0}{2}\right)$ and thus, neglecting the non-negative term $\|\hat{z}_\tau'(\rho) - \hat{z}_\tau'(\sigma)\|_{L^2(\Omega)}^2$, we obtain the estimate

$$\frac{\epsilon}{2} \|\hat{z}_\tau'(t)\|_{L^2(\Omega)}^2 - \frac{\epsilon}{2} \left\|\hat{z}_\tau'\left(\frac{\tau_0}{2}\right)\right\|_{L^2(\Omega)}^2 + \int_{\tau_0}^{\bar{t}_\tau(t)} a_s(\hat{z}_\tau'(r), \hat{z}_\tau'(r))\,\mathrm{d}r$$
$$\leq -\int_{\tau_0}^{\bar{t}_\tau(t)} \frac{1}{\tau(r)} \langle \triangle_{\tau(r)} \mathrm{D}_z \widetilde{\mathcal{I}}(\bar{t}_\tau(r), \bar{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\,\mathrm{d}r \quad \text{for } t \in [0,T]\backslash\{t_0^\tau, \ldots, t_N^\tau\},$$

see Notation 4.1. Adding the squared $L^2(L^2)$-norm of $\hat{z}_\tau'$ on both sides we arrive at

$$\begin{aligned}&\frac{\epsilon}{2} \|\hat{z}_\tau'(t)\|_{L^2(\Omega)}^2 + \int_{\tau_0}^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(r)\|_{\mathcal{Z}}^2\,\mathrm{d}r \\ &\leq \frac{\epsilon}{2} \left\|\hat{z}_\tau'\left(\frac{\tau_0}{2}\right)\right\|_{L^2(\Omega)}^2 + \int_{\tau_0}^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r - \int_{\tau_0}^{\bar{t}_\tau(t)} \frac{1}{\tau(r)} \langle \triangle_{\tau(r)} \mathrm{D}_z \widetilde{\mathcal{I}}(\bar{t}_\tau(r), \bar{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\,\mathrm{d}r.\end{aligned} \tag{4.10}$$

The next goal is to derive for the right-hand side an upper bound that is independent of the time step size τ. Since by assumption we have $\mathrm{D}_z\mathcal{I}(0, z_0) \in L^2(\Omega)$, choosing $t = \frac{\tau_0}{2}$ in (4.7) gives

$$0 = \mathcal{R}_1(\hat{z}_\tau'(\tau_0/2)) + \langle \bar{h}_\tau(\tau_0/2), \hat{z}_\tau'(\tau_0/2)\rangle_{\mathcal{Z}} \geq \epsilon \|\hat{z}_\tau'(\tau_0/2)\|_{L^2(\Omega)}^2 + \langle \mathrm{D}_z\mathcal{I}(\bar{t}_\tau(\tau_0/2), \bar{z}_\tau(\tau_0/2)), \hat{z}_\tau'(\tau_0/2)\rangle_{\mathcal{Z}}.$$

Hence,

$$\begin{aligned}\epsilon \|\hat{z}_\tau'(\tau_0/2)\|_{L^2(\Omega)}^2 &\leq -\langle \mathrm{D}_z\mathcal{I}(0, z_0), \hat{z}_\tau'(\tau_0/2)\rangle_{\mathcal{Z}} - \langle \mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^\tau) - \mathrm{D}_z\mathcal{I}(0, z_0), \hat{z}_\tau'(\tau_0/2)\rangle_{\mathcal{Z}} \\ &\leq \frac{1}{2\epsilon} \|\mathrm{D}_z\mathcal{I}(0, z_0)\|_{L^2(\Omega)}^2 + \frac{\epsilon}{2} \|\hat{z}_\tau'(\tau_0/2)\|_{L^2(\Omega)}^2 - \int_0^{\tau_0} \tau_0^{-1} \langle \mathrm{D}_z\mathcal{I}(t_1^\tau, z_1^\tau) - \mathrm{D}_z\mathcal{I}(0, z_0), \hat{z}_\tau'(\tau_0/2)\rangle_{\mathcal{Z}}\,\mathrm{d}r.\end{aligned}$$

Absorbing the second term on the right-hand side into the term on the left-hand side, and combining the resulting estimate with (4.10) leads to

$$\begin{aligned}&\frac{\epsilon}{2} \|\hat{z}_\tau'(t)\|_{L^2(\Omega)}^2 + \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(r)\|_{\mathcal{Z}}^2\,\mathrm{d}r \\ &\leq \frac{1}{2\epsilon} \|\mathrm{D}_z\mathcal{I}(0, z_0)\|_{L^2(\Omega)}^2 + \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r - \int_0^{\bar{t}_\tau(t)} \frac{1}{\tau(r)} \langle \triangle_{\tau(r)} \mathrm{D}_z \widetilde{\mathcal{I}}(\bar{t}_\tau(r), \bar{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\,\mathrm{d}r.\end{aligned} \tag{4.11}$$

We now derive an estimate for the last term in the same way as in (2.37) in Lemma 2.8. Indeed, for arbitrary $\widetilde{p} \in (2, p)$ let $r, \sigma \in (1, \infty)$ be defined by $r = p\widetilde{p}(p - \widetilde{p})^{-1}$ and $\frac{1}{\sigma} + \frac{1}{r} + \frac{2}{\widetilde{p}} = 1$. Observe that $r > 2$ and $\sigma > 2$. Using (2.37) for $\mathcal{I}_2$ and the fact that f' is Lipschitz by (2.16), for a.a. $\tau \in (0, T)$ we find that

$$\left|\frac{1}{\tau(r)} \langle \triangle_{\tau(r)} \mathrm{D}_z \widetilde{\mathcal{I}}(\bar{t}_\tau(r), \bar{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\right| \leq C \|\hat{z}_\tau'(r)\|_{L^\sigma(\Omega)} \left(\|\hat{z}_\tau'(r)\|_{L^r(\Omega)} + 1\right). \tag{4.12}$$

Let $\sigma_0 = \max\{\sigma, r\} > 2$. Since $s = \frac{d}{2}$, the space $\mathcal{Z}$ is embedded in $L^{\sigma_0}(\Omega)$. Using a Gagliardo-Nirenberg type inequality for Sobolev-Slobodeckij spaces (cf. e.g. [BM01, Cor. 3.2]), it follows with a suitable $\theta \in (0, 1)$ and $\rho > 0$ to be chosen that

$$\|\hat{z}_\tau'(r)\|_{L^{\sigma_0}(\Omega)}^2 \leq c \|\hat{z}_\tau'(r)\|_{L^2(\Omega)}^{2(1-\theta)} \|\hat{z}_\tau'(r)\|_{\mathcal{Z}}^{2\theta} \leq \rho c \|\hat{z}_\tau'(r)\|_{\mathcal{Z}}^2 + c_\rho \|\hat{z}_\tau'(r)\|_{L^2(\Omega)}^2.$$

Therefore we obtain

$$\left|\int_0^{\bar{t}_\tau(t)} \frac{1}{\tau(r)} \langle \triangle_{\tau(r)} \mathrm{D}_z \widetilde{\mathcal{I}}(\bar{t}_\tau(r), \bar{z}_\tau(r)), \hat{z}_\tau'(r)\rangle_{\mathcal{Z}}\,\mathrm{d}r\right| \leq C \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(r)\|_{L^2(\Omega)}^2\,\mathrm{d}r + \frac{1}{2} \int_0^{\bar{t}_\tau(t)} \|\hat{z}_\tau'(r)\|_{\mathcal{Z}}^2\,\mathrm{d}r + C'$$

for some positive constants C and C' independent of τ (and ϵ). Absorbing the term $\frac{1}{2}\int_0^{\bar{t}_\tau(t)} \|\hat{z}'_\tau(r)\|_{\mathcal{Z}}^2 \,\mathrm{d}r$ on the left-hand side of (4.11) we have finally shown that there exist constants $C > 0$ and $C' > 0$ such that for all τ and all $t \in [0,T]\backslash\{t_0^\tau, \ldots, t_N^\tau\}$ it holds

$$\frac{\epsilon}{2}\|\hat{z}'_\tau(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_0^{\bar{t}_\tau(t)} \|\hat{z}'_\tau(r)\|_{\mathcal{Z}}^2 \,\mathrm{d}r \leq C\int_0^{\bar{t}_\tau(t)} \|\hat{z}'_\tau(r)\|_{L^2(\Omega)}^2 \,\mathrm{d}r + \frac{1}{2\epsilon}\|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{L^2(\Omega)}^2 + C'. \quad (4.13)$$

Applying Gronwall's inequality, we conclude that for all τ and all $t \in [0,T]\backslash\{t_0^\tau, \ldots, t_N^\tau\}$

$$\epsilon\|\hat{z}'_\tau(t)\|_{L^2(\Omega)}^2 \leq \left(C' + \frac{1}{2\epsilon}\|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{L^2(\Omega)}^2\right)\exp(C\bar{t}_\tau(t)/\epsilon), \quad (4.14)$$

which after multiplying with ϵ and taking the root, in particular yields (4.6). Then, estimate (4.5) immediately follows from (4.13). □

We can now prove the first part of Theorem 3.3, and pass to the limit in (4.4) as $\tau \to 0$.

Theorem 4.3 (Existence of viscous solutions). *Let $s = d/2$. Assume* (2.1), (2.8)–(2.10), *and* (2.16). *Suppose that the initial datum $z_0 \in \mathcal{Z}$ also fulfills* (3.10), *viz.* $\mathrm{D}_z\mathcal{I}(0,z_0) \in L^2(\Omega)$. *Let $(\hat{z}_\tau)_{\tau>0} \subset H^1(0,T;\mathcal{Z})$ be a family of piecewise affine interpolants constructed from the solutions of* (4.1) *and supplemented with the initial datum z_0.*

Then, for every sequence of fineness-parameters $(\tau^j)_j$ with $\tau^j \searrow 0$ as $j \to \infty$ there exists a (not relabeled) subsequence of $(\hat{z}_{\tau^j})$ and $z \in H^1(0,T;\mathcal{Z})$ such that z is a solution to the Cauchy problem (3.1)–(3.2) *and the following convergences hold as $j \to \infty$:*

$$\hat{z}_{\tau^j} \rightharpoonup z \text{ weakly in } H^1(0,T;\mathcal{Z}), \quad (4.15)$$

$$\int_0^{\bar{t}_{\tau^j}(t)} \mathcal{R}_1(\hat{z}'_{\tau^j}(r))\,\mathrm{d}r \to \int_0^t \mathcal{R}_1(z'(r))\,\mathrm{d}r \quad (4.16)$$

Proof. We split the proof in three steps.

Step 1: compactness. It follows from estimate (4.5) that

$$\|\overline{z}_\tau - \hat{z}_\tau\|_{L^\infty(0,T;\mathcal{Z})}, \|\overline{z}_\tau - \underline{z}_\tau\|_{L^\infty(0,T;\mathcal{Z})} \leq \tau^{1/2}\|\hat{z}'_\tau\|_{L^2(0,T;\mathcal{Z})}. \quad (4.17)$$

Now, in view of (4.5) and standard compactness results, there exist a (not-relabeled) sequence $(\tau^j)_j$ and $z \in H^1(0,T;\mathcal{Z})$ such that, as $j \to \infty$,

$$\hat{z}_{\tau^j} \rightharpoonup z \text{ in } H^1(0,T;\mathcal{Z}), \quad \hat{z}_{\tau^j}(t) \rightharpoonup z(t) \text{ in } \mathcal{Z} \text{ for all } t \in [0,T], \text{ and } \quad \hat{z}_{\tau^j} \to z \text{ in } \mathrm{C}^0([0,T];X) \quad (4.18)$$

for every Banach space X such that $\mathcal{Z} \Subset X$. Due to estimates (4.17), we conclude that, along the same sequence,

$$\overline{z}_{\tau^j}(t),\ \underline{z}_{\tau^j}(t) \rightharpoonup z(t) \text{ in } \mathcal{Z} \text{ for all } t \in [0,T]. \quad (4.19)$$

Step 2: discrete energy equality. Arguing in the very same way as in the proof of [MRS12a, Thm. 4.10], we see that the approximate solutions $\overline{z}_{\tau^j}$, $\underline{z}_{\tau^j}$, $\hat{z}_{\tau^j}$ fulfill the discrete energy identity

$$\begin{aligned}&\int_{\underline{t}_{\tau^j}(s)}^{\bar{t}_{\tau^j}(t)} \left(\mathcal{R}_\epsilon(\hat{z}'_{\tau^j})(r) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\bar{t}_{\tau^j}(r),\overline{z}_{\tau^j}(r)))\right)\mathrm{d}r + \mathcal{I}(\bar{t}_{\tau^j}(t),\overline{z}_{\tau^j}(t))\\ &\quad = \mathcal{I}(\underline{t}_{\tau^j}(s),\underline{z}_{\tau^j}(s)) + \int_{\underline{t}_{\tau^j}(s)}^{\bar{t}_{\tau^j}(t)} \partial_t\mathcal{I}(r,\underline{z}_{\tau^j}(r))\,\mathrm{d}r + \int_{\underline{t}_{\tau^j}(s)}^{\bar{t}_{\tau^j}(t)} \frac{1}{\tau^j(r)}\mathrm{F}(\bar{t}_{\tau^j}(r);\underline{z}_{\tau^j}(r),\overline{z}_{\tau^j}(r))\,\mathrm{d}r\end{aligned} \quad (4.20)$$

where we have used the short-hand notation $\mathrm{F}(t;z,w) := \mathcal{I}(t,z) - \mathcal{I}(t,w) + \langle \mathrm{D}_z\mathcal{I}(t,w), w-z\rangle_{\mathcal{Z}}$. We have the following estimate

$$
\begin{aligned}
|\mathrm{F}(t;z,w)| &= \left| \int_0^1 \langle \mathrm{D}_z\mathcal{I}(t,w) - \mathrm{D}_z\mathcal{I}(t,(1-\sigma)z+\sigma w), w-z\rangle_{\mathcal{Z}} \,\mathrm{d}\sigma \right| \\
&\leq \int_0^1 (1-\sigma) a_s(w-z,w-z)\,\mathrm{d}\sigma + \int_0^1 \left| \langle \mathrm{D}_z\widetilde{\mathcal{I}}(t,w) - \mathrm{D}_z\widetilde{\mathcal{I}}(t,(1-\sigma)z+\sigma w), w-z\rangle_{\mathcal{Z}} \right| \mathrm{d}\sigma \\
&\leq \frac{1}{2}(\|w-z\|_{\mathcal{Z}}^2 + c_8 \|w-z\|_{L^r(\Omega)} \|w-z\|_{\mathcal{Z}}),
\end{aligned}
$$

where the last inequality follows from (2.35), and r is any fixed index in $\left(\frac{3p}{p-2}, +\infty\right)$.

Therefore, the last term on the right-hand side of (4.20) is estimated as follows:

$$
\int_{\underline{t}_{\tau^j}(s)}^{\overline{t}_{\tau^j}(t)} \frac{1}{\tau^j(r)} \left| \mathrm{F}(\overline{t}_{\tau^j}(r); \underline{z}_{\tau^j}(r), \overline{z}_{\tau^j}(r)) \right| \mathrm{d}r \leq C \sup_{t\in(0,T)} \|\overline{z}_{\tau^j}(t) - \underline{z}_{\tau^j}(t)\|_{\mathcal{Z}} \cdot \int_{\underline{t}_{\tau^j}(s)}^{\overline{t}_{\tau^j}(t)} \|\hat{z}_{\tau^j}'(r)\|_{\mathcal{Z}} \,\mathrm{d}r. \tag{4.21}
$$

Step 3: passage to the limit in the discrete energy inequality. Writing (4.20) for $s = 0$ and any $t \in [0,T]$, and taking into account (4.21), we find

$$
\begin{aligned}
&\int_0^{\overline{t}_{\tau^j}(t)} \left(\mathcal{R}_\epsilon(\hat{z}_{\tau^j}'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau^j}(r), \overline{z}_{\tau^j}(r))) \right) \mathrm{d}r + \mathcal{I}(\overline{t}_{\tau^j}(t), \overline{z}_{\tau^j}(t)) \\
&\leq \mathcal{I}(0,z_0) + \int_0^{\overline{t}_{\tau^j}(t)} \partial_t\mathcal{I}(r, \underline{z}_{\tau^j}(r))\,\mathrm{d}r + C \sup_{t\in(0,T)} \|\overline{z}_{\tau^j}(t) - \underline{z}_{\tau^j}(t)\|_{\mathcal{Z}} \cdot \int_0^{\overline{t}_{\tau^j}(t)} \|\hat{z}_{\tau^j}'(r)\|_{\mathcal{Z}} \,\mathrm{d}r.
\end{aligned} \tag{4.22}
$$

We will refer to the integral term on the left-hand side of (4.22) as $I_{\tau^j}^1$, and to the second and third term on the right-hand side as $I_{\tau^j}^2$ and $I_{\tau^j}^3$, respectively. Now, we take the $\liminf$ as $\tau^j \to 0$ of both sides of (4.22). Combining (4.19) with (2.38), we find that $\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau^j}(t), \overline{z}_{\tau^j}(t)) \rightharpoonup \mathrm{D}_z\mathcal{I}(t, z(t))$ in $\mathcal{Z}^*$ for all $t \in [0,T]$. Therefore, also taking into account (4.18) we conclude that $\liminf_{\tau^j\to 0} I_{\tau^j}^1 \geq \int_0^t \mathcal{R}_\epsilon(z'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(r,z(r)))\mathrm{d}r$. In view of (4.18) and the weak lower semicontinuity of the energy $\mathcal{I}$, cf. (2.38), we also have that $\liminf_{\tau^j\to 0} \mathcal{I}(\overline{t}_{\tau^j}(t), \overline{z}_{\tau^j}(t)) \geq \mathcal{I}(t, z(t))$. Again by (4.19), (2.38) and estimate (2.26), with the Lebesgue theorem we find $\lim_{\tau^j\to 0} I_\tau^2 = \int_0^t \partial_t\mathcal{I}(r,z(r))\mathrm{d}r$. Finally,

$$
\lim_{\tau^j\to 0} I_{\tau^j}^3 \leq C \lim_{\tau^j\to 0} \sup_{t\in(0,T)} \|\overline{z}_{\tau^j} - \underline{z}_{\tau^j}\|_{L^\infty(0,T;\mathcal{Z})} \|\hat{z}_{\tau^j}'\|_{L^1(0,T;\mathcal{Z})} = 0, \tag{4.23}
$$

in view of the second estimate in (4.17), combined with (4.5). From the above arguments, we deduce that the limit function $z \in H^1(0,T;\mathcal{Z})$ fulfills the energy inequality (3.8). In view of Proposition 3.2, we conclude that z is a solution to the Cauchy problem (3.1)–(3.2). Finally, in order to obtain (4.16), we pass to the limit in the energy identity (4.20). Observe that, thanks to (4.23), the remainder term on the right-hand side of (4.20) (viz., the third summand) converges to zero as $\tau^j \to 0$. Therefore, in view of the above discussion we obtain the following chain of inequalities

$$
\begin{aligned}
&\int_0^t \left(\mathcal{R}_\epsilon(z'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(r,z(r))) \right) \mathrm{d}r + \mathcal{I}(t,z(t)) \\
&\leq \liminf_{\tau^j\to 0} \left(\int_0^{\overline{t}_{\tau^j}(t)} \left(\mathcal{R}_\epsilon(\hat{z}_{\tau^j}'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau^j}(r), \overline{z}_{\tau^j}(r))) \right) \mathrm{d}r + \mathcal{I}(\overline{t}_{\tau^j}(t), \overline{z}_{\tau^j}(t)) \right) \\
&\leq \limsup_{\tau^j\to 0} \left(\int_0^{\overline{t}_{\tau^j}(t)} \left(\mathcal{R}_\epsilon(\hat{z}_{\tau^j}'(r)) + \mathcal{R}_\epsilon^*(-\mathrm{D}_z\mathcal{I}(\overline{t}_{\tau^j}(r), \overline{z}_{\tau^j}(r))) \right) \mathrm{d}r + \mathcal{I}(\overline{t}_{\tau^j}(t), \overline{z}_{\tau^j}(t)) \right) \\
&\leq \mathcal{I}(0,z_0) + \int_0^t \partial_t\mathcal{I}(r,z(r))\,\mathrm{d}r.
\end{aligned} \tag{4.24}
$$

Applying the chain rule (3.9) for the functional $\mathcal{I}$, and arguing as described in Proposition 3.2, it follows that all inequalities in (4.24) in fact are equalities. From this, (4.16) ensues, see the similar arguments in the proof of [RMS08, Thm. 3.5], and [MRS12b, Thm. 4.4] for more details. □

Remark 4.4 (Proof of the energy inequality under the condition $z_0 \in \mathcal{Z}$). As we have already mentioned in Remark 3.5 if we just assume $z_0 \in \mathcal{Z}$ for the initial datum, taking the limit as $\tau \to 0$ of the time-discrete approximation, we are only able to deduce that there exists a limit curve $z \in L^\infty(0,T;\mathcal{Z}) \cap H^1(0,T;L^2(\Omega))$ fulfilling the energy inequality (3.8), without concluding that z is a solution to the Cauchy problem (3.1)–(3.2). In this connection, let us also mention that we cannot prove (3.8) simply by passing to the limit in the energy inequality (4.22). Indeed, under the sole condition $z_0 \in \mathcal{Z}$ we are not able to obtain the crucial $H^1(0,T;\mathcal{Z})$-estimate (4.5) for $\hat{z}_{\tau^j}$, which guarantees that the remainder term on the right-hand side of (4.22) converges to zero, cf. (4.23).

A possible way to obtain the energy inequality (3.8) for z, is to pass to the limit in an *enhanced* approximate energy inequality for the interpolants of the discrete solutions, which has no remainder term on the right-hand side. Such an inequality was proved for (the time-discrete approximation of) abstract doubly nonlinear evolution equations in [MRS12b]. It involves a kind of *variational* interpolation of the discrete solutions $(z_k^\tau)_{0\le k\le N}$, i.e. the so-called *De Giorgi interpolant*, see also [Amb95, AGS08, RS06].

Proposition 4.5. *In addition to* (2.1)*,* (2.8)*–*(2.10)*, and* (2.16)*, suppose that*

$$\text{the space dimension is } d=2, \text{ hence } s=1 \text{ and } A_s \text{ is the Laplace operator,} \tag{4.25}$$

and the nonlinearities f and g have the following property

$$f(0) \le f(z), \quad g(0) \le g(z) \quad \text{for all } z \le 0. \tag{4.26}$$

Suppose moreover that the initial datum z_0 fulfills (3.10) *and that*

$$z_0(x) \in [0,1] \quad \text{for a.a. } x \in \Omega. \tag{4.27}$$

Then, every viscous solution z constructed via time-discretization also fulfills

$$z(t,x) \in [0,1] \quad \text{for a.a. } x \in \Omega, \text{ for all } t \in [0,T]. \tag{4.28}$$

Proof. Indeed, we shall prove that, starting from z_0 which fulfills (4.27), all solutions of the time-incremental minimization problem fulfill

$$z_k^\tau(x) \in [0,1] \quad \text{for a.a. } x \in \Omega \quad \text{for all } k=1,\ldots,N, \tag{4.29}$$

and then deduce (4.28) by passing to the limit as $\tau \to 0$ in the time-discretization scheme, relying on convergences (4.18), cf. the proof of Theorem 4.3. We shall prove (4.29) by induction on the index k, namely we are going to show that, $z_k^\tau(x) \in [0,1]$ for a.a. $x \in \Omega$ implies that $z_{k+1}^\tau(x) \in [0,1]$ for a.a. $x \in \Omega$.

Indeed, on the one hand, from $\mathcal{R}_1((z_{k+1}^\tau - z_k^\tau)/\tau_k) < +\infty$ we gather that $z_{k+1}^\tau(x) \le z_k^\tau(x) \le 1$ for almost all $x \in \Omega$. On the other hand, it follows from (4.1) (choosing $z = (z_{k+1}^\tau)^+$), that

$$\begin{aligned}
&\mathcal{R}_1\left(z_{k+1}^\tau - z_k^\tau\right) + \frac{\epsilon}{2\tau_k}\int_\Omega |z_{k+1}^\tau(x) - z_k^\tau(x)|^2\,\mathrm{d}x + \frac12\int_\Omega |\nabla z_{k+1}^\tau(x)|^2\,\mathrm{d}x + \int_\Omega f(z_{k+1}^\tau(x))\,\mathrm{d}x + \mathcal{I}_2(t_{k+1}, z_{k+1}^\tau) \\
&\le \mathcal{R}_1\left((z_{k+1}^\tau)^+ - z_k^\tau\right) + \frac{\epsilon}{2\tau_k}\int_\Omega |(z_{k+1}^\tau)^+(x) - z_k^\tau(x)|^2\,\mathrm{d}x + \frac12\int_\Omega |\nabla (z_{k+1}^\tau)^+(x)|^2\,\mathrm{d}x \\
&\qquad + \int_\Omega f((z_{k+1}^\tau)^+(x))\,\mathrm{d}x + \mathcal{I}_2(t_{k+1}, (z_{k+1}^\tau)^+).
\end{aligned} \tag{4.30}$$

Now, with easy calculations one sees that

$$\mathcal{R}_1((z^\tau_{k+1})^+ - z^\tau_k) \le \mathcal{R}_1(z^\tau_{k+1} - z^\tau_k), \qquad \|(z^\tau_{k+1})^+ - z^\tau_k\|^2_{L^2(\Omega)} \le \|z^\tau_{k+1} - z^\tau_k\|^2_{L^2(\Omega)}, \\ \|\nabla(z^\tau_{k+1})^+\|^2_{L^2(\Omega)} \le \|\nabla z^\tau_{k+1}\|^2_{L^2(\Omega)}. \tag{4.31}$$

Furthermore, it follows from assumption (4.26) on f that

$$\int_\Omega f((z^\tau_{k+1})^+(x))\,\mathrm{d}x \le \int_\Omega f(z^\tau_{k+1}(x))\,\mathrm{d}x. \tag{4.32}$$

Moreover, again in view of (4.26),

$$\begin{aligned}\mathcal{I}_2(t_{k+1},(z^\tau_{k+1})^+) &\le \int_\Omega g((z^\tau_{k+1})^+)W(\varepsilon(u^\tau_{k+1}+u_D(t_{k+1})))\,\mathrm{d}x - \langle \ell(t_{k+1}), u^\tau_{k+1}\rangle_{\mathcal{U}} \\ &\le \int_\Omega g(z^\tau_{k+1})W(\varepsilon(u^\tau_{k+1}+u_D(t_{k+1})))\,\mathrm{d}x - \langle \ell(t_{k+1}), u^\tau_{k+1}\rangle_{\mathcal{U}} = \mathcal{I}_2(t_{k+1}, z^\tau_{k+1}),\end{aligned} \tag{4.33}$$

where we have used the short-hand notation $u^\tau_{k+1} := u_{\min}(t_{k+1}, z^\tau_{k+1})$. In view of (4.31)–(4.33), we conclude that $(z^\tau_{k+1})^+$ as well is a minimizer for (4.1). Since the latter minimum problem has a unique solution, we thus have $z^\tau_{k+1} \equiv (z^\tau_{k+1})^+$. Therefore, $z^\tau_{k+1}(x) \ge 0$ for almost all $x \in \Omega$. □

Remark 4.6. Notice that $f(z) = (1-z)^2$ from [Gia05] fulfills (2.8) and (4.26) as well. An example of g which complies with both (2.10) and (4.26) is

$$g \in C^2(\mathbb{R}),\ g \text{ non-decreasing},\ \exists\,\gamma \in (0,1) : \ g(z) = \gamma \text{ for } z \le 0, \quad g(z) = 1 \text{ for } z \ge 1.$$

Functions with this property are often used to model incomplete damage of elastic materials. The value $z = 1$ then describes the undamaged state, whereas $z = 0$ stands for maximal damage. The monotonicity of g reflects the fact that, with increasing damage (i.e. decreasing z), the material becomes weaker.

4.2 A uniform discrete BV-estimate

We now obtain an $L^1(0,T;\mathcal{Z})$-estimate for the derivatives $(\hat{z}'_\tau)_\tau$ (hence an estimate in $\mathrm{BV}([0,T];\mathcal{Z})$ for $(\hat{z}_\tau)_\tau$), with a constant *independent* of both parameters τ and ϵ. Hereafter, we restrict to

$$\text{uniform time steps } \ \tau = \tau^N = T/N$$

and suppose that the parameters τ, ϵ satisfy $\tau \le 2\epsilon$. This is sufficient since we are ultimately interested in obtaining estimates for the limit $\tau \searrow 0$.

Proposition 4.7. *Assume* (2.1), (2.8)–(2.10) *and* (2.16). *Suppose that* $z_0 \in \mathcal{Z}$ *fulfills* $\mathrm{D}_z\mathcal{I}(0,z_0) \in L^2(\Omega)$. *Then, there exists a constant* C_{14} *such that for every* $\epsilon, \tau > 0$ *with* $\tau \le 2\epsilon$ *and the piecewise linear interpolants* $(\hat{z}_\tau)_\tau$ *defined via the solutions* z^τ_k *of* (4.1), *the following estimate holds*

$$\int_0^T \|\hat{z}'_\tau(t)\|_{\mathcal{Z}}\,\mathrm{d}t \le C_{14}\left(T + \sqrt{\epsilon} + \|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{L^2(\Omega)} + \int_0^T \mathcal{R}_1(\hat{z}'_\tau(t))\,\mathrm{d}t\right). \tag{4.34}$$

Proof. The idea is to combine a time-discrete Gronwall estimate with weights (generalizing [NSV00, Lemma 3.17]) with a suitable discrete version of the arguments in the proof of [MZ12, Lemma 3.4].

We start from (4.9), written for $\rho = m_k$ and $\sigma = m_{k-1}$, where $m_k := \frac{1}{2}(t^\tau_{k-1} + t^\tau_k)$ and $k \in \{2,\ldots,N\}$. Adding the term $\|\hat{z}'_\tau(m_k)\|^2_{L^2(\Omega)}$ on both sides, we obtain

$$\begin{aligned}&\frac{\epsilon}{\tau}\langle \hat{z}'_\tau(m_k) - \hat{z}'_\tau(m_{k-1}), \hat{z}'_\tau(m_k)\rangle_{L^2(\Omega)} + \tau^{-1}\langle A_s(\overline{z}_\tau(m_k) - \overline{z}_\tau(m_{k-1})), \hat{z}'_\tau(m_k)\rangle_{\mathcal{Z}} + \|\hat{z}'_\tau(m_k)\|^2_{L^2(\Omega)} \\ &\qquad \le -\tau^{-1}\langle \mathrm{D}_z\widetilde{\mathcal{I}}(t_k, \overline{z}_\tau(m_k)) - \mathrm{D}_z\widetilde{\mathcal{I}}(t_{k-1}, \overline{z}_\tau(m_{k-1})), \hat{z}'_\tau(m_k)\rangle_{\mathcal{Z}} + \|\hat{z}'_\tau(m_k)\|^2_{L^2(\Omega)},\end{aligned} \tag{4.35}$$

where $\widetilde{\mathcal{I}}$ is defined as in (2.33). The left-hand side of (4.35) can be estimated by

$$\text{l.h.s.} \;\geq \frac{\epsilon}{2\tau}\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}\left(\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}-\left\|\hat{z}_\tau'(m_{k-1})\right\|_{L^2(\Omega)}\right)+\left\|\hat{z}_\tau'(m_k)\right\|_{\mathcal{Z}}^2 .$$

In order to estimate the right-hand side, we proceed as follows. With $r,\sigma\in(2,\infty)$ and $\sigma_0=\max\{r,\sigma\}$ as in the proof of Proposition 4.2, analogously to (4.12) we have

$$\text{r.h.s.} \leq C(\left\|\hat{z}_\tau'(m_k)\right\|_{L^{\sigma_0}(\Omega)}^2+\left\|\hat{z}_\tau'(m_k)\right\|_{L^{\sigma_0}(\Omega)}) \leq C(\left\|\hat{z}_\tau'(m_k)\right\|_{L^{\sigma_0}(\Omega)}^2+1),$$

where in the last estimate we have used the Young inequality. Similar to the arguments subsequent to (4.12), by applying a Gagliardo-Nirenberg type inequality and Young's inequality it follows that there exists a $\theta\in(0,1)$ such that for $\rho>0$ to be chosen later we obtain $\left\|\hat{z}_\tau'(m_k)\right\|_{L^{\sigma_0}(\Omega)}^2 \leq C_\rho\left\|\hat{z}_\tau'(m_k)\right\|_{L^1(\Omega)}^2+\rho C\left\|\hat{z}_\tau'(m_k)\right\|_{\mathcal{Z}}^2$. Since by (1.3) it holds $\left\|\hat{z}_\tau'(m_k)\right\|_{L^1(\Omega)}\leq\kappa^{-1}\mathcal{R}_1(\hat{z}_\tau'(m_k))$, suitably tuning the constant ρ we arrive at

$$\text{r.h.s.} \leq \frac{1}{2}\left\|\hat{z}_\tau'(m_k)\right\|_{\mathcal{Z}}^2+C\left(1+\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}\mathcal{R}_1(\hat{z}_\tau'(m_k))\right).$$

Hence, estimate (4.35) yields

$$\begin{aligned}\frac{\epsilon}{2\tau}\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}\left(\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}-\left\|\hat{z}_\tau'(m_{k-1})\right\|_{L^2(\Omega)}\right)+\frac{1}{2}\left\|\hat{z}_\tau'(m_k)\right\|_{\mathcal{Z}}^2&\\ \leq C\left(1+\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}\mathcal{R}_1(\hat{z}_\tau'(m_k))\right)&,\end{aligned}$$

where the constant C is independent of τ,k and ϵ. Multiplying this inequality by $4\tau/\epsilon$ and taking into account that $\left\|\hat{z}_\tau'(m_k)\right\|_{\mathcal{Z}}^2\geq\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}^2$, we arrive at

$$\begin{aligned}2\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}\left(\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}-\left\|\hat{z}_\tau'(m_{k-1})\right\|_{L^2(\Omega)}\right)+\frac{\tau}{\epsilon}\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}^2+\frac{\tau}{\epsilon}\left\|\hat{z}_\tau'(m_k)\right\|_{\mathcal{Z}}^2&\\ \leq\frac{4\tau C}{\epsilon}+\frac{4\tau C}{\epsilon}\left\|\hat{z}_\tau'(m_k)\right\|_{L^2(\Omega)}\mathcal{R}_1(\hat{z}_\tau'(m_k)),&\end{aligned} \tag{4.36}$$

which is valid for all $2\leq k\leq N$. We define now for $0\leq i\leq N-1$

$$a_i=\left\|\hat{z}_\tau'(m_{i+1})\right\|_{L^2(\Omega)},\quad b_i=(\tau/\epsilon)^{\frac{1}{2}}\left\|\hat{z}_\tau'(m_{i+1})\right\|_{\mathcal{Z}},\quad c_i=(4\tau C/\epsilon)^{\frac{1}{2}},\quad d_i=\frac{2\tau C}{\epsilon}\mathcal{R}_1(\hat{z}_\tau'(m_{i+1})),\quad \gamma=\frac{\tau}{2\epsilon}.$$

With this, (4.36) can be rewritten as

$$2a_i(a_i-a_{i-1})+2\gamma a_i^2+b_i^2\leq c_i^2+2a_id_i,$$

which holds for $1\leq i\leq N-1$. Thus, estimate (4.51) in Lemma 4.9 below implies that for all $j\leq N-1$

$$\sum_{i=1}^{j}(1+\gamma)^{2(i-j)-1}b_i^2\leq 2(1+\gamma)^{-2j}a_0^2+2\sum_{i=1}^{j}(1+\gamma)^{2(i-j)-1}c_i^2+4\left(\sum_{i=1}^{j}(1+\gamma)^{i-j-1}d_i\right)^2.$$

Reinserting the explicit values of a_i, b_i and d_i, and using the fact that $c_i^2=8C\gamma$, the above inequality yields for $2\leq n\leq N$ (with $i+1=k$, $n=j+1$):

$$\begin{aligned}&\frac{1}{\epsilon}\sum_{k=2}^{n}\tau(1+\gamma)^{2(k-n)-1}\left\|\hat{z}_\tau'(m_k)\right\|_{\mathcal{Z}}^2\\ &\leq 2(1+\gamma)^{-2n}\left\|\hat{z}_\tau'(m_1)\right\|_{L^2(\Omega)}^2+16C\gamma\sum_{k=2}^{n}(1+\gamma)^{2(k-n)-1}+16C^2\left(\sum_{k=2}^{n}\frac{\tau}{\epsilon}(1+\gamma)^{k-n-1}\mathcal{R}_1(\hat{z}_\tau'(m_k))\right)^2.\end{aligned} \tag{4.37}$$

We now calculate and estimate the second term on the right-hand side of (4.37) explicitly, using that by assumption we have $\gamma \leq 1$:

$$\gamma \sum_{k=2}^{n} (1+\gamma)^{2(k-n)-1} \leq \gamma(1+\gamma)^{-2n-1} \frac{1-(1+\gamma)^{2(n+1)}}{1-(1+\gamma)^2} \leq \frac{1}{2}(1+\gamma) \leq 1. \tag{4.38}$$

Combining (4.38) with (4.37) yields

$$\frac{1}{\epsilon} \sum_{k=2}^{n} \tau (1+\gamma)^{2(k-n)-1} \|\hat{z}'_\tau(m_k)\|^2_{\mathcal{Z}} \leq C \left(1 + (1+\gamma)^{-2n} \|\hat{z}'_\tau(m_1)\|^2_{L^2(\Omega)} + \left(\sum_{k=2}^{n} \frac{\tau}{\epsilon} (1+\gamma)^{k-n-1} \mathcal{R}_1(\hat{z}'_\tau(m_k)) \right)^2 \right), \tag{4.39}$$

which is the discrete counterpart of estimate (3.23) in [MZ12].

We now proceed following the arguments in the proof of [MZ12, Lemma 3.4], translating them in the present time-discrete setting. Let us stress once again that, hereafter, the generic constant $C > 0$ shall be independent of τ and ϵ. We start by observing that, by Hölder inequality,

$$\frac{1}{\epsilon} \sum_{k=2}^{n} \tau (1+\gamma)^{2(k-n)-1} \|\hat{z}'_\tau(m_k)\|_{\mathcal{Z}} \leq \left(\sum_{k=2}^{n} \frac{\tau}{\epsilon} (1+\gamma)^{2(k-n)-1} \right)^{\frac{1}{2}} \left(\sum_{k=2}^{n} \frac{\tau}{\epsilon} (1+\gamma)^{2(k-n)-1} \|\hat{z}'_\tau(m_k)\|^2_{\mathcal{Z}} \right)^{\frac{1}{2}}. \tag{4.40}$$

Recalling that $\frac{\tau}{\epsilon} = 2\gamma \leq 2$ we find with (4.38) that the first factor on the right-hand side is bounded by $\sqrt{2}$. Hence, from (4.40) and (4.39) we deduce

$$\frac{1}{\epsilon} \sum_{k=2}^{n} \tau (1+\gamma)^{2(k-n)-1} \|\hat{z}'_\tau(m_k)\|_{\mathcal{Z}} \leq C \left(1 + (1+\gamma)^{-n} \|\hat{z}'_\tau(m_1)\|_{L^2(\Omega)} + \sum_{k=2}^{n} \frac{\tau}{\epsilon} (1+\gamma)^{k-n-1} \mathcal{R}_1(\hat{z}'_\tau(m_k)) \right). \tag{4.41}$$

Now we multiply both sides of (4.41) by τ and sum over $n = 2, \ldots, N$:

$$\frac{1}{\epsilon} \sum_{n=2}^{N} \tau \sum_{k=2}^{n} \tau (1+\gamma)^{2(k-n)-1} \|\hat{z}'_\tau(m_k)\|_{\mathcal{Z}} \leq C \sum_{n=2}^{N} \tau \left(1 + (1+\gamma)^{-n} \|\hat{z}'_\tau(m_1)\|_{L^2(\Omega)} + \sum_{k=2}^{n} \frac{\tau}{\epsilon} (1+\gamma)^{k-n-1} \mathcal{R}_1(\hat{z}'_\tau(m_k)) \right). \tag{4.42}$$

We discuss the different terms on the left-hand and on the right-hand side of (4.42) separately. Now, we introduce for every $k, n = 2, \ldots, N$ the coefficient c_k^n defined by 1 if $k \leq n$ and 0 if $k > n$. This coefficient will be used below to change the order of the sums. Starting with the left-hand side of (4.42), we have

$$\begin{aligned} \frac{1}{\epsilon} \sum_{n=2}^{N} \tau \sum_{k=2}^{n} \tau (1+\gamma)^{2(k-n)-1} \|\hat{z}'_\tau(m_k)\|_{\mathcal{Z}} &= \frac{1}{\epsilon} \sum_{n=2}^{N} \tau \sum_{k=2}^{N} \tau c_k^n (1+\gamma)^{2(k-n)-1} \|\hat{z}'_\tau(m_k)\|_{\mathcal{Z}} \\ &= \sum_{k=2}^{N} \tau \|\hat{z}'_\tau(m_k)\|_{\mathcal{Z}} \sum_{n=k}^{N} \frac{\tau}{\epsilon} (1+\gamma)^{2(k-n)-1} \\ &= 2 \frac{1+\gamma}{2+\gamma} \sum_{k=2}^{N} \tau \|\hat{z}'_\tau(m_k)\|_{\mathcal{Z}} \left(1 - (1+\gamma)^{2(k-N)-2} \right). \end{aligned} \tag{4.43}$$

Passing to the right-hand side of (4.42), using again the definition of γ we find

$$\sum_{n=2}^{N} \tau(1+\gamma)^{-n} \|\hat{z}_\tau'(m_1)\|_{L^2(\Omega)} \leq 4\epsilon \|\hat{z}_\tau'(m_1)\|_{L^2(\Omega)} . \tag{4.44}$$

Next we discuss the term

$$\begin{aligned}\sum_{n=2}^{N} \tau \sum_{k=2}^{n} \frac{\tau}{\epsilon}(1+\gamma)^{k-n-1}\mathcal{R}_1(\hat{z}_\tau'(m_k)) &= \sum_{k=2}^{N} \tau\mathcal{R}_1(\hat{z}_\tau'(m_k)) \sum_{n=k}^{N} \frac{\tau}{\epsilon}(1+\gamma)^{k-n-1} \\ &\leq 2\sum_{k=2}^{N} \tau\mathcal{R}_1(\hat{z}_\tau'(m_k)),\end{aligned} \tag{4.45}$$

where the last inequality follows from calculations analogous to (4.43). Combining (4.42) with (4.43)–(4.45), and recalling that $\tau = \frac{T}{N}$, so that $\sum_{n=2}^{N} \tau \leq T$, we get

$$\sum_{k=2}^{N} \tau \|\hat{z}_\tau'(m_k)\|_{\mathcal{Z}} \left(1 - (1+\gamma)^{2(k-N)-2}\right) \leq C \left(T + \epsilon \|\hat{z}_\tau'(m_1)\|_{L^2(\Omega)} + \sum_{k=2}^{N} \tau\mathcal{R}_1(\hat{z}_\tau'(m_k))\right),$$

that is

$$\begin{aligned}\sum_{k=2}^{N} \tau \|\hat{z}_\tau'(m_k)\|_{\mathcal{Z}} \leq & C \left(T + \epsilon \|\hat{z}_\tau'(m_1)\|_{L^2(\Omega)} + \sum_{k=2}^{N} \tau\mathcal{R}_1(\hat{z}_\tau'(m_k))\right) \\ &+ \sum_{k=2}^{N} \tau(1+\gamma)^{2(k-N)-2} \|\hat{z}_\tau'(m_k)\|_{\mathcal{Z}} .\end{aligned} \tag{4.46}$$

Using (4.41) to estimate the last term in (4.46), we arrive at

$$\sum_{k=2}^{N} \tau \|\hat{z}_\tau'(m_k)\|_{\mathcal{Z}} \leq C \left(T + \epsilon \|\hat{z}_\tau'(m_1)\|_{L^2(\Omega)} + \sum_{k=2}^{N} \tau\mathcal{R}_1(\hat{z}_\tau'(m_k))\right) . \tag{4.47}$$

Taking into account (4.6) together with $\tau \leq 2\epsilon$, we finally obtain (4.34). □

Combining estimate (4.34) with (4.16) it follows that

Corollary 4.8. *Assume* (2.1), (2.8)–(2.10) *and* (2.16). *Suppose that* $z_0 \in \mathcal{Z}$ *and that* $\mathrm{D}_z\mathcal{I}(0, z_0) \in L^2(\Omega)$. *For* $\epsilon \in (0, \epsilon_0]$ *let* $(z_\epsilon)_\epsilon \in H^1(0, T; \mathcal{Z})$ *be a family of solutions to the Cauchy problem for* (3.1), *which are limits of sequences* $(\hat{z}_{\epsilon,\tau})_{\tau\searrow 0}$ *of solutions to* (4.1) *(as in the statement of Theorem 4.3).*

Then there exist constants C_{15}, $C_{16} > 0$ *such that for all* $\epsilon > 0$ *it holds*

$$\int_0^T \|z_\epsilon'(t)\|_{\mathcal{Z}} \,\mathrm{d}t \leq C_{15} \|\mathrm{D}_z\mathcal{I}(0, z_0)\|_{L^2(\Omega)} + C_{16} \int_0^T (1 + \mathcal{R}_1(z_\epsilon'(t)))\,\mathrm{d}t. \tag{4.48}$$

Estimate (3.12) then follows from inequality (4.48), taking into account that from the energy identity (3.7) it can be deduced that $\sup_{\epsilon>0} \int_0^T \mathcal{R}_1(z_\epsilon'(t))\,\mathrm{d}t \leq C$.

In the proof of Proposition 4.7 we used the following time-discrete Gronwall-type estimate with weights.

Lemma 4.9. *Let* $\{a_k\}_{k=0}^N$ *and* $\{b_k, c_k, d_k\}_{k=1}^N$ *be nonnegative numbers and* $\gamma \geq 0$. *Assume that for* $1 \leq k \leq N$ *it holds*

$$2a_k(a_k - a_{k-1}) + 2\gamma a_k^2 + b_k^2 \leq c_k^2 + 2a_k d_k. \tag{4.49}$$

Then the following estimates are valid for all $n \geq 1$:

$$a_n \leq \left((1+\gamma)^{-2n} a_0^2 + \sum_{k=1}^{n} (1+\gamma)^{2(k-n)-1} c_k^2 \right)^{\frac{1}{2}} + \sum_{k=1}^{n} (1+\gamma)^{k-n-1} d_k \,, \tag{4.50}$$

$$\left(\sum_{k=1}^{n} (1+\gamma)^{2(k-n)-1} b_k^2 \right)^{\frac{1}{2}} \leq \left((1+\gamma)^{-2n} a_0^2 + \sum_{k=1}^{n} (1+\gamma)^{2(k-n)-1} c_k^2 \right)^{\frac{1}{2}} + \sqrt{2} \sum_{k=1}^{n} (1+\gamma)^{k-n-1} d_k \,. \tag{4.51}$$

Remark 4.10. If $\gamma = 0$, then Lemma 4.9 exactly reproduces [NSV00, Lemma 3.17]. Hence, our proof follows closely the steps in [NSV00], introducing the weight $(1+\gamma)^{\alpha}$ in a suitable way.

Proof. We set $a_0 = R_0$ and define for $1 \leq n \leq N$ the quantities

$$R_n = \xi_n + \delta_n, \qquad \delta_n = \sum_{k=1}^{n} (1+\gamma)^{k-n-1} d_k, \tag{4.52}$$

$$\xi_n = \left((1+\gamma)^{-2n} a_0^2 + \sum_{k=1}^{n} (1+\gamma)^{2(k-n)-1} c_k^2 \right)^{\frac{1}{2}} . \tag{4.53}$$

As in [NSV00], we first prove the inequality $a_n \leq R_n$ for all $1 \leq n \leq N$, which then gives (4.50). Since $b_n \geq 0$, from (4.49) we find that

$$(1+\gamma) a_n^2 - (a_{n-1} + d_n) a_n \leq \tfrac{1}{2} c_n^2.$$

Hence, investigating the roots of a_n in the quadratic inequality, we find since $a_n \geq 0$

$$2(1+\gamma) a_n \leq a_{n-1} + d_n + \left((a_{n-1} + d_n)^2 + 2(1+\gamma) c_n^2 \right)^{\frac{1}{2}} . \tag{4.54}$$

Observe that $(1+\gamma)\delta_n = \delta_{n-1} + d_n$ and that $(1+\gamma)\xi_n \geq \xi_{n-1}$. Therefore, from the definition of R_n it follows that

$$\begin{aligned}(1+\gamma) R_n^2 - (R_{n-1} + d_n) R_n &= R_n \Big((1+\gamma)\xi_n + (1+\gamma)\delta_n - \xi_{n-1} - \delta_{n-1} - d_n \Big) \\ &\geq \xi_n \big((1+\gamma)\xi_n - \xi_{n-1} \big)\end{aligned}$$

Using Young's inequality with $\xi_n \xi_{n-1} \leq \frac{1}{2}((1+\gamma)\xi_n^2 + (1+\gamma)^{-1}\xi_{n-1}^2)$ and taking into account the definition of ξ_n, we conclude that

$$\xi_n \big((1+\gamma)\xi_n - \xi_{n-1} \big) \geq \frac{1}{2} \left((1+\gamma)\xi_n^2 - (1+\gamma)^{-1} \xi_{n-1}^2 \right) = \frac{c_n^2}{2}.$$

Hence, we have shown that

$$(1+\gamma) R_n^2 - (R_{n-1} + d_n) R_n \geq \tfrac{1}{2} c_n^2.$$

In the same way as for a_n (cf. (4.54)), we deduce the estimate

$$2(1+\gamma) R_n \geq R_{n-1} + d_n + \left((R_{n-1} + d_n)^2 + 2(1+\gamma) c_n^2 \right)^{\frac{1}{2}} . \tag{4.55}$$

Since $a_0 = R_0$, by induction from (4.54) and (4.55) we have $a_n \leq R_n$ for every $n \leq N$, whence (4.50).

We now prove (4.51). Let $1 \leq n \leq N$. From (4.49),applying the Young inequality to the term $2a_k a_{k-1}$ and taking into account that $a_k \leq R_k$, we find for $1 \leq k \leq n$

$$b_k^2 \leq c_k^2 + 2 R_k d_k + (1+\gamma)^{-1} a_{k-1}^2 - (1+\gamma) a_k^2.$$

Multiplying this inequality with $(1+\gamma)^{2(k-n)-1}$, using that $(1+\gamma)^{k-n}R_k \le R_n$ and summing up we obtain

$$\sum_{k=1}^{n}(1+\gamma)^{2(k-n)-1}b_k^2 \le \sum_{k=1}^{n}(1+\gamma)^{2(k-n)-1}c_k^2 + 2R_n\sum_{k=1}^{n}(1+\gamma)^{k-n-1}d_k + \sum_{k=1}^{n}(1+\gamma)^{2(k-n)-2}a_{k-1}^2 - \sum_{k=1}^{n}(1+\gamma)^{2(k-n)}a_k^2. \quad (4.56)$$

Observe that the last two terms add up to $(1+\gamma)^{-2n}a_0^2 - a_n^2$. Thus, we finally arrive at

$$\sum_{k=1}^{n}(1+\gamma)^{2(k-n)-1}b_k^2 \le \xi_n^2 + 2R_n\delta_n \le \left(\xi_n + \sqrt{2}\delta_n\right)^2,$$

whence (4.51). □

5 Existence of parameterized solutions

Throughout this section, we shall work with a family $(z_\epsilon)_\epsilon \subset H^1(0,T;\mathcal{Z})$ of solutions to the ϵ-viscous Cauchy problem (3.1)–(3.2), for which the L^1-estimate

$$\sup_{\epsilon>0}\int_0^T \|z_\epsilon'(t)\|_{\mathcal{Z}}\,\mathrm{d}t \le C < \infty \quad (5.1)$$

is valid. The existence of such a family is ensured by Theorem 3.3, under condition (3.10) on the initial datum $z_0 \in \mathcal{Z}$.

5.1 The vanishing viscosity analysis

For every $\epsilon > 0$, we consider the graph $\mathrm{Graph}(z_\epsilon) := \{\,(t, z_\epsilon(t))\,;\, t\in[0,T]\,\} \subset [0,T]\times\mathcal{Z}$ and its $\mathcal{Z}$-arclength parameterization

$$s_\epsilon(t) = t + \int_0^t \|z_\epsilon'(r)\|_{\mathcal{Z}}\,\mathrm{d}r. \quad (5.2)$$

For $S_\epsilon = s_\epsilon(T)$ we introduce the functions $\hat{t}_\epsilon : [0,S_\epsilon]\to[0,T]$ and $\hat{z}_\epsilon : [0,S_\epsilon]\to\mathcal{Z}$

$$\hat{t}_\epsilon(s) := s_\epsilon^{-1}(s), \quad \hat{z}_\epsilon(s) := z_\epsilon(\hat{t}_\epsilon(s)) \quad (5.3)$$

and study the limiting behavior as $\epsilon\to0$ of the parameterized trajectories $\{\,(\hat{t}_\epsilon(s),\hat{z}_\epsilon(s))\,;\, s\in[0,S_\epsilon]\,\}$, which fulfill the *normalization condition*

$$\hat{t}_\epsilon'(s) + \|\hat{z}_\epsilon'(s)\|_{\mathcal{Z}} = 1 \quad \text{for a.a. } s\in(0,S_\epsilon). \quad (5.4)$$

Observe that, in view of estimate (5.1), there holds $\sup_{\epsilon>0} S_\epsilon < +\infty$. Therefore, up to a subsequence $S_\epsilon \to S$ as $\epsilon\to0$, with $S\ge T$ (the latter inequality follows from the fact that $s_\epsilon(t)\ge t$). With no loss of generality, we may consider the parameterized trajectories to be defined on the fixed time interval $[0,S]$.

For this passage to the limit, following [MRS09, MRS12a] we adopt an *energetic* viewpoint, i.e. we take the limit of the energy identity fulfilled by the parameterized trajectories $(\hat{t}_\epsilon(s),\hat{z}_\epsilon(s))_{s\in[0,S]}$. Setting

$$\mathrm{d}_2(\xi,\partial\mathcal{R}_1(0)) := \min_{\mu\in\partial\mathcal{R}_1(0)}\sqrt{2\widetilde{\mathcal{R}}_2(\xi-\mu)}$$

the energy identity (3.11) written for the pair $(\hat{t}_\epsilon, \hat{z}_\epsilon)$ on any time interval $(\sigma_1, \sigma_2) \subset [0, S]$ reads

$$\int_{\sigma_1}^{\sigma_2} \Big(\mathcal{R}_1(\hat{z}_\epsilon'(s)) + \frac{\epsilon}{2\hat{t}_\epsilon'(s)} \|\hat{z}_\epsilon'(s)\|_{L^2(\Omega)}^2 + \frac{\hat{t}_\epsilon'(s)}{2\epsilon} \mathrm{d}_2^2(-\mathrm{D}_z\mathcal{I}(\hat{t}_\epsilon(s), \hat{z}_\epsilon(s)), \partial\mathcal{R}_1(0))\Big)\,\mathrm{d}s + \mathcal{I}(\hat{t}_\epsilon(\sigma_2), \hat{z}_\epsilon(\sigma_2)) \\ = \mathcal{I}(\hat{t}_\epsilon(\sigma_1), \hat{z}_\epsilon(\sigma_1)) + \int_{\sigma_1}^{\sigma_2} \partial_t \mathcal{I}(\hat{t}_\epsilon(s), \hat{z}_\epsilon(s))\hat{t}_\epsilon'(s)\mathrm{d}s.$$

The above identity can be also reformulated by means of the functional (cf. [MRS09, Sec. 3.2])

$$\mathcal{M}_\epsilon : (0, +\infty) \times L^2(\Omega) \times [0, +\infty) \to [0, +\infty], \quad \mathcal{M}_\epsilon(\alpha, v, \zeta) := \mathcal{R}_1(v) + \frac{\epsilon}{2\alpha} \|v\|_{L^2(\Omega)}^2 + \frac{\alpha}{2\epsilon}\zeta^2, \tag{5.5}$$

whence

$$\int_{\sigma_1}^{\sigma_2} \mathcal{M}_\epsilon(\hat{t}_\epsilon'(s), \hat{z}_\epsilon'(s), \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}_\epsilon(s), \hat{z}_\epsilon(s)), \partial\mathcal{R}_1(0)))\,\mathrm{d}s + \mathcal{I}(\hat{t}_\epsilon(\sigma_2), \hat{z}_\epsilon(\sigma_2)) \\ = \mathcal{I}(\hat{t}_\epsilon(\sigma_1), \hat{z}_\epsilon(\sigma_1)) + \int_{\sigma_1}^{\sigma_2} \partial_t \mathcal{I}(\hat{t}_\epsilon(s), \hat{z}_\epsilon(s))\hat{t}_\epsilon'(s)\mathrm{d}s. \tag{5.6}$$

For the passage to the limit as $\epsilon \to 0$ in (5.6), we shall rely on the following Γ-convergence/lower semicontinuity result, which was proved in a finite-dimensional setting in [MRS09] (cf. Lemma 3.1 therein).

Lemma 5.1. *Extend the functional* $\mathcal{M}_\epsilon$ (5.5) *to* $[0, +\infty) \times L^2(\Omega) \times [0, +\infty)$ *via*

$$\mathcal{M}_\epsilon(0, v, \zeta) := \begin{cases} 0 & \text{for } v = 0 \text{ and } \zeta \in [0, +\infty), \\ +\infty & \text{for } v \in L^2(\Omega)\backslash\{0\} \text{ and } \zeta \in [0, +\infty). \end{cases}$$

Define $\mathcal{M}_0 : [0, +\infty) \times L^2(\Omega) \times [0, +\infty) \to [0, +\infty]$ *by*

$$\mathcal{M}_0(\alpha, v, \zeta) := \begin{cases} \mathcal{R}_1(v) + \zeta \|v\|_{L^2(\Omega)} & \text{if } \alpha = 0, \\ \mathcal{R}_1(v) + \mathrm{I}_0(\zeta) & \text{if } \alpha > 0, \end{cases} \tag{5.7}$$

where I_0 *denotes the indicator function of the singleton* $\{0\}$. *Then,*
(A) $\mathcal{M}_\varepsilon$ Γ*-converges to* $\mathcal{M}_0$ *on* $[0, +\infty) \times L^2(\Omega) \times [0, +\infty)$ *w.r. to the strong-weak-strong topology, viz.*

Γ*-liminf estimate:*

$$(\alpha_\varepsilon, \zeta_\varepsilon) \to (\alpha, \zeta) \text{ and } v_\varepsilon \rightharpoonup v \text{ in } L^2(\Omega) \implies \mathcal{M}_0(\alpha, v, \zeta) \le \liminf_{\varepsilon \searrow 0} \mathcal{M}_\varepsilon(\alpha_\varepsilon, v_\varepsilon, \zeta_\varepsilon), \tag{5.8a}$$

Γ*-limsup estimate:*

$$\forall\,(\alpha, v, \zeta)\ \exists\,(\alpha_\varepsilon, v_\varepsilon, \zeta_\varepsilon)_{\varepsilon>0} : \begin{cases} (\alpha_\varepsilon, \zeta_\varepsilon) \to (\alpha, \zeta), \quad v_\varepsilon \rightharpoonup v \text{ in } L^2(\Omega) \text{ and} \\ \mathcal{M}_0(\alpha, v, \zeta) \ge \limsup_{\varepsilon \searrow 0} \mathcal{M}_\varepsilon(\alpha_\varepsilon, v_\varepsilon, \zeta_\varepsilon). \end{cases} \tag{5.8b}$$

(B) If $\alpha_\varepsilon \rightharpoonup \bar{\alpha}$ *in* $L^1(a, b)$, $v_\varepsilon \rightharpoonup \bar{v}$ *in* $L^1(a, b; L^2(\Omega))$, *and* $\liminf_{\varepsilon \to 0} \zeta_\varepsilon(s) \ge \bar{\zeta}(s)$ *for a.a.* $s \in (a, b)$, *then*

$$\int_a^b \mathcal{M}_0(\bar{\alpha}(s), \bar{v}(s), \bar{\zeta}(s))\mathrm{d}s \le \liminf_{\varepsilon \to 0} \int_a^b \mathcal{M}_\varepsilon(\alpha_\varepsilon(s), v_\varepsilon(s), \zeta_\varepsilon(s))\mathrm{d}s.$$

The proof can be developed arguing in the same way as in the proof of [MRS09, Lemma 3.1], up to replacing the usage of Ioffe's theorem [Iof77] with its infinite-dimensional version, see e.g. [Val90, Thm. 21]. Moving from this result, and following [MRS09, Def. 3.2], [MRS12a, Def. 5.2], we give the ensuing

Definition 5.2. A pair $(\hat{t},\hat{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ is a *$\mathcal{Z}$-parameterized solution* of (1.6), if it satisfies the *energy identity* for all $0 \le \sigma_1 \le \sigma_2 \le S$

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} \mathcal{M}_0(\hat{t}'(s),\hat{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s + \mathcal{I}(\hat{t}(\sigma_2),\hat{z}(\sigma_2)) \\ = \mathcal{I}(\hat{t}(\sigma_1),\hat{z}(\sigma_1)) + \int_{\sigma_1}^{\sigma_2} \partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))\hat{t}'(s)\mathrm{d}s.\end{aligned} \tag{5.9}$$

We say that a $\mathcal{Z}$-parameterized solution $(\hat{t},\hat{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ is *non-degenerate* if it fulfills

$$\hat{t}'(s) + \|\hat{z}'(s)\|_{\mathcal{Z}} > 0 \quad \text{for a.a. } s \in (0,S). \tag{5.10}$$

We are now in the position of stating the main result of this section.

Theorem 5.3. *Let $s = d/2$. Assume* (2.1), (2.8)–(2.10), *and* (2.16). *Let $(z_\epsilon)_\epsilon \subset H^1(0,T;\mathcal{Z})$ be a family of solutions to the ϵ-viscous problem* (3.1)–(3.2), *for which estimate* (5.1) *is valid, and let $(\hat{t}_\epsilon,\hat{z}_\epsilon)_{\epsilon>0} \subset \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ be defined by* (5.3).

Then, for every sequence $\epsilon_n \searrow 0$ there exist a pair $(\hat{t},\hat{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ and a (not-relabeled) subsequence such that

$$\begin{aligned}&(\hat{t}_{\epsilon_n},\hat{z}_{\epsilon_n}) \overset{*}{\rightharpoonup} (\hat{t},\hat{z}) \text{ in } W^{1,\infty}(0,S;[0,T]\times\mathcal{Z}),\\ &\hat{t}_{\epsilon_n} \to \hat{t} \text{ in } \mathrm{C}^0([0,S];[0,T]), \quad \hat{z}_{\epsilon_n}(s) \rightharpoonup \hat{z}(s) \text{ in } \mathcal{Z} \text{ for all } s \in [0,S],\end{aligned} \tag{5.11}$$

and $(\hat{t},\hat{z})$ is a $\mathcal{Z}$-parameterized solution of (1.6), *fulfilling*

$$\hat{t}'(s) + \|\hat{z}'(s)\|_{\mathcal{Z}} \le 1 \quad \text{for a.a. } s \in (0,S). \tag{5.12}$$

Remark 5.4. At the moment, it remains an open problem to prove that, the limiting $\mathcal{Z}$-parameterized solution in Thm. 5.3 is also *non-degenerate*. Without going into details, we may mention that, in some sense, this is due to the gap between condition (5.10), which involves the $\mathcal{Z}$-norm of $\hat{z}'$, and our *energetic* method for taking the vanishing viscosity limit of (3.1). The mismatch occurs because, neither the viscous energy identity (5.6), nor its limit (5.9) contain information on the term $\|\hat{z}'\|_{\mathcal{Z}}$.

These considerations also suggest that, in order to obtain *non-degenerate* parameterized solutions, it could be necessary to implement on this vanishing viscosity limit the alternative reparameterization techniques which we will discuss in Section 5.3. We plan to address this issue in a future paper.

Remark 5.5. By reparameterizing degenerate $\mathcal{Z}$-parameterized solutions, non-degenerate $\mathcal{Z}$-parameterized solutions can be obtained: Let $(\hat{t},\hat{z}) \in W^{1,\infty}(0,S;\mathbb{R}) \times W^{1,\infty}(0,S;\mathcal{Z})$ be a $\mathcal{Z}$-parameterized solution with $\hat{t}'(\rho) + \|\hat{z}'(\rho)\|_{\mathcal{Z}} \le 1$ for a.a. $\rho \in (0,S)$. Define $m(\rho) := \int_0^\rho \hat{t}'(\sigma) + \|\hat{z}'(\sigma)\|_{\mathcal{Z}}\,\mathrm{d}\sigma$, $r(\mu) := \inf\{\,\rho \ge 0\,;\, m(\rho) = \mu\,\}$ and $\widetilde{t}(\mu) := \hat{t}(r(\mu))$, $\widetilde{z}(\mu) = \hat{z}(r(\mu))$. Clearly, m is monotone and Lipschitz while r is monotone and belongs to $\mathrm{BV}([0,R])$ with $R := r(m(S))$. Moreover, it holds $m(r(\mu)) = \mu$ and for every $\mu \in J(r)$ (where $J(r)$ denotes the jump set of the BV-function r) we have $m(r(\mu-)) = m(r(\mu+))$. Hence, with the chain rule in [AFP00, Theorem 3.96] it follows that

$$\mathrm{d}\mu = \mathrm{D}m(r(\cdot)) = m'(r(\cdot))\widetilde{\mathrm{D}}r + \sum_{\mu\in J(r)} (m(r(\mu+)) - m(r(\mu-)))\delta_\mu = m'(r(\cdot))\widetilde{\mathrm{D}}r. \tag{5.13}$$

Here, $\widetilde{\mathrm{D}}r$ denotes the diffuse part of the distributional derivative $\mathrm{D}r$. The Lipschitz continuity of $\widetilde{t}$ and $\widetilde{z}$ now is an immediate consequence of the above calculations. Indeed, let $0 \le \mu_1 \le \mu_2 \le R$. Then

$$\begin{aligned}0 \le \widetilde{t}(\mu_2) - \widetilde{t}(\mu_1) = \int_{\mu_1}^{\mu_2} \hat{t}'(r(\mu))\widetilde{\mathrm{D}}r(\mu) &= \int_{\mu_1}^{\mu_2} \big(m'(r(\mu)) - \|\hat{z}'(r(\mu))\|_{\mathcal{Z}}\big)\widetilde{\mathrm{D}}r(\mu) \\ &= \int_{\mu_1}^{\mu_2} \mathrm{d}\mu - \int_{\mu_1}^{\mu_2} \|\hat{z}'(r(\mu))\|_{\mathcal{Z}}\,\widetilde{\mathrm{D}}r(\mu) \le \mu_2 - \mu_1,\end{aligned}$$

where the second equality is due to the definition of m and the third one follows from (5.13). The last inequality is due to the monotonicity of r. Similarly we show that $\widetilde{z}$ is Lipschitz continuous. Hence, the pair $(\widetilde{t},\widetilde{z})$ belongs to $W^{1,\infty}(0,R;\mathbb{R})\times W^{1,\infty}(0,R;\mathcal{Z})$ and satisfies $\widetilde{t}'(\mu)+\|\widetilde{z}'(\mu)\|_{\mathcal{Z}}=1$ for a.a. $\mu\in(0,R)$. Finally, it is easy to check that $(\widetilde{t},\widetilde{z})$ satisfies the energy identity (5.9), whence $(\widetilde{t},\widetilde{z})$ is a non-degenerate $\mathcal{Z}$-parameterized solution of (1.6).

The proof of Theorem 5.3 is based on the following result, which is the "parameterized counterpart" to Proposition 3.2. Indeed, it provides an equivalent formulation of (5.9). The reader is referred to [MRS12a, Prop. 5.3] for further characterizations of parameterized solutions.

Lemma 5.6. *Let* $s=d/2$. *Assume* (2.1)*,* (2.8)*–*(2.10)*, and* (2.16)*. Then, a pair* $(\hat{t},\hat{z})\in\mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ *is a* $\mathcal{Z}$*-parameterized solution of* (1.6) *if and only if it satisfies for all* $0\le\sigma\le S$ *the* energy inequality

$$\begin{aligned}\int_0^\sigma \mathcal{M}_0(\hat{t}'(s),\hat{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s+\mathcal{I}(\hat{t}(\sigma),\hat{z}(\sigma))&\\ \le\mathcal{I}(\hat{t}(0),\hat{z}(0))+\int_0^\sigma\partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))\hat{t}'(s)\mathrm{d}s.&\end{aligned}\tag{5.14}$$

Proof. Like in the proof of Proposition 3.2, we observe that, due to the smoothness of the energy functional $\mathcal{I}$ (cf. Corollary 2.9), *any* pair $(\widetilde{t},\widetilde{z})\in\mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ fulfills the "parameterized chain rule" identity

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s))=\partial_t\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s))\widetilde{t}'(s)+\langle\mathrm{D}_z\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s)),\widetilde{z}'(s)\rangle_{\mathcal{Z}}\quad\text{for a.a. }s\in(0,S).\tag{5.15}$$

Now, let $\mu(s)\in\partial\mathcal{R}_1(0)$ satisfy $\left\|-\mathrm{D}_z\mathcal{I}(\widetilde{t}(s)-\mu(s)\right\|_{L^2(\Omega)}=\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s)),\partial\mathcal{R}_1(0)))$ for almost all $s\in(0,S)$ (cf. Lemma 3.1). Then, there holds

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s))+\partial_t\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s))\widetilde{t}'(s)&=\langle-\mathrm{D}_z\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s))-\mu(s),\widetilde{z}'(s)\rangle_{L^2(\Omega)}+\langle\mu(s),\widetilde{z}'(s)\rangle_{\mathcal{Z}}\\ &\le\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s)),\partial\mathcal{R}_1(0)))\,\|\widetilde{z}'(s)\|_{L^2(\Omega)}+\mathcal{R}_1(\widetilde{z}'(s)),\end{aligned}\tag{5.16}$$

where the latter inequality follows from the definition (1.4) of $\partial\mathcal{R}_1(0)$. Hence, let $(\hat{t},\hat{z})$ comply with (5.14). In particular, $\mathcal{M}_0(\hat{t}'(s),\hat{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0)))<\infty$ for a.a. $s\in(0,S)$, which yields that

$$\hat{t}'(s)>0\ \Rightarrow\ \mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0))=0.\tag{5.17}$$

Therefore, for $(\hat{t},\hat{z})$ the following inequality holds

$$\begin{aligned}&\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s)),\partial\mathcal{R}_1(0)))\,\|\widetilde{z}'(s)\|_{L^2(\Omega)}+\mathcal{R}_1(\widetilde{z}'(s))\\ &\le\mathcal{M}_0(\hat{t}'(s),\hat{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0)))\quad\text{for a.a. }s\in(0,S).\end{aligned}\tag{5.18}$$

Combining (5.16) with (5.18) and integrating in time, we deduce from (5.14) the chain of inequalities (ultimately, identities)

$$\begin{aligned}&\int_0^\sigma\mathcal{M}_0(\hat{t}'(s),\hat{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s+\mathcal{I}(\hat{t}(\sigma),\hat{z}(\sigma))\\ &\quad\le\mathcal{I}(\hat{t}(0),\hat{z}(0))+\int_0^\sigma\partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))\hat{t}'(s)\mathrm{d}s\\ &\quad=\int_0^\sigma\mathcal{M}_0(\hat{t}'(s),\hat{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s+\mathcal{I}(\hat{t}(\sigma),\hat{z}(\sigma))\end{aligned}$$

for all $\sigma\in[0,S]$. Then, with the very same arguments as in the proof of Proposition 3.2, we find that $(\hat{t},\hat{z})$ complies with (5.9) for all $0\le\sigma_1\le\sigma_2\le S$. □

Proof of Theorem 5.3. From estimate (5.4), we deduce that there exists $(\hat{t},\hat{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ such that convergences (5.11) hold along some subsequence. Arguing as in the proof of Theorem 4.3 and relying on Corollary 2.9, we then find that, for all $s \in [0,S]$

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{I}(\hat{t}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)) \geq \mathcal{I}(\hat{t}(s),\hat{z}(s)), \quad \mathrm{D}_z\mathcal{I}(\hat{t}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)) \rightharpoonup \mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)) \text{ weakly in } \mathcal{Z}^*,\\ \partial_t\mathcal{I}(\hat{t}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)) \to \partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s)) \quad \text{in } L^p(0,S) \text{ for all } 1 \leq p < \infty.\end{aligned} \tag{5.19}$$

Now, (5.12) follows by taking the limit as $\epsilon_n \to 0$ in (5.4), with a trivial lower semicontinuity argument. Thanks to Lemma 5.1 and convergences (5.19), we have that, for all $0 \leq \sigma \leq S$

$$\begin{aligned}\liminf_{\epsilon_n\to 0}\int_0^\sigma \mathcal{M}_{\epsilon_n}(\hat{t}'_{\epsilon_n}(s),\hat{z}'_{\epsilon_n}(s),&\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}_{\epsilon_n}(s),\hat{z}_{\epsilon_n}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s\\ &\geq \int_0^\sigma \mathcal{M}_0(\hat{t}'(s),\hat{z}'(s),\mathrm{d}_2(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}_1(0)))\,\mathrm{d}s.\end{aligned}$$

Then, combining (5.11) and (5.19), and using that $\hat{z}_\epsilon(0) = z_\epsilon(0) = z_0$ for all $\epsilon > 0$, we pass to the limit in (5.6) written for $\sigma_1 = 0$ and $\sigma_2 = \sigma$. We thus find that the pair $(\hat{t},\hat{z})$ satisfies (5.14) for all $0 \leq \sigma \leq S$. In view of Lemma 5.6, we conclude that $(\hat{t},\hat{z})$ is a parameterized solution. □

5.2 Properties of non-degenerate parameterized solutions

The following characterization of non-degenerate parameterized solutions was proved in [MRS12a, Prop. 5.3, Cor. 5.4]. Adapted to our setting, it reads

Proposition 5.7 (Differential characterization)**.** *Let $s = d/2$. Assume* (2.1)*,* (2.8)*–*(2.10)*, and* (2.16)*. Then, a pair $(\hat{t},\hat{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,S];[0,T]\times\mathcal{Z})$ is a* non-degenerate *$\mathcal{Z}$-parameterized solution of* (1.6)*, if and only if there exists a Borel function $\lambda : (0,S) \to [0,+\infty)$ such that*

$$\begin{cases}\partial\mathcal{R}_1(\hat{z}'(s)) + \lambda(s)\hat{z}'(s) + \mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)) \ni 0,\\ \hat{t}'(s)\lambda(s) = 0\end{cases} \qquad \text{for a.a. } s \in (0,S). \tag{5.20}$$

Remark 5.8 (Mechanical interpretation). As in [MRS12a, Rmk. 5.6] (see also [EM06, MRS09]), from the differential characterization (5.20) of parameterized solutions we may draw the following conclusions on the evolution described by the notion of parameterized solution:

- the regime $(\hat{t}' > 0, \hat{z}' = 0)$ corresponds to *sticking*;
- the regime $(\hat{t}' > 0, \hat{z}' \neq 0)$ corresponds to *rate-independent* evolution: From the second of (5.20) and $\hat{t}'(s) > 0$ we deduce that $\lambda(s) = 0$, hence the first of (5.20) reads

  $$\partial\mathcal{R}_1(\hat{z}'(s)) + \mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)) \ni 0$$

  where only the *rate-independent* dissipation is present;
- when $(\hat{t}' = 0, \hat{z}' \neq 0)$ (note that the latter condition is implied by the non-degeneracy (5.10)), the system has switched to a *viscous regime*. The latter is seen as a jump in the (slow) external time scale, encoded by the time function $\hat{t}$, which is frozen. Since $\hat{t}'(s) = 0$, the second of (5.20) is satisfied and $\lambda(s)$ may be strictly positive. In this case, in the first of (5.20) also *viscous* dissipation is active. Indeed, (5.20) describes the energetic behavior of the system at jump points, see also [MRS09, MRS12a].

5.3 Alternative reparameterization techniques and conclusions

As we have already mentioned in the introduction, in the papers [MRS09, MRS12a] and [EM06, Mie11, MZ12], the vanishing viscosity analysis of rate-independent systems has been developed by reparameterization techniques as well, however based on choices of the parameterization functions different from our own (5.2).

The reparameterization considered in [MRS09, MRS12a] (see also the forthcoming [MRS12c]) would feature, in the present setting, the "energetic quantity"

$$\bar{s}_\epsilon(t) = \int_0^t \left(1 + \mathcal{R}_\epsilon(z_\epsilon'(\tau)) + \mathcal{R}_\epsilon^*(-\mathrm{D}\mathcal{I}(\tau, z_\epsilon(\tau)))\right)\mathrm{d}\tau \tag{5.21}$$

which can be considered as some sort of *energy-dissipation arclength* of the viscous solution z_ε. In fact, under the *sole* assumption $z_0 \in \mathcal{Z}$, from the energy identity (3.7) fulfilled by viscous solutions it is immediate to deduce that $\sup_{\epsilon>0} \bar{s}_\epsilon(T) < +\infty$.

On the other hand, the $L^1(0,T;\mathcal{Z})$-estimate (5.1) for (z_ε') (which can be proved under the additional condition $\mathrm{D}_z\mathcal{I}(0,z_0) \in L^2(\Omega)$) clearly yields that $\sup_\epsilon \int_0^T \|z_\varepsilon'(r)\|_{L^2(\Omega)}\,\mathrm{d}r \leq C < \infty$. In principle, this would also allow us to reparameterize by the $L^2(\Omega)$-*arclength*

$$\widetilde{s}_\epsilon(t) = \int_0^t \left(1 + \|z_\epsilon'(\tau)\|_{L^2(\Omega)}\right)\mathrm{d}\tau \tag{5.22}$$

of the graph of z_ϵ, like in [Mie11, MZ12]. The advantage of the L^2-arclength reparameterization, in comparison with the *energy-dissipation arclength* and the $\mathcal{Z}$-*arclength* repameterizations, is that it leads to a more understandable "parameterized formulation", both on the ϵ-level and in the vanishing viscosity limit. More precisely, setting $\widetilde{t}_\epsilon(s) := \widetilde{s}_\epsilon^{-1}(s)$, $\widetilde{z}_\epsilon(s) := z_\epsilon(\widetilde{t}_\epsilon(s))$ for $s \in [0, \widetilde{s}_\epsilon(T)]$, it can be easily calculated that the pair $(\widetilde{t}_\epsilon, \widetilde{z}_\epsilon)$ fulfills

$$\begin{cases} \widetilde{t}_\epsilon(0) = 0,\ \widetilde{z}_\epsilon(0) = z_0, \quad \widetilde{t}_\epsilon'(s) + \|\widetilde{z}_\epsilon'(s)\|_{L^2(\Omega)} = 1, \\ \partial\mathcal{R}_1(\widetilde{z}_\epsilon'(s)) + \frac{\epsilon}{1-\|\widetilde{z}_\epsilon'(s)\|_{L^2(\Omega)}}\widetilde{z}_\epsilon'(s) + \mathrm{D}_z\mathcal{I}(\widetilde{t}_\epsilon(s), \widetilde{z}_\epsilon(s)) \ni 0 \end{cases} \quad \text{for a.a. } s \in (0, \widetilde{s}_\epsilon(T)). \tag{5.23}$$

As in [Mie11, Section 4.4] (cf. also [EM06, MZ12]), one observes that the ϵ-viscous term is the subdifferential of the potential $\mathcal{V}_\epsilon$, which is defined for every $\eta \in L^2(\Omega)$ as follows:

$$\mathcal{V}_\epsilon(\eta) = \epsilon v\left(\|\eta\|_{L^2(\Omega)}\right) \qquad \text{with } v(\xi) = \begin{cases} -\log(1-\xi) - \xi & \text{if } \xi < 1, \\ +\infty & \text{else.} \end{cases}$$

With $\widetilde{\mathcal{R}}_\epsilon(\eta) := \mathcal{R}_1(\eta) + \mathcal{V}_\epsilon(\eta)$, the differential inclusion in (5.23) can be rewritten as

$$\partial\widetilde{\mathcal{R}}_\epsilon(\widetilde{z}_\epsilon'(s)) + \mathrm{D}_z\mathcal{I}(\widetilde{t}_\epsilon(s), \widetilde{z}_\epsilon(s)) \ni 0 \quad \text{for a.a. } s \in (0, \widetilde{s}_\epsilon(T)). \tag{5.24}$$

Notice that (5.24) has the same structure as the "viscous" doubly nonlinear equation (3.1). It can be checked (cf. with [MZ12, Lemma A.4] in the *reversible* case $\mathcal{R}_1(\cdot) = \|\cdot\|_{L^1(\Omega)}$), that the potential $\widetilde{\mathcal{R}}_\epsilon$ converges monotonously to the limit functional $\widetilde{\mathcal{R}}_0$ with $\widetilde{\mathcal{R}}_0(\eta) = \mathcal{R}_1(\eta)$ if $\|\eta\|_{L^2(\Omega)} \leq 1$ and $+\infty$ else.

Therefore, recalling the results in [Mie11, MZ12], it is to be expected that, up to a subsequence, the pairs $(\widetilde{t}_\epsilon, \widetilde{z}_\epsilon)_\epsilon$ converge to $(\widetilde{t}, \widetilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,\widetilde{S}]; [0,T] \times L^2(\Omega))$ (with $\widetilde{S} = \lim_{\epsilon\to 0} \widetilde{s}_\epsilon(T)$), satisfying

$$\begin{cases} \widetilde{t}(0) = 0,\ \widetilde{t}(S) = T,\ \widetilde{t}'(s) \geq 0, \quad \widetilde{t}'(s) + \|\widetilde{z}'(s)\|_{L^2(\Omega)} \leq 1, \\ \partial\widetilde{\mathcal{R}}_0(\widetilde{z}'(s)) + \mathrm{D}_z\mathcal{I}(\widetilde{t}(s), \widetilde{z}(s)) \ni 0 \end{cases} \quad \text{for a.a. } s \in (0, \widetilde{S}). \tag{5.25}$$

However, at the moment we are not able to prove this convergence result. Our main difficulty in the passage to the limit as $\epsilon \searrow 0$ in (5.24) is related to the unboundedness of the operator $\partial\mathcal{R}_1$. Because of this, it is not possible to perform those comparison estimates in (5.24), which would give a bound in $L^2(0,\widetilde{S};L^2(\Omega))$ for the term $\mathrm{D}_z\mathcal{I}(\widetilde{t}_\epsilon(\cdot),\widetilde{z}_\epsilon(\cdot))$. As it stands, such a term is only estimated in $L^\infty(0,T;\mathcal{Z}^*)$, which is not sufficient for passing from (5.24) to (5.25), since we only have an $L^1(0,T;L^2(\Omega))$-bound for $(\widetilde{z}'_\epsilon)_\epsilon$. Roughly speaking, the terms $\mathrm{D}_z\mathcal{I}(\widetilde{t}_\epsilon,\widetilde{z}_\epsilon)$ and $\widetilde{z}'_\epsilon$ are no longer "in duality": This prevents us from applying the passage to the limit techniques developed in [MZ12] for the "reversible" case. Furthermore, we cannot develop the "energetic" arguments of the proof of Theorem 5.3 any longer. Indeed, in this new setting it would still be possible to prove that $(\widetilde{t}_\epsilon,\widetilde{z}_\epsilon)_\epsilon$ converge in suitable topologies to a pair $(\widetilde{t},\widetilde{z}) \in \mathrm{C}^0_{\mathrm{lip}}([0,\widetilde{S}];[0,T]\times L^2(\Omega))$ which satisfies the energy inequality related to the limit problem (5.25):

$$\begin{aligned} &\mathcal{I}(T,\widetilde{z}(S)) + \int_0^S \widetilde{\mathcal{R}}_0(\widetilde{z}'(s))\,\mathrm{d}s + \int_0^S (\widetilde{\mathcal{R}}_0)^*\left(-\mathrm{D}_z\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s))\right)\,\mathrm{d}s \\ &\le \mathcal{I}(0,\widetilde{z}(0)) + \int_0^S \partial_t\mathcal{I}(\widetilde{t}(s),\widetilde{z}(s))\widetilde{t}'(s)\,\mathrm{d}s. \end{aligned} \tag{5.26}$$

Still, from (5.26) we would not be able to conclude that (5.25) holds via chain rule arguments, like in the proof of Theorem 5.3. In fact, we do not dispose of the "parameterized chain rule" (5.15) any longer, due to the lack of further spatial regularity for $\mathrm{D}_z\mathcal{I}(\widetilde{t}_\epsilon,\widetilde{z}_\epsilon)$.

The vanishing viscosity analysis via the *energy-dissipation arclength* reparameterization would bring forth the same difficulties. Nonetheless, we plan to address these issues in the future, relying on some improved regularity results for the term $\mathrm{D}_z\mathcal{I}(t,z_\epsilon(t))$ in (3.1).

Acknowledgment

D.K. acknowledges the partial financial support by the Robert Bosch Stiftung within the Fast Track Program and the partial support by the Deutsche Forschungsgemeinschaft through the project C32 "Modeling of Phase Separation and Damage Processes in Alloys" of the Research Center MATHEON. R.R. was partially supported by a MIUR-PRIN 2008 grant for the project "Optimal mass transportation, geometric and functional inequalities and applications". C.Z. acknowledges the partial financial support by FP7-IDEAS-ERC-StG Grant # 200497 (BioSMA).

References

[AE84] J.-P. Aubin and I. Ekeland. *Applied Nonlinear Analysis.* John Wiley & Sons Inc., New York, 1984.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[Amb95] L. Ambrosio. Minimizing movements. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 19:191–246, 1995.

[BB08] E. Bonetti and G. Bonfanti. Well-posedness results for a model of damage in thermoviscoelastic materials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25:1187–1208, 2008.

[BFM12] J.-F. Babadjian, G. Francfort, and M.G. Mora. Quasistatic evolution in non-associative plasticity - the cap model. *SIAM J. Math. Anal.*, 44:245–292, 2012.

[BM01] H. Brezis and P. Mironescu. Gagliardo-nirenberg, composition and products in fractional sobolev spaces. *J. Evol. Equ.*, 1(4):387–404, 2001.

[BMR09] G. Bouchitté, A. Mielke, and T. Roubíček. A complete-damage problem at small strains. *ZAMP Z. Angew. Math. Phys.*, 60:205–236, 2009.

[Bre73] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[BS04] E. Bonetti and G. Schimperna. Local existence to Frémond's model for damaging in elastic materials. *Contin. Mech. Thermodyn.*, 16:319–335, 2004.

[BSS05] E. Bonetti, G. Schimperna, and A. Segatti. On a doubly nonlinear model for the evolution of damaging in viscoelastic materials. *J. Differential Equations*, 218:91–116, 2005.

[Cag08] F. Cagnetti. A vanishing viscosity approach to fracture growth in a cohesive zone model with prescribed crack path. *Math. Models Methods Appl. Sci.*, 18(7):1027–1071, 2008.

[DDMM08] G. Dal Maso, A. DeSimone, M. G. Mora, and M. Morini. A vanishing viscosity approach to quasistatic evolution in plasticity with softening. *Arch. Ration. Mech. Anal.*, 189(3):469–544, 2008.

[DDS11] G. Dal Maso, A. DeSimone, and F. Solombrino. Quasistatic evolution for cam-clay plasticity: a weak formulation via viscoplastic regularization and time rescaling. *Calc. Var. Partial Differential Equations*, 40:125–181, 2011.

[DDS12] G. Dal Maso, A. DeSimone, and F. Solombrino. Quasistatic evolution for cam-clay plasticity: properties of the viscosity solutions. *Calc. Var. Partial Differential Equations*, 2012. In press. Online published since August 2011, DOI 10.1007/s00526-011-0443-6.

[EM06] M. Efendiev and A. Mielke. On the rate–independent limit of systems with dry friction and small viscosity. *J. Convex Analysis*, 13:151–167, 2006.

[FG06] G. Francfort and A. Garroni. A variational view of partial brittle damage evolution. *Arch. Ration. Mech. Anal.*, 182(1):125–152, 2006.

[FK06] M. Frémond and N. Kenmochi. Damage problems for viscous locking materials. *Adv. Math. Sci. Appl.*, 16(2):697–716, 2006.

[FKS12] A. Fiaschi, D. Knees, and U. Stefanelli. Young measure quasi-static damage evolution. *Arch. Rational Mech. Anal.*, 203:415–453, 2012.

[FN96] M. Frémond and B. Nedjar. Damage, gradient of damage and principle of virtual power. *Int. J. Solids Struct.*, 33(8):1083–1103, 1996.

[Gia05] A. Giacomini. Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures. *Calc. Var. Partial Differential Equations*, 22(2):129–172, 2005.

[Giu03] E. Giusti. *Direct methods in the calculus of variations.* World Scientific Publishing Co. Inc., 2003.

[GL09] A. Garroni and C. Larsen. Threshold-based quasi-static brittle damage evolution. *Arch. Ration. Mech. Anal.*, 194(2):585–609, 2009.

[Grö89] K. Gröger. A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. *Math. Ann.*, 283(4):679–687, 1989.

[HK10] C. Heinemann and C. Kraus. Existence of weak solutions for Cahn–Hilliard systems coupled with elasticity and damage. Preprint no. 1520, Weierstrass Institute for Applied Analysis and Stochastics, 2010.

[HMW11] R. Herzog, C. Meyer, and G. Wachsmuth. Regularity of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions. *Journal of Mathematical Analysis and Applications*, 382(2):802–813, 2011.

[Iof77] A. D. Ioffe. On lower semicontinuity of integral functionals. I. *SIAM J. Control Optimization*, 15(4):521–538, 1977.

[IT79] A. D. Ioffe and V. M. Tihomirov. *Theory of extremal problems*, volume 6 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, 1979.

[Kac86] L.M. Kachanov. *Introduction to continuum damage mechanics.* Mechanics of Elastic Stability, 10. Dordrecht/Boston/Lancaster: Martinus Nijhoff Publishers, a member of the Kluwer Academic Publishers Group., 1986.

[KMZ08] D. Knees, A. Mielke, and C. Zanini. On the inviscid limit of a model for crack propagation. *Math. Models Methods Appl. Sci.*, 18(9):1529–1569, 2008.

[KRZ11] D. Knees, R. Rossi, and C. Zanini. A vanishing viscosity approach to a rate-independent damage model. Wias-preprint 1633, Weierstrass Institute for Applied Analysis and Stochastics, Berlin, 2011.

[KS12] D. Knees and A. Schröder. Computational aspects of quasi-static crack propagation. *Discrete Contin. Dyn. Syst. Ser. S.*, 2012. To appear.

[KZM10] D. Knees, C. Zanini, and A. Mielke. Crack propagation in polyconvex materials. *Physica D*, 239:1470–1484, 2010.

[LT11] G. Lazzaroni and R. Toader. A model of crack propagation based on viscous approximation. *Math. Models Methods Appl. Sci.*, 21:1–29, 2011.

[Mie03] A. Mielke. Energetic formulation of multiplicative elasto–plasticity using dissipation distances. *Cont. Mech. Thermodynamics*, 15:351–382, 2003.

[Mie05] A. Mielke. Evolution in rate-independent systems (Ch. 6). In C.M. Dafermos and E. Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, pages 461–559. Elsevier B.V., Amsterdam, 2005.

[Mie11] A. Mielke. Differential, energetic, and metric formulations for rate-independent processes. In L. Ambrosio and G. Savaré, editors, *Nonlinear PDEs and Applications. C.I.M.E. Summer School, Cetraro, Italy 2008.*, page 240 pages. Springer, Heidelberg, 2011.

[MR06] A. Mielke and T. Roubíček. Rate-independent damage processes in nonlinear elasticity. *Math. Models Methods Appl. Sci.*, 16:177–209, 2006.

[MRS09] A. Mielke, R. Rossi, and G. Savaré. Modeling solutions with jumps for rate-independent systems on metric spaces. *Discrete Contin. Dyn. Syst.*, 25:585–615, 2009.

[MRS12a] A. Mielke, R. Rossi, and G. Savaré. BV solutions and viscosity approximation of rate-independent systems. *ESAIM Control Optim. Calc. Var.*, 18:36–80, 2012.

[MRS12b] A. Mielke, R. Rossi, and G. Savaré. Nonsmooth analysis of doubly nonlinear evolution equations. *Calc. Var. Partial Differential Equations*, 2012. In press. Online published since January 2012, DOI 10.1007/s00526-011-0482-z.

[MRS12c] A. Mielke, R. Rossi, and G. Savaré. BV solutions to infinite-dimensional rate-independent systems. *In preparation*, 2012.

[MT99] A. Mielke and F. Theil. A mathematical model for rate-independent phase transformations with hysteresis. In H.-D. Alber, R.M. Balean, and R. Farwig, editors, *Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering"*, pages 117–129, Aachen, 1999. Shaker-Verlag.

[MT04] A. Mielke and F. Theil. On rate–independent hysteresis models. *Nonl. Diff. Eqns. Appl. (NoDEA)*, 11:151–189, 2004. (Accepted July 2001).

[MZ12] A. Mielke and S. Zelik. On the vanishing-viscosity limit in parabolic systems with rate-independent dissipation terms. Technical report, 2012. To appear.

[NSV00] R. H. Nochetto, G. Savaré, and C. Verdi. A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations. *Commun. Pure Appl. Math.*, 53(5):525–589, 2000.

[RMS08] R. Rossi, A. Mielke, and G. Savaré. A metric approach to a class of doubly nonlinear evolution equations and applications. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, VII:97–169, 2008.

[RS06] R. Rossi and G. Savaré. Gradient flows of non convex functionals in Hilbert spaces and applications. *ESAIM Control Optim. Calc. Var.*, 12(3):564–614 (electronic), 2006.

[TM10] M. Thomas and A. Mielke. Damage of nonlinearly elastic materials at small strain - existence and regularity results. *ZAMM Z. Angew. Math. Mech.*, 90(2):88–112, 2010.

[TZ09] R. Toader and C. Zanini. An artificial viscosity approach to quasistatic crack growth. *Boll. Unione Mat. Ital. (9)*, 2:1–35, 2009.

[Val90] M. Valadier. Young measures. In *Methods of nonconvex analysis (Varenna, 1989)*, volume 1446 of *Lecture Notes in Math.*, pages 152–188. Springer, Berlin, 1990.