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EXISTENCE AND UNIQUENESS RESULTS FOR A CLASS OF RATE-INDEPENDENT HYSTERESIS PROBLEMS

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In this paper, we address the problem of existence, approximation, and uniqueness of solutions to an abstract doubly nonlinear equation, modeling a rate-independent process with hysteretic behavior. Models of this kind arise in, e.g., plasticity, solid phase transformations, and several other problems in non smooth mechanics. Existence of solutions is proved via passage to the limit in a time-discretization scheme, whereas uniqueness results are obtained by means of convex analysis techniques.

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1. Introduction

Given two functionals $\mathcal{E} : [0,T] \times Z \to \mathbb{R}$ and $\Psi : Z \times Z \to [0,+\infty)$ on a Banach space Z, we consider the following doubly nonlinear evolution equation:

$$\partial_v \Psi(z(t), \dot{z}(t)) + \partial \mathcal{E}(t, z(t)) \ni 0, \quad t \in (0,T). \tag{1.1}$$

Here, $\mathcal{E}$ and Ψ are assumed to be lower semicontinuous, convex in their second arguments and differentiable in their first arguments, and the symbols ∂_v and ∂ both denote the subdifferential w.r.t. the second variable. In fact, $\mathcal{E}$ is the *potential energy* and Ψ the *dissipation functional* associated with a rate-independent process, possibly displaying a hysteretic behavior. Roughly speaking, rate-independence means that the process is insensitive to changes in the time scales. Processes of this kind occur in several branches of applied mathematics, such as plasticity, phase transformations in elastic solids, dry friction on surfaces and many others (see e.g. Ref. 12 and references therein). They may arise as vanishing viscosity limits of systems with strongly separated time scales, whence their hysteretical behavior. On the modeling level, rate-independence is achieved by assuming Ψ to be 1-positively homogeneous w.r.t. its second variable, i.e. $\Psi(z, \lambda v) = \lambda\Psi(z, v)$ for every $\lambda \geq 0$ and $(z, v) \in Z \times Z$. Thus, a solution to (1.1) remains a solution if the time is rescaled.

In the last years, a new energetic approach to the modeling of these problems has been developed in Refs. 17, 18 and 19. The latter work concerns a simplified version of (1.1), obtained by assuming that Ψ *does not* depend on the state z, i.e. $D_z\Psi(z, v) = 0$ for all z, v (in the sequel, we shall denote by $D_z\Psi$ the Gâteau derivative of Ψ w.r.t. the first variable, and by $D\mathcal{E}$ the derivative of $\mathcal{E}$ w.r.t. the second variable). This leads to a special case of the doubly nonlinear problems studied in Refs. 5 and 6 because of the additional rate independence. It is the purpose of this paper to generalize the results in Ref. 18, proving existence, approximation, and uniqueness for (1.1), which includes the state-dependent dissipation functional Ψ. From the very beginning, we will assume the map $z \mapsto \mathcal{E}(t, z)$ to be convex: this is necessary to obtain absolutely continuous solutions. In Sec. 2 we discuss the relations between the doubly nonlinear formulation (1.1) and the corresponding energetic formulation

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, \hat{z}) + \Psi(z(t), \hat{z} - z(t)) \quad \forall\ \hat{z} \in Z, \tag{S_Ψ}$$

$$\mathcal{E}(t, z(t)) + \int_0^t \Psi(z(\tau), \dot{z}(\tau))d\tau = \mathcal{E}(0, z(0)) + \int_0^t \partial_t\mathcal{E}(\tau, z(\tau))d\tau. \tag{E_Ψ}$$

In fact, we will show (cf. Proposition 2.3 later on) that, under suitable conditions, (1.1) and (S_Ψ)–(E_Ψ) are equivalent.

Following Ref. 18, we note that (S_Ψ) is a *stability condition*: in fact, according to (S_Ψ) passing from the state $z(t)$ to the state $\hat{z}$ involves the release of the potential energy $\mathcal{E}(t, z(t)) - \mathcal{E}(t, \hat{z})$, smaller than the dissipated energy $\Psi(z(t), \hat{z} - z(t))$. On the other hand, ($\mathrm{E}_\Psi$) is an energy balance. Note that the formulation (S_Ψ)–(E_Ψ) does not involve the "derivative" of $\mathcal{E}$ w.r.t. the variable z, but only the assumedly smooth power of the external forces $\partial_t\mathcal{E}$. Moreover, in (E_Ψ) one could replace the time derivative of z with (a form) of its derivative in the sense of measures (see Ref. 18), since in non-convex and non-smooth problems the solution z might have jumps. In fact, (S_Ψ)–(E_Ψ) can even be formulated without any linear structure in the state space Z, if we replace $\Psi(\hat{z} - z)$ by a general dissipation distance $\mathcal{D}(z, \hat{z})$, see Sec. 2.3 and Refs. 12 and 13.

In Sec. 3 we show that if $z \mapsto \mathcal{E}(t, z)$ is uniformly convex and Ψ fulfils a Lipschitz continuity condition w.r.t. its first variable, then *any* solution to (1.1) is Lipschitz continuous. In particular, the two conditions

$$\left.\begin{aligned} &\mathcal{E}\left(t, \frac{1}{2}(z_1+z_2)\right) \leq \frac{1}{2}\mathcal{E}(t,z_1) + \frac{1}{2}\mathcal{E}(t,z_2) - \frac{\kappa}{8}\|z_1-z_2\|^2 \\ &|\Psi(z_1,v) - \Psi(z_2,v)| \leq \psi^* \|z_1-z_2\|\|v\| \end{aligned}\right\} \quad \forall\, z_1, z_2, v \in Z$$

lead to the crucial assumption on the state dependence of Ψ, namely

$$\psi^* < \kappa. \tag{1.2}$$

A simple one-dimensional example shows that, without this condition, the existence of continuous solutions may be false.

The existence proof for (1.1) is based on approximation with the discrete *time incremental problem*

$$\begin{cases} z_0 := z(0), \\ z_k \in \operatorname{argmin}\{\mathcal{E}(t_k, z) + \Psi(z_{k-1}, z - z_{k-1}) \mid z \in Z\} \quad \text{for } k = 1, \ldots, N; \end{cases}$$

for suitable partitions $0 = t_0 < t_1 < \cdots < t_N = T$. Actually, we will pass to the limit in the discrete stability condition and in the energy inequality associated with the above minimization problem, and thus obtain the equivalent energetic formulation (S_Ψ)–(E_Ψ). Condition (1.2) is used to provide *a priori* Lipschitz bounds in the form $\|z_k - z_{k-1}\| \leq C^*|t_k - t_{k-1}|$ for equidistant partitions. We argue by weak compactness and lower semicontinuity and exploit crucially a compactness result for Young measures in the framework of the weak topology, recently proved in Ref. 21, see Appendix B.

Let us emphasize that, in proving equivalence of formulations, existence and approximation of solutions, we have developed arguments and techniques quite close to those in Ref. 18. As a matter of fact, loosely speaking the dependence of Ψ on the state variable z brings about relevant analytical difficulties only in the uniqueness issue for (1.1), which we tackle in Sec. 5. The main difficulty in proving uniqueness for the Cauchy problem for (1.1) is its *quasivariational character*, which does not allow us to apply convexity or monotonicity arguments. The only simple uniqueness proof is achieved in the case that the stable sets

$$\mathcal{S}(t) = \{z \in Z \mid \mathcal{E}(t,z) \leq \mathcal{E}(t,\hat{z}) + \Psi(z, \hat{z} - z)\ \forall\, \hat{z} \in Z\}$$

are convex and that $\mathcal{E}$ has the form $\mathcal{E}(t,z) = \hat{\mathcal{E}}(z) - \langle \ell(t), z\rangle$, with $\hat{\mathcal{E}}$ strictly convex, see Theorem 6.5 in Ref. 18. In fact, these conditions hold if $\hat{\mathcal{E}}(t,\cdot)$ is quadratic and Ψ is state-independent. Instead, if either $\hat{\mathcal{E}}$ is general (cf. Ref. 18) or if Ψ is state-dependent, then uniqueness is much more delicate. We explain now that the second case relates exactly to the *quasivariational inequalities* studied in Ref. 3.

Indeed, in view of standard convex analysis results (which will be recalled in Sec. 2 and Appendix A), we may rephrase (1.1) as

$$\dot{z}(t) \in \partial I_{C(z(t))}(-D\mathcal{E}(t, z(t))), \quad t \in (0,T), \tag{1.3}$$

where $\{C(z)\}_{z\in Z}$ is the family of closed convex subsets of Z' related to Ψ by the formula

$$\Psi(z,v) := \sup\{\langle\sigma,v\rangle \mid \sigma\in C(z)\} \quad \text{for all } z,v\in Z,$$

and $I_{C(z)}$ is the indicator function of $C(z)$. Indeed, we may refer to (1.3) as the *sweeping process* formulation of (1.1), as it may be viewed as a generalization of the classical sweeping process

$$\dot z(t) + \partial I_{K(t)}(z(t)) \ni 0, \quad t\in(0,T),$$

$\{K(t)\}_{t\in(0,T)}$ being a family of closed convex subsets of a Hilbert space H. This variational inequality was first analysed in Refs. 15 and 16. In the latter, existence of solutions for the related Cauchy problem was obtained via a suitable time-discretization, whereas uniqueness was proved by a simple variational argument. This variational technique fails as soon as one turns to so-called *quasivariational sweeping processes*

$$\dot z(t) + \partial I_{K(t,z(t))}(z(t)) \ni 0, \quad t\in(0,T),$$

where K depends on the state $z\in Z$. Such processes occur in a variety of applications, ranging from non-smooth mechanics to mathematical economics and convex optimization, see e.g., Ref. 14. As a matter of fact, the dependence of K on the state z essentially destroys the variational structure of the differential inclusion, and rules out the possibility of exploiting monotonicity arguments. See Refs. 9 and 10 for existence results in this context. Uniqueness was obtained only very recently in Ref. 3 for (a generalization of)

$$\dot z(t) \in \partial I_{K(t,z(t))}(\ell(t)-z(t)), \quad t\in(0,T),$$

with $\ell\in C^1([0,T];H)$. The above quasivariational inequality may be translated into a subdifferential form analogous to (1.1), i.e.

$$\partial\widetilde\Psi(t,z(t),\dot z(t)) + D\widetilde{\mathcal E}(t,z(t)) \ni 0, \quad t\in(0,T),$$

by choosing the dissipation potential $\widetilde\Psi$ and the *quadratic energy* as follows:

$$\widetilde\Psi(t,z,v) := \sup\{\langle y,v\rangle \mid y\in K(t,z)\} \quad\text{and}\quad \widetilde{\mathcal E}(t,z) := \frac12\|z\|^2 - \langle\ell(t),z\rangle$$

for all $z,v\in H$ and $t\in(0,T)$. Without giving details, let us point out that the complex proof of uniqueness developed in Ref. 3 is based on careful Lipschitz estimates involving quantities suitably related to Ψ. Moreover, this approach relies on the specific form of the quadratic energy functional.

Our result on uniqueness and continuous dependence for (1.1) combines the ideas of Refs. 3 and 18. Following Ref. 3, we use the auxiliary functional

$$\mathcal B(z,\sigma) = \sup\left\{\langle\sigma,v\rangle - \frac12\Psi(z,v)^2 :\ v\in Z\right\}.$$

Basically, $\mathcal{B}$ measures the distance to the yield surface, defined as the set of (z, σ) fulfilling $\mathcal{B}(z, \sigma) = 1/2$. Along the lines of Ref. 18 we introduce an energetic distance

$$\varrho_{1,2}(t) := \big(\langle D\mathcal{E}(t, z_1(t)) - D\mathcal{E}(t, z_2(t)), z_1(t)) - z_2(t)\rangle\big)^{1/2}.$$

Indeed, $\varrho_{1,2}(t)$ allows for a *one-sided Lipschitz estimate*, which is based on a generalization of the *structure condition* proposed in Ref. 18 and which leads to the final Gronwall-type estimate

$$\begin{aligned}&\frac{d}{dt}\big(\varrho_{1,2}(t) + M_2|\mathcal{B}(z_1(t), \varsigma_1(t)) - \mathcal{B}(z_2(t), \varsigma_2(t))|\big)\\ &\quad\leq M_3\big(\varrho_{1,2}(t) + M_2|\mathcal{B}(z_1(t), \varsigma_1(t)) - \mathcal{B}(z_2(t), \varsigma_2(t))|\big).\end{aligned}$$

In contrast, the stronger assumptions in Ref. 3 lead to *two-sided Lipschitz estimates* and to a much stronger *a priori* estimate of the type

$$\begin{aligned}&\|\dot{z}_1(t) - \dot{z}_2(t)\| + M_2\frac{d}{dt}|\mathcal{B}(z_1(t), \varsigma_1(t)) - \mathcal{B}(z_2(t), \varsigma_2(t))|\big)\\ &\quad\leq M_3\big(\|z_1(t) - z_2(t)\| + M_2|\mathcal{B}(z_1(t), \varsigma_1(t)) - \mathcal{B}(z_2(t), \varsigma_2(t))|\big).\end{aligned}$$

2. Problem Formulations

2.1. *General setup*

In the sequel, $(Z, \|\cdot\|_Z)$ (we will often write $\|\cdot\|$ instead of $\|\cdot\|_Z$) will be a separable Banach space, with dual $(Z', \|\cdot\|_{Z'})$ and duality pairing $\langle\cdot,\cdot\rangle$. We denote by $\mathcal{L}(Z, Z')$ the space of all linear bounded operators from Z to Z'. Let us now state our basic assumptions on the energy functional $\mathcal{E} : [0, T] \times Z \to \mathbb{R}$ and on the dissipation potential $\Psi : Z \times Z \to [0, +\infty]$.

We will assume that

$$\mathcal{E}(t, \cdot) : Z \to \mathbb{R} \quad \text{is convex and l.s.c. for } t \in [0, T], \tag{2.1}$$

and that the function $t \in [0, T] \mapsto \mathcal{E}(t, z)$ is differentiable for all $z \in Z$, with

$$\begin{aligned}&\partial_t\mathcal{E}(\cdot, z) : [0, T] \to \mathbb{R} \text{ measurable,} \quad \text{and} \ \ \exists\, C_0 > 0,\ \exists\, \lambda_0 \in L^1(0, T; [0, \infty)),\\ &\forall\, z \in Z : |\partial_t\mathcal{E}(t, z)| \leq \lambda_0(t)(\mathcal{E}(t, z) + C_0).\end{aligned} \tag{2.2}$$

Hence, (see also Sec. 3 in Ref. 12), $\mathcal{E}$ is bounded from below and absolutely continuous in time, namely $\forall\, t, s \in [0, T]$ and $\forall\, z \in Z$ we have

$$\mathcal{E}(t, z) \geq -C_0 \quad \text{and} \quad \mathcal{E}(t, z) + C_0 \leq \big(\mathcal{E}(s, z) + C_0\big) \exp\left(\left|\int_s^t \lambda_0(\tau) d\tau\right|\right). \tag{2.3}$$

We will denote by $\partial\mathcal{E}(t, \cdot)$ the subdifferential of $\mathcal{E}$ (in the sense of convex analysis) w.r.t. the variable z, i.e.

$$\xi \in \partial\mathcal{E}(t, z) \quad \text{if and only if} \quad \mathcal{E}(t, w) - \mathcal{E}(t, z) \geq \langle\xi, w - z\rangle \quad \forall\, w \in Z. \tag{2.4}$$

For the dissipation potential Ψ, we assume that $\forall\, z \in Z$,

$$\Psi(z,\cdot) : Z \to [0,+\infty) \text{ is convex, positively homogeneous of degree 1} \tag{2.5}$$

$$\exists\, C_\Psi > 0,\ \forall\,(z,v) \in Z \times Z: \ \Psi(z,v) \le C_\Psi \|v\|. \tag{2.6}$$

In particular, by (2.6)

$$D(\Psi(z,\cdot)) = Z \quad \forall\, z \in Z. \tag{2.7}$$

Also, given $(z,v) \in Z \times Z$, $\partial_v \Psi(z,v)$ denotes the subdifferential of the convex function $\Psi(z,\cdot)$ in the point v.

Let us gain some insight into the *geometrical* interpretation of the assumptions on Ψ: indeed, (2.5) yields the triangle inequality

$$\Psi(z, v+\hat{v}) \le \Psi(z,v) + \Psi(z,\hat{v}) \quad \text{for all } z, v, \hat{v} \in Z. \tag{2.8}$$

Actually, (2.8) is a consequence of the fact (equivalent to (2.5) and (2.6)), that for every $z \in Z$, there exists

$$\begin{gathered} \text{a non-empty, closed, and convex set } C(z) \subset Z' \text{ with} \\ \Psi(z,v) := \sup\{\langle \sigma, v\rangle \mid \sigma \in C(z)\} \quad \text{for all } v \in Z. \end{gathered} \tag{2.9}$$

Namely, for every $z \in Z$ $\Psi(z,\cdot)$ is the *support function* of the set $C(z)$: thus, it is easy to see that (2.6) may be equivalently rephrased (cf. Appendix A), as

$$C(z) \subset B^*_{C_\Psi}(0) \quad \text{for all } z \in Z.$$

By standard convex analysis (see Ref. 20), we have for all $v, z \in Z$

$$\partial_v \Psi(z,v) = \operatorname{argmax}\{\langle \sigma, v\rangle \mid \sigma \in C(z)\} \subset C(z), \tag{2.10}$$

$$\partial_v \Psi(z,v) = (\partial I_{C(z)})^{-1}(v). \tag{2.11}$$

In particular,

$$\partial_v \Psi(z,0) = C(z) \quad \forall\, z \in Z. \tag{2.12}$$

In the sequel (cf. especially Sec. 5), we will exploit the representation formula (2.9) by means of some specific convex analysis results, which we recall in Appendix A for the reader's convenience, referring to Chap. 2 in Ref. 11 and Ref. 20 for the proofs and further details.

2.2. *Problem formulations*

As in Refs. 12 and 18, we present different formulations of the Cauchy problem for (1.1). In the sequel, z_0 will be a given element of Z.

Problem 2.1. (Subdifferential formulation) *Find $z \in W^{1,1}(0,T;Z)$ fulfilling the initial condition $z(0) = z_0$ and*

$$\partial_v \Psi(z(t), \dot{z}(t)) + \partial \mathcal{E}(t, z(t)) \ni 0 \quad \textit{for a.e. } t \in (0,T). \tag{SF}$$

The latter differential inclusion means that there exist $\omega, \xi : (0,T) \to Z'$ such that

$$\begin{aligned} &\omega(t) \in \partial_v \Psi(z(t), \dot{z}(t)), \ \ \xi(t) \in \partial \mathcal{E}(t, z(t)) \quad \text{and} \\ &\omega(t) + \xi(t) = 0 \quad \text{for a.e. } t \in (0,T). \end{aligned} \tag{2.13}$$

We may also introduce a local formulation of Problem 2.1.

Problem 2.2. (Local formulation) *Find* $z \in W^{1,1}(0,T;Z)$ *such that* $z(0) = z_0$ *and there exists* $\xi : (0,T) \to Z'$ *such that* for a.e. $t \in (0,T)$ *we have* $\xi(t) \in \partial\mathcal{E}(t, z(t))$ *and*

$$\Psi(z(t), v) + \langle \xi(t), v \rangle \geq 0 \quad \forall\, v \in Z, \tag{$\mathrm{S_{loc}}$}$$

$$\Psi(z(t), \dot{z}(t)) + \langle \xi(t), \dot{z}(t) \rangle \leq 0. \tag{$\mathrm{E_{loc}}$}$$

The proof of the following equivalence result follows closely the proof of Theorem 3.5 in Ref. 18.

Proposition 2.1. *Under the assumptions* (2.1)–(2.2) *on* $\mathcal{E}$ *and* (2.5) *on* Ψ, *the Subdifferential Formulation* 2.1 *and the Local Formulation* 2.2 *are equivalent.*

Proof. Let $z \in W^{1,1}(0,T;Z)$ fulfil (SF). Then, there is a selection $\xi(t)$ of $\partial\mathcal{E}(t, z(t)) \cap (-\partial_v \Psi(z(t), \dot{z}(t)))$ for a.e. $t \in (0,T)$ fulfilling (2.13), which we test by $\dot{z}(t)$, thus obtaining ($\mathrm{E_{loc}}$). We conclude ($\mathrm{S_{loc}}$) by noting that, in view of (2.10) and (2.12), $-\xi(t) \in C(z(t)) = \partial_v \Psi(z(t), 0)$.

Conversely, if a selection $\xi(t) \in \partial\mathcal{E}(t, z(t))$ fulfils ($\mathrm{S_{loc}}$) and ($\mathrm{E_{loc}}$), we easily obtain the variational inequality

$$\Psi(z(t), v) - \Psi(z(t), \dot{z}(t)) \geq \langle -\xi(t), v - \dot{z}(t) \rangle \geq 0 \quad \forall\, v \in Z,$$

yielding $-\xi(t) \in \partial_v \Psi(z(t), \dot{z}(t))$. □

Remark 2.1. Using that $\Psi(z, 0) = 0$ for all $z \in Z$, it is easy to see that for a selection ξ of $\partial\mathcal{E}(\cdot, z(\cdot))$ we have

$$\xi(t) \text{ satisfies } (\mathrm{S_{loc}}) \Leftrightarrow \xi(t) \in \partial\mathcal{E}(t, z(t)) \cap (-\partial_v \Psi(z(t), 0)). \tag{2.14}$$

Moreover, the latter condition implies $\partial_v \Psi(z(t), 0) + \partial\mathcal{E}(t, z(t)) \ni 0$.

Finally, we consider an integral formulation of Problems 2.1 and 2.2. Note that this is not the energetic formulation proposed in Refs. 13, 17 and 18, which will be discussed in Sec. 2.3.

Problem 2.3. (Global formulation) Find $z \in W^{1,1}(0,T;Z)$ with $z(0) = z_0$ such that for all $t \in [0,T]$ the stability condition ($\mathrm{S_\Psi}$) and the energy balance ($\mathrm{E_\Psi}$) hold:

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, \hat{z}) + \Psi(z(t), \hat{z} - z(t)) \quad \forall\, \hat{z} \in Z, \tag{$\mathrm{S_\Psi}$}$$

$$\mathcal{E}(t, z(t)) + \int_0^t \Psi(z(\tau), \dot{z}(\tau)) d\tau = \mathcal{E}(0, z(0)) + \int_0^t \partial_t \mathcal{E}(\tau, z(\tau)) d\tau. \tag{$\mathrm{E_\Psi}$}$$

The following result, which is a version of the chain rule for the subdifferential of convex functionals on Hilbert spaces proved in, e.g., Lemma 3.3 in Ref. 4, will play a crucial role in establishing the links between the Global Formulation 2.3 and the previous formulations 2.1 and 2.2.

Proposition 2.2. *Let* $\mathcal{E} : [0,T] \times Z \to \mathbb{R}$ *comply with* (2.1), (2.2) *and*

$$\begin{aligned} &\exists\, \lambda_1 \in L^1(0,T),\ \forall_{\text{a.e.}} t \in (0,T) \\ &\forall\, z, \hat{z} \in Z : |\partial_t \mathcal{E}(t,z) - \partial_t \mathcal{E}(t,\hat{z})| \le \lambda_1(t) \|z - \hat{z}\|. \end{aligned} \tag{2.15}$$

Furthermore, assume that $z \in W^{1,1}(0,T;Z)$ *and that there exists a selection* g *with*

$$g(t) \in \partial\mathcal{E}(t, z(t)) \quad \textit{for a.e. } t \in (0,T) \quad \textit{and} \quad g \in L^\infty(0,T;Z'). \tag{2.16}$$

Then, the map $t \mapsto \mathcal{E}(t, z(t))$ *is absolutely continuous on* $(0,T)$ *and for every measurable selection* $\zeta(t) \in \partial\mathcal{E}(t,z(t))$ *we have the identity*

$$\frac{d}{dt}\mathcal{E}(t,z(t)) = \langle \zeta(t), \dot{z}(t) \rangle + \partial_t \mathcal{E}(t, z(t)) \quad \textit{for a.e. } t \in (0,T). \tag{2.17}$$

Proof. First, we point out that, in view of (2.2) and (2.15), we have

$$\begin{aligned} \int_0^T |\partial_\tau \mathcal{E}(\tau, z(\tau))|\, d\tau &\le \int_0^T \lambda_1(\tau) \|z(0) - z(\tau)\|\, d\tau + \int_0^T |\partial_\tau \mathcal{E}(\tau, z(0))|\, d\tau \\ &\le \left(\|z\|_{L^\infty(0,T)} + \|z(0)\| \right) \int_0^T \lambda_1(\tau)\, d\tau \\ &\quad + \int_0^T \lambda_0(\tau)\, d\tau (\mathcal{E}(0, z(0)) + C_0) \exp\left(\int_0^T \lambda_0(s)\, ds \right), \end{aligned}$$

where we have also used (2.2) and (2.3) to obtain

$$|\partial_\tau \mathcal{E}(\tau, z(0))| \le \lambda_0(\tau)(\mathcal{E}(\tau, z(0)) + C_0) \le \lambda_0(\tau)(\mathcal{E}(0, z(0)) + C_0) \exp\left(\int_0^T \lambda_0(s)\, ds \right).$$

Thus, we know that the map $t \mapsto \partial_t \mathcal{E}(t, z(t))$ is also in $L^1(0,T)$.

Second, by (2.16), there exists a negligible set $\mathcal{N} \subset (0,T)$ such that for $t \in (0,T) \setminus \mathcal{N}$, $g(t) \in \partial\mathcal{E}(t, z(t))$. Thus, using (2.16) and (2.15), for $s, t \in (0,T) \setminus \mathcal{N}$ with $s \le t$ we have

$$\begin{aligned} \mathcal{E}(t, z(t)) - \mathcal{E}(s, z(s)) &= \mathcal{E}(t, z(t)) - \mathcal{E}(t, z(s)) + \mathcal{E}(t, z(s)) - \mathcal{E}(s, z(s)) \\ &\overset{(2.16)}{\le} \langle g(t), z(t) - z(s) \rangle + \int_s^t \partial_\tau \mathcal{E}(\tau, z(s)) d\tau \\ &\overset{(2.15)}{\le} \langle g(t), z(t) - z(s) \rangle + \int_s^t \lambda_1(\tau) \|z(s) - z(\tau)\| d\tau \\ &\quad + \int_s^t \partial_\tau \mathcal{E}(\tau, z(\tau)) d\tau. \end{aligned} \tag{2.18}$$

In the same way, we obtain the lower estimate

$$\begin{aligned}\mathcal{E}(t,z(t))-\mathcal{E}(s,z(s)) &= \mathcal{E}(t,z(t))-\mathcal{E}(s,z(t))+\mathcal{E}(s,z(t))-\mathcal{E}(s,z(s))\\ &\geq \langle g(s),z(t)-z(s)\rangle-\int_s^t\lambda_1(\tau)\|z(t)-z(\tau)\|d\tau\\ &\quad+\int_s^t\partial_\tau\mathcal{E}(\tau,z(\tau))d\tau. \end{aligned}\tag{2.19}$$

Collecting (2.18) and (2.19) we deduce that for $s,t\notin\mathcal{N}$ with $s\leq t$ we have

$$\begin{aligned}&\left|\mathcal{E}(t,z(t))-\mathcal{E}(s,z(s))-\int_s^t\partial_\tau\mathcal{E}(\tau,z(\tau))\,d\tau\right|\\ &\qquad\leq 2\|z\|_{L^\infty(0,T;Z)}\int_s^t\lambda_1(\tau)\,d\tau+\|g\|_{L^\infty(0,T;Z')}\|z(t)-z(s)\|.\end{aligned}\tag{2.20}$$

Indeed, by continuity (2.20) holds for all $0\leq s\leq t\leq T$, and the absolute continuity of the map $t\mapsto\mathcal{E}(t,z(t))$ hence follows.

Finally, let ζ be an arbitrary selection of $\partial\mathcal{E}(\cdot,z(\cdot))$ satisfying the assumptions of the proposition. Then, the set of points $t_0\in(0,T)$ such that $\frac{d}{dt}\mathcal{E}(t,z(t))|_{t=t_0}$ exists, $\zeta(t_0)\in\partial\mathcal{E}(t_0,z(t_0))$, and t_0 is a Lebesgue point for λ_1 and for the map $t\mapsto\partial_t\mathcal{E}(t,z(t))$ is of full measure. Now, choose such a t_0, consider (2.18) for $s:=t_0-h$ and $t:=t_0$ with $0<h<t_0$, divide it by h and take the limit as $h\searrow 0$. Then, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t,z(t))|_{t=t_0} &= \limsup_{h\searrow 0}\frac{\mathcal{E}(t_0,z(t_0))-\mathcal{E}(t_0-h,z(t_0-h))}{h}\\ &\leq \lim_{h\searrow 0}\left\langle\zeta(t_0),\frac{z(t_0)-z(t_0-h)}{h}\right\rangle\\ &\quad+\lim_{h\searrow 0}\left(\sup_{t_0-h\leq\tau\leq t_0}\|z(\tau)-z(t_0-h)\|\right)\frac{1}{h}\int_{t_0-h}^{t_0}\lambda_1(\tau)\,d\tau\\ &\quad+\lim_{h\searrow 0}\frac{1}{h}\int_{t_0-h}^{t_0}\partial_t\mathcal{E}(\tau,z(\tau))\,d\tau\\ &\leq\langle\zeta(t_0),\dot z(t_0)\rangle+0+\partial_t\mathcal{E}(t_0,z(t_0)).\end{aligned}$$

In the same way, exploiting (2.19) and choosing $s=t_0$ and $t=t_0+h$ this time, we obtain the reverse inequality $\frac{d}{dt}\mathcal{E}(t,z(t))|_{t=t_0}\geq\langle\zeta(t_0),\dot z(t_0)\rangle+\partial_t\mathcal{E}(t_0,z(t_0))$. Thus, we conclude the chain rule formula (2.17) at $t=t_0$. □

Now, we are able to formulate the next equivalence result.

Proposition 2.3. *Assume* (2.1), (2.2), (2.5), (2.6), (2.15) *and* (2.16).

If $z\in W^{1,1}(0,T;Z)$ satisfies the Subdifferential Formulation (SF) in the form (2.13) *with a selection $\xi\in L^\infty(0,T;Z')$ of $t\to\partial\mathcal{E}(t,z(t))$, then z also fulfils the Global Formulation* (S_Ψ) *and* (E_Ψ).

Conversely, any solution $z \in W^{1,1}(0,T;Z)$ *of* (S_Ψ) *and* (E_Ψ) *satisfies the Subdifferential Formulation* (*SF*)*.*

Proof. We will exploit Proposition 2.1 and indeed reduce to proving the equivalence between the Local Formulation 2.2 and the Global Formulation 2.3.

Hence, let ξ be a selection of $\partial\mathcal{E}(\cdot, z(\cdot))$ in $L^\infty(0,T;Z')$ fulfilling $(\mathrm{S}_{\mathrm{loc}})$–$(\mathrm{E}_{\mathrm{loc}})$: then, (E_Ψ) is obtained by integrating in time $(\mathrm{E}_{\mathrm{loc}})$, and using the chain rule formula (2.17), while (S_Ψ) follows by choosing $v := \hat{z} - z(t)$ in $(\mathrm{S}_{\mathrm{loc}})$ for an arbitrary $\hat{z} \in Z$, and recalling the definition of subdifferential (2.4).

For the converse implication, first note that (S_Ψ) implies $(\mathrm{S}_{\mathrm{loc}})$: indeed, in view of (2.7) and Lemma A.1 in Appendix A, (S_Ψ) yields

$$0 \in \partial\mathcal{E}(t, z(t)) + \partial_v \Psi(z(t), 0) \quad \text{for a.e. } t \in (0,T),$$

which can be rephrased as

$$\text{a.e. } t \in (0,T),\ \exists\, \xi(t) \in \partial\mathcal{E}(t, z(t)) \subset Z'\ \forall\, v \in Z : \Psi(z(t), v) + \langle \xi(t), v\rangle \geq 0. \quad (2.21)$$

On the other hand, it is straightforward to check that $(\mathrm{S}_{\mathrm{loc}})$ is equivalent to (2.21).

Second, consider (E_Ψ) and use that $t \mapsto \Psi(z(t), \dot{z}(t))$ is in $L^1(0,T)$ (in view of (2.6) and of $\dot{z} \in L^1(0,T)$), and that $t \mapsto \partial_t\mathcal{E}(t, z(t))$ is in $L^1(0,T)$. This follows from (E_Ψ), (2.2), where $\lambda_0 \in L^1(0,T)$, and the Gronwall lemma, ensuring that $t \mapsto \mathcal{E}(t, z(t))$ is in $L^\infty(0,T)$, whence the estimate for $t \mapsto \partial_t\mathcal{E}(t, z(t))$). Taking t to be a Lebesgue point of these two maps as well as of $\dot{z}$, we obtain, for any $\eta \in \partial\mathcal{E}(t, z(t))$, the estimate

$$\begin{aligned}
&\frac{1}{h}(\mathcal{E}(t+h, z(t+h)) - \mathcal{E}(t, z(t)))\\
&\quad= \frac{1}{h}(\mathcal{E}(t, z(t+h)) - \mathcal{E}(t, z(t))) + \frac{1}{h}(\mathcal{E}(t+h, z(t+h)) - \mathcal{E}(t, z(t+h)))\\
&\quad\geq \left\langle \eta, \frac{1}{h}(z(t+h) - z(t)) \right\rangle + \frac{1}{h}\int_t^{t+h} \partial_s\mathcal{E}(s, z(t+h))\, ds\\
&\quad= \left\langle \eta, \frac{1}{h}(z(t+h) - z(t)) \right\rangle + \partial_t\mathcal{E}(t, z(t)\\
&\qquad+ \frac{1}{h}\int_t^{t+h} (\partial_s\mathcal{E}(s, z(t+h)) - \partial_s\mathcal{E}(t, z(t)))\, ds.
\end{aligned}$$

For $h \searrow 0$, the first term on the right-hand side tends to $\langle \eta, \dot{z}(t)\rangle$, while the last term tends to 0 due to (2.15), and the Lebesgue-point property of t for $\partial_t\mathcal{E}$. Since the derivative of (E_Ψ) gives: $\frac{d}{dt}(\mathcal{E}(t, z(t))) + \Psi(z(t), \dot{z}(t)) = \partial_t\mathcal{E}(t, z(t))$, we arrive at

$$\forall\, \eta \in \partial\mathcal{E}(t, z(t)) : \langle \eta, \dot{z}(t)\rangle + \Psi(z(t), \dot{z}(t)) \leq 0.$$

Inserting $\eta := \xi(t)$, we see that $(\mathrm{E}_{\mathrm{loc}})$ is satisfied as well. $\square$

2.3. *The energetic formulation*

For completeness, we also mention the global energetic approach developed in the series of papers.[13,17–19] To this aim, we associate with the dissipation potential Ψ a global *dissipation distance* $\mathcal{D}$ on Z via

$$\mathcal{D}(z_0, z_1) := \inf \left\{ \mathrm{Diss}_\Psi(\zeta, [0,1]) : \zeta \in C^1([0,1]; Z),\ \zeta(0) = z_0,\ \zeta(1) := z_1 \right\}, \quad (2.22)$$

where the functional Diss_Ψ is defined by

$$\mathrm{Diss}_\Psi(\zeta, [s_0, s_1]) := \int_{s_0}^{s_1} \Psi(\zeta(t), \dot{\zeta}(t))dt. \quad (2.23)$$

Furthermore, given a curve $z : [0,T] \to Z$, and a subinterval $[s,t] \subset [0,T]$, the *total dissipation* of z on $[s,t]$ is defined by

$$\mathrm{Diss}_\mathcal{D}(z; [s,t]) := \sup \left\{ \sum_{j=1}^N \mathcal{D}(z(t_{j-1}), z(t_j)) \;\middle|\; N \in \mathbb{N},\ s = t_0 < t_1 < \cdots < t_N = t \right\}.$$

Under suitable assumptions on Ψ, it is possible to show that $\mathrm{Diss}_\mathcal{D}$ coincides with Diss_Ψ along absolutely continuous curves. However, $\mathrm{Diss}_\mathcal{D}$ is also defined in more general situations.

We can now introduce a derivative-free, energetic formulation of Problem 2.1.

Definition 2.1. (Energetic formulation) A curve $z : [0,T] \to Z$ is called a solution of the rate-independent Problem 2.1 associated with $(\mathcal{E}, \mathcal{D})$ if for all $t \in [0,T]$ the *global stability* $(\mathrm{S}_\mathcal{D})$ and the *energy balance* $(\mathrm{E}_\mathcal{D})$ hold, i.e.

$$\mathcal{E}(t, z(t)) \le \mathcal{E}(t, \hat{z}) + \mathcal{D}(z(t), \hat{z}) \quad \forall\, \hat{z} \in Z, \quad (\mathrm{S}_\mathcal{D})$$

$$\mathcal{E}(t, z(t)) + \mathrm{Diss}_\mathcal{D}(z; [0,t]) = \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, z(s))ds. \quad (\mathrm{E}_\mathcal{D})$$

It is easy to see that if $z \in W^{1,1}((0,T), Z)$ solves $(\mathrm{S}_\mathcal{D})$ and $(\mathrm{E}_\mathcal{D})$, then it also solves (S_Ψ) and (E_Ψ).

3. Temporal Regularity via Uniform Convexity

Throughout this section, we will assume that the energy functional $\mathcal{E} : [0,T] \times Z \to \mathbb{R}$ complies with (2.1) and (2.2).

Here, the crucial condition will be a suitable strict convexity assumption on $\mathcal{E}$. In fact, we require $z \mapsto \mathcal{E}(t, z)$ to be uniformly convex, namely

$$\begin{gathered} \exists\, \kappa > 0, \quad \forall\, z_0, \quad z_1 \in Z, \quad \forall\, t \in [0,T], \quad \forall\, \theta \in [0,1] : \\ \mathcal{E}(t, z_\theta) \le (1-\theta)\mathcal{E}(t, z_0) + \theta \mathcal{E}(t, z_1) - \frac{\kappa}{2}\theta(1-\theta)\|z_0 - z_1\|^2, \end{gathered} \quad (3.1)$$

where $z_\theta := (1-\theta)z_0 + \theta z_1$. Let us stress that condition (3.1) means that $\mathcal{E}$ is κ*-uniformly convex* in the z variable, with a modulus of convexity κ independent of $t \in [0,T]$. In other words, we require the functional $z \mapsto \mathcal{E}(t,z) - \frac{\kappa}{2}\|z\|^2$ to be convex. Note that this implies

$$\mathcal{E}(t,\hat{z}) \geq \mathcal{E}(t,z) + \langle \xi, \hat{z} - z \rangle + \frac{\kappa}{2}\|\hat{z} - z\|^2 \quad \forall\, z,\hat{z} \in Z, \quad \forall\, \xi \in \partial\mathcal{E}(t,z). \tag{3.2}$$

As for Ψ, besides (2.5) and (2.6), we also assume that there exists $\psi^* > 0$ such that

$$|\Psi(z,v) - \Psi(\hat{z},v)| \leq \psi^* \|v\| \|z - \hat{z}\|, \tag{3.3}$$

$$\text{and} \quad \psi^* < \kappa. \tag{3.4}$$

Before stating the main result of this section, we consider a simple example, which shows that our conditions are sharp.

Example 3.1. We consider the case $\mathcal{E}(t,z) = \frac{\kappa}{2}z^2 - \lambda t z$, with $Z = \mathbb{R}$ and fixed $\kappa, \lambda > 0$. The state-dependent dissipation potential takes the form

$$\Psi(z,v) = r(z)|v|, \quad \text{with} \quad r(z) = \begin{cases} 1 + \psi^* & \text{for } z \leq 1, \\ 1 + \psi^*(2 - z) & \text{for } z \in [1,2], \\ 1 & \text{for } z \geq 2, \end{cases}$$

with $\psi^* \geq 0$. For $\psi^* < \kappa$ and for the initial value $z_0 = 0$, a solution can be constructed easily, namely

$$z(t) := \begin{cases} 0 & \text{for } t \in [0, (1+\psi^*)/\lambda], \\ (\lambda t - 1 - \psi^*)/\kappa & \text{for } t \in [(1+\psi^*)/\lambda, (1+\psi^*+\kappa)/\lambda], \\ (\lambda t - 1 - 2\psi^*)/(\kappa - \psi^*) & \text{for } t \in [(1+\psi^*+\kappa)/\lambda, (1+2\kappa)/\lambda], \\ (\lambda t - 1)/\kappa & \text{for } t \geq (1+2\kappa)/\lambda. \end{cases}$$

It is easy to see that the solution is unique. The Lipschitz constant of z is given by $\lambda/(\kappa - \psi^*)$, and hence blows up for $\kappa - \psi^* \searrow 0$.

For $\psi^* \geq \kappa$ there does not exist an absolutely continuous solution. Indeed, any solution must satisfy $z(t) \in \mathcal{S}(t) = [(\lambda t - 1)/\kappa, (\lambda t + 1)/\kappa]$, which is equivalent to $|\kappa z(t) - \lambda t| \leq r(z(t))$. Thus, for $t \geq (1+2\kappa)/\lambda$ we must have $z(t) \geq (\lambda t - 1)/\kappa \geq 2$. However, it is impossible for the solution to move through the z-interval $(1,2)$ in an absolutely continuous fashion, since the relations $0 \in \partial\Psi(z,\dot{z}) + \kappa z - \lambda t$ and $\dot{z} > 0$ imply $r(z) + \kappa z - \lambda t = 0$. Hence, using $\psi^* \geq \kappa > 0$ and differentiating the last expression gives $\dot{z} \leq 0$.

Theorem 3.1. *Assume* (2.1), (2.2), (2.5), (2.6), (2.15), (2.16), (3.1), (3.3) *and* (3.4): *then, any solution* $z \in W^{1,1}(0,T;Z)$ *to Problem* 2.1 *satisfies*

$$\|\dot{z}(t)\| \leq \frac{\lambda_1(t)}{\kappa - \psi^*} \quad \textit{for a.e. } t \in [0,T].$$

In particular, if $\lambda_1 \in L^\infty(0,T)$, *then* $z \in C^{\mathrm{Lip}}(0,T;Z)$.

Proof. We start by noting that any solution z to Problem 2.1 fulfils a stability condition stronger than (S_Ψ), namely

$$\frac{\kappa}{2}\|\hat{z}-z(s)\|^2+\mathcal{E}(s,z(s)) \\ \leq \mathcal{E}(s,\hat{z})+\Psi(z(s),\hat{z}-z(s)) \quad \forall\,\hat{z}\in Z \quad \text{for a.e. } s\in(0,T). \tag{3.5}$$

Indeed, we fix s, out of a negligible set, at which z fulfils $(\mathrm{S}_{\mathrm{loc}})$. On the other hand, we consider (3.2) for $t=s$ and add $\Psi(z(s),\hat{z}-z(s))$ to both sides of the resulting inequality. Then, we use that z fulfils $(\mathrm{S}_{\mathrm{loc}})$ at s, with $\xi(s)\in\partial\mathcal{E}(s,z(s))$ and $v=\hat{z}-z(s)$. Hence, (3.5) follows.

Then, $\forall\,t\in[0,T]$ and for a.e. $s\leq t$ we conclude

$$\begin{aligned}
\frac{\kappa}{2}\|z(t)-z(s)\|^2 &\leq \mathcal{E}(s,z(t))-\mathcal{E}(s,z(s))+\Psi(z(s),z(t)-z(s))\\
&\leq \mathcal{E}(s,z(t))-\mathcal{E}(t,z(t))+\mathcal{E}(t,z(t))-\mathcal{E}(s,z(s))+\int_s^t\Psi(z(s),\dot{z}(\tau))d\tau\\
&= -\int_s^t\partial_t\mathcal{E}(\tau,z(t))d\tau+\int_s^t\partial_t\mathcal{E}(\tau,z(\tau))d\tau-\int_s^t\Psi(z(\tau),\dot{z}(\tau))d\tau\\
&\quad +\int_s^t\Psi(z(s),\dot{z}(\tau))d\tau\\
&\leq \int_s^t\lambda_1(\tau)\|z(t)-z(\tau)\|d\tau+\psi^*\int_s^t\|\dot{z}(\tau)\|\|z(\tau)-z(s)\|d\tau,
\end{aligned}$$

where the first inequality is obtained by choosing $\hat{z}:=z(t)$ in (3.5), the second inequality follows from the convexity of $\Psi(z(s),\cdot)$, the third one from the energy identity (E_Ψ) (fulfilled by z in view of Proposition 2.3), and the last one from (2.15) and (3.3). Note that this estimate is exactly the assumption of the following Lemma 3.1, which concludes the proof. □

Lemma 3.1. *Let $z\in W^{1,1}(0,T;Z)$ and assume that there exist positive constants α and β with $\beta<\alpha$ and a function $\gamma\in L^1(0,T;[0,\infty))$ such that, for all $0\leq s\leq t\leq T$*

$$\frac{\alpha}{2}\|z(t)-z(s)\|^2\leq\int_s^t\gamma(\tau)\|z(t)-z(\tau)\|\,d\tau+\beta\int_s^t\|\dot{z}(\tau)\|\|z(\tau)-z(s)\|\,d\tau. \tag{3.6}$$

Then, we have

$$\|\dot{z}(t)\|\leq\frac{\gamma(t)}{\alpha-\beta}\quad \textit{for a.e. } t\in(0,T). \tag{3.7}$$

Proof. By the mean continuity theorem in L^1-spaces, there exists a full-measure subset $M\subset(0,T)$ such that for all $t\in M$ and for any fixed $\varepsilon>0$ there exists $h_\varepsilon>0$ such that

$$\frac{1}{h}\int_{t-h}^t\|\dot{z}(\tau)-\dot{z}(t)\|\,d\tau<\varepsilon,\quad \frac{1}{h}\int_{t-h}^t|\gamma(\tau)-\gamma(t)|\,d\tau<\varepsilon\quad \forall\,0<h<h_\varepsilon.$$

Hence, for $0 < h < h_\varepsilon$ by easy computations we have

$$\|z(t) - z(t-h)\|^2 = \left\| \int_{t-h}^{t} \dot{z}(\tau)\, d\tau \right\|^2 \geq h^2(\|\dot{z}(t)\| - \varepsilon)^2, \tag{3.8}$$

$$\begin{aligned}\int_{t-h}^{t} \|\dot{z}(\tau)\| \|z(\tau) - z(\tau-h)\|\, d\tau &\leq \frac{1}{2}\int_{t-h}^{t} \frac{d}{d\tau}\left(\int_{\tau-h}^{\tau} \|\dot{z}(\sigma)\|\, d\sigma\right)^2 d\tau \\ &= \frac{1}{2}\left(\int_{t-h}^{t} \|\dot{z}(\tau)\|\, d\tau\right)^2 \\ &\leq \frac{h^2}{2}(\|\dot{z}(t)\| + \varepsilon)^2, \end{aligned} \tag{3.9}$$

$$\begin{aligned}\int_{t-h}^{t} \gamma(\tau)\|z(t) - z(\tau)\|\, d\tau &\leq \int_{t-h}^{t} \gamma(\tau)(t-\tau)(\|\dot{z}(t)\| + \varepsilon)\, d\tau \\ &= (\|\dot{z}(t)\| + \varepsilon)\left(\int_{t-h}^{t} (\gamma(\tau) - \gamma(t))(t-\tau)\, d\tau + \frac{h^2}{2}\gamma(t)\right) \\ &\leq \frac{h^2}{2}(\|\dot{z}(t)\| + \varepsilon)(\gamma(t) + 2\varepsilon). \end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10) with (3.6) and letting $\varepsilon \downarrow 0$, we infer (3.7). □

Remark 3.1. If $\mathcal{E}$ and $\Psi(\cdot, v)$ are sufficiently smooth, the desired Lipschitz estimate in Theorem 3.1 can be obtained from the weakened assumption:

$$\langle D^2\mathcal{E}(t,z)w, w\rangle + D_z\Psi(z,w)[w] \geq \delta\|w\|^2, \tag{3.11}$$

see (5.12). Note that (3.11) is due to (3.1), (3.3) and (3.4), giving $\delta = \kappa - \psi^*$.

Indeed, choose any $s \in (0,T)$ which is a Lebesgue point of $\dot{z}$. Using ($\mathrm{S}_{\mathrm{loc}}$) and ($\mathrm{E}_{\mathrm{loc}}$) with $\xi(t) = D\mathcal{E}(t, z(t))$ and $v = \dot{z}(s)$, the function $\alpha : t \mapsto \Psi(z(t), \dot{z}(s)) + \langle D\mathcal{E}(t,z(t)), \dot{z}(s)\rangle$ satisfies $\alpha(t) \geq 0$ and $\alpha(s) = 0$. Hence, we have

$$0 = \dot{\alpha}(s) = D_z\Psi(z(s), \dot{z}(s))[\dot{z}(s)] + \langle \partial_s D\mathcal{E}(s, z(s)) + D^2\mathcal{E}(s, z(s))\dot{z}(s), \dot{z}(s)\rangle.$$

Now, (3.11) and (2.15) imply

$$\begin{aligned}\lambda_1(s)\|\dot{z}(s)\| &\geq |\langle \partial_s D\mathcal{E}(s, z(s)), \dot{z}(s)\rangle| = -\langle \partial_s D\mathcal{E}(s, z(s)), \dot{z}(s)\rangle \\ &= D_z\Psi(z(s), \dot{z}(s))[\dot{z}(s)] + \langle D^2\mathcal{E}(s, z(s))\dot{z}(s), \dot{z}(s)\rangle \geq \delta\|\dot{z}(s)\|^2,\end{aligned}$$

which is the desired result.

4. An Existence Result

As shown in Ref. 18, there are essentially two ways to establish the existence. In all cases, suitable approximate solutions are constructed via regularization or via time discretization. To obtain solutions, these approximations have to be controlled via *a priori* estimates. One class of existence results is based on compactness arguments, usually by using the weak topology in Banach spaces. It allows us to extract a suitable subsequence which converges to a solution, but does not provide uniqueness of the solution. Another class of existence results is based on a more careful control of the distances of the approximate solutions, in order to show that they form a converging sequence of functions, see e.g. Refs. 3 and 8 and Theorem 7.3 in Ref. 18. Here we follow the first method, and use compactness methods and fairly general conditions. Uniqueness will be established in the following section under much stronger assumptions, and exploiting completely different methods.

4.1. *Statement of the assumptions and the result*

In this section, we will assume that our ambient Banach space

$$Z \text{ is reflexive.} \tag{4.1}$$

We will establish an existence result (cf. Theorem 4.1 later on) for Problem 2.1 essentially under weak continuity conditions on Ψ and on $\mathcal{E}$. Note that weak continuity provides compactness arguments if we obtain additional boundedness conditions, since bounded sequences have weakly convergent subsequences under the reflexivity assumption (4.1).

Let us now enlist all the assumptions on $\mathcal{E}$ and Ψ which will come into play in the proof of Theorem 4.1, referring to the notation of Sec. 2.1. Moreover, we denote by $\partial\mathcal{E} \subset [0,T] \times Z \times Z'$ the graph of the set-valued map $(t,z) \mapsto \partial\mathcal{E}(t,z)$.

Assumptions on the energy functional $\mathcal{E}$*.* We suppose that $\mathcal{E}$ complies with (2.1), (2.2), (2.16); besides, we strengthen the assumption (2.15) by assuming that λ_1 lies in $L^\infty(0,T)$. With $\Lambda_1 := \|\lambda_1\|_{L^\infty}$, we then have

$$\forall\, z, \hat{z} \in Z; \forall\, t \in [0,T] : |\partial_t\mathcal{E}(t,z) - \partial_t\mathcal{E}(t,\hat{z})| \leq \Lambda_1\|z - \hat{z}\|. \tag{4.2}$$

We also assume the strict convexity (3.1), and

$$\text{for a.e. } t \in (0,T) \text{ the map } z \mapsto \partial_t\mathcal{E}(t,z) \text{ is weakly continuous on Z,} \tag{4.3}$$

$$\partial\mathcal{E} \subset [0,T] \times Z \times Z' \text{ is closed in the strong–weak–weak topology.} \tag{4.4}$$

The latter condition means that for any sequence $(t_k, z_k, \sigma_k)_{k\in\mathbb{N}}$ in $\partial\mathcal{E}$ with $t_k \to t$, $z_k \rightharpoonup z$ in Z, and $\sigma_k \rightharpoonup \sigma$ in Z', we have $(t,z,\sigma) \in \partial\mathcal{E}$.

Assumptions on the dissipation functional Ψ*.* We impose (2.5), (2.6), (3.3), (3.4), and the new conditions

$$\Psi : Z \times Z \to [0,\infty) \text{ is weakly lower semicontinuous,} \tag{4.5}$$

$$z \mapsto C(z) \subset Z' \text{ has a closed graph in the weak–weak topology of } Z{\times}Z'. \tag{4.6}$$

(4.6) means that

$$(z_k,\sigma_k) \rightharpoonup (z,\sigma) \text{ in } Z \times Z' \quad \text{and} \quad \sigma_k \in C(z_k) \quad \text{implies } \sigma \in C(z). \tag{4.7}$$

Lemma 4.1. *Let* $\Psi : Z \times Z \to [0,+\infty]$ *fulfil* (2.5), (2.6) *and* (4.5)*. Then,* (4.6) *is equivalent to*

$$\forall\, v \in Z : \Psi(\cdot, v) : Z \to [0,\infty) \textit{ is weakly continuous.} \tag{4.8}$$

Proof. First, we prove (4.6) ⇒ (4.8): in view of (4.5), it is sufficient to show that

$$z_k \rightharpoonup z \Rightarrow \limsup_{k\uparrow\infty} \Psi(z_k, v) \leq \Psi(z, v) \quad \forall\, v \in Z.$$

Indeed, recalling the representation formula (2.9), for any $k > 0$ we have

$$\forall\, v \in Z,\ \forall\, k \in \mathbb{N},\ \exists\, \sigma_{k,v} \in C(z_k) : \langle \sigma_k, v\rangle \leq \Psi(z_k, v) \leq \langle \sigma_k, v\rangle + \frac{1}{k}.$$

Since the sequence $\{\sigma_{k,v}\}_{k\in\mathbb{N}}$ is bounded in Z' by (2.6), we extract subsequences $z_{k_j} \rightharpoonup z$ in Z, $\sigma_{k_j} \rightharpoonup \sigma$ in Z', and by (4.7) conclude that $\sigma \in C(z)$, so that for all $v \in Z$ we have

$$\limsup_{k\uparrow\infty} \Psi(z_k, v) \leq \langle \sigma, v\rangle \leq \Psi(z, v).$$

As for the converse implication, we will show that (4.8) ⇒ (4.7): indeed, we fix a sequence $\{(z_k, \sigma_k)\} \subset Z \times Z'$ in the conditions of (4.7), and we recall that $\sigma_k \in C(z_k) = \partial_v \Psi(z_k, 0)$ is equivalent to

$$\langle \sigma_k, v\rangle \leq \Psi(z_k, v) \quad \forall\, v \in Z.$$

We pass to the limit on both sides of the above inequality and obtain $\sigma \in C(z)$.

□

Remark 4.1. In fact, in the proof of Theorem 4.1 it will be more convenient to use condition (4.6) rather than (4.8). On the other hand, (4.8) is indeed easier to check in the applications: a typical situation in which (4.5) and (4.8) are satisfied occurs when Z is compactly embedded into another Banach space Y, written as $Z \Subset Y$, and Ψ has a continuous extension to all of Y, see the following example.

4.2. *A nontrivial example*

Here, we provide an example which is nontrivial and satisfies all the assumptions of the above theory. This is a typical situation which appears in continuum mechanical models for materials with internal variables whose evolution is rate-independent, see Refs. 7, 12 and 13.

We start with the Banach space $Z = H^1(\Omega; \mathbb{R}^m) = W^{1,2}(\Omega; \mathbb{R}^m)$ where $\Omega \subset \mathbb{R}^d$ is bounded and has a Lipschitz boundary. For the energy functional we use

$$\mathcal{E}(t, z) := \int_\Omega \frac{\alpha_1}{2} |\nabla z|^2 + F(x, z(x))\, dx - \langle \ell(t), z\rangle,$$

where $\ell \in C^1([0, T], Z')$ is typically taken in the form

$$\langle \ell(t), z\rangle = \int_\Omega g(t, x) \cdot z(x)\, dx + \int_{\partial\Omega} h(t, x) \cdot z(x)\, da.$$

The function $F : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ is assumed to be continuous, convex in $z \in \mathbb{R}^m$ and satisfies the bounds

$$c|z|^2 - \beta(x) \leq F(x, z) \leq C|z|^\rho + \beta(x) \quad \forall x \in \Omega,\ \ \forall\, z \in \mathbb{R}^m,$$

where $C, c > 0$, $\beta \in L^1(\Omega)$ and the exponent $\rho \geq 2$ satisfies $\frac{d}{\rho} \geq \frac{d-2}{2}$.

As $H^1(\Omega;\mathbb{R}^m)$ is continuously embedded into $L^\rho(\Omega;\mathbb{R}^m)$, it is easy to see that $\mathcal{E} : [0,T] \times Z \to \mathbb{R}$ is continuous and convex in $z \in H^1(\Omega;\mathbb{R}^m)$, which proves (2.1). Moreover, we have the coercivity estimate

$$\mathcal{E}(t,z) \geq \frac{1}{2}\alpha_2 \|z\|^2_{H^1} - C_\beta \quad \text{with } \alpha_2 = \min\left\{\frac{\alpha_1}{2}, c\right\},$$
$$C_\beta = \int_\Omega |\beta(x)|\,dx + \frac{1}{2\alpha_2}\|\ell\|^2_{L^\infty(0,T;Z')},$$

also taking into account the contribution of the term $\langle \ell(t), z\rangle$.

Moreover, $\partial_t \mathcal{E}(t,z) = -\langle \dot{\ell}(t), z\rangle$ and thus (2.2) holds with $C_0 = \alpha_2 + C_\beta$ and $\lambda_0(t) = \|\dot{\ell}(t)\|_{Z'}/\alpha_2$, while (4.2) follows from $\dot{\ell} \in L^\infty(0,T;Z')$. If we additionally impose that each $F(x,\cdot)$ is α_3-uniformly convex, then $\mathcal{E}(t,\cdot)$ is κ-uniformly convex with $\kappa = \min\{\alpha_1, \alpha_3\}$.

The most difficult condition is the strong–weak–weak closedness of the graph of $\partial\mathcal{E} \subset [0,T] \times Z \times Z'$. First note that

$$\partial\mathcal{E}(t,z) = \{-\alpha_1 \Delta_{\text{Neu}} z + \eta - \ell(t) \in Z' \mid \eta(x) \in \partial F(x, z(x)) \text{ for a.e. } x \in \Omega\},$$

where $\partial F(x,z_1)$ denotes the subdifferential of $F(x,\cdot)$ in the point z_1. For a sequence $(t_k, z_k, w_k) \in \partial\mathcal{E}$, with $t_k \to t$, $z_k \rightharpoonup z$ in Z and $w_k \rightharpoonup w$ in Z', we conclude $\ell(t_k) \to \ell(t)$ in Z' and, by linearity and boundedness, $\Delta_{\text{Neu}} z_k \rightharpoonup \Delta_{\text{Neu}} z$ in Z'. Now, we additionally assume $\frac{d}{\rho} > \frac{d-2}{2}$, such that $H^1(\Omega;\mathbb{R}^m)$ is compactly embedded into $L^\rho(\Omega;\mathbb{R}^m)$. Then, $z_k \to z$ in $L^\rho(\Omega;\mathbb{R}^m)$ (strongly) and, after choosing a subsequence, we may assume $z_k(x) \to z(x)$ in $\mathbb{R}^m$ for a.e. $x \in \Omega$. Now, $\eta_k := w_k + \alpha_1 \Delta_{\text{neu}} z_k + \ell(t_k)$ is a selection for $\partial F(\cdot, z_k(\cdot))$. On the one hand, this implies, via $|\partial F(x,z)| \leq C|z|^{\rho-1} + \beta(x)$, that η_k is bounded in $L^{\rho/(\rho-1)}(\Omega;\mathbb{R}^m)$. On the other hand, $\eta_k \rightharpoonup \eta := w + \alpha_1 \Delta_{\text{neu}} z + \ell(t)$. Hence, we conclude $\eta_k \rightharpoonup \eta$ in $L^{\rho/(\rho-1)}(\Omega;\mathbb{R}^m)$. To conclude $\eta(x) \in \partial F(x, z(x))$ for a.e. $x \in \Omega$, we use that $\mathcal{A} : z \mapsto \partial F(\cdot, z(\cdot))$ is a maximal monotone operator from its domain $L^\rho(\Omega;\mathbb{R}^m)$ into its dual $L^{\rho/(\rho-1)}(\Omega;\mathbb{R}^m)$. However, $\eta_k \in \mathcal{A}(z_k)$, $\eta_k \rightharpoonup \eta$ and $z_k \to z$, then implies $\eta \in \mathcal{A}(z)$, as desired.

The dissipation potential Ψ is taken in the form

$$\Psi(z,v) = \int_\Omega \psi(x, z(x), v(x))\,dx,$$

where the local density $\psi : \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^m \to [0,\infty)$ is continuous. Moreover, each $\psi(x,z,\cdot) : \mathbb{R}^m \to [0,\infty)$ is assumed 1-homogeneous and convex, whence (2.5). Furthermore, we suppose that there exist constants $c_1, \psi_0^* \geq 0$ such that

$$|\psi(x,z_1,v) - \psi(x,z_2,v)| \leq \psi_0^* |z_1 - z_2|\,|v| \quad \text{and} \quad 0 \leq \psi(x,z,v) \leq c_1|v|,$$

so that (2.6) and (3.3) hold. Note that the latter condition, together with convexity and 1-homogeneity, implies $|\psi(z,v_1) - \psi(z,v_2)| \leq c_1|v_1 - v_2|$. For ψ_0^* small enough, (3.4) is also fulfilled.

To establish the weak continuity properties of Ψ, we use that $H^1(\Omega;\mathbb{R}^m)$ is compactly embedded into $Y := L^2(\Omega;\mathbb{R}^m)$. By its definition, we may extend Ψ to all of Y and obtain the estimates

$$|\Psi(z_1,v_1) - \Psi(z_2,v_2)| \le \psi_0^* \|z_1 - z_2\|_Y \|v_1\|_Y + c_1 \|v_1 - v_2\|_{L^1(\Omega)}.$$

With $\|v\|_{L^1} \le \mathrm{vol}(\Omega)^{1/2}\|v\|_{L^2}$, we conclude the continuity of $\Psi : L^2(\Omega;\mathbb{R}^m) \times L^2(\Omega;\mathbb{R}^m) \to [0,\infty)$, which by the above arguments implies conditions (4.5) and (4.8).

4.3. *Time incremental problems and approximate solutions*

Let us consider a partition

$$\mathscr{P}_\tau := \{t_\tau^0 = 0 < t_\tau^1 < \cdots < t_\tau^N = T\}, \quad \tau := \max_{j=1,\ldots,N} \{t_\tau^j - t_\tau^{j-1}\},$$

of the interval $(0,T)$, and let us introduce the following *time incremental problem*, associated with the *time-continuous* Problem 2.1.

Problem 4.1. Given $z_\tau^0 := z_0$, find $z_\tau^1, \ldots, z_\tau^N \in Z$ such that

$$z_\tau^k \in \mathrm{argmin}\{\mathcal{E}(t_\tau^k, z) + \Psi(z_\tau^{k-1}, z - z_\tau^{k-1}) \mid z \in Z\} \quad \text{for } k = 1,\ldots,N. \qquad \textbf{(IP)}$$

It is straightforward to check that, under the present convexity assumptions on $\mathcal{E}$ and Ψ, for every $k = 1,\ldots,N$ the incremental problem (**IP**) admits a solution z_τ^k. Indeed, the solution z_τ^k to (**IP**) is unique, as a consequence of the following:

Lemma 4.2. *Assume* (2.1), (2.5), (3.1) *and* (4.1). *Then, any solution* $\{z_\tau^k\}_{k=0}^N$ *of Problem* 4.1 *fulfils, for* $k = 1,\ldots,N$, *the* variational inequality

$$\frac{\kappa}{2}\|z_\tau^k - \hat{z}\|^2 \le \mathcal{E}(t_\tau^k, \hat{z}) - \mathcal{E}(t_\tau^k, z_\tau^k) + \Psi(z_\tau^{k-1}, \hat{z} - z_\tau^{k-1}) - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) \tag{4.9}$$

for all $\hat{z} \in Z$, *where* κ *is the uniform modulus of convexity of the functional* $\mathcal{E}$, *cf.* (3.1).

Proof. For every $\hat{z} \in Z$, let us set $\hat{z}_\theta := (1-\theta)z_\tau^k + \theta\hat{z}$, $\theta \in [0,1]$. The uniform convexity of the map $z \mapsto \mathcal{E}(t_\tau^k, z)$ and the convexity of $z \mapsto \Psi(z_\tau^{k-1}, z - z_\tau^{k-1})$ yield for every $\theta \in (0,1)$ the estimate

$$\begin{aligned}\frac{\kappa}{2}\theta(1-\theta)\|z_\tau^k - \hat{z}\|^2 \le{}& (1-\theta)\left(\mathcal{E}(t_\tau^k, z_\tau^k) + \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1})\right) + \theta\left(\mathcal{E}(t_\tau^k, \hat{z})\right.\\ &\left. + \Psi(z_\tau^{k-1}, \hat{z} - z_\tau^{k-1})\right) - \mathcal{E}(t_\tau^k, \hat{z}_\theta) - \Psi(z_\tau^{k-1}, \hat{z}_\theta - z_\tau^{k-1}).\end{aligned} \tag{4.10}$$

On the other hand, it follows from (**IP**) that

$$\mathcal{E}(t_\tau^k, z_\tau^k) + \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) \le \mathcal{E}(t_\tau^k, \hat{z}_\theta) + \Psi(z_\tau^{k-1}, \hat{z}_\theta - z_\tau^{k-1}).$$

Substituting the above inequality into (4.10), dividing both sides by θ, and letting $\theta \searrow 0$, we conclude (4.9). □

Corollary 4.1. *For every* $k = 1, \ldots, N$ *the incremental problem* (**IP**) *has a unique solution* $\{z_\tau^k\}_{k=0,\ldots,N}$, *and this solution fulfils the* stability condition

$$\mathcal{E}(t_\tau^k, z_\tau^k) \le \mathcal{E}(t_\tau^k, \hat{z}) + \Psi(z_\tau^{k-1}, \hat{z} - z_\tau^k) \quad \forall\, \hat{z} \in Z. \tag{4.11}$$

Indeed, (4.11) directly follows from (**IP**), also using the triangle inequality (2.8) for Ψ.

Approximate solutions. We can now introduce the piecewise constant interpolants $\overline{Z}_\tau$, $\underline{Z}_\tau : [0,T] \to Z$ and the piecewise linear interpolant $\widehat{Z}_\tau : [0,T] \to Z$ of the discrete solutions $\{z_\tau^k\}_{k=0}^N$ of Problem 4.1, defined by

$$\overline{Z}_\tau(t) := z_\tau^k \quad \text{for } t \in (t_\tau^{k-1}, t_\tau^k], \quad \underline{Z}_\tau(t) := z_\tau^{k-1} \quad \text{for } t \in [t_\tau^{k-1}, t_\tau^k),$$

$$\widehat{Z}_\tau(t) = \frac{t - t_\tau^{k-1}}{t_\tau^k - t_\tau^{k-1}} z_\tau^k + \frac{t_\tau^k - t}{t_\tau^k - t_\tau^{k-1}} z_\tau^{k-1}, \quad t \in [t_\tau^{k-1}, t_\tau^k].$$

Also, let $\bar{\mathsf{t}}_\tau : [0,T] \to [0,T]$ be defined by $\bar{\mathsf{t}}_\tau(0) := 0$ and $\bar{\mathsf{t}}_\tau(t) := t_\tau^k$ for $t \in (t_\tau^{k-1}, t_\tau^k]$. Of course, for every $t \in [0,T]$ we have $\bar{\mathsf{t}}_\tau(t) \downarrow t$ as $\tau \searrow 0$.

By (2.7) and Lemma A.1, the minimization problem (**IP**) yields the subdifferential inclusion

$$\partial_v \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) + \partial\mathcal{E}(t_\tau^k, z_\tau^k) \ni 0, \quad \forall\, k = 1, \ldots, N.$$

Using the 1-homogeneity of the functional $\Psi(z, \cdot)$, we thus obtain

$$\partial_v \Psi(\underline{Z}_\tau(t), \widehat{Z}'_\tau(t)) + \partial\mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{Z}_\tau(t)) \ni 0, \quad \forall\, t \in (t_\tau^{k-1}, t_\tau^k]. \tag{4.12}$$

We can now state our main existence and approximation result for Problem 2.1. After some *a priori* estimates in Sec. 4.4, the proof will be completed in Sec. 4.5.

Theorem 4.1. *Assume* (4.1), *that* $\mathcal{E}$ *complies with* (2.1), (2.2), (2.16), (4.2), (3.1), (4.3), (4.4) *and that* Ψ *fulfils* (2.5), (2.6), (3.3), (3.4), (4.5) *and* (4.6). *Then, the Cauchy Problem* 2.1 *for the Global Formulation* (S_Ψ) *and* (E_Ψ), *supplemented with the stable initial datum* z_0 (*i.e.* ($\mathrm{S}_{\mathrm{loc}}$) *holds for* z_0), *admits a solution.*

Moreover, if $\{\mathcal{P}_{\tau_j}\}$ *is a sequence of uniform time-step partitions of* $[0,T]$ (*i.e.* $t_{\tau_j}^k - t_{\tau_j}^{k-1} = t_{\tau_j}^i - t_{\tau_j}^{i-1} = \tau_j \ \forall k, i$), *with fineness* $\tau_j \searrow 0$ *as* $j \uparrow \infty$ *and* $\{\overline{Z}_{\tau_j}\}$, $\{\underline{Z}_{\tau_j}\}$, $\{\widehat{Z}_{\tau_j}\}$ *are the associated interpolants, there exists a subsequence* $\{\tau_{j_n}\}_n$ *and* *a solution* $z \in W^{1,\infty}(0,T;Z)$ *such that the following convergences hold as* $n \uparrow \infty$*:*

$$\forall\, t \in [0,T] : \widehat{Z}_{\tau_{j_n}}(t) \rightharpoonup z(t) \quad \text{in } Z, \tag{4.13}$$

$$\forall\, t \in [0,T] : \overline{Z}_{\tau_{j_n}}(t),\ \underline{Z}_{\tau_{j_n}}(t) \rightharpoonup z(t) \quad \text{in } Z, \tag{4.14}$$

$$\widehat{Z}_{\tau_{j_n}} \overset{*}{\rightharpoonup} z \quad \text{in } W^{1,\infty}(0,T;Z), \tag{4.15}$$

$$\partial_t \mathcal{E}(\cdot, \underline{Z}_{\tau_{j_n}}(\cdot)) \to \partial_t \mathcal{E}(\cdot, z(\cdot)) \quad \text{in } L^1(0,T), \tag{4.16}$$

$$\forall\ t \in [0,T] : \begin{cases} \mathcal{E}(t, \underline{Z}_{\tau_{j_n}}(t)) \to \mathcal{E}(t, z(t)), \\ \displaystyle\int_0^t \Psi(\underline{Z}_{\tau_{j_n}}(s), \widehat{Z}'_{\tau_{j_n}}(s))\, ds \to \int_0^t \Psi(z(s), \dot{z}(s))\, ds. \end{cases} \tag{4.17}$$

4.4. *A priori estimates for the approximate solutions*

In the sequel, we will denote by C any constant occurring in the estimates, without detailing the quantities C depends on; instead, we will use other symbols for more specific constants.

The following result shows that assumption (4.2) makes the incremental solutions Lipschitz continuous with a uniform bound.

Proposition 4.1. (Lipschitz bounds) *Assume* (4.1), (2.1), (3.1), (4.2), (2.5), (3.3) *and* (3.4). *Let us set*

$$\delta_k := \|z_\tau^k - z_\tau^{k-1}\| \quad \text{for all } k = 1, \ldots, N \ \text{ and } \ \tau > 0. \tag{4.18}$$

Then, for any $k = 1, \ldots, N$ *we have the discrete Lipschitz estimate*

$$\delta_k \le \frac{\Lambda_1}{\kappa - \psi^*}\ \tau. \tag{4.19}$$

Note that (4.19) is the discrete analogue of the Lipschitz continuity estimate proved in Theorem 3.1.

Proof. Let us substitute $\hat{z} := z_\tau^{k-1}$ into (4.9), thus obtaining

$$\frac{\kappa}{2}\|z_\tau^k - z_\tau^{k-1}\|^2 \le \mathcal{E}(t_\tau^k, z_\tau^{k-1}) - \mathcal{E}(t_\tau^k, z_\tau^k) - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}). \tag{4.20}$$

Let us write (4.9) at the $(k-1)$th step: for every $w \in Z$ we have

$$\begin{aligned}\frac{\kappa}{2}\|z_\tau^{k-1} - w\|^2 \le \mathcal{E}(t_\tau^{k-1}, w) - \mathcal{E}(t_\tau^{k-1}, z_\tau^{k-1}) + \Psi(z_\tau^{k-2}, w - z_\tau^{k-2}) \\ - \Psi(z_\tau^{k-2}, z_\tau^{k-1} - z_\tau^{k-2});\end{aligned}$$

let us now choose $w := z_\tau^k$. Adding the resulting inequality and (4.20), we get

$$\begin{aligned}\kappa\|z_\tau^k - z_\tau^{k-1}\|^2 \le \mathcal{E}(t_\tau^k, z_\tau^{k-1}) - \mathcal{E}(t_\tau^k, z_\tau^k) + \mathcal{E}(t_\tau^{k-1}, z_\tau^k) - \mathcal{E}(t_\tau^{k-1}, z_\tau^{k-1}) \\ - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) + \Psi(z_\tau^{k-2}, z_\tau^k - z_\tau^{k-2}) \\ - \Psi(z_\tau^{k-2}, z_\tau^{k-1} - z_\tau^{k-2}).\end{aligned} \tag{4.21}$$

By the triangle inequality (2.8) and by (3.3), we conclude that

$$\begin{aligned}
&\Psi(z_\tau^{k-2}, z_\tau^k - z_\tau^{k-2}) - \Psi(z_\tau^{k-2}, z_\tau^{k-1} - z_\tau^{k-2}) - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) \\
&\quad \le \Psi(z_\tau^{k-2}, z_\tau^k - z_\tau^{k-1}) - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) \\
&\quad \le \psi^* \|z_\tau^{k-1} - z_\tau^{k-2}\| \|z_\tau^k - z_\tau^{k-1}\|.
\end{aligned} \tag{4.22}$$

On the other hand, by (4.2)

$$\begin{aligned}
&\mathcal{E}(t_\tau^k, z_\tau^{k-1}) - \mathcal{E}(t_\tau^k, z_\tau^k) + \mathcal{E}(t_\tau^{k-1}, z_\tau^k) - \mathcal{E}(t_\tau^{k-1}, z_\tau^{k-1}) \\
&\quad = \int_{t_\tau^{k-1}}^{t_\tau^k} (\partial_t \mathcal{E}(\tau, z_\tau^{k-1}) - \partial_t \mathcal{E}(\tau, z_\tau^k))\, d\tau \le \|z_\tau^k - z_\tau^{k-1}\| \int_{t_\tau^{k-1}}^{t_\tau^k} \lambda_1(\tau)\, d\tau \\
&\quad \le \Lambda_1 (t_\tau^k - t_\tau^{k-1}) \|z_\tau^k - z_\tau^{k-1}\|.
\end{aligned} \tag{4.23}$$

Thus, letting $\delta_0 = 0$ and collecting (4.21)–(4.23), we obtain the recurrence relation

$$\delta_k \le \frac{\Lambda_1}{\kappa}(t_\tau^k - t_\tau^{k-1}) + \frac{\psi^*}{\kappa}\delta_{k-1} \quad \forall\, k = 1, \ldots, N, \tag{4.24}$$

whence

$$\delta_k \le \frac{\Lambda_1}{\kappa} \sum_{j=1}^{k} \left(\frac{\psi^*}{\kappa}\right)^{k-j} \left(t_\tau^j - t_\tau^{j-1}\right), \tag{4.25}$$

yielding (4.19) thanks to (3.4). ☐

Proposition 4.2. (*A priori* estimates) *Under the same assumptions of Proposition* 4.1, *the energy estimate*

$$\int_{\mathsf{s}}^{\mathsf{t}} \Psi(\underline{Z}_\tau(r), \widehat{Z}_\tau'(r))dr + \mathcal{E}(\mathsf{t}, \overline{Z}_\tau(\mathsf{t})) \le \mathcal{E}(\mathsf{s}, \overline{Z}_\tau(\mathsf{s})) + \int_{\mathsf{s}}^{\mathsf{t}} \partial_t \mathcal{E}(r, \overline{Z}_\tau(\mathsf{s}))dr \tag{4.26}$$

holds for every pair of nodes $\mathsf{s}, \mathsf{t} \in \mathcal{P}_\tau$, $\mathsf{s} < \mathsf{t}$, *and for all* $t \in [0, T]$ *we have*

$$\begin{gathered}
\max\{\mathcal{E}(t, \overline{Z}_\tau(t)), \mathcal{E}(t, \underline{Z}_\tau(t))\} \le (\mathcal{E}(0, z_0) + C_0)\exp(\Lambda_1 t) - C_0, \\
\int_0^t \Psi(\underline{Z}_\tau(r), \widehat{Z}_\tau'(r))dr \le (\mathcal{E}(0, z_0) + C_0)\exp(\Lambda_1 t).
\end{gathered} \tag{4.27}$$

Further, there exists a constant C *such that for all* $\tau > 0$

$$\|\overline{Z}_\tau\|_{L^\infty(0,T;Z)} \le C, \tag{4.28}$$

$$\|\widehat{Z}_\tau - \overline{Z}_\tau\|_{L^\infty(0,T;Z)} \le \|\overline{Z}_\tau - \underline{Z}_\tau\|_{L^\infty(0,T;Z)} \le \frac{\Lambda_1}{\kappa - \psi^*}\tau. \tag{4.29}$$

In particular, if only uniform step-size partitions are considered we have

$$\|\widehat{Z}_\tau'\|_{L^\infty(0,T;Z)} \le \frac{\Lambda_1}{\kappa - \psi^*} \quad \text{for } \tau \in \{T/k \mid k \in \mathbb{N}\}. \tag{4.30}$$

Proof. It follows from the minimization algorithm (**IP**) and from the 1-homogeneity of Ψ w.r.t. the second variable that for every t_τ^{k-1}, $t_\tau^k \in \mathcal{P}_\tau$

$$\begin{aligned}\mathcal{E}(t_\tau^k, z_\tau^k) &+ (t_\tau^k - t_\tau^{k-1})\Psi\left(z_\tau^{k-1}, \frac{z_\tau^k - z_\tau^{k-1}}{t_\tau^k - t_\tau^{k-1}}\right)\\ &\le \mathcal{E}(t_\tau^k, z_\tau^{k-1})\\ &= \mathcal{E}(t_\tau^{k-1}, z_\tau^{k-1}) + \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t \mathcal{E}(r, z_\tau^{k-1})dr, \end{aligned} \tag{4.31}$$

whence (4.26) by adding up (4.31) on each subinterval of the partition.

We refer to Ref. 12, Cor. 3.3, for the proof of (4.27), obtained through (2.2), (2.3), and the Gronwall lemma. Since $\mathcal{E}$ is bounded from below (cf. (2.3)), it follows from (4.27) that

$$|\mathcal{E}(t, \overline{Z}_\tau(t))| + |\mathcal{E}(t, \underline{Z}_\tau(t))| \le C \quad \forall\, t \in [0, T],$$

whence (4.28), in view of the uniform convexity assumption (3.1).

Finally, the first inequality in (4.29) can be found by trivial calculations, while the second one is a reformulation of (4.19). □

4.5. *Proof of Theorem* 4.1

For the existence proof we restrict ourselves to the approximate solutions $\overline{Z}_\tau$, $\underline{Z}_\tau$, $\widehat{Z}_\tau$ constructed from partitions with uniform time steps. In this case, (4.30) provides equi-continuity of the approximate sequences, and we can apply the Ascoli–Arzelà compactness theorem in the framework of the weak topology on the reflexive space Z. Hence, there exist a subsequence $(\widehat{Z}_{\tau_{j_n}})_{n\in\mathbb{N}}$, which we denote by $(\widehat{Z}_n)_{n\in\mathbb{N}}$ for simplicity, and a limit function $z \in W^{1,\infty}(0, T; Z)$ such that (4.13) and, by (4.29), (4.14) hold for every $t \in [0, T]$ (indeed, the convergences are uniform in t). Standard weak-compactness results further yield (4.15).

Using also $\overline{Z}_n$, $\underline{Z}_n$ and $\bar{\mathfrak{t}}_n$ as short-hands for $\overline{Z}_{\tau_{j_n}}$, $\underline{Z}_{\tau_{j_n}}$ and $\bar{\mathfrak{t}}_{\tau_{j_n}}$, respectively, we see that (4.12) implies the weaker statement

$$\partial_v \Psi(\underline{Z}_n(t), 0) + \partial \mathcal{E}(\bar{\mathfrak{t}}_n(t), \overline{Z}_n(t)) \ni 0 \quad \forall\, t \in (0, T],$$

in view of (2.10) and (2.12). Now, let us keep an arbitrary $t \in (0, T]$ fixed: then, there exists a sequence ξ_n with

$$\xi_n \in \partial \mathcal{E}(\bar{\mathfrak{t}}_n(t), \overline{Z}_n(t)) \cap (-\partial_v \Psi(\underline{Z}_n(t), 0)) \subset B^*_{C_\Psi}(0),$$

where the latter inclusion follows from (2.6). Thus, there exists a weakly convergent subsequence $\xi_{n_k} \rightharpoonup \xi_*$. Using the weak closedness properties (4.4) for $\partial\mathcal{E}$ and (4.6) for $\partial_v \Psi(\cdot, 0)$, as well as the convergences $\bar{\mathfrak{t}}_n(t) \to t$, $\overline{Z}_n(t)) \rightharpoonup z(t)$ and $\underline{Z}_n(t) \rightharpoonup z(t)$, we obtain $\xi_* \in \partial\mathcal{E}(t, z(t))$ and $-\xi_* \in \partial_v \Psi(z(t), 0)$. But this implies

$$\partial_v \Psi(z(t), 0) + \partial \mathcal{E}(t, z(t)) \ni 0,$$

which is equivalent to (S_Ψ) by Proposition 2.1, Remark 2.1 and Proposition 2.3.

To prove the energy balance (E_Ψ), we first establish the one-sided estimate

$$\int_0^t \Psi(z(r), \dot{z}(r))\, dr + \mathcal{E}(t, z(t)) \le \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(r, z(r))\, dr. \tag{4.32}$$

To check this, we start from the discrete energy inequality (4.26), which yields for all $t \in [0, T]$

$$\int_0^{\bar{\mathfrak{t}}_n(t)} \Psi(\underline{Z}_n(r), \widehat{Z}'_n(r))\, dr + \mathcal{E}(\bar{\mathfrak{t}}_n(t), \overline{Z}_n(\bar{\mathfrak{t}}_n(t)))$$
$$\le \mathcal{E}(0, z_0) + \int_0^{\bar{\mathfrak{t}}_n(t)} \partial_t \mathcal{E}(r, \underline{Z}_n(r))\, dr. \tag{4.33}$$

By (4.3) and (4.14) we have $\partial_t \mathcal{E}(r, \underline{Z}_\tau(r)) \to \partial_t \mathcal{E}(r, z(r))$ for all $r \in [0, T]$. Further, in view of (2.2) and (4.27), the integrands are bounded in $L^\infty(0, T)$. Thus, the Lebesgue theorem yields (4.16), so that the integral on the right-hand side of (4.33) converges to $\int_0^t \partial_t \mathcal{E}(r, z(r))\, dr$.

Moreover, using the lower semicontinuity (2.1) of $\mathcal{E}(t, \cdot)$ and the uniform boundedness of $\partial_t \mathcal{E}$, we obtain

$$\liminf_{n\to\infty} (\mathcal{E}(\bar{\mathfrak{t}}_{\tau_n}(t), \overline{Z}_\tau(\bar{\mathfrak{t}}_{\tau_n}(t))) - \mathcal{E}(t, z(t)))$$
$$\ge \lim_{n\to\infty} \int_t^{\bar{\mathfrak{t}}_{\tau_n}(t)} \partial_t \mathcal{E}(r, \overline{Z}_\tau(\bar{\mathfrak{t}}_{\tau_n}(t)))\, dr$$
$$+ \liminf_{n\to\infty} (\mathcal{E}(t, \overline{Z}_\tau(\bar{\mathfrak{t}}_{\tau_n}(t))) - \mathcal{E}(t, z(t))) \ge 0. \tag{4.34}$$

To pass to the limit in the dissipation integral term on the left-hand side of (4.33), we observe that, by (4.28) and (4.30), the sequence $(\underline{Z}_n, \widehat{Z}'_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0, T; Z \times Z)$. Thus, applying Theorem B.1 in the space $X := Z \times Z$, a subsequence $(\underline{Z}_{n_k}, \widehat{Z}'_{n_k})_{k\in\mathbb{N}}$ generates a limiting Young measure $\{\nu_t\}_{t\in(0,T)} \in \mathcal{Y}(0, T; Z \times Z)$. Recalling that Ψ is a weakly normal integrand (cf. Appendix B) on $(0, T) \times Z \times Z$, we thus obtain

$$\liminf_{n\to\infty} \int_0^{\bar{\mathfrak{t}}_{\tau_n}(t)} \Psi(\underline{Z}_\tau(r), \widehat{Z}'_\tau(r))\, dr \ge \int_0^t \left(\int_{Z\times Z} \Psi(z, v)\, d\nu_r(z, v) \right) dr.$$

On the other hand, in view of (4.13), (4.15) and (B.7), for a.e. $t \in (0, T)$ we have $\nu_t = \delta_{z(t)} \otimes \sigma_t$, with $(\sigma_t)_{t\in(0,T)} \in \mathcal{Y}(0, T; Z)$ and

$$\dot{z}(t) = \int_Z v\, d\sigma_t(v) \quad \text{for a.e. } t \in (0, T).$$

Therefore, also by the Jensen inequality we conclude

$$\int_0^t \left(\int_{Z\times Z} \Psi(z, v)\, d\nu_r(z, v) \right) dr = \int_0^t \left(\int_Z \Psi(z(r), v)\, d\sigma_r(v) \right) dr$$
$$\ge \int_0^t \Psi(z(r), \dot{z}(r))\, dr,$$

entailing the following lower semi-continuity result for the dissipation integral:

$$\liminf_{n\uparrow\infty} \int_0^{\bar{\mathsf{t}}_n(t)} \Psi(\underline{Z}_n(r), \widehat{Z}_n'(r))\,dr \geq \int_0^t \Psi(z(r), \dot{z}(r))\,dr. \tag{4.35}$$

Thus, we have shown the convergence for three terms in (4.33), and the desired estimate (4.32) follows.

To obtain the opposite inequality, we use the stability condition (2.21) (equivalent to $(\mathrm{S}_{\mathrm{loc}})$), for $v = \dot{z}(t)$:

$$\Psi(z(r), \dot{z}(r)) + \langle \xi(r), \dot{z}(r) \rangle \geq 0 \quad \text{for a.e. } r \in (0, t),$$

where $\xi(\cdot)$ is a suitable selection of $\partial\mathcal{E}(\cdot, z(\cdot))$. Combining this with the chain rule formula (2.17) (cf. Proposition 2.2), we find

$$\frac{d}{dt}\mathcal{E}(t, z(t)) + \Psi(z(t), \dot{z}(t)) \geq \partial_t \mathcal{E}(t, z(t)) \quad \text{for a.e. } t \in (0, T).$$

Integration of this inequality yields the opposite estimate in (4.32), and we conclude that the equality (E_Ψ) holds.

This concludes the proof of Theorem 4.1. □

Remark 4.2. Condition (4.2) was only assumed for convenience: in fact, you can always rescale the rate-independent problem (1.1), and the following rescaling argument actually shows that we can dispense with (4.2). Indeed, let us set $\tilde{t}(t) := t + \int_0^t |\lambda_1(\tau)|d\tau$, $\tilde{T} := \tilde{t}(T)$, and let us introduce the functionals $\tilde{\mathcal{E}}(s, z) := \mathcal{E}(\tilde{t}^{-1}(s), z)$ for $(s, z) \in [0, \tilde{T}] \times Z$ and $\tilde{\lambda_1}(s) := \lambda_1(\tilde{t}^{-1}(s))$, $s \in (0, \tilde{T})$. Then, the estimate (2.15) gives

$$|\partial_s \mathcal{E}(s, z) - \partial_s \mathcal{E}(s, \hat{z})| \leq \frac{\tilde{\lambda_1}(s)}{1 + |\tilde{\lambda_1}(s)|} \|z - \hat{z}\| \quad \forall\ z, \hat{z} \in Z \quad \text{for a.e. } s \in (0, \tilde{T}).$$

Thus, Theorem 4.1 ensures the existence of a solution $\tilde{z} \in W^{1,\infty}(0, \tilde{T}; Z)$ to Problem 2.1 on the time interval $[0, \tilde{T}]$, yielding by rescaling a solution $z \in W^{1,1}(0, T; Z)$ to our original problem.

The reparametrization can be avoided by taking partitions with time steps adjusted to λ_1. Let $\Lambda_* := \int_0^T \lambda_1(s)\,ds$ and choose t_τ^k such that $\int_0^{t_\tau^k} \lambda_1(s)\,ds = k\Lambda_*/N$. Then, $\tau_N \to 0$ and (4.19) is replaced by $\delta_k \leq \Lambda_*/(N(\kappa - \psi^*))$. We lose the uniform Lipschitz continuity (4.30), but still have an equicontinuity with a modulus of continuity obtained from $t \mapsto \int_0^t \lambda_1(s)\,ds$. Thus, the proof works in this case as well.

It is an open question whether the scheme converges to for any sequence of (non-uniform) partitions with $\tau \to 0$.

5. Uniqueness Results

In this section, we combine the uniqueness results obtained in Ref. 18, Theorem 7.4, and in Ref. 3. In the first work, the case

$$\partial\Psi(\dot{z}(t)) + D\mathcal{E}(t, z(t)) \ni 0$$

is treated, where the dissipation potential Ψ is independent of z but otherwise relatively general. There, the only assumptions on $\Psi : Z \to [0, \infty)$ are convexity, 1-homogeneity and strong continuity (i.e. the upper bound $\Psi(v) \le C_\psi \|v\|$). No smoothness and strict convexity conditions on Ψ are needed. The lower bound $\Psi(v) \ge c_\psi \|v\|$, which is stated in Ref. 18, Eq. (2.2), is used only in the existence part, but not for proving the uniqueness result.

In the second paper, the case

$$\partial\Psi(z(t), \dot{z}(t)) + Az(t) - \ell(t) \ni 0$$

is studied, i.e. the energy is assumed to be quadratic and $A : Z \to Z'$ is an isomorphism. Moreover, the dissipation potential Ψ must be such that $(z, v) \mapsto \Psi(z, v)^2$ lies in $C^{1,\mathrm{Lip}}$ and that $\Psi(z, \cdot)$ is strictly convex. In addition, the severe assumption of lower and upper bounds have to be imposed, namely $c_\psi \|v\| \le \Psi(z, v) \le C_\psi \|v\|$. The lower estimate implies that the elastic domains $C(z) = \partial\Psi(z, 0)$ have non-empty interior, which is not the case in many engineering applications.

In combining the two approaches, we will have to compromise such that at the end the two extreme results will not be covered. However, we believe that our assumptions are somewhat more general and easier to satisfy in particular applications. Moreover, there is potential for future generalizations.

5.1. *Statement of the main assumptions*

To obtain uniqueness, we have to strengthen some of the assumptions for our existence result Theorem 4.1 considerably, whereas other conditions can be weakened.

First of all, in the sequel we assume that

$$Z \text{ is a Hilbert space.} \tag{5.1}$$

Nonetheless, as common practice in mechanics, we will distinguish between the space Z and its dual Z', and keep to the duality pairing $\langle \cdot, \cdot \rangle$ between Z' and Z, instead of using the scalar product; sometimes, we will use the notation $\| \cdot \|$ both for the norm on Z and for the norm on Z'.

Assumptions on the energy functional $\mathcal{E}$. We impose that

$$\begin{gathered} \mathcal{E} \in C^2([0,T] \times Z; \mathbb{R}) \\ \text{and complies with the energetic estimate (2.2),} \end{gathered} \tag{5.2}$$

and that there exist positive constants $C^{\mathcal{E}}_{tz}$, $C^{\mathcal{E}}_{zz}$, $C^{\mathcal{E}}_{zzz}$ and $C^{\mathcal{E}}_{tzz}$ such that

$$\forall\, t \in [0,T],\ \forall\, z \in Z : \|\partial_t D\mathcal{E}(t,z)\| \le C^{\mathcal{E}}_{tz}, \tag{5.3}$$

$$\forall\, t \in [0,T],\ \forall\, z \in Z : \|D^2\mathcal{E}(t,z)\| \le C^{\mathcal{E}}_{zz}, \tag{5.4}$$

$$\forall\, t \in [0,T],\ \forall\, z_1, z_2 \in Z : \|D^2\mathcal{E}(t,z_1) - D^2\mathcal{E}(t,z_2)\| \le C^{\mathcal{E}}_{zzz}\|z_1 - z_2\|, \tag{5.5}$$

$$\forall\, t \in [0,T],\ \forall\, z_1, z_2 \in Z : \|\partial_t D\mathcal{E}(t,z_1) - \partial_t D\mathcal{E}(t,z_2)\| \le C^{\mathcal{E}}_{tzz}\|z_1 - z_2\|. \tag{5.6}$$

(Hence, (2.15) is fulfilled, with $\lambda_1 \in L^\infty(0,T)$.) The main assumption throughout this section is the uniform convexity (3.1) of $\mathcal{E}$, which may now be formulated in terms of the second derivative $H(t,z) = D^2\mathcal{E}(t,z) \in \mathcal{L}(Z,Z')$ as

$$\exists\, \kappa > 0,\ \forall\, t \in [0,T],\ \forall\, z, v \in Z : \langle D^2\mathcal{E}(t,z)v, v\rangle \geq \kappa\|v\|^2. \tag{5.7}$$

This shows that we have adjusted the space Z to fit with the energy.

Assumptions on Ψ. First of all, we assume the basic convexity and 1-homogeneity (2.5) on $\Psi(z,\cdot)$. Further assumptions on the dissipation potential $\Psi : Z\times Z \to [0,\infty)$ will be more involved. Namely, in order to be able to treat reasonable applications, like our example in Sec. 4.2, we introduce an additional Banach space X in which the Hilbert space Z is continuously embedded. In particular, we will use the embeddings

$$Z \subset X \subset X'' \quad \text{and} \quad X' \subset Z'.$$

The typical situation we have in mind is $Z = H^1(\Omega)$ and $X = L^1(\Omega)$, see the example in Sec. 4.2. We will use the estimates

$$\forall\, v \in Z : \|v\|_X \leq C_X\|v\| \quad \text{and} \quad \forall\, \sigma \in X' : \|\sigma\|_{Z'} \leq C_X\|\sigma\|_{X'}. \tag{5.8}$$

We impose upper and lower bounds on Ψ in terms of both norms $\|\cdot\|_X$ and $\|\cdot\|$, cf. (2.6):

$$\exists\, C_X^\Psi \in (0,\infty],\ \exists\, c_X^\Psi > 0,\ \forall\, z, v \in Z : c_X^\Psi\|v\|_X \leq \Psi(z,v) \leq C_X^\Psi\|v\|_X, \tag{5.9}$$

$$\exists\, C_\Psi > 0,\ \exists\, c_\Psi \geq 0,\ \forall\, z, v \in Z : c_\Psi\|v\| \leq \Psi(z,v) \leq C_\Psi\|v\|. \tag{5.10}$$

Note that the cases $c_\Psi = 0$ and $C_X^\Psi = \infty$ are allowed at this stage. Clearly, (5.10) with $c_\Psi > 0$ implies (5.9) with $c_X^\Psi := c_\Psi/C_X$, as well as (5.9) with $C_X^\Psi < \infty$ implies (5.10) with $C_\Psi := C_X^\Psi C_X$. However, for conceptual reasons it is better to keep the constants independent.

On the one hand, from the point of view of the applications, it would be desirable to have $\|\cdot\|_X$ strictly weaker than $\|\cdot\|$, which can only be realized if $c_\Psi = 0$. However, so far, we are unable to establish our uniqueness result without imposing the additional assumption $c_\Psi > 0$, which in fact implies that X and Z are endowed with equivalent norms. On the other hand, it will turn out that most of the estimates can be obtained in terms of weaker estimates, involving c_X^Ψ and C_Ψ only. We conjecture that the assumption $c_\Psi > 0$, which is only used for the proof of Proposition 5.2, is technical and can be avoided by a more careful analysis.

A further condition on Ψ involves the smoothness with respect to the variable z. We assume that for each $v \in Z$ the function $z \mapsto \Psi(z,v)$ is in $C^1(Z)$. Moreover, the Fréchet derivative is bounded as follows

$$\exists\, C_F^\psi > 0,\ \exists\, \sigma \in (0,1],\ \forall\, z, v \in Z : \|D_z\Psi(z,v)\|_{Z'} \leq C_F^\psi\|v\|_X^\sigma\|v\|^{1-\sigma}. \tag{5.11}$$

In Secs. 3 and 4 we have imposed a condition on Ψ (cf. (3.3) and (3.4)), which means that the variations of Ψ with respect to z are weak enough such that the uniform convexity of $\mathcal{E}$ is able to compensate for them. The following weakened version of our previous conditions (3.3) and (3.4) (cf. Remark 3.1) will be central

$$\exists\,\delta > 0,\ \forall\, t \in [0,T],\ \forall\, z, v \in Z : \langle H(t,z)v, v\rangle + D_z\Psi(z,v)[v] \geq \delta\|v\|^2. \tag{5.12}$$

On the other hand, if Ψ is symmetric in v, (5.12) implies the convexity assumption (5.7) with $\kappa = \delta$, as

$$\langle H(t,z)(-v),(-v)\rangle = \langle H(t,z)v, v\rangle \quad \text{and} \quad D_z\Psi(z,(-v))[-v] = -D_z\Psi(z,v)[v].$$

Remark 5.1. Getting further insight into the proof of Theorem 3.1, we see that we can replace the assumptions therein with the new set of assumptions (5.2)–(5.7) on $\mathcal{E}$, and (2.5), (5.8)–(5.12) on Ψ. In this setting, Theorem 3.1 still applies, guaranteeing that any solution to (1.1) is Lipschitz continuous in time, and hence stays inside suitable bounded sets. Thus, the *global* assumptions (5.4)–(5.6) on $\mathcal{E}$ might easily be replaced by suitable *local* estimates. Furthermore, the local versions of (5.4) and (5.6) would then be a mere consequence of the smoothness of $\mathcal{E}$. Still, we have kept to the global estimates to make notation simpler and the estimates more explicit.

The following auxiliary functional $\mathcal{B} : Z \times Z' \to [0,+\infty]$, which was introduced in Ref. 3, plays a central role in the theory:

$$\mathcal{B}(z,\sigma) := \sup\left\{\langle\sigma, v\rangle - \frac{1}{2}\Psi(z,v)^2 \ :\ v \in Z\right\}.$$

The function $\mathcal{B}(z,\cdot) : Z' \to [0,\infty]$ is convex and coercive, but it is finite only if $c_\Psi > 0$. With (5.9) and (5.10), we obtain

$$\frac{\|\sigma\|^2_{Z'}}{2(C_\Psi)^2} \leq \mathcal{B}(z,\sigma) \leq \frac{\|\sigma\|^2_{Z'}}{2(c_\Psi)^2} \quad \text{and} \quad \frac{\|\sigma\|^2_{X'}}{2(C^\Psi_X)^2} \leq \mathcal{B}(z,\sigma) \leq \frac{\|\sigma\|^2_{X'}}{2(c^\Psi_X)^2}. \tag{5.13}$$

Remark 5.2. In view of the convex analysis results of Appendix A, the functionals $\mathcal{B}(z,\cdot)$ can be related to the convex sets $C(z) \subset Z'$ defining Ψ (cf. (2.9)). Indeed, by (2.9) and (A.3), $\Psi(z,\cdot)$ is the Minkowski functional (cf. (A.1)) of the polar set $C(z)^*$ of $C(z)$. Hence, owing to (A.8) we realize that $\mathcal{B}(z,\sigma) = \mathcal{B}_{C(z)}(\sigma)$ for all $z \in Z$ and $\sigma \in Z'$.

We define the *yield surface* $\mathcal{Y}$ and the *admissible domain* $\mathcal{Y}_0$ via

$$\begin{aligned} \mathcal{Y} &:= \left\{(z,\sigma) \in Z \times Z' : \mathcal{B}(z,\sigma) = \frac{1}{2}\right\}, \\ \mathcal{Y}_0 &:= \left\{(z,\sigma) \in Z \times Z' : \mathcal{B}(z,\sigma) \leq \frac{1}{2}\right\}. \end{aligned} \tag{5.14}$$

Also in view of Remark 5.2, note that $(z,\sigma) \in \mathcal{Y}$ if and only if $\sigma \in \partial C(z)$, and $(z,\sigma) \in \mathcal{Y}_0$ if and only if $\sigma \in C(z)$. Moreover, $\mathcal{Y}_0$ is closed and contained in $Z \times B_{1/C_\psi}(0)$. The closedness of $\mathcal{Y}_0$ is indeed equivalent to the fact that the map $z \mapsto C(z)$ has a closed graph in the strong topology of $Z \times Z'$ (cf. assumption (4.6)), and it follows from the lower semicontinuity of the map $(z,\sigma) \mapsto \mathcal{B}(z,\sigma)$.

The subdifferential of $\mathcal{B}(z,\cdot)$ with respect to σ defines a maximal monotone operator (possibly multi-valued) $J(z,\cdot): Z' \to 2^Z$:

$$J(z,\sigma) := \partial_\sigma \mathcal{B}(z,\sigma) \subset Z \quad \forall\, \sigma \in Z'.$$

In the case $c_\Psi = 0$, we may also have $J(z,\sigma) = \emptyset$. In that case, it is sometimes convenient to consider $\mathcal{B}(z,\cdot)$ as a function on X', viz., introducing $\mathcal{B}^X(z,\cdot) := \mathcal{B}(z,\cdot)|_{X'}$. Since $\mathcal{B}^X$ is convex and bounded on bounded sets, it is continuous and the associated subdifferential is nonempty, namely

$$J^X(z,\sigma) = \partial_\sigma^{X'} \mathcal{B}^X(z,\sigma) \subset X'',$$

where $\partial_\sigma^{X'} \mathcal{B}^X(z,\sigma) = \{\eta \in X'' : \forall\, \hat\sigma :\; \mathcal{B}^X(z,\hat\sigma) \geq \mathcal{B}^X(z,\sigma) + {}_{X'}\langle \hat\sigma - \sigma, \eta\rangle_{X''}\}$. From (5.9) and (5.10), we obtain the following estimates

$$\forall\, w \in J(z,\sigma) : \frac{1}{(C_\Psi)^2}\|\sigma\|_{Z'} \leq \|w\| \leq \frac{1}{(c_\Psi)^2}\|\sigma\|_{Z'}, \tag{5.15}$$

$$\forall\, w \in J^X(z,\sigma) : \frac{1}{(C_X^\Psi)^2}\|\sigma\|_{X'} \leq \|w\|_{X''} \leq \frac{1}{(c_X^\Psi)^2}\|\sigma\|_{X'}. \tag{5.16}$$

Indeed, to check for instance the second inequality in (5.15) we use (A.7), Remark 5.2, as well as (5.10) and (5.13), to obtain for all $w \in J(z,\sigma)$

$$c_\Psi \|w\| \leq \Psi(z,w) = \mathcal{M}_{C(z)^*}(w) = \mathcal{M}_{C(z)}(\sigma) \leq \frac{\|\sigma\|_{Z'}}{c_\Psi},$$

whence the desired inequality.

On $[0,T] \times Z\backslash\{0\} \times Z'$ we define another, possibly multi-valued function via

$$V(t,z,\sigma) := \begin{cases} \left\{ \frac{1}{\langle H(t,z)w,w\rangle + D_z\Psi(z,w)[w]}\, w \mid w \in J(z,\sigma)\right\} & \text{if } 0 \notin J(z,\sigma) \neq \emptyset, \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence, $V(t,z,\sigma) \subset Z$ and, since $J(z,\cdot)$ is positively 1-homogeneous, the function $V(t,z,\cdot)$ is (-1)-homogeneous, i.e. $V(z,r\sigma) = \frac{1}{r}V(z,\sigma)$. The importance of this construction is that the elements v in $V(t,z,\sigma)$ satisfy the *a priori* bound $\|v\| \leq C_\Psi/\delta$ if $(z,\sigma) \in \mathcal{Y}$, see Lemma 5.2.

An important and restrictive condition on V is a one-sided Lipschitz continuity, which generalizes the *structure condition* introduced in Sec. 7.2 and App. C of Ref. 18 (but be aware of the different sign convention there):

$$\begin{aligned} &\exists\, L_V > 0,\; \forall\, t \in [0,T] \quad \forall\, (z_1,\sigma_1), (z_2,\sigma_2) \in \mathcal{Y},\; \forall\, v_j \in V(t,z_j,\sigma_j) : \\ &\qquad \langle \sigma_1 - \sigma_2, v_1 - v_2\rangle \geq -L_V\big(\|\sigma_1 - \sigma_2\|^2 + \|z_1 - z_2\|^2\big). \end{aligned} \tag{5.17}$$

The following example shows that the above condition holds in the cases considered in Refs. 3 and 18.

Example 5.1.
Case 1: In the case that Ψ is state-independent, which was treated in Ref. 18, it is possible to allow for Ψ which are not bounded from below, i.e. $c_\Psi = 0$ and Ψ^2 neither smooth nor uniformly convex.

Indeed, we let $C := \partial\Psi(0)$ and obtain $(z,\sigma) \in \mathcal{Y}$ if and only if $\sigma \in \partial C$. Moreover, $(z,\sigma) \in \mathcal{Y}$ implies $J(z,\sigma) \subset N_C(\sigma)$. However, as $N_C(\sigma)$ is a cone, we also have $V(z,\sigma) \subset N_C(\sigma)$. Since the indicator function I_C is convex and lower semicontinuous, its subgradient $\partial I_C = N_C$ is a maximal monotone operator and we conclude $\langle v_1 - v_2, \sigma_1 - \sigma_2 \rangle \geq 0$. Hence, the structure condition (5.17) holds with $L_V = 0$.

Case 2. In Ref. 3, the following special case was considered. The energy takes the form $\mathcal{E}(t,z) = \frac{1}{2}\|z\|^2 - \langle \ell(t), z\rangle$ with $\ell \in C^1([0,T], Z')$. Moreover, we have $X = Z$, i.e. $c_\Psi = c_X^\Psi > 0$ and $C_\Psi = C_X^\Psi < \infty$. The conditions on Ψ are the following. (i) The functional $\Phi : (z,v) \mapsto \Psi(z,v)^2$ lies in $C^2(Z \times Z)$, which implies (5.11). (ii) For each $z \in Z$ the functional $\Phi(z,\cdot)$ is uniformly convex, i.e. $D_v^2\Phi \geq \kappa_0 \mathbf{1}$. In this situation, J and hence V are single-valued maps, which are still C^1 and thus Lipschitz. Therefore, (5.17) follows in the stronger, two-sided version

$$|\langle \sigma_1 - \sigma_2, v_1 - v_2\rangle| \leq L_V \|\sigma_1 - \sigma_2\| \big(\|z_1 - z_2\| + \|\sigma_1 - \sigma_2\|\big).$$

The joint convexity (5.12) reduces to $|D_z\Psi(z,v)[v]| \leq (1-\delta)\|v\|^2$, since $H(t,z) = \mathbf{1}$.

Further assumptions on $\mathcal{E}$. Finally, we need two more conditions on the power of the external forces, which is related to $\partial_t\mathcal{E}(t,z(t))$. For this, introduce the sets $\mathcal{S}_{\rm loc}(t)$ of locally stable states via

$$\mathcal{S}_{\rm loc}(t) := \{z \in Z : 0 \in \partial_v\Psi(z,0) + D\mathcal{E}(t,z)\}.$$

The first condition concerns a certain boundedness, namely

$$\begin{aligned} &\forall\, t \in [0,T],\ \forall\, z \in \mathcal{S}_{\rm loc}(t),\\ &\forall\, w \in J^X(z, -D\mathcal{E}(t,z)) : |{}_{X'}\langle \partial_t D\mathcal{E}(t,z), w\rangle_{X''}| \leq C_\gamma^{\max}. \end{aligned} \tag{5.18}$$

The second condition concerns a Lipschitz estimate, namely

$$\begin{aligned} &\forall\, t \in [0,T],\ \forall\, z_1, z_2 \in \mathcal{S}_{\rm loc}(t),\ \forall\, w_j \in J^X(t, z_j, -D\mathcal{E}(t,z_j)) :\\ &\quad |{}_{X'}\langle \partial_t D\mathcal{E}(t,z_1), w_1\rangle_{X''} - {}_{X'}\langle \partial_t D\mathcal{E}(t,z_2), w_2\rangle_{X''}|\\ &\qquad \leq C_\gamma^{\rm Lip}\big(\|z_1 - z_2\| + |\mathcal{B}(z_1, -D\mathcal{E}(t,z_1)) - \mathcal{B}(z_2, -D\mathcal{E}(t,z_2))|\big). \end{aligned} \tag{5.19}$$

Since generally the union of all $J^X(t,z,-D\mathcal{E}(t,z))$ over $z \in \mathcal{S}_{\rm loc}(t)$ is not bounded in Z, the boundedness (5.18) and the Lipschitz continuity (5.19) are nontrivial. However, assuming $C_X^\Psi < \infty$ and using (5.13) and (5.16), we obtain

$$\|w\|_{X''} \leq \|\sigma\|_{X'}/(c_X^\Psi)^2 \leq C_X^\Psi/(c_X^\Psi)^2 \quad \forall\, w \in J^X(z,\sigma), \quad \forall\,(z,\sigma) \in \mathcal{Y}_0. \tag{5.20}$$

Hence, it suffices to assume that $\partial_t D\mathcal{E} : [0,T] \times Z \to X' \subset Z'$ is bounded and Lipschitz continuous in z. In the typical situation $\mathcal{E}(t,z) = \mathcal{U}(z) - \langle \ell(t), z\rangle$, this is true if $\ell \in C^1([0,T], X')$ holds.

Having introduced all the notation and all the needed assumptions, we are able to formulate the following uniqueness and continuous dependence result.

Theorem 5.1. *Assume* (2.5), (5.1)– (5.7), (5.9) *with* $C_X^\Psi < \infty$, (5.10) *with* $c_\Psi > 0$, (5.11), (5.12) *and* (5.17)–(5.19). *Then, the solution of the Cauchy problem for the subdifferential equation* (1.1) *is unique.*

Moreover, there exists constants $C_1, C_2 > 0$, *which are independent of the constants* c_Ψ *and* C_X^Ψ, *such that any pair of solutions* (z_1, z_2) *satisfies*

$$\|z_1(t) - z_2(t)\| + |\mathcal{B}_1(t) - \mathcal{B}_2(t)| \le C_1 \exp(C_2 t)\big(\|z_1(0) - z_2(0)\| + |\mathcal{B}_1(0) - \mathcal{B}_2(0)|\big),$$

where we have set $\mathcal{B}_j(t) = \mathcal{B}(z_j(t), -D\mathcal{E}(t, z_j(t)))$ *for* $j = 1, 2$.

Note that uniqueness follows without any continuity assumptions on $\mathcal{B}$. However, to derive continuous dependence on the initial data, we need the continuity of the mapping $b : \mathcal{S}_{\mathrm{loc}}(0) \to [0, 1/2]$, defined by $z \mapsto \mathcal{B}(z, -D\mathcal{E}(0, z))$. The following remark shows that it is sufficient to impose $c_\Psi > 0$ in order to obtain Lipschitz continuity of b.

Remark 5.3. Under the assumptions of Theorem 5.1, the function $\mathcal{B} : \mathcal{Y}_0 \to [0, 1/2]$ is locally Lipschitz continuous with respect to the norm topology of $Z \times Z'$. Indeed, for any $(z, \sigma) \in \mathcal{Y}_0$ and $w \in J(z, \sigma)$,

$$\begin{aligned}\|D_z\Psi(z, w)\|_{Z'} &\overset{(1)}{\le} C_F^\psi \|w\|_X^\sigma \|w\|^{1-\sigma} \overset{(2)}{\le} C_X^\sigma C_F^\psi \|w\| \\ &\overset{(3)}{\le} \frac{C_X^\sigma C_F^\psi}{c_\Psi^2} \|\sigma\|_{Z'} \overset{(4)}{\le} \frac{C_X^\sigma C_F^\psi C_\Psi}{c_\Psi^2}.\end{aligned}$$

Here, $\overset{(1)}{\le}$ follows from (5.11), for $\overset{(2)}{\le}$ use (5.8), for $\overset{(3)}{\le}$ use (5.15), and $\overset{(4)}{\le}$ follows from (5.13). In view of (5.22), we thus obtain that $D_z\mathcal{B}$ is bounded on $\mathcal{Y}_0$. Since, by (5.13) and (5.15), the elements w of $J(z, \sigma) = \partial_\sigma \mathcal{B}(z, \sigma)$ satisfy $\|w\| \le \|\sigma\|/(c_\Psi)^2 \le C_\Psi/(c_\Psi)^2$, we also have Lipschitz continuity in σ on $\mathcal{Y}_0$. However, then the Lipschitz norm may depend on c_Ψ.

5.2. *Preliminary results*

Here, we establish some further notation and prove some preliminary results. The proof of Theorem 5.1 will then be completed in the next subsection.

The classical Legendre–Fenchel theory (see also Appendix A), gives the following equivalences:

$$w \in J(z, \sigma) = \partial_\sigma \mathcal{B}(z, \sigma) \Leftrightarrow \sigma \in \Psi(z, w)\partial_v \Psi(z, w), \tag{5.21}$$

$$y \in \partial_z \mathcal{B}(z, \sigma) \Leftrightarrow y = -\Psi(z, w) D_z \Psi(z, w) \text{ for } w \in J(z, \sigma). \tag{5.22}$$

Indeed, the latter relation follows from the identity

$$\mathcal{B}(z, \sigma) = \langle \sigma, w \rangle - \frac{1}{2}\Psi^2(z, w) \quad \forall\, w \in J(z, \sigma),$$

cf. (A.11) in the proof of Proposition A.1. Moreover, in view of the latter result, if $(z, \sigma) \in \mathcal{Y}$, then $J(z, \sigma)$ spans the normal cone of $C(z) = \partial\Psi(z, 0)$ in the point σ, i.e. $\mathrm{N}_{C(z)}(\sigma) = \{rw : r \ge 0,\ w \in J(z, \sigma)\}$.

Lemma 5.1. *In this setting, we have the equivalences*

$$\sigma \in \partial C(z) \Leftrightarrow \mathcal{B}(z, \sigma) = \frac{1}{2} \Leftrightarrow \forall\, w \in J(z, \sigma) : \Psi(z, w) = 1. \tag{5.23}$$

Proof. The proof of the second equivalence follows from the fact that $\Psi(z,\cdot)$ is the Minkowski functional of $C(z)^*$ (cf. Remark 5.2), and from formula (A.7) in Proposition A.1. □

The following *a priori* estimate on the elements in $V(t,z,\sigma)$ will be important below.

Lemma 5.2. *If* (5.10) *and* (5.12) *hold, then we have*

$$\forall\,(t,z,\sigma)\in[0,T]\times\mathcal{Y}: \|v\|\le C_\Psi/\delta \textit{ for all } v\in V(t,z,\sigma).$$

Proof. Using (5.23), any $w\in J(z,\sigma)$ with $(z,\sigma)\in\mathcal{Y}$ satisfies $\Psi(z,w)=1$, and thus (5.10) implies $\|w\|\ge 1/C_\Psi$. However, exploiting (5.12), for $v\in V(t,z,\sigma)$ we have $\|v\|\le\|w\|/(\delta\|w\|^2)\le C_\Psi/\delta$. □

In the sequel, we assume that the functions $z,z_1,z_2\in W^{1,1}([0,T],Z)$ are solutions of our basic equation $0\in\partial_v\Psi(z,\dot z)+D\mathcal{E}(t,z)$. We first collect a few results which hold for all solutions. For this, we will use the notation

$$\varsigma(t)=-D\mathcal{E}(t,z(t))\quad\text{and}\quad\varsigma_j(t)=-D\mathcal{E}(t,z_j(t)),\quad j=1,2,$$

such that the basic equation reads

$$\varsigma(t)\in\partial_v\Psi(z(t),\dot z(t)). \tag{5.24}$$

For the solutions of (5.24), we have (cf. Theorem 3.1 and the estimate (5.13))

$$\begin{aligned}&\|\dot z\|_{L^\infty([0,T],Z)}\le C_{\mathrm{Lip}}:=C^{\mathcal{E}}_{tz}/\delta,\quad \|\varsigma(\cdot)\|_{L^\infty([0,T],X')}\le C^\Psi_X,\\&\|\varsigma(\cdot)\|_{L^\infty([0,T],Z')}\le C_\Psi.\end{aligned} \tag{5.25}$$

Following the arguments in Ref. 3, we observe that any solution $z:[0,T]\to Z$ to (5.24) satisfies

$$\dot z(t)=\lambda(t)v(t),\quad\text{with } v(t)\in V(t,z(t),\varsigma(t))\quad\text{for a.e. } t\in[0,T],$$

for a suitable coefficient $\lambda(t)\ge 0$.

In order to get further insight into this representation formula, we first introduce another representation for $\dot z(t)$. Indeed, we let

$$\hat\alpha(t)=\Psi(z(t),\dot z(t))\quad\text{and}\quad w(t)=\frac{1}{\hat\alpha(t)}\dot z(t)\quad\text{for } t \text{ with } \hat\alpha(t)>0.$$

By construction, $\Psi(z(t),w(t))=1$ if $\hat\alpha(t)>0$. Under these conditions, we also have by the 1-homogeneity of $\Psi(z,\cdot)$ that $\partial_v\Psi(z(t),\dot z(t))=\Psi(z(t),w(t))\partial_v\Psi(z(t),w(t))$. Moreover, under the assumption $\dot z(t)\ne 0$, we conclude by (5.21) that

$$\varsigma(t)\in\partial_v\Psi(z(t),\dot z(t))\Leftrightarrow w(t)\in J(z(t),\varsigma(t)).$$

Collecting these facts, we may infer that for a.e. $t\in(0,T)$ with $\dot z(t)\ne 0$, there holds

$$\dot z(t)=\hat\alpha(t)w(t)\quad\text{with}\quad\begin{cases}\hat\alpha(t)=\Psi(z(t),\dot z(t)),\\ w(t)\in J(z(t),\varsigma(t)).\end{cases}$$

Note that here $w(t) = \frac{1}{\hat{\alpha}(t)}\dot{z}(t) \in Z$. For $t \in [0,T]$ in which $\dot{z}(t)$ is not defined or $\dot{z}(t) = 0$, we may still choose $w(t) \in J^X(z(t), \varsigma(t)) \subset X''$, such that $w : [0,T] \to X''$ is measurable and essentially bounded (see (5.13) and (5.16)), with

$$\|w\|_{L^\infty([0,T],X'')} \le C_X^\Psi/(c_X^\Psi)^2 \quad \text{and} \quad \big(\dot{z}(t) \ne 0 \Rightarrow \|w(t)\|_X \le 1/c_X^\Psi\big), \tag{5.26}$$

the second follows from (5.9) and (5.23). Similarly, for $c_\Psi > 0$ we have

$$\|w\|_{L^\infty([0,T],Z)} \le C_\Psi/(c_\Psi)^2 \quad \text{and} \quad \big(\dot{z}(t) \ne 0 \Rightarrow \|w(t)\| \le 1/c_\Psi\big). \tag{5.27}$$

Since $w(t) \in J(z(t), \varsigma(t))$ is related to $v(t) \in V(t, z(t), \varsigma(t))$ by a scalar, define now the selection $v : [0,T] \to Z$ of V via

$$v(t) = \begin{cases} \dfrac{1}{\langle H(t)w(t), w(t)\rangle + D_z\Psi(z(t), w(t))[w(t)]} w(t) & \text{if } w(t) \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we introduce the scalar function $\alpha : [0,T] \to [0,\infty]$ via

$$\begin{aligned} \alpha(t) &= \hat{\alpha}(t)\big(\langle H(t)w(t), w(t)\rangle + D_z\Psi(z(t), w(t))[w(t)]\big) \\ &= \langle H(t)w(t), \dot{z}(t)\rangle + D_z\Psi(z(t), w(t))[\dot{z}(t)], \end{aligned} \tag{5.28}$$

where we understand the definition such that $\dot{z}(t) = 0$ implies $\alpha(t) = 0$. Therefore, for a.e. $t \in (0,T)$ the coefficient $\lambda(t)$ in our first representation formula coincides with $\alpha(t)$, thus we conclude this crucial representation formula for $\dot{z}(t)$:

$$\dot{z}(t) = \alpha(t)v(t), \quad \text{with } v(t) \in V(t, z(t), \varsigma(t)) \quad \text{for a.e. } t \in [0,T]. \tag{5.29}$$

In addition, we let

$$\beta(t) = \mathcal{B}(z(t), \varsigma(t)) \quad \text{and} \quad \gamma(t) = -\langle \partial_t D\mathcal{E}(t, z(t)), w(t))\rangle. \tag{5.30}$$

The remainder of this subsection will be devoted to the proof (see Proposition 5.2 below) of the central formula

$$\frac{d}{dt}\beta(t) = \frac{d}{dt}\mathcal{B}(z(t), \varsigma(t)) = \gamma(t) - \alpha(t) \tag{5.31}$$

which was obtained in Lemma 5.1 in Ref. 3 under suitable smoothness assumptions on $\mathcal{B}$, and which is at the heart of the theory. In fact, it will only be for the validity of this identity that we need $c_\Psi > 0$.

A chain rule for $\mathcal{B}$. By the definitions of α and γ, the relation (5.31) can be interpreted as a chain rule, which follows from suitable smoothness and convexity properties of $\mathcal{B}$. However, it should be noted that we only need this formula along the true solutions of (5.24) and, thus, there is some hope that relation (5.31) still holds under weaker assumption.

As z is a solution to (5.24), the function β takes values in the interval $[0,1/2]$. In order to discuss the identity (5.31), we make some preparations. First, note the obvious equivalence which holds for a.e. $t\in[0,T]$:

$$\varsigma(t)\in\partial_v\Psi(z(t),\dot z(t))\Leftrightarrow\begin{cases}\langle\varsigma(t),\dot z(t)\rangle=\Psi(z(t),\dot z(t)),\\ \forall\, v\in Z:\langle\varsigma(t),v\rangle\le\Psi(z(t),v).\end{cases}\tag{5.32}$$

The next result goes one step further:

Lemma 5.3. *For a.e. $t\in[0,T]$ we have the identity*

$$\langle\dot\varsigma(t),\dot z(t)\rangle=D_z\Psi(z(t),\dot z(t))[\dot z(t)].$$

Proof. Observe that the right-hand side in (5.32) implies

$$\langle\varsigma(t)-\varsigma(t-h),\dot z(t)\rangle\ge\Psi(z(t),\dot z(t))-\Psi(z(t-h),\dot z(t)).$$

Dividing by $h>0$ and using that z and ς are Lipschitz, with

$$\dot\varsigma(t)=-\partial_tD\mathcal{E}(t,z(t))-D^2\mathcal{E}(t,z(t))\dot z(t)\tag{5.33}$$

(by the chain rule for the C^2-functional $\mathcal{E}$), we find that $\langle\dot\varsigma(t),\dot z(t)\rangle\ge D_z\Psi(z(t),\dot z(t))[\dot z(t)]$. Taking $h<0$ and dividing by $(-h)$ leads to the opposite inequality, and the result follows. ☐

Now we are able to establish some *a priori* estimates on the functions $\hat\alpha$, α and $\|\dot z(t)\|$ as follows.

Proposition 5.1. *Let the conditions* (5.2), (5.4), (5.7), (5.9), (5.10), (5.11), (5.12) *and* (5.18) *hold. Then, for a.e. $t\in[0,T]$ we have the estimate*

$$\hat\alpha(t)\|w(t)\|^2\le C_w,\tag{5.34}$$

or, equivalently,

$$\|\dot z(t)\|^2\le C_w\Psi(z(t),\dot z(t)),$$

where $C_w:=\min\{C_\gamma^{\max}/\kappa,(C_F^\psi/\kappa)^{1/\sigma}C_{\mathrm{Lip}}/c_X^\Psi\}$. Moreover,

$$\|\alpha\|_\infty\le C_\alpha:=\big(C_{zz}^{\mathcal{E}}+C_F^\psi C_X^\sigma\big)C_w.\tag{5.35}$$

Proof. We show the *a priori* estimate (5.34): by construction and from our proof below, it will then be clear that the estimate for $\hat\alpha(t)\|w(t)\|^2$ is equivalent to the estimate for $\|\dot z(t)\|^2$. Indeed, using the convexity condition (5.7) and the chain rule

(5.33) we find, for $\hat\alpha(t)>0$, the estimate

$$\begin{aligned}\kappa\hat\alpha(t)\|w(t)\|^2 &= \frac{\kappa}{\hat\alpha(t)}\|\dot z(t)\|^2 \le \frac{1}{\hat\alpha(t)}\langle H(t,z(t))\dot z(t),\dot z(t)\rangle = \langle H(t,z(t))\dot z(t), w(t)\rangle\\ &= -\langle \dot\varsigma(t)+\partial_t D\mathcal{E}(t,z(t)), w(t)\rangle\\ &\overset{(1)}{=} -D_z\Psi(z(t),w(t))[\dot z(t)] - \langle \partial_t D\mathcal{E}(t,z(t)), w(t)\rangle\\ &\overset{(2)}{\le} C_F^{\psi}\|w(t)\|_X^{\sigma}\hat\alpha(t)\|w(t)\|^{2-\sigma} + C_\gamma^{\max}\\ &\overset{(3)}{=} C_F^{\psi}(1/c_X^{\Psi})^{\sigma}\big(\hat\alpha\|w\|^2\big)^{1-\sigma}\|\dot z\|^{\sigma} + C_\gamma^{\max}\\ &\overset{(4)}{=} C_F^{\psi}(C_{\mathrm{Lip}}/c_X^{\Psi})^{\sigma}\big(\hat\alpha\|w\|^2\big)^{1-\sigma} + C_\gamma^{\max},\end{aligned}$$

which implies the desired result (5.34), since $\rho \le C\rho^{1-\sigma}+D$ entails $\rho \le \min\{C^{1/\sigma}, D\}$. Note that for $\overset{(1)}{=}$ we have used Lemma 5.3 and that $\hat\alpha(t)>0$, for $\overset{(2)}{\le}$ we have used the assumptions (5.11) and (5.18), for $\overset{(3)}{=}$ we used (5.26), and $\overset{(4)}{=}$ ensues from (5.25).

The second result follows simply from the definition (5.28) of α in terms of $\hat\alpha$ and w. □

Now, we formulate the central *chain rule* formula (5.31), stating $\dot\beta=\gamma-\alpha$.

Proposition 5.2. *Let conditions* (5.9) *and* (5.10) *hold with* $C_X^{\Psi}<\infty$ *and* $c_\Psi>0$, *and assume* (5.2), (5.4), (5.7), (5.11), (5.12) *and* (5.18). *Then, the map* $\beta:[0,T]\to[0,1/2]; t\mapsto \mathcal{B}(z(t),\varsigma(t))$ *is absolutely continuous, and* (5.31) *holds, i.e.* $\frac{d}{dt}\beta(t)=\gamma(t)-\alpha(t)$ *for a.e.* $t\in[0,T]$.

Moreover, β *is Lipschitz continuous with a Lipschitz constant* independent *of* c_Ψ, *namely*

$$|\beta(t)-\beta(s)| \le (C_\alpha + C_\gamma^{\max})|t-s| \quad \textit{for all } s,t\in[0,T]. \tag{5.36}$$

In fact, under the above assumptions it should be possible to show that the functional $(z,\sigma)\ni Z\times X'\mapsto \mathcal{B}(z,\sigma)$ satisfies a chain rule along curves $(z,\sigma)\in C^{\mathrm{Lip}}([0,T],Z\times X')$. Then, the result would follow from the equivalence $X\sim Z$, due to $c_\Psi>0$.

Proof. Under the assumption $c_\Psi>0$, we know that $w\in L^\infty([0,T],Z)$, see (5.27). The definition of $\mathcal{B}$ and $w(t)\in\partial_\sigma\mathcal{B}(z(t),\varsigma(t))$ imply

$$\beta(t)=\mathcal{B}(z(t),\varsigma(t)) = \langle\varsigma(t),w(t)\rangle - \frac12\Psi(z(t),w(t))^2 \ge \langle\varsigma(t),\hat w\rangle - \frac12\Psi(z(t),\hat w)^2$$

for all $\hat w\in Z$. Thus, we obtain

$$\beta(t)-\beta(t-h) \le \langle\varsigma(t)-\varsigma(t-h),w(t)\rangle - \frac12\big(\Psi(z(t),w(t))^2-\Psi(z(t-h),w(t))^2\big).$$

Dividing by $h > 0$ and taking the limit $h \searrow 0$, the Lipschitz continuity of z and ς provide the estimate

$$\limsup_{h \searrow 0} \frac{1}{h}\big(\beta(t) - \beta(t-h)\big) \le \langle \dot\varsigma(t), w(t)\rangle - \Psi(z(t), w(t)) D_z\Psi(z(t), w(t))[\dot z(t)].$$

Denote the right-hand side of the above formula by $\hat\beta(t)$: then, a discussion of the cases $\dot z(t) = 0$ and $\dot z(t) \neq 0$ easily shows that $\hat\beta = \gamma - \alpha$ as desired. The same argument leads to $\liminf_{h \searrow 0} \frac{1}{h}\big(\beta(t+h) - \beta(t)\big) \ge \hat\beta(t)$. Thus, if β is absolutely continuous, then its derivative equals $\hat\beta$ a.e. on $[0, T]$.

While the above arguments do not use the condition $c_\Psi > 0$ in an essential way, we will need it now. As mentioned in Remark 5.3, this latter condition implies that $\mathcal{B} : \mathcal{Y}_0 \to [0, 1/2]$ is (globally) Lipschitz w.r.t. the norm topology of $Z \times Z'$, with a Lipschitz constant tending to ∞ for $c_\Psi \searrow 0$. Now, inserting the Lipschitz continuous curve $\Gamma : t \mapsto (z(t), \varsigma(t)) \in Z \times Z'$ we immediately conclude that $\beta = \mathcal{B} \circ \Gamma$ is Lipschitz continuous, and hence absolutely continuous.

Finally, (5.36) is a direct consequence of (5.31), (5.18) and Proposition 5.1. □

5.3. *Proof of the uniqueness result*

Let z_1 and z_2 be two solutions to Problem 2.1, corresponding to the initial data z_1^0 and z_2^0: for a.e. $t \in (0, T)$, we will use the notation

$$\varsigma_i(t) := -D\mathcal{E}(t, z_i(t)), \quad \alpha_i(t),\ \gamma_i(t), \quad J_i(t) := J(z_i(t), \varsigma_i(t)),$$
$$\beta_i(t) := \mathcal{B}(z_i(t), \varsigma_i(t)) \quad H_i(t) := H(t, z_i(t))$$

for the quantities previously defined and related to the solution z_i, $i = 1, 2$. Moreover, recalling the representation formula (5.29), we have for $i = 1, 2$

$$\dot z_i(t) = \alpha_i(t) v_i(t), \quad \text{with } v_i(t) \in V(t, z_i(t), \varsigma_i(t)) \quad \text{for a.e. } t \in [0, T].$$

Following Ref. 3, our first step will be to show a crucial estimate for the quantities α_i, β_i and γ_i, $i = 1, 2$, in Lemma 5.4 below. Indeed, the proof of this lemma, which we present here for the sake of completeness, is analogous to the argument developed for Lemma 5.2 in Ref. 3.

Lemma 5.4. *Assume* (5.2), (5.4), (5.7), (5.9) *with* $C_X^\Psi < \infty$, (5.10) *with* $c_\Psi > 0$, (5.11), (5.12) *and* (5.18). *Let* z_1, z_2 *be two solutions to Problem* 2.1. *Then,*

$$|\alpha_1(t) - \alpha_2(t)| + \frac{d}{dt}|\beta_1 - \beta_2|(t) \le |\gamma_1(t) - \gamma_2(t)| \quad \textit{for a.e. } t \in (0, T). \tag{5.37}$$

Proof. Preliminarily, note that there exists a negligible set $\mathcal{N} \subset (0, T)$ such that for $t \in (0, T) \setminus \mathcal{N}$ the quantities α_i, β_i, γ_i, and the derivatives $\dot z_i$, $i = 1, 2$ are well defined. From now on, we will always consider t in $(0, T) \setminus \mathcal{N}$. Then, we distinguish three cases.

(1) Assume $\dot{z}_1(t) = \dot{z}_2(t) = 0$. Then, $\alpha_i(t) = 0$ and, by (5.31), $\frac{d}{dt}\beta_i(t) = \gamma_i(t)$, so that

$$\frac{d}{dt}|\beta_1 - \beta_2|(t) \leq \left|\frac{d}{dt}\beta_1(t) - \frac{d}{dt}\beta_2(t)\right| = |\gamma_1(t) - \gamma_2(t)|$$

and (5.37) holds.

(2) Let $\dot{z}_1(t), \dot{z}_2(t) \neq 0$. Then, $(z_i(t), \varsigma_i(t)) \in \mathcal{Y}$, whence $\beta_i(t) = 1/2 = \max_{s\in[0,T]} \beta_i(s)$, so that $\frac{d}{dt}\beta_1(t) = \frac{d}{dt}\beta_2(t) = 0$ and thus $\frac{d}{dt}|\beta_1 - \beta_2|(t) = 0$. Moreover, owing to (5.31) we have $\alpha_i(t) = \gamma_i(t)$, so that (5.37) holds as well.

(3) Assume $\dot{z}_1(t) = 0$ and $\dot{z}_2(t) \neq 0$. Then, $\alpha_1(t) = 0$, $\frac{d}{dt}\beta_1(t) = \gamma_1(t)$, while $\beta_2(t) = 1/2$, $\frac{d}{dt}\beta_2(t) = 0$ and $\alpha_2(t) = \gamma_2(t) > 0$. In particular, $|\alpha_1(t) - \alpha_2(t)| = \alpha_2(t)$ and $|\beta_1(t) - \beta_2(t)| = 1/2 - \beta_1(t)$. Whence

$$|\alpha_1(t) - \alpha_2(t)| + \frac{d}{dt}|\beta_1 - \beta_2|(t) = \alpha_2(t) - \frac{d}{dt}\beta_1(t) = \gamma_2(t) - \gamma_1(t) \leq |\gamma_2(t) - \gamma_1(t)|.$$

Again, (5.37) holds.

As the cases $\dot{z}_1(t) \neq 0$ and $\dot{z}_2(t) = 0$ are similar, (5.37) is established in all cases.

□

Energy estimates. Mimicking the approach to uniqueness developed in Sec. 7.2 of Ref. 18, for all $t \in [0, T]$, we introduce the *energetic* quantity

$$\varrho_{1,2}(t) := \sqrt{\langle D\mathcal{E}(t, z_1(t)) - D\mathcal{E}(t, z_2(t)), z_1(t) - z_2(t)\rangle}.$$

Owing to the κ-uniform convexity and to the smoothness of $\mathcal{E}(t, \cdot)$ we have

$$\sqrt{\kappa}\|z_1(t) - z_2(t)\| \leq \varrho_{1,2}(t) \leq \sqrt{C^{\mathcal{E}}_{zz}}\|z_1(t) - z_2(t)\|. \tag{5.38}$$

Our ultimate aim is to derive a Gronwall-type estimate for $\varrho_{1,2}(t) + M_2|\beta_1(t) - \beta_2(t)|$ (cf. (5.46) below). Our technique will combine (5.37) with suitable *energy* estimates (analogous to the ones in Sec. 7.2 in Ref. 18), obtained by only exploiting the smoothness assumptions on $\mathcal{E}$ and (5.12).

We split the proof of Theorem 5.1 in intermediate steps. In the next two lemmas, we derive a fundamental one-sided Lipschitz estimate from the structure condition (5.17).

Lemma 5.5. *Assume* (5.2), (5.4), (5.7), (5.9) *with* $C^{\Psi}_X < \infty$, (5.10) *with* $c_\Psi > 0$, (5.11), (5.12), (5.17) *and* (5.18). *Then, for any two solutions* z_1, z_2 *to Problem* 2.1 *there holds*, for a.e. $t \in (0, T)$:

$$\langle \varsigma_2(t) - \varsigma_1(t), \dot{z}_1(t) - \dot{z}_2(t)\rangle \\ \leq C_\alpha L_V\big((C^{\mathcal{E}}_{zz})^2 + 1\big)\|z_1(t) - z_2(t)\|^2 + \frac{C_\Psi C^{\mathcal{E}}_{zz}}{\delta}|\alpha_1(t) - \alpha_2(t)|\|z_1(t) - z_2(t)\|. \tag{5.39}$$

Proof. Of course (5.39) is trivially true when $\dot{z}_1(t) = \dot{z}_2(t) = 0$.

Let us then assume that $\dot{z}_1(t) \neq 0 \neq \dot{z}_2(t)$, which implies $\alpha_1(t) > 0$ and $\alpha_2(t) > 0$. Using (5.29), we get

$$\begin{aligned}\langle \varsigma_2(t) - \varsigma_1(t), \dot{z}_1(t) - \dot{z}_2(t)\rangle &= \langle \varsigma_2(t) - \varsigma_1(t), \alpha_1(t)v_1(t) - \alpha_2(t)v_2(t)\rangle \\ &= \alpha_1(t)\langle \varsigma_2(t) - \varsigma_1(t), v_1(t) - v_2(t)\rangle \\ &\quad + \langle \varsigma_2(t) - \varsigma_1(t), (\alpha_1(t) - \alpha_2(t))v_2(t)\rangle.\end{aligned}$$

Let us now estimate the latter two summands separately. For the first summand the *structure condition* (5.17) gives

$$\begin{aligned}\alpha_1(t)\langle \varsigma_2(t) - \varsigma_1(t), v_1(t) - v_2(t)\rangle &\leq \alpha_1(t)L_V\big(\|\varsigma_1(t) - \varsigma_2(t)\|^2_{Z'} + \|z_1(t) - z_2(t)\|^2\big) \\ &\leq C_\alpha L_V\big((C^{\mathcal{E}}_{zz})^2 + 1\big)\|z_1(t) - z_2(t)\|^2. \qquad (5.40)\end{aligned}$$

The last estimate follows from (5.4) and the *a priori* estimate (5.35) for α_1. For the second summand we have

$$\begin{aligned}&\langle \varsigma_2(t) - \varsigma_1(t), (\alpha_1(t) - \alpha_2(t))v_2(t)\rangle \\ &\quad \leq \|v_2(t)\|\|\varsigma_2(t) - \varsigma_1(t)\||\alpha_1(t) - \alpha_2(t)| \\ &\quad \leq (C_\Psi C^{\mathcal{E}}_{zz})/\delta\|z_1(t) - z_2(t)\||\alpha_1(t) - \alpha_2(t)|, \qquad (5.41)\end{aligned}$$

where we have used Lemma 5.2 and (5.4) again. Adding (5.40) and (5.41), (5.39) follows.

In the case $\dot{z}_1(t) \neq 0$ and $\dot{z}_2(t) = 0$ we have $\alpha_2(t) = 0$, and we estimate as follows:

$$\begin{aligned}\langle \varsigma_2(t) - \varsigma_1(t), \dot{z}_1(t) - \dot{z}_2(t)\rangle &= \langle \varsigma_2(t) - \varsigma_1(t), (\alpha_1(t) - \alpha_2(t))v_1(t)\rangle \\ &\leq \|\varsigma_2(t) - \varsigma_1(t)\|\|v_1(t)\||\alpha_1(t) - \alpha_2(t)| \\ &\leq (C_\Psi C^{\mathcal{E}}_{zz})/\delta\|z_1(t) - z_2(t)\||\alpha_1(t) - \alpha_2(t)|.\end{aligned}$$

Thus, the desired estimate (5.39) is established. □

Proposition 5.3. *Assume* (5.2)–(5.7), (5.9) *with* $C^\Psi_X < \infty$, (5.10) *with* $c_\Psi > 0$, (5.11), (5.12), (5.17) *and* (5.18). *Then, for any two solutions* z_1, z_2 *to Problem* 2.1, *there holds for a.e.* $t \in (0, T)$

$$\frac{d}{dt}\varrho_{1,2}(t) \leq \frac{1}{\varrho_{1,2}(t)}\left(M_1\|z_1(t) - z_2(t)\|^2 + \frac{C_\Psi C^{\mathcal{E}}_{zz}}{\delta}|\alpha_1(t) - \alpha_2(t)|\|z_1(t) - z_2(t)\|\right), \qquad (5.42)$$

where $M_1 = C_\alpha L_V\big((C^{\mathcal{E}}_{zz})^2 + 1\big) + \frac{1}{2}C^{\mathcal{E}}_{tzz} + \frac{1}{2}C^{\mathcal{E}}_{zzz}C_{\mathrm{Lip}}$.

The constant M_1 indeed shows that we combine the ideas of Ref. 18, which deals with the case $L_V = 0$, and Ref. 3, which is restricted to $C^{\mathcal{E}}_{zzz} = C^{\mathcal{E}}_{tzz} = 0$.

Proof. Elementary computations yield

$$\frac{d}{dt}\varrho_{1,2}(t) = \frac{1}{2\varrho_{1,2}(t)}\big(\langle\varsigma_2(t)-\varsigma_1(t),\dot z_1(t)-\dot z_2(t)\rangle + \langle\dot\varsigma_2(t)-\dot\varsigma_1(t),z_1(t)-z_2(t)\rangle\big)$$
$$= \frac{1}{2\varrho_{1,2}(t)}\big(2\langle\varsigma_2(t)-\varsigma_1(t),\dot z_1(t)-\dot z_2(t)\rangle + \mathcal{T}_1+\mathcal{T}_2\big), \qquad (5.43)$$

where we used the chain rule $\dot\varsigma_j(t) = -\partial_t D\mathcal{E}(t,z_j(t)) - H_j(t)\dot z_j(t)$, and the abbreviations

$$\mathcal{T}_1 = \langle\partial_t D\mathcal{E}(t,z_1(t)) - \partial_t D\mathcal{E}(t,z_2(t)), z_1(t)-z_2(t)\rangle$$
$$\mathcal{T}_2 = \langle A_1(t),\dot z_1(t)\rangle - \langle A_2(t),\dot z_2(t)\rangle, \text{ with}$$
$$A_j(t) = H_j(t)(z_{3-j}(t)-z_j(t)) + \varsigma_{3-j}(t)-\varsigma_j(t).$$

The term $\mathcal{T}_1$ is easily estimated using (5.6), namely $|\mathcal{T}_1| \le C^{\mathcal{E}}_{tzz}\|z_1(t)-z_2(t)\|^2$. For $\mathcal{T}_2$ we use $\|\dot z_j(t)\| \le C_{\mathrm{Lip}}$, the identity

$$A_j(t) = \int_0^1 \big(D^2\mathcal{E}(t,z_j) - D^2\mathcal{E}(t,z_j+s(z_{3-j}-z_j))\big)\big(z_{3-j}-z_j\big)\,ds,$$

and the assumption (5.5) to obtain $|\mathcal{T}_2| \le C^{\mathcal{E}}_{zzz}C_{\mathrm{Lip}}\|z_1(t)-z_2(t)\|^2$.

Using the estimate from the previous Lemma 5.5 and adding the estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ gives the desired result. □

As $\|z_1(t)-z_2(t)\| \le \kappa^{-1/2}\varrho_{1,2}(t)$, we arrive at the estimate

$$\frac{d}{dt}\varrho_{1,2}(t) \le \frac{M_1}{\kappa}\varrho_{1,2}(t) + M_2|\alpha_1(t)-\alpha_2(t)| \quad \text{where } M_2 = \frac{C_\Psi C^{\mathcal{E}}_{zz}}{\delta\sqrt{\kappa}}. \qquad (5.44)$$

The importance of the function $\mathcal{B}$ lies in the fact that the relation $\frac{d}{dt}\beta = \gamma - \alpha$ leads to the estimate (5.37), which allows us to estimate $|\alpha_1(t)-\alpha_2(t)|$ in terms of $|\gamma_1(t)-\gamma_2(t)|$. In fact, multiplying (5.37) by M_2 times and adding it to (5.44) leads to a cancellation, so that we find

$$\frac{d}{dt}\varrho_{1,2}(t) + M_2\frac{d}{dt}|\beta_1-\beta_2|(t) \le \frac{M_1}{\kappa}\varrho_{1,2}(t) + M_2|\gamma_1(t)-\gamma_2(t)|. \qquad (5.45)$$

As γ_j does not depend on the time derivative $\dot z_j$, it behaves much better and allows for a Lipschitz estimate.

Using (5.30) and (5.19), we infer that for a.e. $t\in(0,T)$

$$|\gamma_1(t)-\gamma_2(t)| \le C^{\mathrm{Lip}}_\gamma\big(\|z_1(t)-z_2(t)\| + |\beta_1(t)-\beta_2(t)|\big).$$

Inserting this into (5.45) and recalling (5.38) yields the Gronwall estimate

$$\frac{d}{dt}\varrho_{1,2}(t) + M_2\frac{d}{dt}|\beta_1-\beta_2|(t) \le M_3\big(\varrho_{1,2}(t) + M_2|\beta_1-\beta_2|(t)\big) \qquad (5.46)$$

for a.e. $t\in(0,T)$, where $M_3 = \max\Big\{\frac{M_1}{\kappa} + \frac{M_2 C^{\mathrm{Lip}}_\gamma}{\sqrt{\kappa}}, M_2 C^{\mathrm{Lip}}_\gamma\Big\}$.

Since $m(t) := \varrho_{1,2}(t) + M_2|\beta_1 - \beta_2|(t)$ satisfies the Gronwall estimate $\frac{d}{dt}m(t) \leq M_3 m(t)$, we have $m(t) \leq m(0)\exp(M_3 t)$ for $t \in [0,T]$. Using (5.38), we obtain the desired estimate of our main Theorem 5.1, namely

$$\|z_1(t) - z_2(t)\| + |\beta_1(t) - \beta_2(t)| \leq C_1 \exp(C_2 t)\big(\|z_1(0) - z_2(0)\| + |\beta_1(0) - \beta_2(0)|\big)$$

with $C_1 = \max\{1/\sqrt{\kappa}, 1/M_2\} \max\{\sqrt{C^{\mathcal{E}}_{zz}}, M_2\}$ and $C_2 = M_3$.

Thus, Theorem 5.1 is proved. □

Appendix A. Convex Analysis Tools

Henceforth, $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ will denote a generic separable and reflexive Banach space, with dual $(\mathcal{X}^*, \|\cdot\|^*_{\mathcal{X}})$. In the next lines, we collect for completeness some convex analysis notions and results which can be found in Refs. 3 and 11.

Let $\mathcal{C} \subset \mathcal{X}$ be a non-empty, closed, convex set - we denote by $\partial\mathcal{C}$ its boundary, and by $I_{\mathcal{C}}$ its indicator function. The *polar set* $\mathcal{C}^* \subset \mathcal{X}^*$ to $\mathcal{C}$ is

$$\mathcal{C}^* := \{y \in \mathcal{X}^* : \langle y, x\rangle \leq 1 \quad \forall\, x \in \mathcal{C}\}.$$

Note that, due to the reflexivity of $\mathcal{X}$, $(\mathcal{C}^*)^* = \mathcal{C}$. We also define the *Minkowski functional* $\mathcal{M}_{\mathcal{C}} : \mathcal{X} \to [0, +\infty]$ of $\mathcal{C}$ by

$$\mathcal{M}_{\mathcal{C}}(x) := \inf\left\{s > 0 : \frac{1}{s}x \in \mathcal{C}\right\}. \tag{A.1}$$

Observe that

$$\mathcal{M}_{\mathcal{C}}(x) \leq 1 \Leftrightarrow x \in \mathcal{C} \quad \text{and} \quad \mathcal{M}_{\mathcal{C}}(w) = 1 \Leftrightarrow w \in \partial\mathcal{C}. \tag{A.2}$$

The following crucial relation between the support function (cf. (2.9)) of $\mathcal{C}$ (of $\mathcal{C}^*$, resp.) and the Minkowski functional of its polar $\mathcal{C}^*$ (of $\mathcal{C}$, resp.) follows by elementary computations:

$$\mathcal{M}_{\mathcal{C}}(x) = \sup\{\langle y, x\rangle : y \in \mathcal{C}^*\}, \quad \mathcal{M}_{\mathcal{C}^*}(y) = \sup\{\langle y, x\rangle : x \in \mathcal{C}\} \tag{A.3}$$

for all $x \in \mathcal{X}$, and $y \in \mathcal{X}^*$, whence

$$\partial\mathcal{M}_{\mathcal{C}}(x) = \operatorname{argmax}\{\langle y, x\rangle \mid y \in \mathcal{C}^*\}, \quad \partial\mathcal{M}_{\mathcal{C}^*}(y) = \operatorname{argmax}\{\langle y, x\rangle \mid x \in \mathcal{C}\} \tag{A.4}$$

for all $x \in \mathcal{X}$, $y \in \mathcal{X}^*$. Furthermore, note that

$$\langle y, x\rangle \leq \mathcal{M}_{\mathcal{C}}(x)\,\mathcal{M}_{\mathcal{C}^*}(y) \quad \forall\, x \in \mathcal{X},\, y \in \mathcal{X}^*. \tag{A.5}$$

Finally, let us introduce the *convex* functional $\mathcal{B}_{\mathcal{C}} : \mathcal{X} \to [0, +\infty]$ by

$$\mathcal{B}_{\mathcal{C}}(x) := \frac{1}{2}\mathcal{M}^2_{\mathcal{C}}(x), \tag{A.6}$$

and analogously we define $\mathcal{B}_{\mathcal{C}^*}$. In view of (A.2), we have $\mathcal{B}_{\mathcal{C}}(x) = \frac{1}{2}$ if and only if $x \in \partial\mathcal{C}$. We denote by $J_{\mathcal{C}}$ the subdifferential of $\mathcal{B}_{\mathcal{C}}$: of course $J_{\mathcal{C}} : \mathcal{X} \to 2^{\mathcal{X}^*}$ is a maximal monotone operator. The following result subsumes Lemma 2.21, Prop. 2.25, and Ex. 2.26 in Ref. 11. For the reader's convenience, we report here the proof: note that it is not required to assume $\mathcal{C}$ to contain/to be contained in any ball.

Proposition A.1. *Let $\mathcal{C}$ be a non-empty, closed, and convex set. Then,*

$$\mathcal{M}_{\mathcal{C}^*}(y) = \mathcal{M}_{\mathcal{C}}(x) \quad \forall\, x \in \mathcal{C},\ y \in J_{\mathcal{C}}(x) \tag{A.7}$$

and $\mathcal{B}_{\mathcal{C}^*}$ coincides with the Legendre–Fenchel transform of $\mathcal{B}_{\mathcal{C}}$, and analogously for $\mathcal{B}_{\mathcal{C}}$, namely

$$\begin{cases} \mathcal{B}_{\mathcal{C}^*}(y) = \sup_{x\in\mathcal{X}}\{\langle y, x\rangle - \mathcal{B}_{\mathcal{C}}(x)\} \\ \mathcal{B}_{\mathcal{C}}(x) = \sup_{y\in\mathcal{X}^*}\{\langle y, x\rangle - \mathcal{B}_{\mathcal{C}^*}(y)\} \end{cases} \quad \text{for all } x \in \mathcal{X},\ y \in \mathcal{X}^*. \tag{A.8}$$

Moreover, for every $x \in \partial\mathcal{C}$

$$J_{\mathcal{C}}(x) \subset \partial I_{\mathcal{C}}(x), \tag{A.9}$$

$$\forall\, y \in \partial I_{\mathcal{C}}(x) \setminus \{0\} : \langle y, x\rangle = \mathcal{M}_{\mathcal{C}^*}(y) \text{ and } y \in \mathcal{M}_{\mathcal{C}^*}(y) J_{\mathcal{C}}(x). \tag{A.10}$$

Proof. Thanks to (A.4), for any $x \in \mathcal{C}$ and $y \in J_{\mathcal{C}}(x) = \mathcal{M}_{\mathcal{C}}(x)\partial\mathcal{M}_{\mathcal{C}}(x)$ there exists $y' \in \mathcal{C}^*$ such that $y = \mathcal{M}_{\mathcal{C}}(x)\, y'$ and $\mathcal{M}_{\mathcal{C}}(x) = \langle y', x\rangle$, whence, in view of (A.5), we obtain $\mathcal{M}_{\mathcal{C}^*}(y') \geq 1$. As $y' \in \mathcal{C}^*$, necessarily $\mathcal{M}_{\mathcal{C}^*}(y') = 1$, so that, by the positive homogeneity of $\mathcal{M}_{\mathcal{C}^*}$,

$$\mathcal{M}_{\mathcal{C}^*}(y) = \mathcal{M}_{\mathcal{C}}(x)\, \mathcal{M}_{\mathcal{C}^*}(y') = \mathcal{M}_{\mathcal{C}}(x)$$

as desired.

We may now check, for instance, the second of (A.8): one inequality is a trivial consequence of (A.5). To show the converse inequality, we note that for any $x \in \mathcal{X}$ and $y \in J_{\mathcal{C}}(x)$

$$\langle y, x\rangle - \mathcal{B}_{\mathcal{C}^*}(y) = \mathcal{M}_{\mathcal{C}}(x)^2 - \frac{1}{2}\mathcal{M}_{\mathcal{C}^*}(y)^2 = \frac{1}{2}\mathcal{M}_{\mathcal{C}}(x)^2 = \mathcal{B}_{\mathcal{C}}(x), \tag{A.11}$$

where the first equality follows from the previous computations and from the very definition of $\mathcal{B}_{\mathcal{C}^*}$, and the second one is due to (A.7).

Finally, (A.9) can be trivially checked using the definition. On the other hand, following the proof of Prop. 2.25 in Ref. 11, we note that for all $y \in \partial I_{\mathcal{C}}(x) \setminus \{0\}$ $\langle y, x\rangle \geq \langle y, z\rangle$ for all $z \in \mathcal{C}$, so that $y/\langle y, x\rangle \in \mathcal{C}^*$. Therefore, by the positive homogeneity of $\mathcal{M}_{\mathcal{C}^*}$ we have $\mathcal{M}_{\mathcal{C}^*}(y) \leq \langle y, x\rangle$. Since $\mathcal{M}_{\mathcal{C}}(x) = 1$ by $x \in \partial\mathcal{C}$, we gather from (A.5) that $\langle y, x\rangle = \mathcal{M}_{\mathcal{C}}(x)\mathcal{M}_{\mathcal{C}^*}(y) = \mathcal{M}_{\mathcal{C}^*}(y)$. The latter relation and (A.5) again yield

$$\left\langle \frac{y}{\mathcal{M}_{\mathcal{C}^*}(y)}, x - z \right\rangle \geq \mathcal{M}_{\mathcal{C}}(x) - \mathcal{M}_{\mathcal{C}}(z) \quad \forall\, z \in \mathcal{X},$$

so that $y/\langle y, x\rangle \in \partial\mathcal{M}_{\mathcal{C}}(x)$ and (A.10) follows. □

In the end, we also recall the following lemma (see e.g. Cor. IV.6 in Ref. 1).

Lemma A.1. *Let $\psi_1, \psi_2 : \mathcal{X} \to (-\infty, +\infty]$ be two proper, convex and l.s.c. functionals. If $0 \in \operatorname{int}(D(\psi_1) - D(\psi_2))$, then $D(\partial(\psi_1 + \psi_2)) = D(\partial\psi_1) \cap D(\partial\psi_2)$, and*

$$\partial(\psi_1 + \psi_2)(x) = \partial\psi_1(x) + \partial\psi_2(x).$$

Appendix B. Young Measures and the Weak Topology

Although all the following definitions could be given in the general framework of a separable metric space, we will restrict to the setting of the separable reflexive Banach space $\mathcal{X}$, since reflexivity plays indeed a crucial role in the proof of Theorem B.1.

Notation. We denote by $\mathcal{B}(\mathcal{X})$ the Borel σ-algebra of $\mathcal{X}$, while $\mathcal{L}$ is the σ-algebra of the Lebesgue measurable subsets of $(0,T)$, and $\mathcal{L}\otimes\mathcal{B}(\mathcal{X})$ is the product σ-algebra on $(0,T)\times\mathcal{X}$. A $\mathcal{L}\otimes\mathcal{B}(\mathcal{X})$-measurable function $h:(0,T)\times\mathcal{X}\to(-\infty,+\infty]$ is a *normal integrand* if

$$v\mapsto h_t(v):=h(t,v)\quad \text{is l.s.c. on } X \text{ for a.e. } t\in(0,T). \tag{B.1}$$

We say that a $\mathcal{L}\otimes\mathcal{B}(\mathcal{X})$-measurable functional $h:(0,T)\times\mathcal{X}\to(-\infty,+\infty]$ is a *weakly normal integrand* if

$$v\mapsto h_t(v)=h(t,v)\quad \text{is weakly l.s.c. for a.e. } t\in(0,T). \tag{B.2}$$

Definition B.1. (Time-dependent parametrized measures) A *parametrized measure* in $\mathcal{X}$ is a family $\boldsymbol{\nu}:=\{\nu_t\}_{t\in(0,T)}$ of Borel probability measures on $\mathcal{X}$ such that

$$t\in(0,T)\mapsto\nu_t(B)\quad\text{is } \mathcal{L}\text{-measurable}\quad\forall\, B\in\mathcal{B}(\mathcal{X}). \tag{B.3}$$

We denote by $\mathcal{Y}(0,T;\mathcal{X})$ the set of all parametrized measures.

The following compactness result for Young measures is proved in Ref. 21 (cf. Theorem 3.2 therein), in the case in which $\mathcal{X}$ is a Hilbert space. Actually, its proof could be straightforwardly adapted to the Banach space case, and it is a direct consequence of the so-called Fundamental Theorem for Young measures, see Theorem 1 in Ref. 2.

Theorem B.1. (The fundamental theorem for weak topologies) *Let* $\{v_n\}_{n\in\mathbb{N}}$ *be a* bounded *sequence in* $L^p(0,T;\mathcal{X})$, *for some* $p>1$. *Then there exists a subsequence* $k\mapsto v_{n_k}$ *and a parametrized measure* $\boldsymbol{\nu}=\{\nu_t\}_{t\in(0,T)}\in\mathcal{Y}(0,T;\mathcal{X})$ *such that for a.e.* $t\in(0,T)$

$$\nu_t \text{ is concentrated on the set } L(t):=\bigcap_{p=1}^{\infty}\overline{\{v_{n_k}(t):k\ge p\}}^{w} \tag{B.4}$$

of the weak limit points *of* $\{v_{n_k}(t)\}$, *and*

$$\liminf_{k\to\infty}\int_0^T h(t,v_{n_k}(t))\,dt\ge\int_0^T\left(\int_{\mathcal{X}}h(t,\xi)\,d\nu_t(\xi)\right)dt \tag{B.5}$$

for every weakly normal *integrand* h *such that* $\{h^-(\cdot,v_{n_k}(\cdot))\}$ *is uniformly integrable. In particular,*

$$\int_0^T\left(\int_{\mathcal{X}}|\xi|^p\,d\nu_t(\xi)\right)dt\le\liminf_{k\to\infty}\int_0^T|v_{n_k}(t)|^p\,dt<+\infty, \tag{B.6}$$

and, setting $v(t) := \int_{\mathcal{X}} \xi \, d\nu_t(\xi)$, *we have*

$$v_{n_k} \rightharpoonup v \text{ in } L^p(0,T;\mathcal{X}) \text{ if } p < \infty. \tag{B.7}$$

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