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Analysis of a nonlinear degenerating PDE system for phase transitions in thermoviscoelastic materials

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Abstract

We address the analysis of a nonlinear and degenerating PDE system, proposed by M. Frémond for modelling phase transitions in viscoelastic materials subject to thermal effects. The system features an internal energy balance equation, governing the evolution of the absolute temperature ϑ, an evolution equation for the phase change parameter χ, and a stress–strain relation for the displacement variable $\mathbf{u}$. The main novelty of the model is that the equations for χ and $\mathbf{u}$ are coupled in such a way as to take into account the fact that the properties of the viscous and of the elastic parts influence the phase transition phenomenon in different ways. However, this brings about an elliptic degeneracy in the equation for $\mathbf{u}$ which needs to be carefully handled.

In this paper, we first prove a *local* (in time) well-posedness result for (a suitable initial–boundary value problem for) the above mentioned PDE system, in the (spatially) three-dimensional setting. Secondly, we restrict to the one-dimensional case, in which, for the same initial–boundary value problem, we indeed obtain a *global* well-posedness theorem.

MSC: 76A10; 35L80; 35A05; 35B45; 35B50; 80A22

Keywords: Nonlinear and degenerating PDE system; Local and global existence; Uniqueness; Phase transitions; Thermoviscoelastic materials

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1. Introduction

This paper is concerned with the analysis of the initial–boundary value problem for the following PDE system:

$$\vartheta_t + \chi_t \vartheta - \Delta\vartheta = g \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$\chi_t - \Delta\chi + W'(\chi) = \vartheta - \vartheta_c + \frac{|\varepsilon(\mathbf{u})|^2}{2} \quad \text{in } \Omega \times (0, T), \tag{1.2}$$

$$\mathbf{u}_{tt} - \operatorname{div}\big((1-\chi)\varepsilon(\mathbf{u}) + \chi\varepsilon(\mathbf{u}_t)\big) = \mathbf{f} \quad \text{in } \Omega \times (0, T), \tag{1.3}$$

which describes a phase transition phenomenon for a two-phase viscoelastic system, occupying a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N = 1, 2, 3$, during a time interval $[0, T]$. The state variables are the *absolute* temperature ϑ of the system (ϑ_c being the equilibrium temperature), and the order parameter χ, standing for the local proportion of one of the two phases (for example, in a melting–solidification process we shall have $\chi = 0$ in the solid phase and $\chi = 1$ in the liquid phase). The symbol $\mathbf{u}$ denotes the vector of the small displacements.

We shall specify the boundary conditions with which system (1.1)–(1.3) is supplemented in Section 2, where we detail the derivation of this PDE system following the approach proposed by M. Frémond (cf. [14] and [15]) to the modelling of phase transitions. Here, we just point out that (1.1) is the internal energy balance equation, where Δ is the Laplace operator (with respect to the space variables), and g a known heat source. Eq. (1.3), ruling the evolution of the displacement $\mathbf{u}$, is the classical balance equation for macroscopic movements (also known as *stress–strain relation*, see (2.3)–(2.5) later on), and accounts for accelerations as well. As usual, $\varepsilon(\mathbf{u})$ is the linearized symmetric strain tensor, which in the (spatially) three-dimensional case is given by $\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2$, $i, j = 1, 2, 3$ (with the commas we denote space derivatives), while the symbol div stands both for the scalar and for the vectorial divergence operator. Further, the term $\mathbf{f}$ on the right-hand side may be interpreted as an exterior volume force applied to the body. Following Frémond's perspective, (1.1) and (1.3) are coupled with the equation of microscopic movements for the phase variable χ (cf. [14, p. 5]), which, within this theory, is derived from of a particular choice of the free-energy functional and of the pseudo-potential of dissipation (cf. the following formulas (2.1) and (2.2)). In (1.2), $|\varepsilon(\mathbf{u})|^2$ is a short-hand for the colon product $\varepsilon(\mathbf{u}):\varepsilon(\mathbf{u})$, while the potential W in (1.2) is given by the sum of a smooth nonconvex function $\hat{\gamma}$ and of a convex function $\hat{\beta}$ with bounded domain. In the sequel, we shall take the domain of $\hat{\beta}$ to be contained in $[0, 1]$. Note that, in this way, the values outside $[0, 1]$ (which indeed are not physically meaningful for the order parameter χ, denoting a phase proportion) are excluded. However, in some situations our analysis could be extended to the case in which the domain of $\hat{\beta}$ is the whole half-line $[0, +\infty)$, see Remark 3.3 for further comments. Typical examples of functionals which we can include in our analysis are the logarithmic potential

$$W(r) := r\ln(r) + (1-r)\ln(1-r) - c_1 r^2 - c_2 r - c_3 \quad \forall r \in (0, 1), \tag{1.4}$$

where c_1 and c_2 are positive constants, as well as the double obstacle potential, given by the sum of the indicator function $I_{[0,1]}$ with a nonconvex $\hat{\gamma}$.

The main mathematical difficulties encountered in the study of system (1.1)–(1.3) are related to the degenerating character of Eq. (1.3) and to the nonlinear features of Eqs. (1.1)–(1.2). The

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former is due to the presence of the terms $(1-\chi)$ and χ in front of the elasticity and viscosity contributions, respectively; in fact, such terms vanish as $\chi \nearrow 1$ and $\chi \searrow 0$, making the related elliptic operator degenerate. Moreover, the term $W'(\chi)$ and the quadratic terms $1/2|\varepsilon(\mathbf{u})|^2$ and $\chi_t\vartheta$ occurring in (1.1)–(1.2) give a strongly nonlinear character to the system, so that its analysis turns out to be nontrivial.

In fact, in the case $\Omega \subset \mathbb{R}^3$ we shall only prove a *local well-posedness* result for system (1.1)–(1.3), supplemented with suitable initial and boundary conditions (which shall be specified in Section 2 later on). We shall also analyze (1.1)–(1.3) in the (spatially) one-dimensional case, i.e. when Ω is a bounded interval of $\mathbb{R}$. In such a framework, we shall obtain a *global well-posedness* result for the same initial–boundary value problem.

Before entering into details, let us briefly review some related literature. First of all, we aim at pointing out that most of the models for phase transition phenomena do not feature a balance equation for macroscopic movements. In this regard, starting from the pioneering paper [9], there is a comprehensive literature on the models of phase change with microscopic movements proposed by Frémond (we refer to the PhD thesis [34] and the references therein). In particular, the system coupling (1.1) with (1.2) is usually derived within Frémond's approach by choosing a free energy functional Ψ and a pseudo-potential of dissipation Φ different from our own. In fact, the models arising from the "traditional" choices for Ψ and Φ do not take into account the different properties of the viscous and elastic parts of the system (for example, a viscous liquid portion coexisting with an elastic solid portion in the case of a solid–liquid system undergoing a melting–solidification process). Therefore, there is no coupling between system (1.1)–(1.2) and the equation for macroscopic movements, which is thus neglected. Nonetheless, in the case in which $\mathbf{u}$ is fixed, (1.1)–(1.2) still needs to be carefully handled, mainly because of the term $\chi_t\vartheta$ in (1.1). As a matter of fact, to our knowledge, no global-in-time well-posedness result has yet been obtained for the initial–boundary value problem related to (1.1)–(1.2) in the three-dimensional case. Instead, a global existence result has been proved for (a generalization of) (1.1)–(1.2) in the one-dimensional case in [25,26].

On the contrary, in our modelling viewpoint the coexisting viscous and elastic properties of the system are given a distinguished role, under the working assumption that they indeed influence the phase transition process. This reflects in the analytical expression of the functionals Ψ and Φ (see (2.1) and (2.2) later on), as a result of which the term $1/2|\varepsilon(\mathbf{u})|^2$ appears in the equation for the phase parameter on the one hand. On the other hand, the χ-dependence in the stress–strain relation leads to the aforementioned degeneracy of the elliptic operator therein. We believe that the latter is the most peculiar feature of system (1.1)–(1.3) in comparison with other models (both for phase transition and for damaging phenomena), which take into account visco-elastic effects.

There is by now a rich literature on this kind of models. Among others, we would like to cite the papers [3–6,8,16,23]. The analysis of a thermoviscoelastic system not subject to a phase transition has been tackled in [3,4], in which a linear viscoelastic equation for the displacement $\mathbf{u}$ and an internal energy balance equation for ϑ are considered. The latter parabolic equation has a quadratic contribution in $\varepsilon(\mathbf{u}_t)$ on the right-hand side. Due to the highly nonlinear character of the system, only a local well-posedness result (proved in [3]) is available in the three-dimensional case, while in [4] an asymptotic analysis is performed. However, in this framework no degeneracy of the elliptic operator in the equation for $\mathbf{u}$ is allowed.

The papers [5,6,8,23] instead address models for damaging phenomena. In this case, the phase variable χ is related to the local proportion of damaged material. Hence χ is forced to take values in $[0,1]$, with the convention that $\chi = 0$ when the body is completely damaged, and $\chi = 1$ in the damage-free case. While [23] focuses on the quasistatic *reversible* evolution of damage in visco-plastic materials, in [5,6,8], the irreversibility of the damaging process is rendered by forcing the microscopic velocity χ_t to take nonpositive values. Thus, the equation for the order parameter (which is otherwise analogous to (1.2), with the quadratic term $|\varepsilon(\mathbf{u})|^2/2$ on the right-hand side) features an additional nonsmooth nonlinearity (namely, the subdifferential of the indicator function of $(-\infty, 0]$ acting on χ_t), and has a *doubly nonlinear character*. On the other hand, in [6,8] the equation for macroscopic movements displays a different degeneracy in the elliptic operators. Indeed, their coefficients vanish only as $\chi \searrow 0$, contrary to the twofold degeneracy of Eq. (1.3). Within this framework, in [6,8] *local* well-posedness results are proved for the resulting PDE system. Furthermore, damage in thermoviscoelastic systems is analyzed in [5]. Therein, a strongly nonlinear equation for the absolute temperature ϑ is coupled with a *nondegenerate* stress–strain relation for $\mathbf{u}$ and with a doubly nonlinear equation for χ. Again because of its several nonlinearities, only a local existence and uniqueness theorem has been obtained for this system.

Finally, we would like to mention the paper [16], concerned with the analysis of a PDE system describing phase change models with the possibility of voids. For example, in solid–liquid phase transitions voids can be bubbles in ice, which may have been forming during the water freezing. In this case, if χ_1 represents the local proportion of the liquid phase and χ_2 the local proportion of the solid phase, we do not have the relation $\chi_1 + \chi_2 = 1$ anymore, but only $\chi_1 + \chi_2 \in [0, 1]$. Hence, we cannot get rid of one of the two variables and we must deal with a vectorial phase equation instead of (1.2). Such equation, the internal energy balance, and the (*nondegenerate*) displacement equation have to be further coupled with a mass balance equation, which can instead be neglected in case of phase transitions without voids.

Let us now briefly describe our own results. As already mentioned, in the three-dimensional case we are going to obtain a local well-posedness result (Theorem 1), in a suitable functional framework, for (the initial–boundary value problem related to) a generalization of system (1.1)–(1.3) (see (3.18)–(3.20)). To this aim, we shall require the initial value χ_0 of the phase parameter to be separated from the potential barriers, namely

$$\min_{x\in\overline{\Omega}} \chi_0(x) > 0, \tag{1.5}$$

$$\max_{x\in\overline{\Omega}} \chi_0(x) < 1. \tag{1.6}$$

Note that a stronger form of (1.5) was also assumed in [6,8], in which the authors confined their results to the case $\chi_0 \equiv 1$. In fact, arguing as in [6,8] we shall exploit (1.5)–(1.6) to ensure that the solution component χ *locally* stays away from both the potential barriers, i.e. there exists $\sigma \in (0, 1)$ such that

$$\chi \geqslant \sigma > 0 \quad \text{in } \Omega \times [0, \overline{T}], \tag{1.7}$$

$$\chi \leqslant 1 - \sigma < 1 \quad \text{in } \Omega \times [0, \overline{T}], \tag{1.8}$$

for some $\overline{T} < T$. Therefore, during that time interval neither the coefficient of $\varepsilon(\mathbf{u}_t)$ nor the one of $\varepsilon(\mathbf{u})$ degenerates, and system (1.1)–(1.3) is (locally) well-posed. However, since (1.7) and (1.8) are only *local in time*, it is not possible to extend the local solution to a global one. In this connection, we may observe that, in principle, one would expect that (1.7) is sufficient to ensure the local well-posedness of (1.1)–(1.3). For, the sole (1.7) entails that the main part of the elliptic operator in (1.3) does not degenerate. In this perspective, only condition (1.5) would be needed. Indeed, the other separation condition (1.6) (implying (1.8)) apparently has a purely technical role. Together with the assumption that the convex part $\hat{\beta}$ of the potential W be locally Lipschitz on $(0, 1)$ (which is true in the case of the logarithmic potential (1.4) and for the indicator function), (1.8) is exploited to obtain enhanced regularity estimates on χ. It turns out that we could dispense with (1.6) by slightly strengthening our assumptions on W (cf. Remark 3.3).

In the one-dimensional case, instead, with the sole *one-sided* condition (1.5) we shall obtain a *global* well-posedness theorem for (a generalization of) (1.1)–(1.3), see Theorem 2. Indeed, in this simplified framework we shall deduce from (1.5) the separation inequality

$$\chi \geqslant \delta > 0 \quad \text{in } \Omega \times [0, T] \tag{1.9}$$

(for some constant $\delta > 0$), in a *global* form. This rules out the degeneracy of (1.3) *globally in time*. The crucial (1.9) shall be proved by combining (1.5) with the requirement that

$$\lim_{x \to 0^+} W'(x) = -\infty. \tag{1.10}$$

In fact, in this framework we can allow for a more general $\hat{\beta}$, leading to the presence of a truly multivalued nonlinearity in (1.2), and (1.10) can be accordingly formulated in this possibly multivalued case, see (3.35) later on. Relying on (1.5) and on (1.10), we shall prove (1.9) with a maximum principle argument (cf. Lemma 5.6). We refer the reader to Section 5.1, where the proof of Theorem 2 is outlined with some detail. There, we also illustrate the reason why the argument leading to global existence in one dimension does not carry over to the three-dimensional case.

Let us point out that *global* separation inequalities of the same kind as (1.9) have been obtained with a similar comparison technique in [27], in the framework of the analysis of the viscous Cahn–Hilliard equation (see, e.g., [13,20,30]) with the logarithmic potential (1.4). We may also recall [21], where a separation inequality for the order parameter in the Penrose–Fife phase transition model (cf. [31]), is proved by means of a Moser iteration argument. However, the implementation of these techniques could not be reproduced for the systems analyzed in [6,8], essentially due to the doubly nonlinear character of the equation for the damage parameter χ (more precisely, due to the presence of a subdifferential operator acting on χ_t). Finally, we mention that such separation inequalities play a key role in the study of the convergence to equilibrium, for large times, of some phase transition systems with singular potentials in the papers [18,19], where Łojasiewicz–Simon techniques are used.

In this regard, in the future it would be interesting to study the long-time behavior of system (1.1)–(1.3) in the one-dimensional case. To our knowledge, such a problem is indeed open, both in the direction of determining global attractors for bundles of trajectories, and of characterizing the ω-limit sets of single trajectories. Further, we think that it would be worthwhile to address the local well-posedness of the full system for phase transitions with the possibility of voids, like in [16].

Plan of the paper. In Section 2 we detail the derivation of (a generalization of) Eqs. (1.1)–(1.3). In Section 3, we introduce our notation and the main results of this paper: the local well-posedness in three dimensions for a suitable variational formulation of the initial–boundary value problem related to system (1.1)–(1.3), whose proof is performed in Section 4, and the global well-posedness in one dimension, proved in Section 5.

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2. The model

This section is devoted to the introduction of the model: in particular, we shall review the derivation developed in [15, Chapter 4] and show how system (1.1)–(1.3) follows from the generalized Principle of Virtual Power introduced by Frémond (cf. [14]).

Free energy and pseudo-potential of dissipation. The thermomechanical equilibrium of the system is described by its free energy functional Ψ, which depends on the state variables, namely the absolute temperature ϑ, the order parameter χ (i.e. the local proportion of the viscous phase), its gradient $\nabla\chi$, and the linearized symmetric strain tensor $\varepsilon(\mathbf{u})$. Then, we take the following expression (cf. [14]) for the volumetric free energy Ψ:

$$\Psi\big(\vartheta, \varepsilon(\mathbf{u}), \chi, \nabla\chi\big) = c_V\vartheta(1-\log\vartheta) - \frac{\lambda}{\vartheta_c}(\vartheta-\vartheta_c)\chi + \frac{(1-\chi)\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} + W(\chi) + \frac{\nu}{2}|\nabla\chi|^2, \tag{2.1}$$

where $(1-\chi)$ represents the local proportion of the nonviscous phase and $\mathcal{R}_e$ is a symmetric positive definite elasticity tensor. The constants c_V, ϑ_c, and $\nu > 0$ are, respectively, the specific heat, the equilibrium temperature, and the interfacial energy coefficient, while λ stands for the latent heat of the system. Moreover, the term $W(\chi) + (\nu/2)|\nabla\chi|^2$ is a *mixture or interaction free-energy*. Hereafter, for simplicity we shall set $c_V = \nu = \lambda/\vartheta_c = 1$ and we shall incorporate the term $\vartheta_c\chi$ in $W(\chi)$. Indeed, since in the following the potential W shall be given by the sum of a convex (possibly nonsmooth) part and of a smooth (possibly nonconvex) function (cf. formula (3.14)), we may suppose without loss of generality that $\vartheta_c\chi$ contributes to the latter. We include dissipation in the model by following Moreau's approach (cf. [14] and references therein) through a pseudo-potential of dissipation Φ depending on the dissipative variables $\nabla\vartheta$, χ_t, $\varepsilon(\mathbf{u}_t)$. In particular, we take

$$\Phi\big(\nabla\vartheta, \chi_t, \varepsilon(\mathbf{u}_t)\big) = \frac{1}{2}|\chi_t|^2 + \frac{\chi}{2}\varepsilon(\mathbf{u}_t)\mathcal{R}_v\varepsilon(\mathbf{u}_t) + \frac{|\nabla\vartheta|^2}{2\vartheta}, \tag{2.2}$$

where for the sake of simplicity all physical parameters have been set equal to 1 and $\mathcal{R}_v$ is a symmetric and positive definite viscosity matrix.

Remark 2.1. Let us point out that the definition of Ψ and Φ yields that, e.g. in melting phenomena, when the system is in the solid phase (i.e. $\chi = 0$) viscous effects are not present in the model, while when the system is in the liquid phase (i.e. $\chi = 1$) we do not have elasticity effects. In the intermediate case, the model takes into account the influence of both effects, which is the main novelty of this approach to phase transitions. Indeed, we could include more general functions $a(\chi)$ and $b(\chi)$ in formulas (2.1) and (2.2), respectively, with a and b sufficiently regular functions such that $a(\chi) + b(\chi) = 1$ for all $\chi \in (0,1)$, $a(\chi) \to 0$ for $\chi \nearrow 1$, $a(\chi) \to 1$ for $\chi \searrow 0$, and, conversely, $b(\chi) \to 1$ for $\chi \nearrow 1$, $b(\chi) \to 0$ for $\chi \searrow 0$. Nonetheless, for simplicity we shall confine our analysis to the meaningful case in which $a(\chi) = 1-\chi$ and $b(\chi) = \chi$.

The equation for the macroscopic motions. The equation of macroscopic motion (1.3) is provided by the principle of virtual power (cf. [14]) and can be written as

$$\mathbf{u}_{tt} - \operatorname{div}\sigma = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{2.3}$$

where $\mathbf{f}$ stands for the exterior volume force and σ is the stress tensor. On account of (2.1), (2.2), and of the well-known constitutive law

$$\sigma = \sigma^{\mathrm{nd}} + \sigma^{\mathrm{d}} = \frac{\partial \Psi}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \Phi}{\partial \varepsilon(\mathbf{u}_t)}, \tag{2.4}$$

the tensor σ can be written as

$$\sigma = (1-\chi)\mathcal{R}_e\varepsilon(\mathbf{u}) + \chi\mathcal{R}_v\varepsilon(\mathbf{u}_t) \quad \text{in } \Omega \times (0,T). \tag{2.5}$$

Then, the equilibrium equation (2.3) turns out to be exactly

$$\mathbf{u}_{tt} - \operatorname{div}\big((1-\chi)\mathcal{R}_e\varepsilon(\mathbf{u}) + \chi\mathcal{R}_v\varepsilon(\mathbf{u}_t)\big) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{2.6}$$

which is in fact a generalization of (1.3). We shall supplement (2.6) with a nondisplacement prescription on the boundary of Ω

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T), \tag{2.7}$$

yielding a *pure displacement* boundary value problem for $\mathbf{u}$, according to the terminology of [11]. However, our analysis carries over to other kinds of boundary conditions on $\mathbf{u}$, like Neumann conditions (*pure traction* problem) or mixed Dirichlet–Neumann conditions (*displacement–traction* problem), see Remark 3.4.

The equation for the microscopic motions. The evolution of the phase variable χ is ruled by an equation related to the microscopic motions which occur in the phase change. This equation as well is derived from a generalization of the principle of virtual power (cf. [14]). Let B (a density of energy function) and $\mathbf{H}$ (an energy flux vector) represent the internal microscopic forces responsible for the mechanically induced heat sources, and let us denote by B^{d} and $\mathbf{H}^{\mathrm{d}}$ their dissipative parts, and by B^{nd} and $\mathbf{H}^{\mathrm{nd}}$ their nondissipative parts. Thus, using (2.1)–(2.2) and deducing from (2.1) that the nondissipative part of the entropy flux $\mathbf{Q}^{\mathrm{nd}}$ is zero, the standard constitutive relations yield

$$B = B^{\mathrm{nd}} + B^{\mathrm{d}} = \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t} = -\vartheta - \frac{\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} + W'(\chi) + \chi_t, \tag{2.8}$$

$$\mathbf{H} = \mathbf{H}^{\mathrm{nd}} + \mathbf{H}^{\mathrm{d}} = \mathbf{H}^{\mathrm{nd}} = \frac{\partial \Psi}{\partial \nabla\chi} = \nabla\chi. \tag{2.9}$$

Then, if the volume amount of mechanical energy provided to the domain by the external actions (which do not involve macroscopic motions) is zero, the equation for the microscopic motions can be written as

$$B - \operatorname{div}\mathbf{H} = 0 \quad \text{in } \Omega \times (0,T). \tag{2.10}$$

With trivial computations, from (2.8)–(2.10) we derive

$$\chi_t - \Delta\chi + W'(\chi) = \vartheta + \frac{\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} \quad \text{in } \Omega \times (0,T), \tag{2.11}$$

generalizing (1.2). Moreover, if the surface amount of mechanical energy provided by the external local surface actions (not involving macroscopic motions) is zero as well, then the natural boundary condition for this equation of motion is

$$\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T),$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. Thus, we obtain the homogeneous Neumann boundary condition on χ

$$\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{2.12}$$

The internal energy balance. Finally, the energy balance equation reads

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H} \cdot \nabla\chi_t \quad \text{in } \Omega \times (0,T), \tag{2.13}$$

where e, the (specific) internal energy, is linked to the free energy Ψ by the standard Helmholtz relation

$$e = \Psi + \vartheta s, \quad s = -\frac{\partial\Psi}{\partial\vartheta}, \tag{2.14}$$

in which we have denoted by s the entropy of the system. On the right-hand side of (2.13) there appears the heat source g and the mechanically induced heat sources, related to macroscopic and microscopic stresses. The heat flux $\mathbf{q}$ is defined by the following constitutive relation:

$$\mathbf{q} = -\vartheta\frac{\partial\Phi}{\partial\nabla\vartheta} = -\nabla\vartheta. \tag{2.15}$$

We couple (2.13) with a no-flux boundary condition on $\partial\Omega$

$$\mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{2.16}$$

Using (2.14), the definition (2.1) of Ψ, and the constitutive laws (2.4), (2.8), (2.9), and (2.15), we end up with the internal energy balance:

$$\vartheta_t + \chi_t\vartheta - \Delta\vartheta = g + \chi\varepsilon(\mathbf{u}_t)\mathcal{R}_v\varepsilon(\mathbf{u}_t) + |\chi_t|^2 \quad \text{in } \Omega \times (0,T). \tag{2.17}$$

Thermodynamical consistency. Let us now point out that our model complies with the Second Principle of Thermodynamics: in fact, the following form of the Clausius–Duhem inequality

$$s_t + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) - \frac{g}{\vartheta} \geqslant 0 \tag{2.18}$$

holds true. To check (2.18), it is sufficient to note that the internal energy balance (2.17) can be expressed in terms of the entropy s in this way:

$$\vartheta\left(s_t + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) - \frac{g}{\vartheta}\right) = \sigma^{\mathrm{d}} : \varepsilon(\mathbf{u}_t) + B^{\mathrm{d}}\chi_t - \frac{\mathbf{q}}{\vartheta} \cdot \nabla\vartheta, \tag{2.19}$$

σ^{d} being the dissipative part of the stress tensor σ. Note that the right-hand side of (2.19) turns out to be nonnegative because $(\sigma^{\mathrm{d}}, B^{\mathrm{d}}, -\mathbf{q}/\vartheta) \in \partial\Phi(\mathbf{u}_t, \chi_t, \nabla\vartheta)$, and Φ is convex in all of its variables. Therefore, (2.18) ensues from the positivity of ϑ.

Now, using the small perturbation assumption (cf. [17]), we may suppose that the dissipative heat sources in the energy balance are small with respect to the external heating g. Thus, we can neglect the higher order dissipative terms on the right-hand side of (2.17). In this way, we derive Eq. (1.1) for the absolute temperature ϑ. Moreover, the no-flux boundary condition (2.16) leads to a homogeneous Neumann boundary condition on ϑ:

$$\partial_{\mathbf{n}}\vartheta = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{2.20}$$

3. Main results

In the following, we shall provide a variational framework for studying system (1.1), (2.6), (2.11), supplemented with the boundary conditions (2.7), (2.12), (2.20), and with the following initial conditions

$$\vartheta(0) = \vartheta_0 \quad \text{in } \Omega, \tag{3.1}$$

$$\chi(0) = \chi_0 \quad \text{in } \Omega, \tag{3.2}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \mathbf{u}_t(0) = \mathbf{v}_0 \quad \text{in } \Omega, \tag{3.3}$$

where ϑ_0, χ_0, $\mathbf{u}_0$, and $\mathbf{v}_0$ are suitable known initial data for the problem.

3.1. Notation

Throughout the paper, given a Banach space X, we shall denote by $\|\cdot\|_X$ its norm and by $\mathrm{C}^0_{\mathrm{w}}([0,T];X)$ the space of weakly continuous functions with values in X; further, we shall use the symbol X for the space X^3 as well.

Hereafter, we shall suppose that $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, is a bounded connected domain, with Lipschitz continuous boundary $\partial\Omega$. We shall denote by $H^1_0(\Omega)$, $H^2_0(\Omega)$, and $H^2_N(\Omega)$ the following spaces

$$H^1_0(\Omega) := \left\{v \in H^1(\Omega):\ v = 0 \text{ on } \partial\Omega\right\}, \qquad H^2_0(\Omega) := \left\{v \in H^2(\Omega):\ v = 0 \text{ on } \partial\Omega\right\},$$
$$H^2_N(\Omega) := \left\{v \in H^2(\Omega):\ \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega\right\},$$

endowed with the norms of $H^1(\Omega)$ and $H^2(\Omega)$, respectively. Furthermore, we identify $L^2(\Omega)$ with its dual space $L^2(\Omega)'$, so that $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)'$ with dense and continuous embeddings. We shall use the symbols $|\cdot|$ and $(\cdot,\cdot)$ for the norm and scalar product on $L^2(\Omega)$, while $\langle\cdot,\cdot\rangle$ shall stand both for the duality pairing between $H^1(\Omega)'$ and $H^1(\Omega)$ and for the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Preliminaries of mathematical elasticity. As usual in elasticity theory, we assume the material to be homogeneous and isotropic, so that the elasticity matrix $\mathcal{R}_e$ in relation (2.1) may be represented by

$$\mathcal{R}_e\varepsilon(\mathbf{u}) = \lambda\,\mathrm{tr}\big(\varepsilon(\mathbf{u})\big)\mathbf{1} + 2\mu\varepsilon(\mathbf{u}),$$

where $\lambda, \mu > 0$ are the so-called Lamé constants and $\mathbf{1}$ is the identity matrix. In order to state the variational formulation of the initial–boundary value problem for (1.1), (2.6), (2.11), we need to introduce the bilinear forms related to the χ-dependent elliptic operators appearing in (2.6). Hence, let $\eta : \Omega \to [0,1]$ be a measurable function and let us consider the continuous bilinear symmetric forms $a_\eta, b_\eta : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a_\eta(\mathbf{u},\mathbf{v}) := \lambda \int_\Omega \eta\,\mathrm{div}(\mathbf{u})\,\mathrm{div}(\mathbf{v}) + 2\mu \sum_{i,j=1}^{3} \int_\Omega \eta\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega),$$

$$b_\eta(\mathbf{u},\mathbf{v}) := \sum_{i,j=1}^{3} \int_\Omega \eta b_{ij}\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega), \tag{3.4}$$

where (b_{ij}) is the viscosity matrix $\mathcal{R}_v$, cf. (2.2). Just for the sake of simplicity but without loss of generality, we let $b_{ij} = 1$ for $i, j = 1, 2, 3$. On the other hand, as $\eta(x) \leqslant 1$ for all $x \in \Omega$, there exists some positive constant K_a, only depending on λ and μ, such that

$$\big|a_\eta(\mathbf{u},\mathbf{v})\big| \leqslant K_a \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)} \quad \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega). \tag{3.5}$$

Furthermore, by Korn's inequality (see e.g. [11, Theorem 6.3-3]), the forms $a_\eta(\cdot,\cdot)$ and $b_\eta(\cdot,\cdot)$ are $H_0^1(\Omega)$-elliptic, i.e., there exist $C_a, C_b > 0$ such that for all $\mathbf{u} \in H_0^1(\Omega)$

$$a_\eta(\mathbf{u},\mathbf{u}) \geqslant \inf_{x\in\Omega}\big(\eta(x)\big) C_a \|\mathbf{u}\|^2_{H^1(\Omega)}, \tag{3.6}$$

$$b_\eta(\mathbf{u},\mathbf{u}) \geqslant \inf_{x\in\Omega}\big(\eta(x)\big) C_b \|\mathbf{u}\|^2_{H^1(\Omega)}. \tag{3.7}$$

We shall also need the following elliptic regularity result (see e.g. [28, p. 260]): there exist constants $C_\gamma, C_\delta > 0$ such that

$$C_\gamma \|\mathbf{v}\|_{H^2(\Omega)} \leqslant \big|\mathrm{div}\big(\varepsilon(\mathbf{v})\big)\big| \leqslant C_\delta \|\mathbf{v}\|_{H^2(\Omega)} \quad \forall \mathbf{v} \in H_0^2(\Omega). \tag{3.8}$$

We denote by $\mathcal{H}(\eta\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $\mathcal{K}(\eta\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$ the operators associated with a_η and b_η, respectively, namely

$$\big\langle \mathcal{H}(\eta\mathbf{v}), \mathbf{w}\big\rangle = a_\eta(\mathbf{v},\mathbf{w}) \quad \text{and} \quad \big\langle \mathcal{K}(\eta\mathbf{v}), \mathbf{w}\big\rangle = b_\eta(\mathbf{v},\mathbf{w}) \quad \forall \mathbf{v},\mathbf{w} \in H_0^1(\Omega).$$

It can be checked via an approximation argument that the following regularity result holds:

$$\text{if } \eta \in H^2(\Omega) \text{ and } \mathbf{v} \in H_0^2(\Omega), \quad \text{then } \mathcal{H}(\eta\mathbf{v}), \mathcal{K}(\eta\mathbf{v}) \in L^2(\Omega). \tag{3.9}$$

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Further, we introduce the operator $A : H^1(\Omega) \to H^1(\Omega)'$ realizing the Laplace operator $-\Delta$ with homogeneous Neumann boundary conditions, defined by

$$\langle Au, v\rangle := (\nabla u, \nabla v) \quad \forall u, v \in H^1(\Omega).$$

We denote by J the duality operator $A + I : H^1(\Omega) \to H^1(\Omega)'$ (I being the identity operator); in the sequel, we shall make use of the relations

$$\langle Ju, u\rangle = \|u\|^2_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \qquad \left\langle J^{-1}v, v\right\rangle = \|v\|^2_{H^1(\Omega)'} \quad \forall v \in H^1(\Omega)'. \tag{3.10}$$

Finally, we recall here the celebrated Gagliardo–Nirenberg inequality (cf. [29, p. 125]) in a simplified form: for all $q, r \in [1, +\infty]$, and for all $v \in L^q(\Omega)$ such that $\nabla v \in L^r(\Omega)$, there holds

$$\|v\|_{L^p(\Omega)} \leqslant C\|v\|^{\theta}_{W^{1,r}(\Omega)}\|v\|^{1-\theta}_{L^q(\Omega)}, \tag{3.11}$$

with

$$\frac{1}{p} = \theta\left(\frac{1}{r} - \frac{1}{N}\right) + (1-\theta)\frac{1}{q}, \quad 0 \leqslant \theta \leqslant 1,$$

the positive constant C depending only on N, q, r, θ. We shall also make use of the continuous Sobolev embeddings

$$\text{if } \Omega \subset \mathbb{R}^3, \quad H^1(\Omega) \subset L^p(\Omega), \quad H^2(\Omega) \subset W^{1,p}(\Omega) \quad \text{for } 1 \leqslant p \leqslant 6, \tag{3.12}$$

$$\text{if } \Omega = (0,\ell) \subset \mathbb{R}, \quad H^1(0,\ell) \subset L^\infty(0,\ell), \quad H^2(0,\ell) \subset W^{1,\infty}(0,\ell). \tag{3.13}$$

3.2. Local existence and uniqueness results

In the sequel, we shall assume that the potential W in (1.2) is given by

$$W = \hat{\beta} + \hat{\gamma}, \tag{3.14}$$

where

$$\hat{\gamma} \in \mathrm{C}^2\big([0,1]\big) \quad \text{and} \tag{3.15}$$

$$\overline{\mathrm{dom}(\hat{\beta})} = [0,1], \quad \hat{\beta} : \mathrm{dom}(\hat{\beta}) \to \mathbb{R} \text{ is proper, l.s.c., convex,} \tag{3.16}$$

$$\hat{\beta} \in \mathrm{C}^{1,1}_{\mathrm{loc}}(0,1). \tag{3.17}$$

We point out that the latter requirement means that for all $0 < \rho < 1$, β is Lipschitz continuous on the interval $[\rho, 1-\rho]$.

Remark 3.1. For example, both the logarithmic function $\hat{\beta}(r) = r\ln(r) + (1-r)\ln(1-r)$, for $r \in (0,1)$ (cf. (1.4)), and the indicator function $\hat{\beta} = I_{[0,1]}$ of the interval $[0,1]$ fulfil (3.16)–(3.17), the latter yielding that the maximal monotone operator $\partial\hat{\beta}$ is single-valued and locally Lipschitz continuous on $(0,1)$. Now, the solution component χ we are going to find (see Theorem 1) shall

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take values in (0, 1). Hence, for the sake of simplicity, throughout this section and Section 4 we shall treat both $\partial\hat{\beta}$ and ∂W as functions β and W'. We also set $\gamma := \hat{\gamma}'$, so that (3.14) yields

$$W' = \beta + \gamma.$$

We can now detail the initial–boundary value problem we are interested in.

Problem 1. Find functions $\vartheta, \chi : \Omega \times [0, T] \to \mathbb{R}$ and $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^3$ fulfilling the initial conditions (3.1)–(3.3),

$$\chi(x,t) \in \mathrm{dom}(\beta) \quad \text{and} \quad \vartheta(x,t) > 0 \quad \text{for a.e. } (x,t) \in \Omega \times (0,T),$$

and the equations

$$\vartheta_t + \chi_t \vartheta + A\vartheta = g \quad \text{a.e. in } \Omega \times (0,T), \tag{3.18}$$

$$\chi_t + A\chi + \beta(\chi) + \gamma(\chi) = \vartheta + \frac{\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} \quad \text{a.e. in } \Omega \times (0,T), \tag{3.19}$$

$$\mathbf{u}_{tt} + \mathcal{H}\big((1-\chi)\mathbf{u}\big) + \mathcal{K}(\chi\mathbf{u}_t) = \mathbf{f} \quad \text{a.e. in } \Omega \times (0,T). \tag{3.20}$$

Our assumptions on the data are

$$g \in H^1\big(0,T;L^2(\Omega)\big), \qquad g(x,t) \geqslant 0 \quad \text{for a.e. } (x,t) \in \Omega \times (0,T), \tag{3.21}$$

$$\mathbf{f} \in L^2\big(0,T;L^2(\Omega)\big), \tag{3.22}$$

$$\vartheta_0 \in H^2_N(\Omega) \quad \text{and} \quad \min_{x\in\overline{\Omega}} \vartheta_0(x) > 0, \tag{3.23}$$

$$\chi_0 \in H^2_N(\Omega), \tag{3.24}$$

$$\mathbf{u}_0 \in H^2_0(\Omega), \qquad \mathbf{v}_0 \in H^1_0(\Omega). \tag{3.25}$$

Finally, we shall assume that the initial datum χ_0 is “separated from the potential barriers”:

$$\min_{x\in\overline{\Omega}} \chi_0(x) > 0, \tag{3.26a}$$

$$\max_{x\in\overline{\Omega}} \chi_0(x) < 1. \tag{3.26b}$$

It follows from (3.16)–(3.17) and from (3.26a)–(3.26b) that

$$\hat{\beta}(\chi_0), \beta(\chi_0) \in L^\infty(\Omega). \tag{3.27}$$

Theorem 1. *Assume* (3.15)–(3.17) *and* (3.21)–(3.26b). *Then, there exist* $\widehat{T} \in (0, T]$, *a constant* $\varsigma > 0$, *and a unique triplet* $(\vartheta, \chi, \mathbf{u})$ *with the regularity*

$$\vartheta \in H^2\big(0,\widehat{T};H^1(\Omega)'\big)\cap W^{1,\infty}\big(0,\widehat{T};L^2(\Omega)\big)\cap H^1\big(0,\widehat{T};H^1(\Omega)\big)\cap L^\infty\big(0,\widehat{T};H^2_N(\Omega)\big), \tag{3.28}$$

$$\chi \in H^2\big(0,\widehat{T};H^1(\Omega)'\big)\cap W^{1,\infty}\big(0,\widehat{T};L^2(\Omega)\big)\cap H^1\big(0,\widehat{T};H^1(\Omega)\big)\cap L^\infty\big(0,\widehat{T};H^2_N(\Omega)\big), \tag{3.29}$$

$$\mathbf{u} \in H^2\big(0,\widehat{T};L^2(\Omega)\big)\cap W^{1,\infty}\big(0,\widehat{T};H^1_0(\Omega)\big)\cap H^1\big(0,\widehat{T};H^2_0(\Omega)\big), \tag{3.30}$$

solving Problem 1 *on the interval* $(0,\widehat{T})$*, and fulfilling the inequalities*

$$\min_{x\in\overline{\Omega}} \vartheta(x,t) > 0 \quad \forall t\in[0,\widehat{T}], \tag{3.31}$$

$$0<\varsigma\leqslant\chi(x,t)\leqslant 1-\varsigma<1 \quad \forall (x,t)\in\Omega\times(0,\widehat{T}). \tag{3.32}$$

Note that, by [1, Section 1.3] and [35, Lemma III.1.4], (3.28)–(3.30) yield the further regularity $\vartheta,\chi\in \mathrm{C}^1([0,\widehat{T}];L^2(\Omega))\cap \mathrm{C}^0_{\mathrm{w}}([0,\widehat{T}];H^2_N(\Omega))$ and $\mathbf{u}\in \mathrm{C}^1([0,\widehat{T}];H^1_0(\Omega))$. Furthermore, thanks to (3.9) and (3.29)–(3.30), one has that $\mathcal{H}((1-\chi)\mathbf{u}),\mathcal{K}(\chi\mathbf{u}_t)\in L^2(\Omega)$, in agreement with the requirement that (3.20) holds a.e. in $\Omega\times(0,\widehat{T})$.

Remark 3.2. In fact, it is also possible to prove (see Proposition 4.8) that the above local solution triplet $(\vartheta,\chi,\mathbf{u})$ depends continuously on the data g, $\mathbf{f}$ and on the initial data of the problem. We shall discuss the dependence of the life-time $\widehat{T}$ on the initial datum χ_0 in Remark 4.6 later on.

Remark 3.3. As we already mentioned in the Introduction, separation condition (3.26b) only has a technical role. In fact, it is functional to proving that the component χ of the solution is separated from *both* the potential barriers, and this is necessary in order to perform further regularity estimates on Eqs. (3.18)–(3.19). The latter are needed to carry out the Schauder fixed point procedure with which we shall prove Theorem 1.

Indeed, it would be possible to dispense with (3.26b) by strengthening (3.17), namely requiring that

$$\text{for all } \rho>0 \quad \beta \text{ is a Lipschitz continuous function on } [\rho,1]. \tag{3.33}$$

Under the latter conditions, both the existence and the uniqueness statements of Theorem 1 can be proved assuming the sole separation condition (3.26a) on the initial datum χ_0: roughly speaking, the idea is that (3.33) is tailored to the situation in which the component χ is only separated from 0. Furthermore, in this framework it would not be necessary any longer to require $\hat{\beta}$ to have a bounded domain. We refer to Remarks 4.7 and 4.9 for further details.

Remark 3.4. The proof of our local well-posedness Theorem 1 could be carried out with suitable modifications in the case of Neumann boundary conditions on $\mathbf{u}$, as well. We would also be able to handle the case of Neumann conditions on a portion Γ_0 of $\partial\Omega$ and Dirichlet conditions on $\Gamma_1:=\partial\Omega\setminus\Gamma_0$ ($|\Gamma_0|,|\Gamma_1|>0$), provided that the closures of the sets Γ_0 and Γ_1 do not intersect. Indeed, without the latter geometric condition, the elliptic regularity results ensuring the (crucial) $H^2_0(\Omega)$-regularity of $\mathbf{u}$ may fail to hold, see [11, Chapter VI, Section 6.3].

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3.3. A global well-posedness result in the one-dimensional case

Throughout this section, we take the domain Ω to be a bounded interval of $\mathbb{R}$ and, for simplicity, we set

$$\Omega = (0, \ell), \quad \text{for some } \ell > 0. \tag{3.34}$$

In this framework, the existence of a global solution to the initial–boundary value problem for system (1.1), (2.6), (2.11) can be proved in the case in which $\beta = \partial\hat{\beta}$ is a general multivalued operator, whereas we obtain uniqueness only in the case in which β is single-valued. Indeed, the basic assumptions on the potential W are (3.14)–(3.16), and the following "coercivity" condition

$$\lim_{x\to 0^+} \beta^0(x) = -\infty, \tag{3.35}$$

where for all $r \in \mathrm{dom}(\beta)$ $\beta^0(r)$ denotes the element of minimal norm in $\beta(r)$. Note that (3.35) (which corresponds to the *strong coercivity condition* of [18]), in fact rules out the case in which $\hat{\beta}$ is the indicator function of $[0,1]$, but is fulfilled by the logarithmic potential (1.4). Furthermore, in this case we shall require the initial datum χ_0 to fulfil only the one-sided separation condition (3.26a).

We shall treat the displacement as a vectorial unknown in this one-dimensional setting as well. Accordingly, we shall keep the notation $\mathcal{H}$ and $\mathcal{K}$ for the elliptic operators appearing in Eq. (2.6), although their definition considerably simplifies (cf. (3.4)). Moreover, we shall continue to denote the realization of the Laplace operator $-\partial^2_{xx}$ with homogeneous Neumann boundary conditions by the symbol A.

For the sake of clarity, we now re-state the variational formulation of the initial–boundary value problem for (1.1), (2.6), (2.11) in the case $N = 1$.

Problem 2. Find $(\vartheta, \chi, \xi, \mathbf{u})$ with

$$\vartheta \in W^{1,\infty}\big(0,T;L^2(0,\ell)\big) \cap H^1\big(0,T;H^1(0,\ell)\big) \cap L^\infty\big(0,T;H^2_N(0,\ell)\big), \tag{3.36}$$

$$\chi \in W^{1,\infty}\big(0,T;L^2(0,\ell)\big) \cap H^1\big(0,T;H^1(0,\ell)\big) \cap L^\infty\big(0,T;H^2_N(0,\ell)\big),$$

$$\chi \in \mathrm{dom}(\beta) \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{3.37}$$

$$\xi \in L^\infty\big(0,T;L^2(0,\ell)\big), \qquad \xi \in \beta(\chi) \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{3.38}$$

$$\mathbf{u} \in H^2\big(0,T;L^2(0,\ell)\big) \cap W^{1,\infty}\big(0,T;H^1_0(0,\ell)\big) \cap H^1\big(0,T;H^2_0(0,\ell)\big), \tag{3.39}$$

complying with initial conditions (3.1)–(3.3), Eqs. (3.18) and (3.20) a.e. in $(0,\ell)\times(0,T)$, as well as

$$\chi_t + A\chi + \xi + \gamma(\chi) = \vartheta + \frac{\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{3.40}$$

and such that

$$\min_{(x,t)\in[0,\ell]\times[0,T]} \vartheta(x,t) > 0, \qquad \min_{(x,t)\in[0,\ell]\times[0,T]} \chi(x,t) > 0. \tag{3.41}$$

Theorem 2. *In the setting of* (3.34)*, assume* (3.15)–(3.16)*,* (3.21)–(3.25)*, the one-sided separation condition* (3.26a)*,* (3.35)*, and that*

$$\hat{\beta}(\chi_0) \in L^1(0,\ell), \tag{3.42a}$$

$$\beta^0(\chi_0) \in L^2(0,\ell). \tag{3.42b}$$

Then, there exist $\delta > 0$, depending on the potential W defined in (3.14) *and on the initial datum χ_0, another constant $\theta_* > 0$, depending on the problem data, and a quadruple $(\vartheta, \chi, \xi, \mathbf{u})$ solving Problem* 2*, such that ϑ and χ fulfil*

$$\vartheta(x,t) \geqslant \theta_* > 0 \quad \forall (x,t) \in [0,\ell] \times [0,T], \tag{3.43}$$

$$\chi(x,t) \geqslant \delta > 0 \quad \forall (x,t) \in [0,\ell] \times [0,T]. \tag{3.44}$$

Suppose in addition that

$$\beta : \mathrm{dom}(\beta) \to \mathbb{R} \text{ is a single-valued function complying with (3.33).}$$

Then, the triple $(\vartheta, \chi, \mathbf{u})$ is the unique *solution to Problem* 2*, and χ has the further regularity*

$$\chi \in H^2\big(0,T; H^1(0,\ell)'\big). \tag{3.45}$$

In fact, as in the case of system (3.18)–(3.20), under the additional assumption (3.33) the solution triple $(\vartheta, \chi, \mathbf{u})$ depends continuously on the initial data, on g, and on $\mathbf{f}$, in the sense specified by Proposition 4.8.

Throughout the following sections, we shall denote by the symbol C (whose meaning may vary even within the same line) most of the positive constants occurring in the various estimates, and we shall often neglect the constants related to Sobolev embeddings. On the other hand, we shall use symbols like M_i, ℓ_i, and S_i, $i = 0, 1, \ldots$, for constants having a specific meaning.

4. Proof of Theorem 1

Strategy of the proof of Theorem 1. Following the approach of [8], we first of all tackle a "regularized" version of system (3.18)–(3.20). Namely, using (3.26a)–(3.26b) we fix a constant

$$\varsigma \in (0,1) \quad \text{such that} \quad \varsigma \leqslant \frac{2}{3} \min\Big\{ \min_{x \in \overline{\Omega}} \chi_0(x), 1 - \max_{x \in \overline{\Omega}} \chi_0(x) \Big\}, \tag{4.1}$$

and we introduce the truncation operator

$$T_\varsigma(r) := \max\{r, \varsigma\} \quad \forall r \in \mathbb{R}. \tag{4.2}$$

Hence, we consider the following PDE system

$$\begin{cases} \vartheta_t + \chi_t \vartheta + A\vartheta = g & \text{a.e. in } \Omega \times (0,T), \\ \chi_t + A\chi + \beta(\chi) + \gamma(\chi) = \vartheta + \dfrac{\varepsilon(\mathbf{u}) \mathcal{R}_e \varepsilon(\mathbf{u})}{2} & \text{a.e. in } \Omega \times (0,T), \\ \mathbf{u}_{tt} + \mathcal{H}\big(T_\varsigma(1-\chi)\mathbf{u}\big) + \mathcal{K}\big(T_\varsigma(\chi)\mathbf{u}_t\big) = \mathbf{f} & \text{a.e. in } \Omega \times (0,T), \end{cases} \tag{4.3}$$

in which the viscous and elastic coefficients in (3.20) are truncated by means of T_ς, thus ruling out the degeneracy of the associated elliptic operator. We shall prove the existence of a local-in-time solution to (4.3) (complying with the initial conditions (3.1)–(3.3) and enjoying the regularity (3.28)–(3.30)), by a Schauder fixed point argument, which we briefly sketch.

For a fixed $\mathsf{t} \in (0, T]$ (which shall be specified later on) and a fixed constant $R > 0$, let us introduce the spaces

$$\begin{aligned} \mathcal{O}_\mathsf{t} &:= \bigl\{\vartheta \in H^1\bigl(0,\mathsf{t}; L^2(\Omega)\bigr):\ \|\vartheta\|_{H^1(0,\mathsf{t};L^2(\Omega))} \leqslant R\bigr\}, \\ \mathcal{U}_\mathsf{t} &:= \bigl\{\mathbf{u} \in H^1\bigl(0,\mathsf{t}; W_0^{1,4}(\Omega)\bigr):\ \|\mathbf{u}\|_{H^1(0,\mathsf{t};W_0^{1,4}(\Omega))} \leqslant R\bigr\}. \end{aligned} \tag{4.4}$$

In the following, we shall construct an operator $\mathcal{T}$, which maps $\mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}}$ into itself for a suitable time $0 < \widehat{T} \leqslant T$, in such a way that any fixed point of $\mathcal{T}$ yields a solution to the PDE system (4.3), supplemented with (3.1)–(3.3). In Proposition 4.5 below, we shall prove that $\mathcal{T}$ is compact and continuous w.r.t. the topology of $H^1(0,\widehat{T}; L^2(\Omega)) \times H^1(0,\widehat{T}; W_0^{1,4}(\Omega))$. Hence, by the Schauder theorem $\mathcal{T}$ admits (at least) a fixed point $(\bar\vartheta, \bar{\mathbf{u}})$ on $\mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}}$, whence the existence of a solution $(\bar\vartheta, \bar\chi, \bar{\mathbf{u}})$ to the Cauchy problem for (4.3) on the interval $[0, \widehat{T}]$.

In fact, in the proof of Proposition 4.5 we shall also show that the solution component $\bar\chi$ complies with the separation property (3.32). Hence, the triple $(\bar\vartheta, \bar\chi, \bar{\mathbf{u}})$ shall turn out to be a local solution to (3.18)–(3.20) on $[0, \widehat{T}]$. In this way, we shall conclude the proof of the existence statement of Theorem 1 (cf. the end of Section 4.2). Finally, our local uniqueness result is a consequence of Proposition 4.8 later on.

4.1. Construction of the Schauder operator

First step. We consider the following initial–boundary value problem associated with Eq. (3.19).

Problem 4.1. Given $(\bar\vartheta, \bar{\mathbf{u}}) \in \mathcal{O}_\mathsf{t} \times \mathcal{U}_\mathsf{t}$, find $\chi : \Omega \times [0,\mathsf{t}] \to \mathbb{R}$ fulfilling initial condition (3.2) and the equation

$$\chi_t + A\chi + \beta(\chi) + \gamma(\chi) = \bar\vartheta + \frac{\varepsilon(\bar{\mathbf{u}})\mathcal{R}_e\varepsilon(\bar{\mathbf{u}})}{2} \quad \text{a.e. in } \Omega \times (0,\mathsf{t}). \tag{4.5}$$

Lemma 4.2. *Assume* (3.15)–(3.17)*,* (3.24)*,* (3.26a)–(3.26b)*. Then, there exists a constant $M_0 > 0$, depending on R and on the problem data, but independent of $\mathsf{t} \in (0, T]$, such that for all $(\bar\vartheta, \bar{\mathbf{u}}) \in \mathcal{O}_\mathsf{t} \times \mathcal{U}_\mathsf{t}$ Problem* 4.1 *admits a unique solution $\chi \in W^{1,\infty}(0,\mathsf{t}; L^2(\Omega)) \cap H^1(0,\mathsf{t}; H^1(\Omega)) \cap L^\infty(0,\mathsf{t}; H^2_N(\Omega))$, with*

$$\|\chi\|_{W^{1,\infty}(0,\mathsf{t};L^2(\Omega))\cap H^1(0,\mathsf{t};H^1(\Omega))\cap L^\infty(0,\mathsf{t};H^2_N(\Omega))} \leqslant M_0. \tag{4.6}$$

We now introduce the set

$$\begin{aligned} \mathcal{X}_\mathsf{t} := \bigl\{\chi \in W^{1,\infty}\bigl(0,\mathsf{t}; L^2(\Omega)\bigr) \cap H^1\bigl(0,\mathsf{t}; H^1(\Omega)\bigr) \cap L^\infty\bigl(0,\mathsf{t}; H^2_N(\Omega)\bigr)&: \\ \|\chi\|_{W^{1,\infty}(0,\mathsf{t};L^2(\Omega))\cap H^1(0,\mathsf{t};H^1(\Omega))\cap L^\infty(0,\mathsf{t};H^2_N(\Omega))} \leqslant M_0\bigr\}&. \end{aligned} \tag{4.7}$$

In view of Lemma 4.2, the solution operator $\mathcal{T}_1$ associated with Problem 4.1 is well-defined and maps $\mathcal{O}_{\mathrm{t}} \times \mathcal{U}_{\mathrm{t}}$ into $\mathcal{X}_{\mathrm{t}}$ for all $\mathrm{t} \in (0, T]$.

Second step. We consider the following initial–boundary value problem associated with Eqs. (3.18) and the truncated version of Eq. (3.20).

Problem 4.3. Given $\overline{\chi} \in \mathcal{X}_{\mathrm{t}}$, find $\vartheta : \Omega \times [0, \mathrm{t}] \to \mathbb{R}$ and $\mathbf{u} : \Omega \times [0, \mathrm{t}] \to \mathbb{R}^3$ fulfilling initial conditions (3.1) and (3.3) and the equations

$$\vartheta_t + \overline{\chi}_t \vartheta + A\vartheta = g \quad \text{a.e. in } \Omega \times (0, \mathrm{t}), \tag{4.8}$$

$$\mathbf{u}_{tt} + \mathcal{H}\big(T_\varsigma(1 - \overline{\chi})\mathbf{u}\big) + \mathcal{K}\big(T_\varsigma(\overline{\chi})\mathbf{u}_t\big) = \mathbf{f} \quad \text{a.e. in } \Omega \times (0, \mathrm{t}). \tag{4.9}$$

We have the following result:

Lemma 4.4. *Assume* (3.21)–(3.23) *and* (3.25). *Then, there exists a constant $M_1 > 0$, only depending on M_0, on the constant ς defined in* (4.1)*, and on the problem data, but independent of* $\mathrm{t} \in (0, T]$, *such that for all $\overline{\chi} \in \mathcal{X}_{\mathrm{t}}$ there exists a unique solution* $(\vartheta, \mathbf{u})$ *to Problem* 4.3 *on* $(0, \mathrm{t})$, $\mathbf{u}$ *with the regularity* (3.30) *and*

$$\vartheta \in W^{1,\infty}\big(0, \mathrm{t}; L^2(\Omega)\big) \cap H^1\big(0, \mathrm{t}; H^1(\Omega)\big) \cap L^\infty\big(0, \mathrm{t}; H^2_N(\Omega)\big), \tag{4.10}$$

fulfilling

$$\begin{cases} \|\vartheta\|_{W^{1,\infty}(0,\mathrm{t};L^2(\Omega))\cap H^1(0,\mathrm{t};H^1(\Omega))\cap L^\infty(0,\mathrm{t};H^2_N(\Omega))} \\ \|\mathbf{u}\|_{H^2(0,\mathrm{t};L^2(\Omega))\cap W^{1,\infty}(0,\mathrm{t};H^1_0(\Omega))\cap H^1(0,\mathrm{t};H^2_0(\Omega))} \end{cases} \leqslant M_1. \tag{4.11}$$

Note that, thanks to (3.12), one has $H^1(0, \mathrm{t}; H^2_0(\Omega)) \subset H^1(0, \mathrm{t}; W^{1,4}_0(\Omega))$. Thus, we associate with Problem 4.3 a solution operator

$$\mathcal{T}_2 : \mathcal{X}_t \to H^1\big(0, \mathrm{t}; L^2(\Omega)\big) \times H^1\big(0, \mathrm{t}; W^{1,4}_0(\Omega)\big),$$

so that the composition operator $\mathcal{T} := \mathcal{T}_2 \circ \mathcal{T}_1$ maps $\mathcal{O}_{\mathrm{t}} \times \mathcal{U}_{\mathrm{t}}$ into the above product space. The proof of the following result is postponed to the next section, along with the proof of Lemmas 4.2 and 4.4.

Proposition 4.5. *Assume* (3.15)–(3.17) *and* (3.21)–(3.26b). *Then, there exist* $0 < \widehat{T} \leqslant T$ *and two positive constants M_2 and M_3, only depending on R, on ς, and on the problem data, such that*:

1. *For all* $(\overline{\vartheta}, \bar{\mathbf{u}}) \in \mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}}$, *setting* $\overline{\chi} := \mathcal{T}_1(\overline{\vartheta}, \bar{\mathbf{u}})$, *we have* $\overline{\chi} \in H^2(0, \widehat{T}; H^1(\Omega)') \cap W^{1,\infty}(0, \widehat{T}; L^2(\Omega)) \cap H^1(0, \widehat{T}; H^1(\Omega)) \cap L^\infty(0, \widehat{T}; H^2_N(\Omega))$, *with*

$$\|\overline{\chi}\|_{H^2(0,\widehat{T};H^1(\Omega)')\cap W^{1,\infty}(0,\widehat{T};L^2(\Omega))\cap H^1(0,\widehat{T};H^1(\Omega))\cap L^\infty(0,\widehat{T};H^2_N(\Omega))} \leqslant M_2, \tag{4.12}$$

and $\overline{\chi}$ fulfils

$$0 < \varsigma \leqslant \overline{\chi}(x, t) \leqslant 1 - \varsigma < 1 \quad \forall (x, t) \in \Omega \times [0, \widehat{T}]. \tag{4.13}$$

2. *For all* $(\overline{\vartheta}, \bar{\mathbf{u}}) \in \mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}}$, *setting* $(\vartheta, \mathbf{u}) = \mathcal{T}(\overline{\vartheta}, \bar{\mathbf{u}})$, *we have*

$$\|\vartheta\|_{H^2(0,\widehat{T};H^1(\Omega)')\cap W^{1,\infty}(0,\widehat{T};L^2(\Omega))\cap H^1(0,\widehat{T};H^1(\Omega))\cap L^\infty(0,\widehat{T};H^2_N(\Omega))} \leqslant M_3, \tag{4.14}$$

and estimate (4.11) *holds for* $\mathbf{u}$. *Further,*

$$\text{the operator } \mathcal{T} \text{ maps } \mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}} \text{ into itself.} \tag{4.15}$$

3. $\mathcal{T} : \mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}} \to \mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}}$ *is compact and continuous with respect to the topology of* $H^1(0, \widehat{T}; L^2(\Omega)) \times H^1(0, \widehat{T}; W^{1,4}_0(\Omega))$.

4.2. Proofs

Throughout this section, the symbol ℓ_i, $i = 0, 1, \ldots, 12$, shall denote some positive constants depending on R, on the problem data, and possibly on the constant ς (see (4.1)) (in which case, ℓ_i explodes for $\varsigma \downarrow 0$).

Proof of Lemma 4.2. We notice that for every $(\overline{\vartheta}, \bar{\mathbf{u}}) \in \mathcal{O}_{\mathrm{t}} \times \mathcal{U}_{\mathrm{t}}$ the term on the right-hand side of (4.5) (which shall be denoted as ω throughout this proof), fulfils

$$\omega \in H^1\big(0, \mathrm{t}; L^2(\Omega)\big). \tag{4.16}$$

In particular, $\omega \in L^2(0, \mathrm{t}; L^2(\Omega))$: therefore, thanks to [12, Lemma 3.3] (based on the theory of maximal monotone operators in Hilbert and Banach spaces [2,10]), Problem 4.1 has a unique solution

$$\chi \in H^1\big(0, \mathrm{t}; L^2(\Omega)\big) \cap \mathrm{C}^0\big([0, \mathrm{t}]; H^1(\Omega)\big) \cap L^2\big(0, \mathrm{t}; H^2_N(\Omega)\big). \tag{4.17}$$

Further, in view of (3.15), (4.17) entails that

$$\gamma(\chi) \in H^1\big(0, \mathrm{t}; L^2(\Omega)\big). \tag{4.18}$$

Since we have not been able to find any precise reference for the further regularity (3.37), we now briefly sketch its proof. Indeed, we test (4.5) by $(A\chi + \beta(\chi))_t$ and integrate in time (note that at this stage such an estimate is only formal, but it can be made rigorous for example by approximating β with its Yosida regularization). Thus, for $t \in (0, \mathrm{t}]$ we have

$$\int_0^t |\nabla\chi_t|^2 + \frac{1}{2}\big|A\chi(t) + \beta\big(\chi(t)\big)\big|^2 + \int_0^t \int_\Omega \beta'(\chi)|\chi_t|^2 \leqslant \|\chi_0\|^2_{H^2(\Omega)} + \big|\beta(\chi_0)\big|^2 + I_0, \tag{4.19}$$

where, integrating by parts, we estimate

$$
\begin{aligned}
I_0 &= \left| \int_0^t \int_\Omega \big(\omega - \gamma(\chi)\big)\big(A\chi + \beta(\chi)\big)_t \right| \\
&\leqslant \int_0^t \int_\Omega \big|\big(\omega_t - \gamma'(\chi)\chi_t\big)\big(A\chi + \beta(\chi)\big)\big| + \int_\Omega \big|\big(\omega(t) - \gamma\big(\chi(t)\big)\big)\big(A\chi(t) + \beta\big(\chi(t)\big)\big)\big| \\
&\quad + \int_\Omega \big|\big(\omega(0) - \gamma(\chi_0)\big)\big(A\chi_0 + \beta(\chi_0)\big)\big| \\
&\leqslant \frac{1}{4}\big(\|\chi_0\|^2_{H^2(\Omega)} + \big|\beta(\chi_0)\big|^2 + \big|A\chi(t) + \beta\big(\chi(t)\big)\big|^2\big) + 2\|\omega + \gamma(\chi)\|^2_{C^0(0,\mathrm{t};L^2(\Omega))} \\
&\quad + \frac{1}{2}\left(\int_0^t \big|A\chi + \beta(\chi)\big|^2 + \|\omega + \gamma(\chi)\|^2_{H^1(0,\mathrm{t};L^2(\Omega))}\right),
\end{aligned}
\tag{4.20}
$$

the latter inequality following from (4.16) and (4.18). Collecting (4.19)–(4.20), recalling (3.27), and applying the Gronwall Lemma (see, e.g., [10, Lemma A.4]), we easily deduce that there exists a constant $\ell_0 > 0$ such that

$$
\big\|A\chi + \beta(\chi)\big\|_{L^\infty(0,\mathrm{t};L^2(\Omega))} + \|\chi_t\|_{L^2(0,\mathrm{t};H^1(\Omega))} \leqslant \ell_0. \tag{4.21}
$$

Now, using the monotonicity of β we infer

$$
\big\|A\chi + \beta(\chi)\big\|^2_{L^\infty(0,\mathrm{t};L^2(\Omega))} \geqslant \|A\chi\|^2_{L^\infty(0,\mathrm{t};L^2(\Omega))} + \big\|\beta(\chi)\big\|^2_{L^\infty(0,\mathrm{t};L^2(\Omega))}, \tag{4.22}
$$

so that (4.21) and well-known elliptic regularity results yield an estimate for χ in the space $L^\infty(0,\mathrm{t}; H^2_N(\Omega))$. Moreover, from (4.19) we get a bound for $\|\chi_t\|_{L^2(0,\mathrm{t};H^1(\Omega))}$. Finally, by a comparison in (4.5) we estimate $\|\chi_t\|_{L^\infty(0,\mathrm{t};L^2(\Omega))}$, and (4.6) ensues. □

Proof of Lemma 4.4. Since Eqs. (4.8) and (4.9) are not coupled, we shall tackle them separately.

Analysis of (4.9). The well-posedness on a generic interval $(0,\mathrm{t})$ of the Cauchy problem for (4.9) has already been proved in [8, Lemma 3.4], where, however, the authors considered the case of a scalar displacement u for the sake of simplicity. Hence, we shall adapt some of the computations of [8, Lemma 3.4] to the present vectorial case. As pointed out in [8], for any $\overline{\chi} \in \mathcal{X}_\mathrm{t}$ existence and uniqueness of a solution $\mathbf{u}$ to the Cauchy problem for (4.9), with the regularity (3.30), follows from standard well-posedness results for parabolic equations. Indeed, the key point is that, thanks to (3.6)–(3.7), all the operators $\mathcal{H}(T_\varsigma(1-\overline{\chi}(t))\cdot), \mathcal{K}(T_\varsigma(\overline{\chi}(t))\cdot) : H^1_0(\Omega) \to H^{-1}(\Omega)$ are elliptic, uniformly w.r.t. $t \in [0,\mathrm{t}]$. Then, in order to prove (4.11), we first test (4.9) by $\mathbf{u}_t$ and integrate over the interval $(0,t)$. Exploiting (3.5) and (3.7), and recalling (3.22), with elementary computations we find

$$\frac{1}{2}\left|\mathbf{u}_t(t)\right|^2+\varsigma C_b\int_0^t\|\mathbf{u}_t\|_{H^1(\Omega)}^2\leqslant\frac{1}{2}|\mathbf{v}_0|^2+\frac{\varsigma C_b}{4}\int_0^t\|\mathbf{u}_t\|_{H^1(\Omega)}^2+\frac{K_a^2}{\varsigma C_b}\int_0^t\|\mathbf{u}\|_{H^1(\Omega)}^2+\frac{1}{2}\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2+\frac{1}{2}\|\mathbf{u}_t\|_{L^2(0,T;L^2(\Omega))}^2. \tag{4.23}$$

We note that

$$\int_0^t\left\|\mathbf{u}(s)\right\|_{H^1(\Omega)}^2\,\mathrm{d}s\leqslant 2T\left(\|\mathbf{u}_0\|_{H^1(\Omega)}^2+\int_0^t\left(\int_0^s\left\|\mathbf{u}_t(r)\right\|_{H^1(\Omega)}^2\,\mathrm{d}r\right)\mathrm{d}s\right). \tag{4.24}$$

Thus, applying the Gronwall Lemma we deduce that there exists $\ell_1>0$ such that

$$\|\mathbf{u}_t\|_{L^\infty(0,\mathrm{t};L^2(\Omega))\cap L^2(0,\mathrm{t};H_0^1(\Omega))}\leqslant\ell_1. \tag{4.25}$$

Next, we multiply (4.9) by $-\mathrm{div}(\varepsilon(\mathbf{u}_t))$ and integrate in time. Formal computations (which could be made rigorous by suitably regularizing the test function) yield for all $t\in(0,\mathrm{t})$

$$-\int_0^t\int_\Omega\mathbf{u}_{tt}\cdot\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)=\frac{1}{2}\left|\varepsilon\big(\mathbf{u}_t(t)\big)\right|^2-\frac{1}{2}\left|\varepsilon(\mathbf{v}_0)\right|^2, \tag{4.26}$$

$$-\int_0^t\int_\Omega\mathcal{H}\big(T_\varsigma(1-\overline{\chi})\mathbf{u}\big)\cdot\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)=I_1+I_2, \tag{4.27}$$

where (Δ stands for the vectorial Laplace operator, too)

$$\begin{aligned}|I_1|&=\left|\lambda\int_0^t\int_\Omega T_\varsigma(1-\overline{\chi})\Delta(\mathbf{u}_t)\cdot\nabla\big(\mathrm{div}(\mathbf{u})\big)+2\mu\int_0^t\int_\Omega T_\varsigma(1-\overline{\chi})\,\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\cdot\mathrm{div}\big(\varepsilon(\mathbf{u})\big)\right|\\&\leqslant C\int_0^t\|\mathbf{u}_t\|_{H^2(\Omega)}\|\mathbf{u}\|_{H^2(\Omega)}\leqslant C\int_0^t\|\mathbf{u}\|_{H^2(\Omega)}^2+\frac{\varsigma C_\gamma^2}{8}\int_0^t\|\mathbf{u}_t\|_{H^2(\Omega)}^2\end{aligned} \tag{4.28}$$

thanks to (3.8), and

$$|I_2|=\left|\lambda\int_0^t\int_\Omega\mathrm{div}(\mathbf{u})\nabla\big(T_\varsigma(1-\overline{\chi})\big)\cdot\Delta(\mathbf{u}_t)+2\mu\int_0^t\int_\Omega\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\cdot\varepsilon(\mathbf{u})\nabla\big(T_\varsigma(1-\overline{\chi})\big)\right|$$

$$\leqslant C\int_0^t \|\mathbf{u}_t\|_{H^2(\Omega)}\|\nabla\overline{\chi}\|_{L^3(\Omega)}\big(\big\|\mathrm{div}(\mathbf{u})\big\|_{L^6(\Omega)}+\big\|\varepsilon(\mathbf{u})\big\|_{L^6(\Omega)}\big)$$

$$\leqslant C\int_0^t \|\mathbf{u}_t\|_{H^2(\Omega)}\|\mathbf{u}\|_{H^2(\Omega)}\|\overline{\chi}\|_{H^2(\Omega)}\leqslant CM_0^2\int_0^t \|\mathbf{u}\|^2_{H^2(\Omega)}+\frac{\varsigma C_\gamma^2}{8}\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega)}. \tag{4.29}$$

The latter estimate as well follows from (3.8), from the fact that $\overline{\chi}\in\mathcal{X}_t$, and from the continuous embedding (3.12). Furthermore, we have

$$-\int_0^t\int_\Omega \mathcal{K}\big(T_\varsigma(\overline{\chi})\mathbf{u}_t\big)\cdot\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)=I_3+I_4, \tag{4.30}$$

where, again by (3.8),

$$\begin{aligned}I_3&=\int_0^t\int_\Omega T_\varsigma(\overline{\chi})\,\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\cdot\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\\&\geqslant\varsigma\int_0^t\big|\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\big|^2\geqslant\varsigma C_\gamma^2\int_0^t\|\mathbf{u}_t\|^2_{H^2(\Omega)},\end{aligned} \tag{4.31}$$

$$\begin{aligned}|I_4|&=\left|\int_0^t\int_\Omega \mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\cdot\varepsilon(\mathbf{u}_t)\nabla\big(T_\varsigma(\overline{\chi})\big)\right|\\&\leqslant\int_0^t\int_\Omega\big|\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\big|\big|\varepsilon(\mathbf{u}_t)\big|^{1/2}\big|\varepsilon(\mathbf{u}_t)\big|^{1/2}\big|\nabla\big(T_\varsigma(\overline{\chi})\big)\big|\\&\leqslant\frac{\varsigma C_\gamma^2}{8}\int_0^t\|\mathbf{u}_t\|^2_{H^2(\Omega)}+C\int_0^t\big\|\varepsilon(\mathbf{u}_t)\big\|_{L^6(\Omega)}\big|\varepsilon(\mathbf{u}_t)\big|\|\nabla\overline{\chi}\|^2_{L^6(\Omega)}\\&\leqslant\frac{\varsigma C_\gamma^2}{4}\int_0^t\|\mathbf{u}_t\|^2_{H^2(\Omega)}+CM_0^4\int_0^t\big|\varepsilon(\mathbf{u}_t)\big|^2.\end{aligned} \tag{4.32}$$

Finally,

$$-\int_0^t\int_\Omega \mathbf{f}\cdot\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big)\leqslant C\|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))}+\frac{\varsigma C_\gamma^2}{4}\int_0^t\|\mathbf{u}_t\|^2_{H^2(\Omega)}. \tag{4.33}$$

We add (4.26)–(4.33), estimate the integral terms containing $\|\mathbf{u}\|^2_{H^2(\Omega)}$ in the same way as in (4.24), and apply the Gronwall Lemma to conclude that

$$\|\mathbf{u}_t\|_{L^\infty(0,\mathrm{t};H^1(\Omega))\cap L^2(0,\mathrm{t};H^2_0(\Omega))} \leqslant \ell_2 \tag{4.34}$$

for some positive constant ℓ_2. The remaining estimate for $\mathbf{u}_{tt}$ in the space $L^2(0,\mathrm{t};L^2(\Omega))$ easily follows by comparison, and (4.11) ensues.

Analysis of (4.8). As for Eq. (4.8), using that $\overline{\chi}_t \in L^2(0,\mathrm{t};H^1(\Omega))$ one checks that the assumptions of [1, Theorem 3.2] are satisfied, hence there exists a unique function

$$\begin{aligned} &\vartheta \in H^1\big(0,\mathrm{t};H^1(\Omega)'\big) \cap \mathrm{C}^0\big([0,\mathrm{t}];L^2(\Omega)\big) \cap L^2\big(0,\mathrm{t};H^1(\Omega)\big),\\ &\text{with } \|\vartheta\|_{H^1(0,\mathrm{t};H^1(\Omega)')\cap \mathrm{C}^0([0,\mathrm{t}];L^2(\Omega))\cap L^2(0,\mathrm{t};H^1(\Omega))} \leqslant \ell_3 \end{aligned} \tag{4.35}$$

for some $\ell_3 > 0$, fulfilling (a suitably weak formulation of) Eq. (4.8), supplemented with the initial condition (3.1). The further regularity (4.10) of ϑ may be inferred by testing (4.8) by ϑ_t and integrating in time (this formal estimate may be made rigorous on a suitable regularization scheme). Hence, we easily deduce

$$\begin{aligned} \int_0^t |\vartheta_t|^2 + \frac{1}{2}\big|\nabla\vartheta(t)\big|^2 &\leqslant \frac{1}{2}|\nabla\vartheta_0|^2 + \int_0^t |g|^2 + \frac{1}{4}\int_0^t |\vartheta_t|^2 + \int_0^t \|\overline{\chi}_t\|_{L^6(\Omega)}\|\vartheta\|_{L^3(\Omega)}|\vartheta_t| \\ &\leqslant \|\vartheta_0\|^2_{H^1(\Omega)} + \|g\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + \frac{1}{2}\int_0^t |\vartheta_t|^2 + \int_0^t \|\overline{\chi}_t\|^2_{L^6(\Omega)}|\nabla\vartheta|^2 + \int_0^t \|\overline{\chi}_t\|^2_{L^6(\Omega)}|\vartheta|^2, \end{aligned}$$

where we have also used (3.12). Recalling (4.35) and applying the Gronwall Lemma, we obtain that there exists $\ell_4 > 0$ such that

$$\|\vartheta\|_{H^1(0,\mathrm{t};L^2(\Omega))\cap L^\infty(0,\mathrm{t};H^1(\Omega))} \leqslant \ell_4. \tag{4.36}$$

Then, it is not difficult to see that $\overline{\chi}_t\vartheta \in L^2(0,\mathrm{t};L^2(\Omega))$. By comparison, we deduce an estimate for $A\vartheta$ in the same space, yielding by standard elliptic regularity results that

$$\|\vartheta\|_{L^2(0,\mathrm{t};H^2_N(\Omega))} \leqslant \ell_5. \tag{4.37}$$

We may now test (4.8) by $A\vartheta_t$ (again, the following computations may be made rigorous by a regularization procedure) and integrate in time, thus obtaining

$$\int_0^t \|\vartheta_t\|^2_{H^1(\Omega)} + \frac{1}{2}\big|A\vartheta(t)\big|^2 \leqslant \frac{1}{2}\|\vartheta_0\|^2_{H^2(\Omega)} + \int_0^t |\vartheta_t|^2 + I_5 + I_6 + I_7, \tag{4.38}$$

where with an easy integration by parts we estimate

$$I_5 = \left|\int_0^t gA\vartheta_t\right| \leqslant \left|\int_0^t g_t A\vartheta\right| + \left|\int_\Omega g(t)A\vartheta(t)\right| + \left|\int_\Omega g(0)A\vartheta(0)\right|$$

$$\leqslant \|\vartheta_0\|^2_{H^2(\Omega)} + C\|g\|^2_{H^1(0,T;L^2(\Omega))} + \frac{1}{2}\int_0^{\mathrm{t}} \|\vartheta\|^2_{H^2(\Omega)} + \frac{1}{4}\big|A\vartheta(t)\big|^2 \tag{4.39}$$

while, recalling the continuous embedding (3.12), we have

$$\begin{aligned} I_6 &= \left|\int_0^t\int_\Omega \nabla\vartheta\overline{\chi}_t\nabla\vartheta_t\right| \leqslant \int_0^t \|\nabla\vartheta\|_{L^6(\Omega)}\|\overline{\chi}_t\|_{L^3(\Omega)}|\nabla\vartheta_t| \\ &\leqslant \int_0^t \|\overline{\chi}_t\|^2_{H^1(\Omega)}\|\vartheta\|^2_{H^2(\Omega)} + \frac{1}{4}\int_0^t \|\vartheta_t\|^2_{H^1(\Omega)} \end{aligned} \tag{4.40}$$

and, finally, again due to (3.12)

$$\begin{aligned} I_7 &= \left|\int_0^t\int_\Omega \vartheta\nabla\overline{\chi}_t\nabla\vartheta_t\right| \leqslant \int_0^t \|\vartheta\|_{L^\infty(\Omega)}|\nabla\overline{\chi}_t||\nabla\vartheta_t| \\ &\leqslant \int_0^t \|\overline{\chi}_t\|^2_{H^1(\Omega)}\|\vartheta\|^2_{H^2(\Omega)} + \frac{1}{4}\int_0^t \|\vartheta_t\|^2_{H^1(\Omega)}. \end{aligned} \tag{4.41}$$

Then, collecting (4.38)–(4.41), exploiting (3.21), the previous estimates (4.36)–(4.37), and the fact that $\|\overline{\chi}_t\|_{L^2(0,\mathrm{t};H^1(\Omega))} \leqslant M_0$, upon an application of the Gronwall Lemma we conclude an estimate for ϑ in $H^1(0,\mathrm{t};H^1(\Omega)) \cap L^\infty(0,\mathrm{t};H^1_0(\Omega))$, which in turn yields that $\|\overline{\chi}_t\vartheta\|_{L^\infty(0,\mathrm{t};L^2(\Omega))} \leqslant C$. Then, arguing by comparison in (4.8), we conclude that $\|\vartheta_t\|_{L^\infty(0,\mathrm{t};L^2(\Omega))} \leqslant C$, whence the desired (4.11). □

Proof of Proposition 4.5.

Proof of Claim 1. As a first step, we show that there exists a time $0 < \overline{T} \leqslant T$ for which (4.13) holds by repeating the argument devised in the proof of [8, Theorem 2.2]. Namely, we note that for all $\mathrm{t} \in (0,T]$ and for all $\overline{\chi} \in \mathcal{T}_1(\mathcal{O}_\mathrm{t} \times \mathcal{U}_\mathrm{t})$

$$\begin{aligned} \big\|\overline{\chi}(t) - \chi_0\big\|_{H^2(\Omega)} &\leqslant M_0 + \|\chi_0\|_{H^2(\Omega)} \quad \forall t \in [0,\mathrm{t}], \\ \big\|\overline{\chi}(t) - \chi_0\big\|_{H^1(\Omega)} &\leqslant t^{1/2}\|\partial_t\overline{\chi}\|_{L^2(0,T;H^1(\Omega))} \leqslant M_0 t^{1/2} \quad \forall t \in [0,\mathrm{t}]. \end{aligned}$$

Combining this with the interpolation estimate $\|v\|_{L^\infty(\Omega)} \leqslant C\|v\|^{2/3}_{H^2(\Omega)}\|v\|^{1/3}_{H^1(\Omega)}$ (see (3.11)), we find

$$\big\|\overline{\chi}(t) - \chi_0\big\|_{L^\infty(\Omega)} \leqslant \ell_6\big(M_0 + \|\chi_0\|_{H^2(\Omega)}\big)^{2/3} M_0^{1/3} t^{1/6}$$

for some suitable positive constant ℓ_6. Hence, there exists some $0 < \overline{T} \leqslant T$ for which

$$\big\|\overline{\chi}(t) - \chi_0\big\|_{L^\infty(\Omega)} \leqslant \frac{\varsigma}{2} \quad \forall t \in [0,\overline{T}], \tag{4.42}$$

whence we readily deduce (4.13) in view of the choice of ς (see (4.1)). Using (3.15)–(3.17) and (4.13), we infer that there exists a positive constant ℓ_7 such that

$$\left\|\beta'(\overline{\chi})+\gamma'(\overline{\chi})\right\|_{L^\infty(\Omega\times(0,\overline{T}))}\leqslant \ell_7. \tag{4.43}$$

From now on, we perform some formal estimates, which again may be rigorously devised on a suitable regularization scheme. Indeed, we differentiate (4.5), test it by $J^{-1}(\overline{\chi}_{tt})$, and integrate in time. Elementary computations and (3.10) then yield

$$\int_0^t \|\overline{\chi}_{tt}\|^2_{H^1(\Omega)'}\leqslant I_8+I_9+I_{10}+I_{11}, \tag{4.44}$$

where, in view of (4.6) and (4.43), we have

$$\begin{aligned} I_8&=\left|\int_0^t\int_\Omega \nabla\overline{\chi}_t\nabla J^{-1}(\overline{\chi}_{tt})\right|\leqslant \|\overline{\chi}_t\|^2_{L^2(0,\overline{T};H^1(\Omega))}+\frac{1}{4}\|\overline{\chi}_{tt}\|^2_{L^2(0,t;H^1(\Omega)')}\\ &\leqslant M_0^2+\frac{1}{4}\|\overline{\chi}_{tt}\|^2_{L^2(0,t;H^1(\Omega)')}, \end{aligned} \tag{4.45}$$

$$\begin{aligned} I_9&=\left|\int_0^t\int_\Omega \big(\beta'(\overline{\chi})+\gamma'(\overline{\chi})\big)\overline{\chi}_t J^{-1}(\overline{\chi}_{tt})\right|\\ &\leqslant \left\|\beta'(\overline{\chi})+\gamma'(\overline{\chi})\right\|_{L^\infty(\Omega\times(0,t))}\left\|J^{-1}(\overline{\chi}_{tt})\right\|_{L^2(0,t;L^2(\Omega))}\|\overline{\chi}_t\|_{L^2(0,t;L^2(\Omega))}\\ &\leqslant \frac{1}{4}\|\overline{\chi}_{tt}\|^2_{L^2(0,t;H^1(\Omega)')}+\ell_7^2\|\overline{\chi}_t\|^2_{L^2(0,t;L^2(\Omega))}, \end{aligned} \tag{4.46}$$

$$I_{10}=\left|\int_0^t\int_\Omega \overline{\vartheta}_t J^{-1}(\overline{\chi}_{tt})\right|\leqslant \frac{1}{8}\|\overline{\chi}_{tt}\|^2_{L^2(0,t;H^1(\Omega)')}+2R^2, \tag{4.47}$$

$$\begin{aligned} I_{11}&=\left|\int_0^t\int_\Omega \varepsilon(\bar{\mathbf{u}}_t)\mathcal{R}_e\varepsilon(\bar{\mathbf{u}})J^{-1}(\overline{\chi}_{tt})\right|\\ &\leqslant C\left\|\varepsilon(\bar{\mathbf{u}})\right\|_{L^\infty(0,t;L^4(\Omega))}\left\|\varepsilon(\bar{\mathbf{u}}_t)\right\|_{L^2(0,t;L^4(\Omega))}\|\overline{\chi}_{tt}\|_{L^2(0,t;H^1(\Omega)')}\\ &\leqslant \frac{1}{8}\|\overline{\chi}_{tt}\|^2_{L^2(0,t;H^1(\Omega)')}+CR^4. \end{aligned} \tag{4.48}$$

Collecting (4.44)–(4.48), we conclude that

$$\|\overline{\chi}\|_{H^2(0,\overline{T};H^1(\Omega)')}\leqslant \ell_8. \tag{4.49}$$

Proof of Claim 2. First, we improve estimate (4.11) for ϑ. To this aim, exploiting (4.12) we perform on Eq. (4.8) the same (formal) regularity estimates we have just developed on (4.5). Namely, we differentiate (4.8), test it by $J^{-1}(\vartheta_{tt})$, and integrate in time. Now, it can be readily checked that

$$\begin{aligned}\left\|\partial_t(\vartheta\overline{\chi}_t)\right\|_{H^1(\Omega)'} &\leqslant \|\vartheta_t\overline{\chi}_t\|_{H^1(\Omega)'} + \|\vartheta\overline{\chi}_{tt}\|_{H^1(\Omega)'} \\ &\leqslant C\|\vartheta_t\|_{L^3(\Omega)}|\overline{\chi}_t| + C\|\overline{\chi}_{tt}\|_{H^1(\Omega)'}\big(\|\vartheta\|_{L^\infty(\Omega)} + \|\nabla\vartheta\|_{L^4(\Omega)}\big). \quad (4.50)\end{aligned}$$

Hence, from the previous estimates we deduce that $\partial_t(\vartheta\overline{\chi}_t) \in L^2(0,\overline{T};H^1(\Omega)')$. Therefore, arguing in the same way as throughout (4.44)–(4.48), exploiting (3.21) as well as estimates (4.11) and (4.12), we easily get

$$\|\vartheta\|_{H^2(0,\overline{T};H^1(\Omega)')} \leqslant \ell_9, \qquad (4.51)$$

thus concluding (4.14). Now, in order to prove (4.15), we notice that

$$\|\vartheta\|_{H^1(0,t;L^2(\Omega))} \leqslant t^{1/2}\|\vartheta\|_{W^{1,\infty}(0,t;L^2(\Omega))} \leqslant t^{1/2}M_3 \quad \forall t \in [0,\overline{T}]. \qquad (4.52)$$

On the other hand, arguing as in [8], we exploit the Gagliardo–Nirenberg inequality (cf. (3.11)) to deduce from (4.11) that

$$\|\mathbf{u}_t\|_{L^{8/3}(0,\overline{T};W_0^{1,4}(\Omega))} \leqslant \ell_{11}. \qquad (4.53)$$

Thus, by using Hölder inequality and again (4.11), we conclude that there exists some $\ell_{12} > 0$ such that

$$\|\mathbf{u}\|_{H^1(0,t;W_0^{1,4}(\Omega))} \leqslant t^{1/4}\ell_{12} \quad \forall t \in [0,\overline{T}]. \qquad (4.54)$$

Then, (4.52) and (4.54) entail that

$$\exists 0 < \widehat{T} \leqslant \overline{T} \quad \text{s.t.} \quad \left\|\mathcal{T}(\overline{\vartheta},\bar{\mathbf{u}})\right\|_{H^1(0,\widehat{T};L^2(\Omega))\times H^1(0,\widehat{T};W_0^{1,4}(\Omega))} \leqslant R \quad \forall(\overline{\vartheta},\bar{\mathbf{u}}) \in \mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}}. \qquad (4.55)$$

Proof of Claim 3. It follows from estimates (4.11), (4.14), and from [33, Theorem 4, Corollary 5] that $\mathcal{T}$ is a compact operator with respect to the topology of the space

$$H^1\big(0,\widehat{T};L^2(\Omega)\big) \times H^1\big(0,\widehat{T};W_0^{1,4}(\Omega)\big).$$

In order to show that $\mathcal{T}$ is continuous, we fix a sequence

$$\big\{(\overline{\vartheta}_n,\bar{\mathbf{u}}_n)\big\} \text{ strongly converging to } (\overline{\vartheta},\bar{\mathbf{u}}) \quad \text{in } H^1\big(0,\widehat{T};L^2(\Omega)\big) \times H^1\big(0,\widehat{T};W_0^{1,4}(\Omega)\big), \quad (4.56)$$

and we let $\chi_n := \mathcal{T}_1(\overline{\vartheta}_n,\bar{\mathbf{u}}_n)$ for all $n \in \mathbb{N}$. Hence, we deduce from estimate (4.12) that the sequence $\{\chi_n\}$ is bounded in $H^2(0,\widehat{T};H^1(\Omega)') \cap W^{1,\infty}(0,\widehat{T};L^2(\Omega)) \cap H^1(0,\widehat{T};H^1(\Omega)) \cap L^\infty(0,\widehat{T};H^2_N(\Omega))$. Furthermore, by (4.13) we have

$$0 < \varsigma \leqslant \chi_n(x,t) \leqslant 1-\varsigma \quad \forall (x,t) \in \Omega \times [0,\widehat{T}],$$

which implies, thanks to (3.17), that the sequence $\{\beta(\chi_n)\}$ is bounded in $L^2(0,\widehat{T};L^2(\Omega))$. Again using [33, Theorem 4, Corollary 5] and the Ascoli–Arzelà compactness theorem, we obtain that there exist a subsequence, which we do not relabel, and a function $\overline{\chi}$ such that the following convergences hold as $n \uparrow \infty$ for all $1 \leqslant p < \infty$ and for all $\rho > 0$

$$\begin{aligned} \chi_n \to \overline{\chi} \quad & \text{in } \mathrm{C}^1\big(0,\widehat{T};H^1(\Omega)'\big) \cap \mathrm{C}^0\big(0,\widehat{T};H^{2-\rho}(\Omega)\big), \\ \chi_n \to \overline{\chi} \quad & \text{in } W^{1,p}\big(0,\widehat{T};L^2(\Omega)\big) \cap H^1\big(0,\widehat{T};H^{1-\rho}(\Omega)\big) \cap L^p\big(0,\widehat{T};H^2_N(\Omega)\big), \\ \chi_n \rightharpoonup^* \overline{\chi} \quad & \text{in } H^2\big(0,\widehat{T};H^1(\Omega)'\big) \cap W^{1,\infty}\big(0,\widehat{T};L^2(\Omega)\big) \cap H^1\big(0,\widehat{T};H^1(\Omega)\big) \\ & \cap L^\infty\big(0,\widehat{T};H^2_N(\Omega)\big). \end{aligned} \tag{4.57}$$

Hence, $\overline{\chi}$ fulfils the initial condition (3.2) and the separation property (3.32). Furthermore, by the strong-weak closedness of the maximal monotone operator induced by β on $L^2(0,\widehat{T};L^2(\Omega))$, we also conclude that, along the same subsequence,

$$\beta(\chi_n) \rightharpoonup \beta(\overline{\chi}) \quad \text{in } L^2\big(0,\widehat{T};L^2(\Omega)\big). \tag{4.58}$$

Using (4.56)–(4.58), it is easy to pass to the limit in Eq. (4.5) and conclude that

$$\overline{\chi} = \mathcal{T}_1(\overline{\vartheta},\bar{\mathbf{u}}).$$

Therefore, we infer that convergences (4.57) and (4.58) hold along the *whole* sequence $\{\chi_n\}$.

We now consider the sequence $(\vartheta_n,\mathbf{u}_n) := \mathcal{T}_2(\chi_n) = \mathcal{T}(\overline{\vartheta}_n,\bar{\mathbf{u}}_n)$. In view of estimates (4.11) and (4.14), of [33, Theorem 4, Corollary 5], and of the Ascoli theorem, we deduce that there exist suitable subsequences (which we do not relabel) of $\{\vartheta_n\}$ and $\{\mathbf{u}_n\}$ and two limit functions ϑ and $\mathbf{u}$ such that for all $1 \leqslant p < \infty$ and for all $\rho > 0$

$$\begin{aligned} \mathbf{u}_n \to \mathbf{u} \quad & \text{in } H^1\big(0,\widehat{T};H^{2-\rho}(\Omega)\big) \cap W^{1,p}\big(0,\widehat{T};H^1(\Omega)\big) \cap \mathrm{C}^1\big(0,\widehat{T};H^{1-\rho}(\Omega)\big), \\ \mathbf{u}_n \rightharpoonup^* \mathbf{u} \quad & \text{in } H^2\big(0,\widehat{T};L^2(\Omega)\big) \cap W^{1,\infty}\big(0,\widehat{T};H^1(\Omega)\big) \cap H^1\big(0,\widehat{T};H^2_0(\Omega)\big), \end{aligned} \tag{4.59}$$

while for $\{\vartheta_n\}$ and ϑ the same convergences as in (4.57) hold true. In particular, $\mathbf{u}_n \to \mathbf{u}$ in $H^1(0,\widehat{T};W^{1,4}_0(\Omega))$, whence

$$(\vartheta_n,\mathbf{u}_n) \to (\vartheta,\mathbf{u}) \quad \text{in } H^1\big(0,\widehat{T};L^2(\Omega)\big) \times H^1\big(0,\widehat{T};W^{1,4}_0(\Omega)\big). \tag{4.60}$$

It follows from (4.59) that $\mathbf{u}$ complies with the initial condition (3.3). Combining (4.57) and (4.59) and arguing in the same way as in the proof of [8, Theorem 3.5], we infer that the pair $(\mathbf{u},\overline{\chi})$ satisfies Eq. (4.9) on $\Omega \times (0,\widehat{T})$. In the same way, convergences (4.57) for $\{\chi_n\}$ and $\{\vartheta_n\}$ (yielding, in particular, that $\vartheta_n \partial_t \chi_n \to \vartheta \partial_t \overline{\chi}$ in $L^2(0,\widehat{T};L^2(\Omega))$), enable to conclude that $(\vartheta,\overline{\chi})$ fulfils (4.8) on $\Omega \times (0,\widehat{T})$, and that ϑ complies with the initial condition (3.1). In the end, we deduce that

$$(\vartheta,\mathbf{u}) = \mathcal{T}_2(\overline{\chi}) = \mathcal{T}(\overline{\vartheta},\bar{\mathbf{u}}),$$

and that (4.60) holds along the whole sequence $\{(\vartheta_n, \mathbf{u}_n)\}$. We have thus shown that the operator $\mathcal{T}$ is continuous. $\square$

Remark 4.6. It follows from the calculations developed in the proofs of Lemmas 4.2, 4.4, and Proposition 4.5 that the life-time $\widehat{T}$ specified by (4.55) tends to 0 when the separation inequality (4.1) degenerates (indeed, constants ℓ_1, ℓ_2, ℓ_7–ℓ_9, and ℓ_{12} explode as $\varsigma \downarrow 0$). On the other hand, with a simple argument one verifies that for any $\Upsilon > 0$ and for any $H^2(\Omega)$-ball of "separated" data

$$\mathcal{X}_\Upsilon = \Big\{\chi \in H^2(\Omega)\colon \ \|\chi\|_{H^2(\Omega)} \leqslant \Upsilon, \ \min_{x\in\overline{\Omega}} \chi(x) > 0, \ \max_{x\in\overline{\Omega}} \chi(x) < 1\Big\},$$

there exists a constant $\varsigma_\Upsilon \in (0,1)$ such that

$$\varsigma_\Upsilon \leqslant \frac{2}{3}\min\Big\{\min_{x\in\overline{\Omega}} \chi(x), 1 - \max_{x\in\overline{\Omega}} \chi(x)\Big\} \quad \forall \chi \in \mathcal{X}_\Upsilon,$$

i.e. (4.1) holds uniformly in $\mathcal{X}_\Upsilon$. Now, letting the initial datum χ_0 vary in the set $\mathcal{X}_\Upsilon$, in view of the above calculations one finds a lower bound for the life-time $\widehat{T}$ in terms of ς_Υ.

Conclusion of the proof of local existence. It follows from Proposition 4.5 and from the Schauder fixed point theorem that $\mathcal{T}$ has a fixed point $(\vartheta, \mathbf{u})$ in $\mathcal{O}_{\widehat{T}} \times \mathcal{U}_{\widehat{T}}$. Hence, setting $\chi := \mathcal{T}_1(\vartheta, \mathbf{u})$, the triplet $(\vartheta, \chi, \mathbf{u})$ is, by construction, a solution of the Cauchy problem associated with the truncated system (4.3), enjoying the regularity (3.28)–(3.30). Furthermore, since χ fulfils the separation property (4.13), $(\vartheta, \chi, \mathbf{u})$ is in fact a solution of Problem 1.

In the end, we prove the positivity estimate (3.31) for any solution $(\vartheta, \chi, \mathbf{u})$ of Problem 1 in the spaces (3.28)–(3.30). To this aim, we first show that

$$\vartheta(x,t) \geqslant 0 \quad \text{for all } (x,t) \in \Omega \times [0, \widehat{T}]. \tag{4.61}$$

Indeed, we test (3.18) by $-(\vartheta)^-$ ($(\cdot)^-$ denoting the negative part) and integrate in time. With trivial computations we get

$$\begin{aligned}\frac{1}{2}\big|\vartheta^-(t)\big|^2 + \int_0^t \big|\nabla\vartheta^-\big|^2 &\leqslant \frac{1}{2}\big|\vartheta_0^-\big|^2 - \int_0^t\int_\Omega g\vartheta^- + \int_0^t\int_\Omega \vartheta^-\chi_t\vartheta^- \\ &\leqslant I_{12} := \int_0^t |\chi_t|\big\|\vartheta^-\big\|_{L^3(\Omega)}\big\|\vartheta^-\big\|_{L^6(\Omega)},\end{aligned} \tag{4.62}$$

where the second inequality follows from the positivity of g and from the fact that $\vartheta_0^-(x) = 0$ for all $x \in \Omega$ (cf. (3.21) and (3.23)). Using that (see, e.g., [24, Lemma 5.1, p. 58])

$$\forall \varepsilon > 0 \quad \exists C_\varepsilon > 0 \quad \big\|\vartheta^-\big\|_{L^3(\Omega)} \leqslant \varepsilon\big\|\vartheta^-\big\|_{H^1(\Omega)} + C_\varepsilon\big|\vartheta^-\big|,$$

we conclude that

$$
\begin{aligned}
|I_{12}| &\leqslant \|\chi\|_{W^{1,\infty}(0,\widehat{T};L^2(\Omega))} \int_0^t \|\vartheta^-\|_{L^3(\Omega)} \|\vartheta^-\|_{L^6(\Omega)} \\
&\leqslant \frac{1}{4}\int_0^t \|\vartheta^-\|^2_{H^1(\Omega)} + C\int_0^t |\vartheta^-|^2 \leqslant \frac{1}{4}\int_0^t |\nabla\vartheta^-|^2 + C\int_0^t |\vartheta^-|^2. \qquad (4.63)
\end{aligned}
$$

Combining (4.62) and (4.63) and applying the Gronwall Lemma we infer (4.61).

Now, following [32, Section 4] (see also [5, Section 3]), for any $h > 0$ we set $\vartheta_h := \max\{\vartheta, h\}$, test (3.18) by $-\vartheta_h^{-2}$ and integrate in time. Noting that

$$
\begin{cases}
-\int_0^t \int_\Omega \vartheta_t \vartheta_h^{-2} = \int_\Omega \vartheta_h^{-1}(t) - \int_\Omega \vartheta_h^{-1}(0), \\
-\int_0^t \int_\Omega \nabla\vartheta \cdot \nabla\big(\vartheta_h^{-2}\big) = 8\int_0^t \big|\nabla\vartheta_h^{-1/2}\big|^2
\end{cases}
$$

(see [32] for details) we obtain

$$
\int_\Omega \vartheta_h^{-1}(t) + 8\int_0^t \big|\nabla\vartheta_h^{-1/2}\big|^2 \leqslant \int_\Omega \vartheta_h^{-1}(0) - \int_0^t\int_\Omega g\vartheta_h^{-2} + \int_0^t\int_\Omega \vartheta\chi_t\vartheta_h^{-2} \leqslant \int_\Omega \vartheta_0^{-1} + I_{13}, \qquad (4.64)
$$

the second inequality following from the fact that $0 < \vartheta_0(x) \leqslant \vartheta_h(x,0)$ for all $x \in \Omega$. Now, since

$$
0 \leqslant \vartheta(x,t) \leqslant \vartheta_h(x,t) \quad \forall (x,t) \in \Omega \times [0,\widehat{T}],
$$

we estimate

$$
\begin{aligned}
I_{13} &= \left|\int_0^t\int_\Omega \vartheta\chi_t\vartheta_h^{-2}\right| \leqslant \int_0^t\int_\Omega \vartheta|\chi_t|\vartheta_h^{-2} \leqslant \int_0^t\int_\Omega |\chi_t|\vartheta_h^{-1} \\
&\leqslant \int_0^t \|\chi_t\|_{L^4(\Omega)} \big\|\vartheta_h^{-1/2}\big\|_{L^4(\Omega)} \big|\vartheta_h^{-1/2}\big| \leqslant C\int_0^t \big(\|\chi_t\|^2_{H^1(\Omega)} + 1\big)\big|\vartheta_h^{-1/2}\big|^2 + 4\int_0^t \big|\nabla\vartheta_h^{-1/2}\big|^2. \qquad (4.65)
\end{aligned}
$$

Hence, we combine (4.64) and (4.65) and apply the Gronwall Lemma: recalling the regularity (3.29) for χ and the positivity assumption (3.23) on ϑ_0, we conclude that there exists a constant C, independent of h, such that

$$
\big\|\vartheta_h^{-1}\big\|_{L^\infty(0,\widehat{T};L^1(\Omega))} \leqslant C. \qquad (4.66)
$$

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Finally, we let $h \downarrow 0$: by the monotone convergence theorem, we conclude estimate (4.66) for ϑ^{-1} as well, whence

$$\vartheta(x,t) > 0 \quad \text{for all } (x,t) \in \Omega \times [0, \widehat{T}].$$

Since for all $t \in [0, \widehat{T}]$ we have $\vartheta(t) \in H^2_N(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$, (3.31) follows.

Remark 4.7. Let us highlight that the double separation condition (3.26a)–(3.26b) on χ_0 has been fully exploited to deduce the "two-sided" separation inequality (3.32), which has on the other hand enabled us to prove the crucial estimate (4.43). The latter is indeed the key ingredient for obtaining (4.49), whence the enhanced regularity estimates (4.51) for ϑ, needed for carrying out the fixed point procedure.

These considerations point to the essentially technical character of separation condition (3.26b). In fact, as mentioned in Remark 3.3, we could do without it if we replaced (3.17) by (3.33). In that case, exploiting (4.42) and (3.26a) we would deduce only the "one-sided" separation inequality

$$\chi(x,t) \geqslant \varsigma > 0 \quad \forall (x,t) \in \Omega \times [0, \widehat{T}], \tag{4.67}$$

which, combined with (3.33), would still guarantee (4.43) and thus lead to (4.51). A close examination of the above proofs also show that, within this frame, it is not necessary to have an upper constraint on the values of the χ component of the solution, and we can allow $\mathrm{dom}(\hat{\beta}) = (0, +\infty)$.

4.3. Local uniqueness

We may finally state our continuous dependence result for Problem 1.

Proposition 4.8. *Under assumptions* (3.15)–(3.17)*, let* $(g_i, \mathbf{f}_i, \vartheta_0^i, \chi_0^i, \mathbf{u}_0^i, \mathbf{v}_0^i)$*,* $i = 1, 2$*, be two sets of data for Problem* 1*, complying with* (3.21)–(3.25)*, and, accordingly, let* $(\vartheta_i, \chi_i, \mathbf{u}_i)$*,* $i = 1, 2$*, be two associated solution triplets on some interval* $(0, \mathrm{t})$*,* $\mathrm{t} \in (0, T]$*, such that the functions* $\mathbf{u}_i$*,* $i = 1, 2$*, enjoy the regularity* (3.30) *and* (ϑ_i, χ_i) *fulfil*

$$\begin{cases} \vartheta_i \in H^1\big(0,\mathrm{t}; L^2(\Omega)\big) \cap L^\infty\big(0,\mathrm{t}; H^1(\Omega)\big) \cap L^2\big(0,\mathrm{t}; H^2_N(\Omega)\big), \\ \chi_i \in H^1\big(0,\mathrm{t}; H^1(\Omega)\big) \cap L^\infty\big(0,\mathrm{t}; H^2_N(\Omega)\big), \end{cases} \quad i = 1, 2. \tag{4.68}$$

Further, suppose that for some $\nu > 0$

$$0 < \nu \leqslant \chi_i(x,t) \leqslant 1 - \nu \quad \forall (x,t) \in \Omega \times [0, \mathrm{t}] \quad \textit{for } i = 1, 2. \tag{4.69}$$

Set

$$M_4 := \max_{i=1,2} \big\{ \|\chi_i\|_{H^1(0,\mathrm{t}; H^1(\Omega))} + \|\mathbf{u}_i\|_{H^1(0,\mathrm{t}; H^2_0(\Omega))} + \|\vartheta_i\|_{L^2(0,\mathrm{t}; H^2_N(\Omega))} \big\}.$$

Then there exists a positive constant S_0, depending on M_4, ν, T, and Ω, such that

$$
\begin{aligned}
&\|\mathbf{u}_1 - \mathbf{u}_2\|_{W^{1,\infty}(0,\mathrm{t};L^2(\Omega))\cap H^1(0,\mathrm{t};H^1(\Omega))} + \|\chi_1 - \chi_2\|_{H^1(0,\mathrm{t};L^2(\Omega))\cap L^\infty(0,\mathrm{t};H^1(\Omega))} \\
&\qquad + \|\vartheta_1 - \vartheta_2\|_{L^\infty(0,\mathrm{t};L^2(\Omega))\cap L^2(0,\mathrm{t};H^1(\Omega))} \\
&\quad \leqslant S_0\big(\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_{H^1(\Omega)} + |\mathbf{v}_0^1 - \mathbf{v}_0^2| + \|\chi_0^1 - \chi_0^2\|_{H^1(\Omega)} + |\vartheta_0^1 - \vartheta_0^2| \\
&\qquad + \|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(0,T;H^{-1}(\Omega))} + \|g_1 - g_2\|_{L^2(0,T;H^1(\Omega)')}\big). \qquad (4.70)
\end{aligned}
$$

Note that this continuous dependence result holds under regularity requirements on (ϑ, χ) which are weaker than (3.28)–(3.29).

Remark 4.9. It will be clear from the proof below that the continuous dependence estimate (4.70) also holds in the case in which the functions χ_1 and χ_2 only comply with the "one-sided" inequality (4.67), provided that β fulfils the Lipschitz continuity condition (3.33).

Proof of Proposition 4.8. Since some of the computations we are going to develop are similar to the ones contained in the proof of [8, Theorem 2.4], we do not detail all of them, referring to [8] for details. Let $(\vartheta_i, \chi_i, \mathbf{u}_i)$, $i = 1, 2$, be two solution triplets like in the above statement and set $(\vartheta, \chi, \mathbf{u}) := (\vartheta_1 - \vartheta_2, \chi_1 - \chi_2, \mathbf{u}_1 - \mathbf{u}_2)$. Clearly, the triplet $(\vartheta, \chi, \mathbf{u})$ fulfils a.e. in $\Omega \times (0, \mathrm{t})$

$$\vartheta_t + \partial_t \chi_1 \vartheta + \chi_t \vartheta_2 + A\vartheta = g_1 - g_2, \qquad (4.71)$$

$$\chi_t + A\chi + \beta(\chi_1) - \beta(\chi_2) + \gamma(\chi_1) - \gamma(\chi_2) = \vartheta + \frac{\varepsilon(\mathbf{u}_1)\mathcal{R}_e\varepsilon(\mathbf{u}_1)}{2} - \frac{\varepsilon(\mathbf{u}_2)\mathcal{R}_e\varepsilon(\mathbf{u}_2)}{2}, \qquad (4.72)$$

$$\mathbf{u}_{tt} + \mathcal{H}\big((1 - \chi_1)\mathbf{u}\big) - \mathcal{H}(\chi \mathbf{u}_2) + \mathcal{K}(\chi_1 \mathbf{u}_t) + \mathcal{K}(\chi\, \partial_t \mathbf{u}_2) = \mathbf{f}_1 - \mathbf{f}_2. \qquad (4.73)$$

Now, we test (4.73) by $\mathbf{u}_t$ and integrate in time. Recalling (4.69) and (3.7), it is not difficult to infer

$$\frac{1}{2}|\mathbf{u}_t(t)|^2 + C_b\nu \int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)} \leqslant \frac{1}{2}|\mathbf{v}_0^1 - \mathbf{v}_0^2|^2 + \int_0^t \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^{-1}(\Omega)}\|\mathbf{u}_t\|_{H^1(\Omega)} + I_{14} + I_{15}, \qquad (4.74)$$

where, also exploiting (3.12),

$$
\begin{aligned}
I_{14} &= -\int_0^t \big\langle \mathcal{H}\big((1 - \chi_1)\mathbf{u}\big), \mathbf{u}_t\big\rangle \leqslant C \int_0^t \|1 - \chi_1\|_{L^\infty(\Omega)}\, \|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{u}_t\|_{H^1(\Omega)} \\
&\leqslant \frac{C_b\nu}{4} \int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)} + C \int_0^t \|\mathbf{u}\|^2_{H^1(\Omega)}, \qquad (4.75)
\end{aligned}
$$

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$$
\begin{aligned}
I_{15} &= \int_0^t \big\langle \mathcal{H}(\chi \mathbf{u}_2) - \mathcal{K}(\chi \partial_t \mathbf{u}_2), \mathbf{u}_t \big\rangle \\
&\leqslant C \int_0^t \big(\|\mathbf{u}_2\|_{W^{1,4}(\Omega)} + \|\partial_t \mathbf{u}_2\|_{W^{1,4}(\Omega)} \big) \|\chi\|_{L^4(\Omega)} \|\mathbf{u}_t\|_{H^1(\Omega)} \\
&\leqslant \frac{C_b \nu}{4} \int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)} + C \int_0^t \big(\|\mathbf{u}_2\|_{H^2(\Omega)} + \|\partial_t \mathbf{u}_2\|_{H^2(\Omega)} \big)^2 \|\chi\|^2_{H^1(\Omega)}. \qquad (4.76)
\end{aligned}
$$

Noting that (cf. with (4.24))

$$
\big\|\mathbf{u}(t)\big\|^2_{H^1(\Omega)} \leqslant 2 \big\|\mathbf{u}_0^1 - \mathbf{u}_0^2\big\|^2_{H^1(\Omega)} + 2t \int_0^t \big\|\mathbf{u}_t(r)\big\|^2_{H^1(\Omega)} \,\mathrm{d}r, \qquad (4.77)
$$

we obtain from (4.74)–(4.76), also recalling Korn's inequality, that

$$
\begin{aligned}
&\frac{1}{2}\big|\mathbf{u}_t(t)\big|^2 + \frac{C_b \nu}{2} \int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)} \\
&\quad \leqslant \frac{1}{2}\big|\mathbf{v}_0^1 - \mathbf{v}_0^2\big|^2 + C\|\mathbf{f}_1 - \mathbf{f}_2\|^2_{L^2(0,T;H^{-1}(\Omega))} \\
&\qquad + \frac{C_b \nu}{4} \int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)} + C\big\|\mathbf{u}_0^1 - \mathbf{u}_0^2\big\|^2_{H^1(\Omega)} + C \int_0^t \left(\int_0^s \big\|\mathbf{u}_t(r)\big\|^2_{H^1(\Omega)} \,\mathrm{d}r \right) \mathrm{d}s \\
&\qquad + C \int_0^t \big(\|\mathbf{u}_2\|_{H^2(\Omega)} + \|\partial_t \mathbf{u}_2\|_{H^2(\Omega)} \big)^2 \|\chi\|^2_{H^1(\Omega)}. \qquad (4.78)
\end{aligned}
$$

Next, we test (4.72) by χ_t and integrate the resulting equation in time. With elementary computations, also taking into account the Lipschitz continuity of γ, as well as (4.69) and the local Lipschitz continuity of β, we get

$$
\int_0^t |\chi_t|^2 + \frac{1}{2}\big\|\chi(t)\big\|^2_{H^1(\Omega)} \leqslant \frac{1}{2}\big\|\chi_0^1 - \chi_0^2\big\|^2_{H^1(\Omega)} + \int_0^t |\vartheta|^2 + \frac{3}{4}\int_0^t |\chi_t|^2 + C\int_0^t |\chi|^2 + I_{16},
$$

with

$$
I_{16} := \int_0^t \int_\Omega \left(\frac{\varepsilon(\mathbf{u}_1)\mathcal{R}_e \varepsilon(\mathbf{u}_1)}{2} - \frac{\varepsilon(\mathbf{u}_2)\mathcal{R}_e \varepsilon(\mathbf{u}_2)}{2} \right) \chi_t
$$

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$$= -\int_0^t \int_\Omega \big(\varepsilon(\partial_t \mathbf{u}_1) - \varepsilon(\partial_t \mathbf{u}_2)\big)\mathcal{R}_e \varepsilon(\mathbf{u}_1)\chi - \int_0^t \int_\Omega \varepsilon(\partial_t \mathbf{u}_2)\mathcal{R}_e\big(\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\big)\chi$$
$$+ \int_\Omega \frac{\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))}{2}\mathcal{R}_e \varepsilon\big(\mathbf{u}_1(t)\big)\chi(t) + \int_\Omega \varepsilon\big(\mathbf{u}_2(t)\big)\mathcal{R}_e \frac{\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))}{2}\chi(t)$$
$$- \int_\Omega \frac{\varepsilon(\mathbf{u}_0^1) - \varepsilon(\mathbf{u}_0^2)}{2}\mathcal{R}_e \varepsilon\big(\mathbf{u}_0^1\big)\big(\chi_0^1 - \chi_0^2\big) - \int_\Omega \varepsilon\big(\mathbf{u}_0^2\big)\mathcal{R}_e \frac{\varepsilon(\mathbf{u}_0^1) - \varepsilon(\mathbf{u}_0^2)}{2}\big(\chi_0^1 - \chi_0^2\big)$$
$$\leqslant \frac{1}{2}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)} + \frac{1}{2}\int_0^t \|\mathbf{u}\|^2_{H^1(\Omega)} + C\int_0^t \big(\big\|\varepsilon(\mathbf{u}_1)\big\|^2_{H^1(\Omega)} + \big\|\varepsilon(\partial_t \mathbf{u}_2)\big\|^2_{H^1(\Omega)}\big)\|\chi\|^2_{H^1(\Omega)}$$
$$+ \frac{1}{4}\big\|\chi(t)\big\|^2_{H^1(\Omega)} + C\big(\big\|\varepsilon(\mathbf{u}_1)\big\|^2_{L^\infty(0,\mathrm{t};H^1(\Omega))} + \big\|\varepsilon(\mathbf{u}_2)\big\|^2_{L^\infty(0,\mathrm{t};H^1(\Omega))}\big)\big\|\mathbf{u}(t)\big\|^2_{H^1(\Omega)}$$
$$+ C\big(\big\|\chi_0^1 - \chi_0^2\big\|^2_{H^1(\Omega)} + \big\|\mathbf{u}_0^1 - \mathbf{u}_0^2\big\|^2_{H^1(\Omega)}\big),$$

where the second equality follows from integration by parts and simple algebraic manipulations. Therefore, estimating the terms containing $\|\mathbf{u}\|^2_{H^1(\Omega)}$ by means of (4.77), we end up with

$$\int_0^t |\chi_t|^2 + \big\|\chi(t)\big\|^2_{H^1(\Omega)} \leqslant \overline{C}\Bigg(\big\|\chi_0^1 - \chi_0^2\big\|^2_{H^1(\Omega)} + \big\|\mathbf{u}_0^1 - \mathbf{u}_0^2\big\|^2_{H^1(\Omega)} + \int_0^t |\vartheta|^2 + \int_0^t |\chi|^2$$
$$+ \int_0^t \big(\big\|\varepsilon(\mathbf{u}_1)\big\|^2_{H^1(\Omega)} + \big\|\varepsilon(\partial_t \mathbf{u}_2)\big\|^2_{H^1(\Omega)}\big)\|\chi\|^2_{H^1(\Omega)}$$
$$+ \int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)} + \int_0^t \int_0^s \big\|\mathbf{u}_t(r)\big\|^2_{H^1(\Omega)}\,\mathrm{d}r\,\mathrm{d}s\Bigg), \tag{4.79}$$

where the constant $\overline{C}$ depends on M_4 as well. Finally, we test (4.71) by ϑ. Integrating in time, we get

$$\frac{1}{2}\big|\vartheta(t)\big|^2 + \int_0^t |\nabla\vartheta|^2 \leqslant \frac{1}{2}\big|\vartheta_0^1 - \vartheta_0^2\big|^2 + \int_0^t \|g_1 - g_2\|_{H^1(\Omega)'}\|\vartheta\|_{H^1(\Omega)}$$
$$+ \int_0^t \int_\Omega |\partial_t \chi_1||\vartheta|^2 + \int_0^t \int_\Omega |\chi_t||\vartheta_2|\vartheta|$$
$$\leqslant \frac{1}{2}\big|\vartheta_0^1 - \vartheta_0^2\big|^2 + \int_0^t \|g_1 - g_2\|^2_{H^1(\Omega)'} + \varpi\int_0^t |\chi_t|^2 + \frac{1}{2}\int_0^t |\nabla\vartheta|^2$$
$$+ \int_0^t \bigg(\|\partial_t \chi_1\|^2_{H^1(\Omega)} + C_\varpi\|\vartheta_2\|^2_{H^2(\Omega)} + \frac{1}{2}\bigg)|\vartheta|^2, \tag{4.80}$$

for some suitable $\varpi > 0$ (C_ϖ being the related constant via the Young inequality). Finally, we add (4.78), (4.79) (multiplied by a positive constant m such that $m\overline{C} \leqslant (C_b\nu)/8$), and (4.80), in which we choose $0 < \varpi < m/4$. Applying the Gronwall Lemma, we conclude the continuous dependence estimate for $\|\chi_1 - \chi_2\|_{H^1(0,\mathrm{t};L^2(\Omega))\cap L^\infty(0,\mathrm{t};H^1(\Omega))}$, for $\|\vartheta_1 - \vartheta_2\|_{L^\infty(0,\mathrm{t};L^2(\Omega))\cap L^2(0,\mathrm{t};H^1(\Omega))}$, and for $\|\mathbf{u}_t^1 - \mathbf{u}_t^2\|_{L^\infty(0,\mathrm{t};L^2(\Omega))\cap L^2(0,\mathrm{t};H_0^1(\Omega))}$. Integrating in time, we obtain the estimate for $\|\mathbf{u}_1 - \mathbf{u}_2\|_{W^{1,\infty}(0,\mathrm{t};L^2(\Omega))\cap H^1(0,\mathrm{t};H^1(\Omega))}$ as well, and (4.70) ensues. □

5. Global well-posedness for Problem 2

5.1. Strategy of the proof of Theorem 2

In order to prove the existence of a global solution to Problem 2, i.e. in the (spatially) one-dimensional case, we shall combine a Schauder fixed point argument (yielding the existence of a local solution), with a careful extension procedure.

First step: *Existence of a local solution.* Using (3.26a) and (3.35), we fix a constant $\delta > 0$ such that

$$\chi_0(x) \geqslant \delta > 0 \quad \text{and} \quad \beta^0(\delta) + \gamma(\delta) < 0, \tag{5.1}$$

and we consider the truncated PDE system

$$\vartheta_t + \chi_t\vartheta + A\vartheta = g \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{5.2a}$$

$$\begin{cases} \chi_t + A\chi + \xi + \gamma(\chi) = \vartheta + \dfrac{\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} & \text{a.e. in } (0,\ell)\times(0,T), \\ \xi \in \beta(\chi) & \text{a.e. in } (0,\ell)\times(0,T), \end{cases} \tag{5.2b}$$

$$\mathbf{u}_{tt} + \mathcal{H}\big((1-\chi)\mathbf{u}\big) + \mathcal{K}\big(T_\delta(\chi)\mathbf{u}_t\big) = \mathbf{f} \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{5.2c}$$

where T_δ is the truncation operator defined by (4.2). Note that, contrary to the case of system (4.3), here we truncate only one of the (degenerating) coefficients in the elliptic operator appearing in (5.2c). Yet, in this case as well the degeneracy of the *main* part of the elliptic operator is ruled out.

We shall prove the existence of a local solution $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ to the Cauchy problem for system (5.2) on some interval $[0, T_0]$, enjoying the regularity (3.36)–(3.39), the positivity properties (3.41), and fulfilling

$$\hat{\chi}(x,t) \geqslant \delta > 0 \quad \forall (x,t) \in [0,\ell]\times[0,T_0]. \tag{5.3}$$

Hence, we shall conclude that the quadruple $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ is in particular a local solution to Problem 2. We shall split the proof of this local existence result in two steps.

First, in Section 5.2 we shall obtain the existence of a local solution $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ to the Cauchy problem for the truncated system (5.2) by a Schauder fixed point argument (see Proposition 5.4), following the lines of the one developed in Sections 4.1–4.2. In particular, we shall construct the Schauder operator in two phases: we shall start by solving (5.2b) with ϑ and $\mathbf{u}$ fixed, and then proceed to solving (5.2a) and (5.2c) with χ fixed, see Section 5.2 below. However, relying on the present one-dimensional setting, we shall be in the position of developing the fixed point procedure in a functional framework *weaker* than the one in Section 4.1 (compare (4.4) with (5.4)).

Remark 5.1. Due to this weaker fixed point setting, proving the compactness of the solution operator associated with system (5.2) shall involve performing estimates on the solution component ϑ weaker than the ones performed in Section 4.2 (cf. the proof of Proposition 4.5). In particular, it will not be necessary to derive Eq. (5.2a). Therefore, the proof of local existence will be disentangled from the achievement of a two-sided separation inequality like (4.13) for the solution component χ. This brings about a major advantage: for proving local existence, we do not have to require the initial datum to be separated from 1 (cf. (3.26b)) anymore. The latter fact is important in view of extending the local solution beyond its life-time T_0 by global estimates and the standard ODE continuation argument: such a procedure can be indeed implemented without having the solution component χ at time T_0 separated from 1. Indeed, if the proof of local existence were based on the achievement of a separation inequality for the local solution χ both from 0 and from 1, in order to carry out the above mentioned extension procedure one should also prove that for all $t \in [0, T_0]$ $\chi(\cdot, t)$ stays separated from both potential barriers by a constant invariant from the initial time (we shall call *global in time* such a separation inequality, see also Remark 5.7). Note that this is not the case with the separation inequality (3.32) found in Theorem 1: the separation constant ς defined in (4.1) at the final time $\widehat{T}$ is strictly smaller than the one at time $t = 0$. In fact, we dispose of a method for obtaining *global in time* separation inequalities from below, only, see the proof of Lemma 5.6. As we have already said, this shall be sufficient in the present one-dimensional framework.

Secondly, in Section 5.3 we shall prove some further regularity for the triplet $(\hat{\vartheta}, \hat{\chi}, \hat{\xi})$, the positivity of $\hat{\vartheta}$, and the one-sided separation inequality (5.3) for $\hat{\chi}$ (the latter by an application of the weak maximum principle and substantially exploiting the "coercivity" condition (3.35)). We refer to Remark 5.7, highlighting the *global character* of (5.3).

Second step: *Extension procedure.* After finding a local solution $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ to Problem 2, we shall prove the existence of a *global* solution by a technique which is essentially tailored to extending to the whole interval $[0, T]$ the local solution $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ *along with* the separation inequality (3.44), see Remark 5.10. This procedure shall be developed at length in Section 5.5 and substantially relies on some *global estimates* (proved in Lemma 5.9) for those solutions to Problem 2 fulfilling the separation inequality (5.25), which in fact plays a key role.

While existence holds in the general case in which $\beta = \partial\hat{\beta}$ is a multivalued operator, we have been able to obtain uniqueness only for β single-valued, as a consequence of Proposition 4.8. Finally, we shall conclude the proof of Theorem 2 by showing the enhanced regularity (3.45).

5.2. *The Schauder fixed point argument revisited*

We fix $\mathrm{t} \in (0, T]$ and $R > 0$, and consider the balls

$$\begin{aligned} \mathcal{O}_{\mathrm{t}}^{\sharp} &:= \left\{\vartheta \in L^2\left(0, \mathrm{t}; L^2(0, \ell)\right): \|\vartheta\|_{L^2(0,\mathrm{t}; L^2(0,\ell))} \leqslant R\right\}, \\ \mathcal{U}_{\mathrm{t}} &:= \left\{\mathbf{u} \in H^1\left(0, \mathrm{t}; W_0^{1,4}(0, \ell)\right): \|\mathbf{u}\|_{H^1(0,\mathrm{t}; W_0^{1,4}(0,\ell))} \leqslant R\right\}. \end{aligned} \tag{5.4}$$

The following result parallels Lemma 4.2.

Lemma 5.2. *Assume* (3.15)–(3.16)*,* (3.24)*,* (3.34)*, and* (3.42a)*. Then, there exists a constant* $M_5 > 0$*, depending on* R *and on the problem data but independent of* $\mathrm{t} \in (0, T]$*, such that for all* $(\overline{\vartheta}, \bar{\mathbf{u}}) \in \mathcal{O}_{\mathrm{t}}^{\sharp} \times \mathcal{U}_{\mathrm{t}}$ *there exists a unique pair* (χ, ξ)*, with*

$$\begin{aligned}
&\chi \in H^1\big(0,\mathrm{t}; L^2(0,\ell)\big) \cap \mathrm{C}^0\big([0,\mathrm{t}]; H^1(0,\ell)\big) \cap L^2\big(0,\mathrm{t}; H^2_N(0,\ell)\big),\\
&\quad \chi \in \mathrm{dom}(\beta) \quad \textit{a.e. in } (0,\ell)\times(0,\mathrm{t}),\\
&\quad \xi \in L^2\big(0,\mathrm{t}; L^2(0,\ell)\big), \qquad \xi \in \beta(\chi) \quad \textit{a.e. in } (0,\ell)\times(0,\mathrm{t}),
\end{aligned} \tag{5.5}$$

fulfilling initial condition (3.2)*, the equation*

$$\chi_t + A\chi + \xi + \gamma(\chi) = \overline{\vartheta} + \frac{\varepsilon(\bar{\mathbf{u}})\mathcal{R}_e\varepsilon(\bar{\mathbf{u}})}{2} \quad \textit{a.e. in } (0,\ell)\times(0,\mathrm{t}), \tag{5.6}$$

and the estimate

$$\|\chi\|_{H^1(0,\mathrm{t};L^2(0,\ell))\cap L^\infty(0,\mathrm{t};H^1(0,\ell))\cap L^2(0,\mathrm{t};H^2_N(0,\ell))} + \|\xi\|_{L^2(0,\mathrm{t};L^2(0,\ell))} \leqslant M_5. \tag{5.7}$$

In fact, as it will be clear from the forthcoming proof, the statement of the above results holds true under the weaker condition that $\chi_0 \in H^1(0,\ell)$. We also point out that, thanks to inequality (3.11), the solution χ has the further regularity

$$\chi \in L^8\big(0,\mathrm{t}; W^{1,4}(0,\ell)\big). \tag{5.8}$$

Thanks to Lemma 5.2, for all $\mathrm{t} \in (0, T]$ the solution operator $\mathcal{T}_1^{\sharp}$ associated with the Cauchy problem for (5.6) is well defined on $\mathcal{O}_{\mathrm{t}}^{\sharp} \times \mathcal{U}_{\mathrm{t}}$, which is mapped into the set

$$\begin{aligned}
\mathcal{X}_{\mathrm{t}}^{\sharp} := \big\{&\chi \in H^1\big(0,\mathrm{t}; L^2(0,\ell)\big) \cap L^\infty\big(0,\mathrm{t}; H^1(0,\ell)\big) \cap L^2\big(0,\mathrm{t}; H^2_N(0,\ell)\big):\\
&\|\chi\|_{H^1(0,\mathrm{t};L^2(0,\ell))\cap L^\infty(0,\mathrm{t};H^1(0,\ell))\cap L^2(0,\mathrm{t};H^2_N(0,\ell))} \leqslant M_5\big\}.
\end{aligned}$$

We now solve (the Cauchy problem for) system (5.2a), (5.2c) with a fixed $\overline{\chi} \in \mathcal{X}_{\mathrm{t}}^{\sharp}$.

Lemma 5.3. *Assume* (3.21)–(3.23)*,* (3.25)*, and* (3.34)*. Then, there exists a constant* $M_6 > 0$*, only depending on* R*, on the constant* δ *specified by* (5.1)*, and on the problem data, but independent of* $\mathrm{t} \in (0, T]$*, such that for all* $\overline{\chi} \in \mathcal{X}_{\mathrm{t}}^{\sharp}$ *there exists a unique pair* $(\vartheta, \mathbf{u})$*,* $\mathbf{u}$ *with the regularity* (3.30) *and*

$$\vartheta \in H^1\big(0,\mathrm{t}; L^2(0,\ell)\big) \cap \mathrm{C}^0\big([0,\mathrm{t}]; H^1(0,\ell)\big) \cap L^2\big(0,\mathrm{t}; H^2_N(0,\ell)\big), \tag{5.9}$$

fulfilling

$$\vartheta_t + \overline{\chi}_t \vartheta + A\vartheta = g \quad \textit{a.e. in } (0,\ell) \times (0,\mathrm{t}), \tag{5.10}$$

$$\mathbf{u}_{tt} + \mathcal{H}\big((1-\overline{\chi})\mathbf{u}\big) + \mathcal{K}\big(T_\delta(\overline{\chi})\mathbf{u}_t\big) = \mathbf{f} \quad \textit{a.e. in } (0,\ell) \times (0,\mathrm{t}), \tag{5.11}$$

initial conditions (3.1) *and* (3.3)*, and the estimate*

$$\begin{cases} \|\vartheta\|_{H^1(0,\mathrm{t};L^2(0,\ell))\cap L^\infty(0,\mathrm{t};H^1(0,\ell))\cap L^2(0,\mathrm{t};H^2_N(0,\ell))} \\ \|\mathbf{u}\|_{H^2(0,\mathrm{t};L^2(0,\ell))\cap W^{1,\infty}(0,\mathrm{t};H^1_0(0,\ell))\cap H^1(0,\mathrm{t};H^2_0(0,\ell))} \end{cases} \leqslant M_6. \tag{5.12}$$

Again, we note that, in order to prove the above result, it is in fact sufficient to require $\vartheta_0 \in H^1(0,\ell)$ and $g \in L^2(0,T;L^2(0,\ell))$.

We conclude that it is possible to associate with the Cauchy problem for (5.10)–(5.11) a solution operator $\mathcal{T}_2^\sharp : \mathcal{X}_\mathrm{t}^\sharp \to L^2(0,\mathrm{t}; L^2(0,\ell)) \times H^1(0,\mathrm{t}; W_0^{1,4}(0,\ell))$. Therefore, the composition operator $\mathcal{T}^\sharp := \mathcal{T}_2^\sharp \circ \mathcal{T}_1^\sharp$ maps $\mathcal{O}_\mathrm{t}^\sharp \times \mathcal{U}_\mathrm{t}$ into the above product space.

Proposition 5.4. *Assume* (3.15)–(3.16)*,* (3.21)–(3.25)*,* (3.34)*,* (3.42a)*. Then, there exists* $0 < T_0 \leqslant T$ *such that*

$$\textit{the operator } \mathcal{T}^\sharp \textit{ maps } \mathcal{O}_{T_0}^\sharp \times \mathcal{U}_{T_0} \textit{ into itself.} \tag{5.13}$$

Further, $\mathcal{T}^\sharp : \mathcal{O}_{T_0}^\sharp \times \mathcal{U}_{T_0} \to \mathcal{O}_{T_0}^\sharp \times \mathcal{U}_{T_0}$ *is compact and continuous with respect to the topology of* $L^2(0,T_0;L^2(0,\ell)) \times H^1(0,T_0;W_0^{1,4}(0,\ell))$.

5.2.1. Proofs

Proof of Lemma 5.2. We notice that for every $(\overline{\vartheta}, \overline{\mathbf{u}}) \in \mathcal{O}_\mathrm{t} \times \mathcal{U}_\mathrm{t}$ the term on the right-hand side of (4.5) is in $L^2(0,\mathrm{t};L^2(0,\ell))$. Then, by the same considerations as in the proof of Lemma 4.2 (which trivially adapt to the case in which β is a multivalued operator), we conclude the thesis. □

Proof of Lemma 5.3. *Analysis of* (5.11). The well-posedness on a generic interval $(0,\mathrm{t})$ of the Cauchy problem for (5.11) can be inferred following the same lines as in the analysis of Eq. (4.9), cf. Lemma 4.4. Further, the proof of estimate (4.25) goes through in this setting by the same computations as in (4.23)–(4.24). Therefore, here we shall just show how the proof of estimate (4.34) (hence of the estimate for $\mathbf{u}_{tt}$ in $L^2(0,\mathrm{t};L^2(0,\ell))$ by comparison), adapts to this case, in which $\overline{\chi}$ has the regularity (5.5), weaker than (4.6). Again, we (formally) test (5.11) by $-\mathrm{div}(\varepsilon(\mathbf{u}_t))$ and integrate in time. We repeat (4.26) and we estimate

$$-\int_0^t \int_0^\ell \mathcal{H}\big((1-\overline{\chi})\mathbf{u}\big)\,\mathrm{div}\big(\varepsilon(\mathbf{u}_t)\big) = I_{17} + I_{18}. \tag{5.14}$$

Now, denoting by $\partial_x \mathbf{u}$, $\partial^2_{xx}\mathbf{u}$, $\partial^2_{xx}\mathbf{u}_t$ the vectors of the first/second derivatives of $\mathbf{u}$, $\mathbf{u}_t$ w.r.t. x, we have with straightforward computations

$$\begin{aligned}
|I_{17}| &\leqslant C\int_0^t \|\partial_x \overline{\chi}\|_{L^2(0,\ell)} \big\|\partial^2_{xx}\mathbf{u}_t\big\|_{L^2(0,\ell)} \|\partial_x \mathbf{u}\|_{L^\infty(0,\ell)} \\
&\leqslant C\|\overline{\chi}\|_{L^\infty(0,\mathrm{t};H^1(0,\ell))}\int_0^t \|\mathbf{u}_t\|_{H^2(0,\ell)}\|\mathbf{u}\|_{H^2(0,\ell)} \\
&\leqslant \epsilon\int_0^t \|\mathbf{u}_t\|^2_{H^2(0,\ell)} + C_{1,\epsilon}\int_0^t \|\mathbf{u}\|^2_{H^2(0,\ell)},
\end{aligned} \tag{5.15}$$

where the second inequality follows from (3.13), while ϵ will be specified later on, and $C_{1,\epsilon}$ also depends on the constant M_5 in (5.7). In the same way, we compute

$$\begin{aligned}
|I_{18}| &\leqslant C\int_0^t \|\overline{\chi}\|_{L^\infty(0,\ell)} \big\|\partial^2_{xx}\mathbf{u}\big\|_{L^2(0,\ell)} \big\|\partial^2_{xx}\mathbf{u}_t\big\|_{L^2(0,\ell)} \\
&\leqslant C\|\overline{\chi}\|_{L^\infty(0,\mathrm{t};H^1(0,\ell))}\int_0^t \|\mathbf{u}\|_{H^2(0,\ell)}\|\mathbf{u}_t\|_{H^2(0,\ell)} \\
&\leqslant \epsilon\int_0^t \|\mathbf{u}_t\|^2_{H^2(0,\ell)} + C_{2,\epsilon}\int_0^t \|\mathbf{u}\|^2_{H^2(0,\ell)},
\end{aligned} \tag{5.16}$$

the third passage again due to (3.13). Then, following (4.30), we calculate

$$-\int_0^t\int_0^\ell \mathcal{K}\big(T_\delta(\overline{\chi})\mathbf{u}_t\big)\operatorname{div}\big(\varepsilon(\mathbf{u}_t)\big) = I_{19} + I_{20}. \tag{5.17}$$

Indeed,

$$I_{19} \geqslant C\int_0^t\int_0^\ell T_\delta(\overline{\chi})\big|\partial^2_{xx}\mathbf{u}_t\big|^2 \geqslant \varLambda\int_0^t \|\mathbf{u}_t\|^2_{H^2_0(0,\ell)}, \tag{5.18}$$

where the second inequality follows by standard elliptic regularity results and the constant $\varLambda$ therein also depends on δ. Further,

$$\begin{aligned}
|I_{20}| &\leqslant C\int_0^t\int_0^\ell \big|\partial_x T_\delta(\overline{\chi})\big||\partial_x\mathbf{u}_t|\big|\partial^2_{xx}\mathbf{u}_t\big| \\
&\leqslant \epsilon\int_0^t \|\mathbf{u}_t\|^2_{H^2_0(0,\ell)} + C_{3,\epsilon}\int_0^t \big\|\partial_x T_\delta(\overline{\chi})\big\|^2_{L^4(0,\ell)}\|\partial_x\mathbf{u}_t\|_{L^2(0,\ell)}\|\partial_x\mathbf{u}_t\|_{L^\infty(0,\ell)} \\
&\leqslant 2\epsilon\int_0^t \|\mathbf{u}_t\|^2_{H^2_0(0,\ell)} + C_{4,\epsilon}\int_0^t \big\|\partial_x T_\delta(\overline{\chi})\big\|^4_{L^4(0,\ell)}\|\partial_x\mathbf{u}_t\|^2_{L^2(0,\ell)},
\end{aligned} \tag{5.19}$$

where in the second passage we have exploited the same trick as in (4.32), and, in the third one, we have again used (3.13), so that $C_{4,\epsilon}$ depends on the embedding constant of $H^2(0,\ell) \subset W^{1,\infty}(0,\ell)$ as well. Finally, we repeat (4.33). We now combine the latter with (4.26) and (5.14)–(5.19), in which we choose $\epsilon \leqslant \Lambda/8$. Then, handling the integral terms containing $\|\mathbf{u}\|^2_{H^2_0(0,\ell)}$ in the same way as in (4.24), and applying the Gronwall Lemma, we obtain the desired estimate

$$\|\mathbf{u}_t\|_{L^\infty(0,\mathrm{t};H^1_0(0,\ell))\cap L^2(0,\mathrm{t};H^2_0(0,\ell))} \leqslant C.$$

Analysis of (5.10). In this one-dimensional framework, the regularity $\overline{\chi}_t \in L^2(0,\mathrm{t};L^2(0,\ell))$ guarantees that the assumptions of [1, Theorem 3.2] are fulfilled. Thus, we again conclude that the Cauchy problem for (5.10) (in its weak formulation) has a unique solution $\vartheta \in H^1(0,\mathrm{t};H^1(0,\ell)') \cap \mathrm{C}^0([0,\mathrm{t}];L^2(0,\ell)) \cap L^2(0,\mathrm{t};H^1(0,\ell))$. In order to prove (5.9), like in the proof of Lemma 4.4 we test (5.10) by ϑ_t and integrate in time. Adding $1/2|\vartheta(t)|^2$ to both sides, we obtain

$$\begin{aligned}
&\int_0^t |\vartheta_t|^2 + \frac{1}{2}\big\|\vartheta(t)\big\|^2_{H^1(0,\ell)} \\
&\quad\leqslant \frac{1}{2}\|\vartheta_0\|^2_{H^1(0,\ell)} + \frac{1}{2}\big|\vartheta(t)\big|^2 + \int_0^t |g|^2 + \frac{1}{4}\int_0^t |\vartheta_t|^2 + \int_0^t |\overline{\chi}_t|\|\vartheta\|_{L^\infty(0,\ell)}|\vartheta_t| \\
&\quad\leqslant \frac{1}{2}\|\vartheta_0\|^2_{H^1(0,\ell)} + \frac{1}{2}\big|\vartheta(t)\big|^2 + \int_0^t |g|^2 + \frac{1}{2}\int_0^t |\vartheta_t|^2 + C\int_0^t |\overline{\chi}_t|^2\|\vartheta\|^2_{H^1(0,\ell)},
\end{aligned} \tag{5.20}$$

where the above constant C also takes into account the embedding constant of $H^1(0,\ell) \subset L^\infty(0,\ell)$, see (3.13). Then, applying the Gronwall Lemma, we conclude an estimate for ϑ in $H^1(0,\mathrm{t};L^2(0,\ell)) \cap L^\infty(0,\mathrm{t};H^1(0,\ell))$, using which it may be inferred that $\overline{\chi}_t\vartheta \in L^2(0,\mathrm{t};L^2(0,\ell))$. Arguing in the same way as in the proof of Lemma 4.4, we then deduce an estimate for $\vartheta \in L^2(0,\mathrm{t};H^2_N(0,\ell))$, leading to (5.12). □

Proof of Proposition 5.4. In order to prove the first part of the statement, we argue in the same way as in the proof of Proposition 4.5: thanks to (5.12), (4.54) holds in this setting as well, and, in place of (4.52), we have

$$\|\vartheta\|_{L^2(0,\mathrm{t};L^2(\Omega))} \leqslant \mathrm{t}^{1/2}\|\vartheta\|_{L^\infty(0,\mathrm{t};L^2(\Omega))} \leqslant \mathrm{t}^{1/2}M_6 \quad \forall \mathrm{t} \in (0,T].$$

Combining this with (4.54), we infer that there exists T_0 for which (5.13) is fulfilled.

Thanks to estimates (5.7) and (5.12) and the aforementioned [33, Theorem 4, Corollary 5], $\mathcal{T}^\sharp$ is a compact operator w.r.t. the topology of $L^2(0,T_0;L^2(0,\ell)) \times H^1(0,T_0;W^{1,4}_0(0,\ell))$. We show that $\mathcal{T}^\sharp$ is continuous in the same way as in the proof of Claim 3 in Proposition 4.5, namely we tackle the continuity of operators $\mathcal{T}^\sharp_1$ and $\mathcal{T}^\sharp_2$ separately.

First, we fix a sequence

$$\big\{(\overline{\vartheta}_n,\bar{\mathbf{u}}_n)\big\} \text{ converging to } (\overline{\vartheta},\bar{\mathbf{u}}) \quad \text{in } L^2\big(0,T_0;L^2(0,\ell)\big) \times H^1\big(0,T_0;W^{1,4}_0(0,\ell)\big), \tag{5.21}$$

and we let $\chi_n := \mathcal{T}_1^\sharp(\overline{\vartheta}_n, \bar{\mathbf{u}}_n)$ for all $n \in \mathbb{N}$; accordingly, we consider the associated sequence $\{\xi_n\}$. By (5.7), $\{\chi_n\}$ is bounded in $H^1(0,T_0;L^2(0,\ell)) \cap L^\infty(0,T_0;H^1(0,\ell)) \cap L^2(0,T_0;H^2_N(0,\ell))$ and $\{\xi_n\}$ in $L^2(0,T_0;L^2(0,\ell))$. Again using [33, Theorem 4, Corollary 5] and the Ascoli–Arzelà compactness theorem, we obtain that there exist a subsequence, which we do not relabel, and a pair $(\overline{\chi}, \bar{\xi})$ such that the following convergences hold as $n \uparrow \infty$ for all $1 \leqslant p < \infty$ and for all $\rho > 0$

$$\begin{aligned}
\chi_n \to \overline{\chi} \quad & \text{in } \mathrm{C}^0\big(0,T_0;H^{1-\rho}(0,\ell)\big) \cap L^p\big(0,T_0;H^1(0,\ell)\big) \cap L^2\big(0,T_0;H^{2-\rho}(0,\ell)\big),\\
\chi_n \rightharpoonup^* \overline{\chi} \quad & \text{in } H^1\big(0,T_0;L^2(0,\ell)\big) \cap L^\infty\big(0,T_0;H^1(0,\ell)\big) \cap L^2\big(0,T_0;H^2_N(0,\ell)\big),\\
\xi_n \rightharpoonup \bar{\xi} \quad & \text{in } L^2\big(0,T_0;L^2(0,\ell)\big).
\end{aligned} \tag{5.22}$$

Hence, $\overline{\chi}$ fulfils initial condition (3.2) and, again by the strong-weak closedness of β, we conclude that $\bar{\xi} \in \beta(\overline{\chi})$ a.e. in $(0,\ell)\times(0,T_0)$. Using (5.21), it is easy to pass to the limit in Eq. (5.6) and conclude that $\overline{\chi} = \mathcal{T}_1^\sharp(\overline{\vartheta}, \bar{\mathbf{u}})$, so that convergences (5.22) hold along the *whole* sequence $\{\chi_n\}$.

Concerning $\mathcal{T}_2^\sharp$, we let $(\vartheta_n, \mathbf{u}_n) := \mathcal{T}_2^\sharp(\chi_n) = \mathcal{T}(\overline{\vartheta}_n, \bar{\mathbf{u}}_n)$. In view of estimate (5.12), one concludes that $\{\mathbf{u}_n\}$ converges to some $\mathbf{u} \in H^2(0,T_0;L^2(0,\ell)) \cap W^{1,\infty}(0,T_0;H^1_0(0,\ell)) \cap H^1(0,T_0;H^2_0(0,\ell))$, up to a subsequence, in the topologies specified by (4.59), whereas $\{\vartheta_n\}$ converges to some $\vartheta \in H^1(0,T_0;L^2(0,\ell)) \cap L^\infty(0,T_0;H^1(0,\ell)) \cap L^2(0,T_0;H^2_N(0,\ell))$ in the sense specified by the first two lines of (5.22). In particular,

$$(\vartheta_n, \mathbf{u}_n) \to (\vartheta, \mathbf{u}) \quad \text{in } L^2\big(0,T_0;L^2(0,\ell)\big) \times H^1\big(0,T_0;W^{1,4}_0(0,\ell)\big). \tag{5.23}$$

Thanks to (5.22) and (4.59), we conclude that the limit pair $(\mathbf{u}, \overline{\chi})$ satisfies (5.11) and that $\mathbf{u}$ complies with initial condition (3.3). In the same way, convergences (5.22) for $\{\chi_n\}$ and $\{\vartheta_n\}$ enable to conclude that $(\vartheta, \overline{\chi})$ fulfils (5.10), and that ϑ fulfils (3.1). In the end, we have that

$$(\vartheta, \mathbf{u}) = \mathcal{T}_2^\sharp(\overline{\chi}) = \mathcal{T}^\sharp(\overline{\vartheta}, \bar{\mathbf{u}}), \tag{5.24}$$

and that (5.22) and (4.59) hold along the whole sequences $\{\vartheta_n\}$ and $\{\mathbf{u}_n\}$. In view of (5.23)–(5.24), we find that the operator $\mathcal{T}^\sharp$ is continuous. □

Remark 5.5. It follows from Proposition 5.4 and the Schauder fixed point theorem that the Cauchy problem for the truncated system (5.2) has a local solution $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ on the time interval $(0,T_0)$, with the regularity (5.9) for $\hat{\vartheta}$, (5.5) for $(\hat{\chi}, \hat{\xi})$, and (3.30) for $\hat{\mathbf{u}}$. Let us point out that, as we have already mentioned, it could be possible to prove this intermediate result under weaker assumption on the data ϑ_0, χ_0, and g.

5.3. Regularity and lower bounds for local solutions

Now, we prove that the solution components $\hat{\vartheta}$, $\hat{\chi}$, and $\hat{\xi}$ in fact have some further regularity. As a by-product, we obtain a "global-in-time" lower bound both for $\hat{\vartheta}$ and for $\hat{\chi}$.

Lemma 5.6. *Assume* (3.15)–(3.16), (3.21)–(3.25), (3.26a), (3.34), (3.35), (3.42a)–(3.42b). *Then, the components* $\hat\vartheta$, $\hat\chi$, *and* $\hat\xi$ *of the local solution* $(\hat\vartheta, \hat\chi, \hat\xi, \hat{\mathbf{u}})$ *to Problem* 2 *on* $(0, T_0)$ *have the further regularity* (3.36)–(3.38). *Moreover,*

$$\hat\chi(x,t) \geqslant \delta > 0 \quad \forall (x,t) \in [0,\ell] \times [0,T_0] \tag{5.25}$$

(δ *being the constant specified by* (5.1)), *and there exists a constant* $\theta_* > 0$, *depending on the problem data, such that*

$$\hat\vartheta(x,t) \geqslant \theta_* > 0 \quad \forall (x,t) \in [0,\ell] \times [0,T_0]. \tag{5.26}$$

Remark 5.7. Separation inequality (5.25) for $\hat\chi$ has a substantially different character that (4.13), obtained in the course of the proof of the local existence Theorem 1. Indeed, in the latter case the solution life-time depends on the separation constant for the initial datum χ_0, see also Remark 4.6, and, on the other hand, from the initial time $t = 0$ up to the final time the separation inequality has “deteriorated,” see (4.1) and (4.42). These are the reasons why it was not possible to extend inequality (4.13) to the whole time interval $[0, T]$.

On the contrary, inequality (5.26) holds invariantly since time $t = 0$. As it will be clear from the proof below, one could in fact prove that, *for any* local solution $(\tilde\vartheta, \tilde\chi, \tilde\xi, \tilde{\mathbf{u}})$ to the Cauchy problem for system (5.2) on some interval (0, t), there holds

$$\tilde\chi(x,t) \geqslant \delta > 0 \quad \forall (x,t) \in [0,\ell] \times [0,\mathrm{t}].$$

In this sense, (5.25) is a *global-in-time* separation inequality.

Proof of Lemma 5.6. First of all, we sketch the (formal) estimates needed to conclude the further regularity of $\hat\chi$ and $\hat\xi$. We test (5.2b) by $(A\hat\chi + \hat\xi)_t$ and integrate in time. Now, due to the regularities (3.30) and (5.9) of $\hat{\mathbf{u}}$ and $\hat\vartheta$, the right-hand side of (5.2b) is in $H^1(0, T_0; L^2(0,\ell))$. Then, the very same computations developed throughout (4.19)–(4.22) enable us to conclude an estimate for $\hat\chi$ in $H^1(0, T_0; H^1(0,\ell)) \cap L^\infty(0, T_0; H^2_N(0,\ell))$ and for $\hat\xi$ in $L^\infty(0, T_0; L^2(0,\ell))$. Arguing by comparison in (5.2b), we finally infer regularity (3.37) for $\hat\chi$.

Next, we (formally) test (5.2a) by $(A\hat\vartheta)_t$ and integrate on some interval $[0, t]$, $t \in (0, T_0]$. Adding $\int_0^t |\hat\vartheta_t|^2$ to both sides, we get (compare with (4.38))

$$\int_0^t \|\hat\vartheta_t\|^2_{H^1(0,\ell)} + \frac{1}{2}\big|A\hat\vartheta(t)\big|^2 \leqslant \frac{1}{2}\|\vartheta_0\|^2_{H^2(0,\ell)} + \int_0^t |\hat\vartheta_t|^2 + I_{21} + I_{22} + I_{23}, \tag{5.27}$$

where $I_{21} = |\int_0^t g(A\hat\vartheta)_t|$ is estimated with an easy integration by parts, in the same way as in (4.39), while, recalling (3.13), we have

$$I_{22} = \left| \int_0^t \int_0^\ell \hat{\vartheta}_x \hat{\chi}_t (\hat{\vartheta}_x)_t \right| \leqslant \int_0^t \|\hat{\vartheta}_x\|_{L^\infty(0,\ell)} |\hat{\chi}_t| \left|(\hat{\vartheta}_x)_t\right|$$

$$\leqslant C \int_0^t |\hat{\chi}_t|^2 \|\hat{\vartheta}\|^2_{H^2(0,\ell)} + \frac{1}{4} \int_0^t \|\hat{\vartheta}_t\|^2_{H^1(0,\ell)} \tag{5.28}$$

and, finally, again due to (3.13) we estimate

$$I_{23} = \left| \int_0^t \int_0^\ell \hat{\vartheta} (\hat{\chi}_x)_t (\hat{\vartheta}_x)_t \right| \leqslant \int_0^t \|\hat{\vartheta}\|_{L^\infty(0,\ell)} \left|(\hat{\chi}_x)_t\right| \left|(\hat{\vartheta}_x)_t\right|$$

$$\leqslant C \int_0^t \|\hat{\chi}_t\|^2_{H^1(0,\ell)} \|\hat{\vartheta}\|^2_{H^1(0,\ell)} + \frac{1}{4} \int_0^t \|\hat{\vartheta}_t\|^2_{H^1(0,\ell)}. \tag{5.29}$$

Then, collecting (5.27)–(5.29), exploiting (3.21), the previous estimate (5.12) on $\hat{\vartheta}$, and the fact that $\|\hat{\chi}_t\|_{L^2(0,T_0;H^1(0,\ell))} \leqslant C$, upon an application of the Gronwall Lemma we conclude an estimate for $\hat{\vartheta}$ in $H^1(0,T_0;H^1(0,\ell)) \cap L^\infty(0,T_0;H^2_N(0,\ell))$, hence in $W^{1,\infty}(0,T_0;L^2(0,\ell))$ by comparison.

In order to prove inequality (5.26), we employ a refined version of the maximum principle for parabolic equations proved in [22]. Indeed, since $\hat{\chi}_t$ is estimated in $L^2(0,T_0;L^\infty(0,\ell))$ by (3.37) and (3.13), applying [22, Proposition 3.6, p. 10] we get

$$\hat{\vartheta}(x,t) \geqslant \min_{x \in [0,\ell]} \vartheta_0(x) \exp\left(- \int_0^{T_0} \left\|\hat{\chi}_t(s)\right\|_{L^\infty(0,\ell)} \mathrm{d}s \right).$$

We are now in the position of proving inequality (5.25). Recalling our choice of δ (5.1), we subtract the term $\beta^0(\delta) + \gamma(\delta)$ from both sides of (3.40), test the resulting equation by $-(\hat{\chi} - \delta)^-$, and integrate on some interval $(0,t)$, $t \in (0,T_0)$. Elementary computations yield

$$\frac{1}{2}\left|(\hat{\chi}(t) - \delta)^-\right|^2 + \int_0^t \left|\left((\hat{\chi} - \delta)^-\right)_x\right|^2 - \int_0^t \int_0^\ell \left(\hat{\xi} - \beta^0(\delta)\right)(\hat{\chi} - \delta)^-$$

$$= \frac{1}{2}\left|(\chi_0 - \delta)^-\right|^2 + \int_0^t \int_0^\ell \left(\gamma(\hat{\chi}) - \gamma(\delta)\right)(\hat{\chi} - \delta)^-$$

$$- \int_0^t \int_0^\ell \left(\hat{\vartheta} + \frac{\varepsilon(\hat{\mathbf{u}})\mathcal{R}_e\varepsilon(\hat{\mathbf{u}})}{2} - \beta^0(\delta) - \gamma(\delta) \right)(\hat{\chi} - \delta)^-. \tag{5.30}$$

Note that the third term on the left-hand side of (5.30) is nonnegative by the monotonicity of β. In the same way, the last term on the right-hand side is nonpositive since $\hat{\vartheta} + \varepsilon(\hat{\mathbf{u}})\mathcal{R}_e\varepsilon(\hat{\mathbf{u}}) \geqslant 0$,

$\mathcal{R}_e$ being positive definite, and thanks to (5.1), yielding that $(\chi_0(x)-\delta)^- = 0$ for all $x \in (0,\ell)$. Hence, using that $\gamma \in \mathrm{C}^1([0,1])$ we deduce from (5.30) that

$$\frac{1}{2}\big|\big(\hat{\chi}(t)-\delta\big)^-\big|^2 \leqslant \|\gamma'\|_{L^\infty(0,1)} \int_0^t \big|\big(\hat{\chi}(s)-\delta\big)^-\big|^2 \,\mathrm{d}s$$

and the Gronwall Lemma leads to $|(\hat{\chi}(t)-\delta)^-| = 0$ for all $t \in [0,T_0]$, whence (5.25). □

5.4. Global estimates

The notion of δ-separated solution. Thanks to the separation inequality (5.25) proved in Lemma 5.6, the (local) solution component $\hat{\chi}$ fulfils

$$T_\delta(\hat{\chi})(x,t) = \hat{\chi}(x,t) \quad \forall (x,t) \in [0,\ell] \times [0,T_0].$$

As a result, the quadruple $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ is in fact a solution of Problem 2 in $(0,T_0)$. As we have already mentioned, inequality (5.25) shall play a crucial role for extending $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$ to a solution on the whole interval $(0,T)$. We now introduce a notion of local solution which generalizes the properties of $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$, retaining (5.25).

Definition 5.8. We say that a quadruple $(\vartheta, \chi, \xi, \mathbf{u})$ is a *δ-separated solution* of Problem 2 on some interval $(0,\mathrm{t})$, $0 < \mathrm{t} \leqslant T$, if

$$\begin{aligned}
&(\vartheta,\chi,\xi,\mathbf{u}) \text{ has the regularity (3.36)–(3.39), solves Problem 2 on } (0,\ell)\times(0,\mathrm{t}),\\
&\chi \text{ satisfies (5.25) on } (0,\mathrm{t}), \text{ and } \min_{(x,t)\in[0,\ell]\times[0,\mathrm{t}]} \vartheta(x,t) > 0.
\end{aligned}$$

Global estimates for δ-separated solutions.

Lemma 5.9 *(Global estimates). In the setting of* (3.34)*, assume* (3.15)–(3.16)*,* (3.21)–(3.25)*,* (3.26a)*,* (3.35)*, and* (3.42a)–(3.42b)*. Then, there exists a constant $M_7 > 0$, only depending on the problem data and on δ but* independent *of* $\mathrm{t} \in (0,T]$*, such that for any δ*-separated solutions $(\vartheta, \chi, \xi, \mathbf{u})$ *of Problem* 2 *on the interval* $(0,\mathrm{t})$ *there holds*

$$\begin{aligned}
&\|\chi\|_{W^{1,\infty}(0,\mathrm{t};L^2(0,\ell))\cap H^1(0,\mathrm{t};H^1(0,\ell))\cap L^\infty(0,\mathrm{t};H^2_N(0,\ell))} + \|\xi\|_{L^\infty(0,\mathrm{t};L^2(0,\ell))}\\
&\quad + \|\vartheta\|_{W^{1,\infty}(0,\mathrm{t};L^2(0,\ell))\cap H^1(0,\mathrm{t};H^1(0,\ell))\cap L^\infty(0,\mathrm{t};H^2_N(0,\ell))}\\
&\quad + \|\mathbf{u}\|_{H^2(0,\mathrm{t};L^2(0,\ell))\cap W^{1,\infty}(0,\mathrm{t};H^1_0(0,\ell))\cap H^1(0,\mathrm{t};H^2_0(0,\ell))}\\
&\leqslant M_7\big(1 + \|\chi_0\|_{H^2(0,\ell)} + \big\|\hat{\beta}(\chi_0)\big\|_{L^1(0,\ell)} + \big|\beta^0(\chi_0)\big| + \|\vartheta_0\|_{H^2(0,\ell)}\\
&\quad + \|\mathbf{u}_0\|_{H^2(0,\ell)} + \|\mathbf{v}_0\|_{H^1(0,\ell)} + \|g\|_{H^1(0,T;L^2(0,\ell))} + \|\mathbf{f}\|_{L^2(0,T;L^2(0,\ell))}\big).
\end{aligned} \tag{5.31}$$

Proof. Throughout this proof, we shall denote by S_i, $i = 1,\ldots,$ some positive constant only depending on δ and on the quantities $\|\chi_0\|_{H^2(0,\ell)}$, $\|\hat{\beta}(\chi_0)\|_{L^1(0,\ell)}$, $|\beta^0(\chi_0)|$, $\|\vartheta_0\|_{H^2(0,\ell)}$, $\|\mathbf{u}_0\|_{H^2(0,\ell)}$, $\|\mathbf{v}_0\|_{H^1(0,\ell)}$, $\|g\|_{H^1(0,T;L^2(0,\ell))}$, and $\|\mathbf{f}\|_{L^2(0,T;L^2(0,\ell))}$, but independent of t.

First estimate. We test (3.20) by $\mathbf{u}_t$ and integrate in time. Exploiting (3.5), (3.7), and the separation inequality (5.25), and developing the very same computations as in the proof of Lemma 4.4 (see (4.23)–(4.25)) we obtain that there exists a positive constant S_1 such that

$$\|\mathbf{u}\|_{W^{1,\infty}(0,t;L^2(0,\ell))\cap H^1(0,t;H_0^1(0,\ell))} \leqslant S_1. \tag{5.32}$$

Second estimate. We test (3.18) by 1, (3.40) by χ_t, add the resulting relations and integrate in time. Two terms cancel out and, recalling the positivity of ϑ (according to Definition 5.8), we get

$$\begin{aligned} &\big\|\vartheta(t)\big\|_{L^1(0,\ell)} + \int_0^t |\chi_t|^2 + \frac{1}{2}\big|\chi_x(t)\big|^2 + \int_0^\ell \hat{\beta}\big(\chi(t)\big) \\ &\quad \leqslant \|\vartheta_0\|_{L^1(0,\ell)} + \frac{1}{2}\|\chi_0\|^2_{H^1(0,\ell)} + \big\|\hat{\beta}(\chi_0)\big\|_{L^1(0,\ell)} + \|g\|_{L^1(0,T;L^1(0,\ell))} + I_{24} + I_{25}, \end{aligned} \tag{5.33}$$

where

$$\begin{aligned} I_{24} &:= \int_0^t \big|\gamma(\chi)\big||\chi_t| \leqslant \int_0^t \big(\|\gamma'\|_{L^\infty(0,1)}|\chi| + \big|\gamma(0)\big|\ell^{1/2}\big)|\chi_t| \\ &\leqslant \frac{1}{4}\int_0^t |\chi_t|^2 + C\left(|\chi_0|^2 + \int_0^t \left(\int_0^s |\chi_t|^2\right) \mathrm{ds}\right) \end{aligned} \tag{5.34}$$

and, by a trivial integration by parts,

$$\begin{aligned} I_{25} &:= \int_0^t \int_0^\ell \chi_t \frac{\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} \\ &= -\int_0^t\int_0^\ell \chi\varepsilon(\mathbf{u}_t)\mathcal{R}_e\varepsilon(\mathbf{u}) + \int_0^\ell \chi(t)\frac{\varepsilon(\mathbf{u}(t))\mathcal{R}_e\varepsilon(\mathbf{u}(t))}{2} - \int_0^\ell \chi_0 \frac{\varepsilon(\mathbf{u}_0)\mathcal{R}_e\varepsilon(\mathbf{u}_0)}{2} \\ &\leqslant \frac{C}{2}\|\mathbf{u}\|^2_{L^\infty(0,t;H_0^1(0,\ell))} + \frac{C}{4}\int_0^t \|\mathbf{u}\|^2_{H^1(0,\ell)} + \frac{1}{4}\int_0^t \|\mathbf{u}_t\|^2_{H^1(0,\ell)}. \end{aligned} \tag{5.35}$$

On the other hand, by the convexity of $\hat{\beta}$ there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\int_0^\ell \hat{\beta}\big(\chi(t)\big) \geqslant -c_1\big\|\chi(t)\big\|_{L^1(0,\ell)} - c_2 \geqslant -C. \tag{5.36}$$

Combining (5.33)–(5.36), recalling estimate (5.32) and applying the Gronwall Lemma, we deduce that for some positive constant S_2

$$\|\vartheta\|_{L^\infty(0,\mathrm{t};L^1(0,\ell))}+\|\chi\|_{H^1(0,\mathrm{t};L^2(0,\ell))\cap L^\infty(0,\mathrm{t};H^1(0,\ell))}\leqslant S_2. \tag{5.37}$$

Third estimate. On account (3.15), we get

$$\left\|\gamma(\chi)\right\|_{L^\infty(0,\mathrm{t};L^\infty(0,\ell))}\leqslant C.$$

Combining this bound with estimates (5.32) and (5.37) and arguing by comparison in Eq. (3.40), we deduce that

$$\|A\chi+\xi\|_{L^2(0,\mathrm{t};L^1(0,\ell))}\leqslant C.$$

Now, with an easy argument based on the monotonicity of the operator β, it is not difficult to prove that $\|-A\chi+\xi\|_{L^1(\Omega)}$ bounds the norm $\|-A\chi\|_{L^1(\Omega)}$ (cf. also with (4.22)). Hence, we conclude that

$$\|\chi\|_{L^2(0,\mathrm{t};W^{2,1}(0,\ell))}\leqslant C,$$

which, via the Gagliardo–Nirenberg inequality (3.11), leads us to

$$\|\chi\|_{L^4(0,\mathrm{t};W^{1,4}(0,\ell))}\leqslant S_3. \tag{5.38}$$

Fourth estimate. We test (3.20) by $-\mathrm{div}(\varepsilon(\mathbf{u}_t))$ and integrate in time. We refer to the proof of Lemma 5.3 for the full development of the calculations (see (5.14)–(5.19)). Here, we just point out that (5.18) carries over to this framework as well, thanks to the separation inequality (5.25). Moreover, after repeating (5.14)–(5.19) we are in the position of applying the Gronwall Lemma in the same way as in the proof of Lemma 5.3, since the term $\|\partial_x T_\delta(\chi)\|^4_{L^4(0,\ell)}$ appearing in the last integral on the right-hand side of (5.19) is estimated in $L^1(0,\mathrm{t})$ due to (5.38). Therefore, we conclude a bound for the norm of $\mathbf{u}_t$ in $L^\infty(0,\mathrm{t};\,H^1_0(0,\ell))\cap L^2(0,\mathrm{t};\,H^2_0(0,\ell))$. Hence, by a comparison argument in (3.20) we have

$$\|\mathbf{u}\|_{H^2(0,\mathrm{t};L^2(0,\ell))\cap W^{1,\infty}(0,\mathrm{t};H^1_0(0,\ell))\cap H^1(0,\mathrm{t};H^2_0(0,\ell))}\leqslant S_4. \tag{5.39}$$

Fifth estimate. We multiply (3.18) by $(\vartheta+\vartheta_t)$ and integrate in time. With straightforward calculations, we get

$$\begin{aligned}&\frac{1}{2}\left\|\vartheta(t)\right\|^2_{H^1(0,\ell)}+\int_0^t|\vartheta_t|^2+\int_0^t|\nabla\vartheta|^2\\&\quad\leqslant\frac{1}{2}\|\vartheta_0\|^2_{H^1(0,\ell)}+2\|g\|^2_{L^2(0,T;L^2(0,\ell))}+\frac{1}{4}\int_0^t|\vartheta_t|^2+\frac{1}{4}\int_0^t|\vartheta|^2+I_{26}+I_{27},\end{aligned}$$

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where, also exploiting (3.13),

$$I_{26} = \int_0^t |\chi_t| \|\vartheta\|_{L^\infty(0,\ell)} |\vartheta_t| \leqslant C \int_0^t |\chi_t|^2 \|\vartheta\|^2_{H^1(0,\ell)} + \frac{1}{4} \int_0^t |\vartheta_t|^2,$$

$$I_{27} = \int_0^t |\chi_t| \|\vartheta\|_{L^\infty(0,\ell)} |\vartheta| \leqslant C \int_0^t |\chi_t| \|\vartheta\|^2_{H^1(0,\ell)}.$$

Collecting the above estimates, recalling estimate (5.37) for χ_t in $L^2(0, \mathrm{t}; L^2(0, \ell))$ and applying the Gronwall Lemma, we find

$$\|\vartheta\|_{H^1(0,\mathrm{t};L^2(0,\ell)) \cap L^\infty(0,\mathrm{t};H^1(0,\ell))} \leqslant S_5. \tag{5.40}$$

Sixth estimate. We (formally) test (3.40) by $(A\chi + \xi)_t$ and integrate in time. Hence, we develop the very same computations as in the proof of Lemma 5.6 (see also (4.19)–(4.22) in the proof of Lemma 4.2). Taking into account (5.39) and (5.40), we easily infer an estimate for χ in $H^1(0, \mathrm{t}; H^1(0, \ell)) \cap L^\infty(0, \mathrm{t}; H^2_N(0, \ell))$. A comparison in (3.40) yields

$$\|\chi\|_{W^{1,\infty}(0,\mathrm{t};L^2(0,\ell)) \cap H^1(0,\mathrm{t};H^1(0,\ell)) \cap L^\infty(0,\mathrm{t};H^2_N(0,\ell))} + \|\xi\|_{L^\infty(0,\mathrm{t};L^2(0,\ell))} \leqslant S_6. \tag{5.41}$$

Seventh estimate. We (formally) test (3.18) by $(A\vartheta)_t$ and integrate in time. We repeat the very same calculations (5.27)–(5.29) in the proof of Lemma 5.6. Relying on estimate (5.41) for χ, we arrive at

$$\|\vartheta\|_{W^{1,\infty}(0,\mathrm{t};L^2(0,\ell)) \cap H^1(0,\mathrm{t};H^1(0,\ell)) \cap L^\infty(0,\mathrm{t};H^2_N(0,\ell))} \leqslant S_7. \qquad \square \tag{5.42}$$

5.5. *Conclusion of the proof of Theorem 2*

Existence of a global solution. We now introduce the set

$$\mathscr{T} := \big\{ \mathrm{t} \in (0, T]: \text{ there exists a } \delta\text{-}\textit{separated solution} \text{ on } (0, \mathrm{t}) \big\}. \tag{5.43}$$

Of course, $\mathscr{T} \neq \emptyset$, as $T_0 \in \mathscr{T}$ thanks to Proposition 5.4 and Lemma 5.6. We let $T^* = \sup \mathscr{T}$ and, without loss of generality, suppose that $T^* > T_0$. In the following lines, we shall first prove that

$$T^* \in \mathscr{T} \tag{5.44}$$

(so that $T^* = \max \mathscr{T}$), and secondly that

$$T^* = T. \tag{5.45}$$

In this way, we shall conclude the existence of a *global* δ-separated solution $(\vartheta, \chi, \xi, \mathbf{u})$ to Problem 2 on $(0, T)$.

Proof of (5.44). By definition of T^*, there exists a sequence $\{\mathrm{t}_n\} \subset (0, T^*]$, with $\mathrm{t}_n \nearrow T^*$, such that for all $n \in \mathbb{N}$ there exists a δ-separated solution $(\vartheta_n, \chi_n, \xi_n, \mathbf{u}_n)$ on $(0, \mathrm{t}_n)$. Clearly, there exists $\bar{n} \in \mathbb{N}$ such that $\mathrm{t}_n > T_0$ for $n \geqslant \bar{n}$: however, since we do not yet dispose of any uniqueness result for Problem 2, the quadruple $(\vartheta_n, \chi_n, \xi_n, \mathbf{u}_n)$ need not be an extension of the local solution $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$. Let us now extend $(\vartheta_n, \chi_n, \xi_n, \mathbf{u}_n)$ to the interval $[0, T^*]$ by setting

$$\tilde{\vartheta}_n(t) := \begin{cases} \vartheta_n(t), & t \in [0, \mathrm{t}_n], \\ \vartheta_n(\mathrm{t}_n), & t \in (\mathrm{t}_n, T^*], \end{cases}$$
$$\tilde{\chi}_n(t) := \begin{cases} \chi_n(t), & t \in [0, \mathrm{t}_n], \\ \chi_n(\mathrm{t}_n), & t \in (\mathrm{t}_n, T^*], \end{cases}$$
$$\tilde{\xi}_n(t) := \begin{cases} \xi_n(t), & t \in [0, \mathrm{t}_n], \\ \xi^*, & t \in (\mathrm{t}_n, T^*], \end{cases}$$
$$\tilde{\mathbf{u}}_n(t) := \begin{cases} \mathbf{u}_n(t), & t \in [0, \mathrm{t}_n], \\ \partial_t \mathbf{u}_n(\mathrm{t}_n)(t - \mathrm{t}_n) + \mathbf{u}_n(\mathrm{t}_n), & t \in (\mathrm{t}_n, T^*], \end{cases}$$

for some $\xi^* \in L^2(\Omega)$. The global estimates of Lemma 5.9 yield that there exists a positive constant M_8 such that

$$\left\{ \begin{array}{l} \|\tilde{\vartheta}_n\|_{W^{1,\infty}(0,T^*;L^2(0,\ell))\cap H^1(0,T^*;H^1(0,\ell))\cap L^\infty(0,T^*;H^2_N(0,\ell))} \\ \|\tilde{\chi}_n\|_{W^{1,\infty}(0,T^*;L^2(0,\ell))\cap H^1(0,T^*;H^1(0,\ell))\cap L^\infty(0,T^*;H^2_N(0,\ell))} \\ \|\tilde{\xi}_n\|_{L^\infty(0,T^*;L^2(0,\ell))} \\ \|\tilde{\mathbf{u}}_n\|_{H^2(0,T^*;L^2(0,\ell))\cap W^{1,\infty}(0,T^*;H^1_0(0,\ell))\cap H^1(0,T^*;H^2_0(0,\ell))} \end{array} \right. \leqslant M_8.$$

Hence, [33, Theorem 4, Corollary 5] imply that there exists a quadruple $(\vartheta^*, \chi^*, \xi^*, \mathbf{u}^*)$ such that, along a (not relabelled) subsequence, $\tilde{\vartheta}_n$ converges to ϑ^* and $\tilde{\chi}_n$ converges to χ^* in the topologies specified by (5.22), $\tilde{\mathbf{u}}_n$ converges to $\mathbf{u}^*$ in the spaces (4.59), and $\tilde{\xi}_n \rightharpoonup^* \xi^*$ in $L^\infty(0, T^*; L^2(0, \ell))$. Hence, the quadruple $(\vartheta^*, \chi^*, \xi^*, \mathbf{u}^*)$ fulfils Eqs. (3.18), (3.20) and (3.40), and has the regularity (3.36)–(3.39). Furthermore, ϑ^*, χ^*, and $\mathbf{u}^*$ comply with initial conditions (3.1)–(3.3). Now, by the first of (5.22) and the continuous Sobolev embedding $H^{1-\rho}(0, \ell) \subset \mathrm{C}^0([0, \ell])$ for $0 \leqslant \rho \leqslant 1/2$, we have in particular $\tilde{\vartheta}_n \to \vartheta^*$ and $\tilde{\chi}_n \to \chi^*$ in $\mathrm{C}^0([0, T^*]; \mathrm{C}^0([0, \ell]))$, so that we infer

$$\min_{(x,t)\in[0,\ell]\times[0,T^*]} \vartheta^*(x,t) > 0, \qquad \chi^*(x,t) \geqslant \delta \quad \forall (x,t) \in \Omega \times \bigl[0, T^*\bigr]. \tag{5.46}$$

It remains to prove that $\xi^* \in \beta(\chi^*)$ a.e. in $\Omega \times (0, T^*)$. In order not to overburden the paper, we prefer to omit the proof here. This can be performed exactly like in [7], where a careful passage to the limit technique has been developed in the proof of [7, Lemma 4.12]. In this way, we conclude that $(\vartheta^*, \chi^*, \xi^*, \mathbf{u}^*)$ is in fact a δ-separated solution to Problem 2 on the interval $(0, T^*)$, and (5.44) ensues.

Proof of (5.45). Suppose now by contradiction that $T^* < T$. Now, due to regularity (3.36)–(3.39), the quadruple $(\vartheta^*(T^*), \chi^*(T^*), \mathbf{u}^*(T^*), \partial_t \mathbf{u}^*(T^*))$ provides a set of admissible initial data for Problem 2. Furthermore, by (5.46), $\chi(T^*)$ complies with inequality (5.25). Therefore, arguing in the same way as throughout Sections 5.2–5.3 (cf. Proposition 5.4 and Lemma 5.6), we find that Problem 2, supplemented with the initial conditions

$$\vartheta(T^*) = \vartheta^*\bigl(T^*\bigr) \quad \text{in } \Omega, \qquad \chi(T^*) = \chi^*\bigl(T^*\bigr) \quad \text{in } \Omega, \qquad \mathbf{u}\bigl(T^*\bigr) = \mathbf{u}^*\bigl(T^*\bigr) \quad \text{in } \Omega,$$

has a δ-separated solution $(\vartheta_\sharp, \chi_\sharp, \xi_\sharp, \mathbf{u}_\sharp)$ on some interval $[T^*, T^*+\eta]$. Then, we set

$$\tilde{\vartheta}(t) := \begin{cases} \vartheta^*(t), & t \in [0, T^*], \\ \vartheta_\sharp(t), & t \in (T^*, T^*+\eta], \end{cases}$$
$$\tilde{\chi}(t) := \begin{cases} \chi^*(t), & t \in [0, T^*], \\ \chi_\sharp(t), & t \in (T^*, T^*+\eta], \end{cases}$$
$$\tilde{\xi}(t) := \begin{cases} \xi^*(t), & t \in [0, T^*], \\ \xi_\sharp(t), & t \in (T^*, T^*+\eta], \end{cases}$$
$$\tilde{\mathbf{u}}(t) := \begin{cases} \mathbf{u}^*(t), & t \in [0, T^*], \\ \mathbf{u}_\sharp(t), & t \in (T^*, T^*+\eta], \end{cases}$$

and we easily conclude that $(\tilde{\vartheta}, \tilde{\chi}, \tilde{\xi}, \tilde{\mathbf{u}})$ is a δ-separated solution to Problem 2 on $[0, T^*+\eta]$. Thus, $T^*+\eta \in \mathscr{T}$, which is a contradiction.

Uniqueness and further regularity. Thanks to Remark 4.9, under the additional assumption (3.33) the continuous dependence estimate of Proposition 4.8 extends to this case as well. Thus, the global solution found in the previous step is also unique. Finally, the further regularity (3.45) can be proved arguing exactly in the same way as in the proof Proposition 4.5: namely, we differentiate (3.40) (in which β is now single-valued), test it by $J^{-1}(\chi_{tt})$, and integrate in time (cf. (4.44)–(4.49) for the detailed computations).

Remark 5.10. A common way to extend a local solution (of some PDE evolutionary system with unique solutions) to a global one (on the interval $(0, T)$) is to consider the maximal extension of the local solution and, exploiting some suitable *global estimates*, deduce that it must be defined on the whole $(0, T)$ by the same contradiction argument we illustrated in Step 2.

However, in this case, since for β multivalued we do not dispose of a uniqueness result for Problem 2, in Step 2 we have had to develop a slightly finer construction of the global solution, carefully handling regularity and δ-separation properties. Note that the resulting global solution is not, *a priori*, an extension of the previously found local solution $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$.

On the other hand, if in addition we assume (3.33) on β, Problem 2 turns out to have a *unique* global solution, which is *a posteriori* a prolongation of $(\hat{\vartheta}, \hat{\chi}, \hat{\xi}, \hat{\mathbf{u}})$.

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References

[1] C. Baiocchi, Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert, Ann. Mat. Pura Appl. (4) 76 (1967) 233–304.
[2] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff International Publishing, Leiden, 1976.
[3] E. Bonetti, G. Bonfanti, Existence and uniqueness of the solution to a 3D thermoviscoelastic system, Electron. J. Differential Equations 2003 (50) (2003), 15 pp. (electronic).
[4] E. Bonetti, G. Bonfanti, Asymptotic analysis for vanishing acceleration in a thermoviscoelastic system, Abstr. Appl. Anal. 2005 (2) (2005) 105–120.

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[5] E. Bonetti, G. Bonfanti, Well-posedness results for a model of damage in thermoviscoelastic materials, Ann. Inst. H. Poincaré Anal. Non Linéaire (2008), doi: 10.1016/j.anihpc.2007.05.009.

[6] E. Bonetti, G. Schimperna, Local existence to Frémond's model for damaging in elastic materials, Contin. Mech. Thermodyn. 16 (2004) 319–335.

[7] E. Bonetti, G. Bonfanti, R. Rossi, Global existence for a contact problem with adhesion, Math. Methods Appl. Sci. 31 (2008) 1–36.

[8] E. Bonetti, G. Schimperna, E. Segatti, On a doubly nonlinear model for the evolution of damaging in viscoelastic materials, J. Differential Equations 218 (2005) 91–116.

[9] G. Bonfanti, M. Frémond, F. Luterotti, Global solution to a nonlinear system for irreversible phase changes, Adv. Math. Sci. Appl. 10 (2000) 1–24.

[10] H. Brezis, Opérateurs Maximaux Monotones et Sémi-groupes de Contractions dans les Espaces de Hilbert, North-Holland Math. Stud., vol. 5, North-Holland Publishing Co./American Elsevier Publishing Co., Inc., Amsterdam, London/New York, 1973.

[11] P. Ciarlet, Mathematical Elasticity. Vol. I. Three-Dimensional Elasticity, Stud. Math. Appl., vol. 20, North-Holland Publishing Co., Amsterdam, 1988.

[12] P. Colli, P. Laurençot, Weak solutions to the Penrose–Fife phase field model for a class of admissible heat flux laws, Phys. D 111 (1998) 311–334.

[13] J.F. Blowey, C.M. Elliott, The Cahn–Hilliard gradient theory for phase separation with nonsmooth free energy. I. Mathematical analysis, European J. Appl. Math. 2 (1991) 233–280.

[14] M. Frémond, Non-Smooth Thermomechanics, Springer-Verlag, Berlin, 2002.

[15] M. Frémond, Phase change in mechanics, in: Lecture Notes of XXX Scuola estiva di Fisica Matematica, Ravello, 2005, in press.

[16] M. Frémond, E. Rocca, Well-posedness of a phase transition model with the possibility of voids, Math. Models Methods Appl. Sci. 16 (2006) 559–586.

[17] P. Germain, Cours de méchanique des milieux continus. Tome I: Théorie générale, Masson et Cie, Éditeurs, Paris, 1973.

[18] M. Grasselli, H. Petzeltová, G. Schimperna, Long time behavior of solutions to the Caginalp system with singular potential, Z. Anal. Anwend. 25 (2006) 51–72.

[19] M. Grasselli, H. Petzeltová, G. Schimperna, Convergence to stationary solutions for a parabolic–hyperbolic phase-field system, Commun. Pure Appl. Anal. 5 (2006) 827–838.

[20] M.E. Gurtin, Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance, Phys. D 92 (1996) 178–192.

[21] W. Horn, J. Sprekels, S. Zheng, Global existence of smooth solutions to the Penrose–Fife model for Ising ferromagnets, Adv. Math. Sci. Appl. 6 (1996) 227–241.

[22] P. Krejčí, E. Rocca, J. Sprekels, Nonlocal temperature-dependent phase-field models for non-isothermal phase transitions, WIAS Preprint No. 1006, Berlin, 2005. Appeared in: J. London Math. Soc. 76 (2007) 197–210.

[23] K.L. Kuttler, Quasistatic evolution of damage in an elastic–viscoplastic material, Electron. J. Differential Equations 2005 (147) (2005), 25 pp. (electronic).

[24] J.L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod, Gauthier–Villars, Paris, 1969.

[25] F. Luterotti, U. Stefanelli, Existence result for the one-dimensional full model of phase transitions, Z. Anal. Anwend. 21 (2002) 335–350.

[26] F. Luterotti, U. Stefanelli, Errata and addendum to: "Existence result for the one-dimensional full model of phase transitions" [Z. Anal. Anwend. 21 (2002) 335–350], Z. Anal. Anwend. 22 (2003) 239–240.

[27] A. Miranville, S. Zelik, Robust exponential attractors for Cahn–Hilliard type equations with singular potentials, Math. Methods Appl. Sci. 27 (2004) 545–582.

[28] J. Nečas, Les méthodes directes en théorie des équations elliptiques, Masson et Cie, Éditeurs, Paris, 1967.

[29] L. Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Super. Pisa (3) 13 (1959) 115–162.

[30] A. Novick-Cohen, On the viscous Cahn–Hilliard equation, in: Material Instabilities in Continuum Mechanics, Edinburgh, 1985–1986, Oxford Sci. Publ., Oxford Univ. Press, New York, 1988, pp. 329–342.

[31] O. Penrose, P.C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transitions, Phys. D 43 (1990) 44–62.

[32] G. Schimperna, U. Stefanelli, Positivity of the temperature for phase transitions with micro-movements, Nonlinear Anal. Real World Appl. 8 (2007) 257–266.

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[33] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4) 146 (1987) 65–96.
[34] U. Stefanelli, Models of phase change with microscopic movements, PhD thesis, University of Pavia, 2003.
[35] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, third ed., Stud. Math. Appl., vol. 2, North-Holland Publishing Co., Amsterdam, 1984.

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