A degenerating PDE system for phase transitions and damage

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Abstract

In this paper, we analyze a PDE system arising in the modeling of phase transition and damage phenomena in thermoviscoelastic materials. The resulting evolution equations in the unknowns ϑ (absolute temperature), $\mathbf{u}$ (displacement), and χ (phase/damage parameter) are strongly nonlinearly coupled. Moreover, the momentum equation for $\mathbf{u}$ contains χ-dependent elliptic operators, which degenerate at the *pure phases* (corresponding to the values $\chi = 0$ and $\chi = 1$), making the *whole* system degenerate.

That is why, we have to resort to a suitable weak solvability notion for the analysis of the problem: it consists of the weak formulations of the heat and momentum equation, and, for the phase/damage parameter χ, of a generalization of the principle of virtual powers, partially mutuated from the theory of rate-independent damage processes.

To prove an existence result for this weak formulation, an approximating problem is introduced, where the elliptic degeneracy of the displacement equation is ruled out: in the framework of damage models, this corresponds to allowing for *partial damage* only. For such an approximate system, global-in-time existence and well-posedness results are established in various cases. Then, the passage to the limit to the degenerate system is performed via suitable variational techniques.

Key words: Phase transitions, damage phenomena, thermoviscoelastic materials, elliptic degenerate operators, nonlocal operators, global existence of weak solutions, continuous dependence.

AMS (MOS) subject classification: 35K65, 35K92, 35R11, 80A17, 74A45.

1 Introduction

We consider the following PDE system

$$\mathsf{c}(\vartheta)\vartheta_t + \chi_t\vartheta + \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \mathrm{div}(\mathbf{d}(x,\nabla\chi)) + W'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega\times(0,T), \tag{1.3}$$

which describes a thermoviscoelastic system occupying a reference domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, supplemented with suitable initial and boundary conditions. The symbols ϑ and $\mathbf{u}$ respectively denote the absolute temperature of the system and the vector of *small displacements*. Depending on the choices of the functions a and b, we obtain a model

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- for *phase transitions*: in this case, χ is the order parameter, standing for the local proportion of one of the two phases;
- for *damage*: in this case, χ is the damage parameter, assessing the soundness of the material.

We will assume that χ takes values between 0 and 1, choosing 0 and 1 as reference values:

- for the *pure phases* in phase change models (for example, $\chi = 0$ stands for the solid phase and $\chi = 1$ for the liquid one in solid-liquid phase transitions, and one has $0 < \chi < 1$ in the so-called *mushy regions*);
- for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, in damage models, while $0 < \chi < 1$ corresponds to *partial damage*.

1.1 The model

Let us now briefly illustrate the derivation of the PDE system (1.1)–(1.3). We shall systematically refer for more details to [45], where we dealt with the case of phase transitions in thermoviscoelastic materials, and just underline here the main differences with respect to the discussion in [45].

Equation (1.2), governing the evolution of the displacement $\mathbf{u}$, is the classical balance equation for macroscopic movements (also known as the *stress-strain relation*), in which inertial effects are taken into account as well. It is derived from the principle of virtual power (cf. [18]), which yields

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{1.4}$$

where the symbol div stands both for the scalar and for the vectorial divergence operator, σ is the stress tensor, and $\mathbf{f}$ an exterior volume force. For σ, we adopt the well-known constitutive law

$$\sigma = \sigma^{\mathrm{nd}} + \sigma^{\mathrm{d}} = \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)}, \tag{1.5}$$

with $\varepsilon(\mathbf{u})$ the linearized symmetric strain tensor, which in the (spatially) three-dimensional case is given by $\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2,\ i,j = 1,2,3$ (with the commas we denote space derivatives). Hence, the explicit expression of σ depends on the form of the free energy functional $\mathcal{F}$ and of the pseudopotential of dissipation $\mathcal{P}$. The former is a function of the state variables, namely χ, its gradient $\nabla\chi$, the absolute temperature ϑ, and the linearized symmetric strain tensor $\varepsilon(\mathbf{u})$. According to Moreau's approach (cf. [18] and references therein), we include dissipation in the model by means of the latter potential, which depends on the dissipative variables $\nabla\vartheta$, χ_t, and $\varepsilon(\mathbf{u}_t)$. We will make precise our choice for $\mathcal{F}$ and $\mathcal{P}$ below, cf. (1.14) and (1.17).

We shall supplement (1.4) with a zero Dirichlet boundary condition on the boundary of Ω

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T)\,, \tag{1.6}$$

yielding a *pure displacement* boundary value problem for $\mathbf{u}$, according to the terminology of [12]. However, our analysis carries over to other kinds of boundary conditions on $\mathbf{u}$, see Remark 2.9.

Following Frémond's perspective, (1.2) is coupled with the equation of microscopic movements for the phase variable χ (cf. [18, p. 5]), leading to (1.3). Let B (a density of energy function) and $\mathbf{H}$ (an energy flux vector) represent the internal microscopic forces responsible for the mechanically induced heat sources, and let us denote by B^{d} and $\mathbf{H}^{\mathrm{d}}$ their dissipative parts, and by B^{nd} and $\mathbf{H}^{\mathrm{nd}}$ their non-dissipative parts. Standard constitutive relations yield

$$B = B^{\mathrm{nd}} + B^{\mathrm{d}} = \frac{\partial \mathcal{F}}{\partial \chi} + \frac{\partial \mathcal{P}}{\partial \chi_t}\,, \tag{1.7}$$

$$\mathbf{H} = \mathbf{H}^{\mathrm{nd}} + \mathbf{H}^{\mathrm{d}} = \frac{\partial \mathcal{F}}{\partial \nabla\chi} + \frac{\partial \mathcal{P}}{\partial \nabla\chi_t}\,. \tag{1.8}$$

Then, if the volume amount of mechanical energy provided to the domain by the external actions (which do not involve macroscopic motions) is zero, the equation for the microscopic motions can be written as

$$B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega \times (0,T), \tag{1.9}$$

where B and $\mathbf{H}$ will be specified according to the expression of $\mathcal{F}$ and $\mathcal{P}$. The natural boundary condition for this equation of motion is

$$\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T),$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. Thus (cf. (1.14)) we obtain the homogeneous Neumann boundary condition on χ

$$\partial_{\mathbf{n}} \chi = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.10}$$

Finally, equation (1.1) is derived from the internal energy balance

$$e_t + \operatorname{div} \mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H} \cdot \nabla \chi_t \quad \text{in } \Omega \times (0,T), \tag{1.11}$$

where g denotes a heat source and e and $\mathbf{q}$ are obtained from $\mathcal{F}$ and $\mathcal{P}$ by means of the standard constitutive relations

$$e = \mathcal{F} - \vartheta \frac{\partial \mathcal{F}}{\partial \vartheta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla \vartheta}. \tag{1.12}$$

We couple equation (1.11) with a no-flux boundary condition:

$$\mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T)$$

implying (cf. (1.17)) the homogeneous Neumann boundary condition

$$\partial_{\mathbf{n}} \vartheta = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.13}$$

From the above relations and the following choices for the free energy functional and of the pseudopotential of dissipation (cf. (1.14) and (1.17)), we derive the PDE system (1.1)–(1.3) within the *small perturbation assumption* [23] (i.e. neglecting the quadratic terms $|\chi_t|^2 + a(\chi)\varepsilon(\mathbf{u}_t)\mathrm{R}_v\varepsilon(\mathbf{u}_t)$ on the right-hand side of the heat equation). In agreement with Thermodynamics (cf. [18, 20] and [19, Sec. 4, 6]), we choose the volumetric free energy $\mathcal{F}$ of the form

$$\mathcal{F}(\vartheta, \varepsilon(\mathbf{u}), \chi, \nabla\chi) = \int_\Omega \left(f(\vartheta) + b(\chi) \frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \phi(x, \nabla\chi) + W(\chi) - \vartheta\chi - \rho\vartheta \operatorname{tr}(\varepsilon(\mathbf{u})) \right) \mathrm{d}x\,, \tag{1.14}$$

where f is a concave function of ϑ. Notice that the symmetric, positive-definite elasticity tensor R_e is pre-multiplied by a function b of the phase/damage parameter χ. In particular,

- in the case of phase transitions in viscoelastic materials, a meaningful choice for b is $b(\chi) = 1 - \chi$, or a function vanishing at 1 [19, Sec. 4.5, pp. 42-43]. This reflects the fact that we have the full elastic contribution of $b(\chi)\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})$ only in the non-viscous phase, and that such a contribution is null in the viscous one (i.e. when $\chi = 1$);
- for damage models a significant choice is instead $b(\chi) = \chi$ (cf. [20] and [19, Sec. 6.2, pp. 102-103] for further comments on this topic). The term $\chi\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2}$ represents the classical elastic contribution in which the stiffness of the material decreases as χ approaches 0, i.e. during the evolution of damage.

The term $\phi(x, \nabla\chi) + W(\chi)$ is a *mixture* or *interaction free-energy*. We shall suppose that $\phi : \Omega \times \mathbb{R}^d \to [0, +\infty)$ is a normal integrand, such that for almost all $x \in \Omega$ the function $\phi(x, \cdot) : \mathbb{R}^d \to [0, +\infty)$ is convex,

C^1, with p-growth, and $p > d$. Hence, the field $\mathbf{d}(x,\cdot) = \nabla\phi(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, $x \in \Omega$, leads to a p-Laplace type operator in (1.3). The prototypical example is $\phi(x,\nabla\chi) = \frac{1}{p}|\nabla\chi|^p$, yielding $\mathbf{d}(x,\nabla\chi) := |\nabla\chi|^{p-2}\nabla\chi$. Let us point out that the gradient of χ accounts for interfacial energy effects in phase transitions, and for the influence of damage at a material point, undamaged in its neighborhood, in damage models. In this sense we can say that the term $\frac{1}{p}|\nabla\chi|^p$ models nonlocality of the phase transition or the damage process, i.e. the feature that a particular point is influenced by its surrounding. In damage, this leads to possible hardening or softening effects (cf. also [9] for further comments on this topic). Gradient regularizations of p-Laplacian type are often adopted in the mathematical papers on damage (see for example [6, 7, 27, 38, 39, 41]), and in the modeling literature as well (cf., e.g., [18, 20, 34]). In a different context, a p-Laplacian elliptic regularization with $p > d$ has also been exploited in [1], in order to study a diffuse interface model for the flow of two viscous incompressible Newtonian fluids in a bounded domain.

In the following, we will also scrutinize another kind of elliptic regularization in (1.3), given by the *nonlocal* s-Laplacian operator on the Sobolev-Slobodeckij space $W^{s,2}(\Omega)$, hereafter denoted by A_s (cf. (2.35) later on for its precise definition). Recently, fractional Laplacian operators have been widely investigated (cf., e.g., [11, 52] and the references therein), and used in connection with real-world applications, such as thin obstacle problems, finance, material sciences, but also phase transition and damage phenomena (cf., e.g. [24] and [31]). For analytical reasons, we will have to assume $s > d/2$, which ensures the (compact) embedding $W^{s,2}(\Omega) \Subset C^0(\overline{\Omega})$, in the same way as $W^{1,p}(\Omega) \Subset C^0(\overline{\Omega})$ for $p > d$. This property will play a crucial role in the *degenerate* limit to complete damage, as it did in [39] within the rate-independent context, cf. Remark 7.5 for more details.

As for the potential W, we suppose that

$$W = \widehat{\beta} + \widehat{\gamma}, \tag{1.15}$$

with $\widehat{\beta} : \mathbb{R} \to (-\infty, +\infty]$ convex and possibly nonsmooth, and $\widehat{\gamma} : \mathbb{R} \to \mathbb{R}$ smooth and possibly nonconvex. We will take the domain of $\widehat{\beta}$ to be contained in $[0,1]$. Note that, in this way, the values outside $[0,1]$ (which indeed are not physically meaningful for the order parameter χ, denoting a phase or damage proportion) are excluded. Typical examples of functionals which we can include in our analysis are the logarithmic potential

$$W(r) = r\ln(r) + (1-r)\ln(1-r) - c_1 r^2 - c_2 r - c_3 \quad \text{for } r \in (0,1), \tag{1.16}$$

where c_1 and c_2 are positive constants, as well as the sum of the indicator function $\widehat{\beta} := I_{[0,1]}$ with a *nonconvex* $\widehat{\gamma}$. In such a case, in (1.3) the derivative W' needs to be understood as the subdifferential $\partial W = \partial\widehat{\beta} + \widehat{\gamma}'$ in the sense of convex analysis.

The term $\rho\vartheta\mathrm{tr}(\varepsilon(\mathbf{u}))$ in (1.14) accounts for the thermal expansion of the system, with the thermal expansion coefficient ρ assumed to be constant (cf., e.g., [32]). Indeed, one could consider more general functions ρ depending, e.g., on the phase parameter χ and vanishing when $\chi = 0$. This would be meaningful especially in damage models, where the terms associated with deformations should disappear once the material is completely damaged (cf., e.g., [6]). We will discuss the mathematical difficulties attached to this extension in Section 1.2.

For the pseudo-potential $\mathcal{P}$, following [19, Sec. 4, 6] we take

$$\mathcal{P}(\nabla\vartheta, \chi_t, \varepsilon(\mathbf{u}_t)) = \frac{\mathsf{K}(\vartheta)}{2}|\nabla\vartheta|^2 + \frac{1}{2}|\chi_t|^2 + \mu I_{(-\infty,0]}(\chi_t) + a(\chi)\frac{\varepsilon(\mathbf{u}_t)\mathrm{R}_v\varepsilon(\mathbf{u}_t)}{2}\,, \tag{1.17}$$

where R_v is a symmetric and positive definite viscosity tensor, premultiplied by a function a of χ. In particular, for phase change models, one can take for example $a(\chi) = \chi$. The underlying physical interpretation is that the viscosity term $\chi\varepsilon(\mathbf{u}_t)\mathrm{R}_v\varepsilon(\mathbf{u}_t)$ vanishes when we are in the non-viscous phase, i.e. in the solid phase $\chi = 0$. Also in damage models the choice $a(\chi) = \chi$ is considered, cf. e.g. [39].

The heat conductivity function K will be assumed continuous; for the analysis of system (1.1)–(1.3), we will need to impose some compatibility conditions on the growth of $\mathsf{K}(\vartheta)$ and of the heat capacity function $\mathsf{c}(\vartheta) = -\vartheta f''(\vartheta)$ in (1.1), see Hypothesis (II) in Section 2.2. Furthermore, in (1.17) $\mu \geq 0$ is a non-negative coefficient: for $\mu > 0$ we encompass in our model the *unidirectionality* constraint $\chi_t \leq 0$ a.e. in $\Omega \times (0, T)$. In fact, throughout the paper we are going to use the term *irreversible* in connection with the case in which the process under consideration is unidirectional, which is indeed typical of damage phenomena.

With straightforward computations, from (1.7)–(1.9) and using the form of the free energy functional (1.14) and of the pseudopotential of dissipation (1.17), we derive equations (1.1)–(1.3), neglecting the quadratic contributions in the velocities on the right-hand side in (1.1) by means of the aforementioned *small perturbation assumption* [23]. This is a simplification needed from the analytical point of view in order to solve the problem. Indeed, in a forthcoming paper we plan to tackle the PDE system (1.1)–(1.3), featuring in addition these quadratic terms in the temperature equation. To do so, we are going to resort to specific techniques, partially mutuated from [17], however confining the analysis to some particular cases.

In fact, to our knowledge only few results are available on diffuse interface models in thermoviscoelasticity (i.e. also accounting for the evolution of the displacement variables, besides the temperature and the order parameter): among others, we quote [21, 22, 45, 46]. In all of these papers, the small perturbation assumption is adopted. For, without it in the spatial three-dimensional case existence results seem to be out of reach, at the moment, even when the equation for displacements is neglected (whereas the existence of solutions to the *full* phase change model in the unknowns ϑ and χ has been obtained in $1D$ in [35]). This has led to the development of suitable *weak solvability* notions to handle (the usually neglected) quadratic terms, like in [17] (where however $\mathbf{u}$ is still taken constant). Also in [49], a PDE system coupling the displacement and the temperature equation (with quadratic nonlinearities) and a *rate-independent* flow rule for an internal dissipative variable χ (such as the damage parameter) has been analyzed. Rate-independence means that the evolution equation for χ has no longer the *gradient flow* structure of (1.3): the term χ_t therein is replaced by $\mathrm{Sign}(\chi_t)$, viz. in the pseudo-potential $\mathcal{P}$, instead of the quadratic contribution $\frac{1}{2}|\chi_t|^2$ we have the 1-homogeneous dissipation term $|\chi_t|$. In the frame of the (weak) *energetic formulation* for rate-independent systems [37], suitably adapted to the temperature-dependent case, in [49] existence results have been obtained. A temperature-dependent, *full* model for (rate-dependent) damage has been addressed in [6] as well, with local-in-time existence results.

1.2 Mathematical difficulties and related literature

The main difficulties attached to the analysis of system (1.1)–(1.3) are:

1) the *elliptic degeneracy* of the momentum equation (1.2): in particular, we allow for the positive coefficients $a(\chi)$ and $b(\chi)$ to tend to zero simultaneously;

2) the *highly nonlinear coupling* between the single equations, resulting in the the quadratic terms $\chi_t\vartheta$, $\vartheta\,\mathrm{div}(\mathbf{u}_t)$, and $|\varepsilon(\mathbf{u})|^2$ in the heat and phase equations (1.1) and (1.3), respectively;

3) the *poor regularity* of the temperature variable, which brings about difficulties in dealing with the coupling between equations (1.1) and (1.2) when we consider the thermal expansion terms (i.e. we take $\rho \neq 0$);

4) the *doubly nonlinear* character of (1.3), due to the nonsmooth graph $\partial\widehat{\beta}$ and the nonlinear operator $-\mathrm{div}(\mathbf{d}(\nabla\chi)) \sim -\Delta_p\chi$ (which on the other hand has a key regularizing role). Furthermore, if we set $\mu > 0$ in (1.3) to enforce an irreversible evolution for χ, the simultaneous presence of the terms $-\mathrm{div}(\mathbf{d}(\nabla\chi))$ and $\partial I_{(-\infty,0]}(\chi_t)$ makes it difficult to derive suitable estimates for χ, also due to the low regularity of the right-hand side of (1.3).

We now partially survey how each of these problems has been handled in the recent literature.

As for **1)**, in [45, 46] we have focused on the *phase transition* case, in which $a(\chi) = \chi$ and $b(\chi) = 1-\chi$. We have proved the local-in time (in the $3D$-setting) and the global-in-time (in the $1D$-setting) well-posedness of a system in thermoviscoelasticity analogous to (1.1)–(1.3) (with the Laplacian instead of the $p-$Laplacian in (1.3), in the case $\mu = 0$ and $\rho = 0$, and for constant heat capacity and heat conductivity in (1.1)). The main idea in [45, 46] to handle the possible elliptic degeneracy of (1.2) is in fact to *prevent it*. Specifically, we have shown that, if the initial datum χ_0 stays away from the values points 0 and 1, so does χ during this evolution, guaranteeing that the operators in (1.2) are uniformly elliptic. This *separation property* is proved by exploiting a sufficient coercivity of W at the thresholds 0 and 1, which for example holds true for the logarithmic potential (1.16).

In [7, 8] an *isothermal* (irreversible) model for damage has been considered: therein, because of the elliptic degeneracy of (1.2), the authors only prove a local-in-time existence result. For (isothermal) *rate-independent* damage models [38, 9, 39, 41], the results change significantly: in this realm, only poor time-regularity of the solution component χ is to be expected, because the 1-homogeneous dissipation contribution in χ_t to $\mathcal{P}$ just ensures BV-estimates for the function $t \mapsto \chi(x, t)$. That is why, one has to resort to the aforementioned notion of *energetic solution* [37], in which no time-derivatives of χ are featured. Therefore, this concept is very flexible for analysis, and has allowed for handling the (degenerate) case of *complete damage* in [9, 39] by means of a specially devised formulation we will refer to later.

Concerning problem **2)**, as already mentioned existence results have been obtained in [17] for a *full* model of phase transitions (in the reversible case $\mu = 0$ and for constant $\mathbf{u}$), even featuring the term $|\chi_t|^2$ on the right-hand side of the temperature equation. Therein, a suitable notion of weak solution is addressed, consisting of the phase equation, coupled with a total energy balance and a weak entropy inequality, for which existence is proved by relying on an iterative regularization procedure. This technique cannot be applied to system (1.1)–(1.3). Nonetheless, let us mention that a key assumption in [17] is a suitable growth of the heat conductivity K. Following [49, 47], here we will combine it with conditions on the heat capacity coefficient c to handle the quadratic nonlinearities $\chi_t\vartheta$ and $\vartheta\,\mathrm{div}(\mathbf{u}_t)$ in (1.1).

Due to the lack of "good" a priori estimates for ϑ mentioned in **3)**, we will not be able to encompass in our analysis the case of a non-constant thermal expansion coefficient ρ, e.g. $\rho(\chi) = \chi$, which would still be interesting for damage [6]. Indeed, such a choice would lead to an additional term of the type $\vartheta\chi_t\,\mathrm{div}(\mathbf{u})$ in the heat equation, which we would not be able to handle without resorting to further regularizations, and possibly proving only local-in-time existence results. Nonetheless, let us stress that, especially in case of phase transition phenomena, the choice of a constant ρ is quite reasonable (cf., e.g., [32]).

As for **4)**, in [27] (dealing with Cahn-Hilliard systems coupled with elasticity and damage processes; see also [28]), the authors have devised a weak formulation of (1.3) (in the irreversible case $\mu = 1$) which has allowed them to circumvent its triply nonlinear character. Such a formulation strongly relies on the special choice $\widehat{\beta}(\chi) = I_{[0,+\infty)}(\chi)$ (which, joint with the irreversibility constraint, still ensures that χ takes values in the meaningful interval $[0, 1]$, provided that $\chi_0 \in [0, 1]$). It consists of a *one-sided* variational inequality (i.e. with test functions having a fixed sign), and of an *energy inequality*, see (1.19) later.

1.3 Our results

Unlike [45, 46], here we shall not enforce separation of χ from the threshold values 0 and 1, and accordingly we will allow for general initial configurations of χ. Then, it is not to be expected that either of the coefficients $a(\chi)$ and $b(\chi)$ stay away from 0, which results in the elliptic degeneracy of the displacement equation (1.2). To handle it, we shall approximate system (1.1)–(1.3) with a non-degenerating one, where we replace (1.2) with

$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + (b(\chi) + \delta)\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0, T), \qquad \text{for } \delta > 0. \tag{1.18}$$

Let us note that, to rule out the degeneracy, it is sufficient to truncate away from zero only the coefficient $a(\chi)$ of the *viscous* part of the elliptic operator in the momentum equation. However, for technical reasons which will become apparent in Section 7 (cf. Rmk. 7.2), when addressing the asymptotic analysis as $\delta \downarrow 0$ to the *degenerate* limit, we will need to truncate the coefficient $b(\chi)$ as well, resulting in (1.18). In the analysis of (1.18), we will distinguish the cases $\rho = 0$ and $\rho \neq 0$: let us stress that, in the latter, there is an additional coupling between the heat and the momentum balance equations, which needs to be carefully handled and indeed requires strengthening of some of our assumptions. Furthermore, to avoid overburdening the paper we will tackle the case $\rho \neq 0$ only for the *reversible* system (i.e. with $\mu = 0$). More specifically, in Theorems 1, 4 and 5, we will establish global-in-time existence results for the non-degenerating system (1.1, 1.18, 1.3) with $\rho = 0$, both in the *reversible* and in the *irreversible* cases. We will work under quite general assumptions on c and K, basically requiring that c and K are bounded from below and above by the sum of a bounded function and function behaving like a small power of ϑ (cf. Hypotheses (I) and (II) in Sec. 2.2). In Theorem 2, we will handle the case $\rho \neq 0$, $\mu = 0$ and prove the existence of global solutions to (1.1, 1.18, 1.3), under the more restrictive assumption that K is bounded from below and above by a function behaving like $\vartheta^{2+\nu}$, with $0 \leq (d-2)/(d+2) < \nu < 1$, cf. Hypothesis (VIII). A continuous dependence result, yielding uniqueness of solutions, for the non-degenerating *isothermal* reversible system, possibly with $\rho \neq 0$, will be given in Theorem 3. Finally, we will address the degenerate limit $\delta \downarrow 0$ in Theorem 6 in a less general setting, in particular confining ourselves to the case $\rho = 0$. In what follows, we give more details on Thms. 1–6.

Our first main result Thm. 1 states the existence of solutions to system (1.1, 1.18, 1.3) with $\rho = 0$, in the *reversible* case $\mu = 0$, with the heat equation (1.1) suitably reformulated by means of an *enthalpy* transformation (cf. Sec. 2.2), switching from the temperature variable ϑ to the enthalpy w. Already in the proof of this global-in-time existence result, a key role is played by the aforementioned p-growth assumption on the function ϕ (1.14) with $p > d$. In fact, it enables us to derive an estimate for χ in $L^\infty(0, T; W^{1,p}(\Omega))$, which in turns allows for a suitable regularity estimate on the displacement variable $\mathbf{u}$, leading to a global-in-time bound on the quadratic nonlinearity $|\varepsilon(\mathbf{u})|^2$ on the right-hand side of (1.3). For further details we refer to the proof of Thm. 1, developed by passing to the limit in a carefully designed time-discretization scheme and exploiting BOCCARDO&GALLOUËT-type estimates on ϑ.

Relying on the stronger Hyp. (VIII), in the case $\rho \neq 0$, $\mu = 0$ we will obtain enhanced estimates on (the sequence, constructed by time discretization, approximating) ϑ, cf. also Remark 2.10 later on. These bounds and the related enhanced convergences will enable us to handle the (passage to the limit in the time discretization of the) thermal expansion terms in (1.1) and (1.18). In this way, we will conclude the proof of the existence Theorem 2 for system (1.1, 1.18, 1.3) in the case $\rho \neq 0$.

In the reversible and *isothermal* case, continuous dependence of the solutions on the initial and problem data is proved in Thm. 3 under a slightly more restrictive condition on the field ϕ, which is however satisfied in the prototypical case of the p-Laplacian operator.

As already mentioned, in the *irreversible* case $\mu > 0$ a major difficulty in the analysis of system (1.1, 1.18, 1.3) stems from the simultaneous presence in (1.3) of the multivalued operators $\partial I_{(-\infty,0]}(\chi_t)$ and $\beta(\chi) = \partial\widehat{\beta}(\chi)$, (cf. (1.15)), as well as of the p-Laplacian type operator $-\mathrm{div}(\mathbf{d}(x, \nabla\chi))$, which still has a key role in providing global-in-time estimates for $|\varepsilon(\mathbf{u})|^2$. To tackle this problem, following the approach of [27] we restrict to the yet meaningful case $\widehat{\beta} = I_{[0,+\infty)}$ and consider a suitable weak formulation of

(1.3). It consists (cf. Definition 2.13 later on) of the *one-sided* variational inequality

$$\int_\Omega \Big(\chi_t(t)\varphi + \mathbf{d}(x,\nabla\chi(t))\cdot\nabla\varphi + \xi(t)\varphi + \gamma(\chi(t))\varphi + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2}\varphi - \vartheta(t)\varphi\Big)\,\mathrm{d}x \geq 0$$
$$\text{for all } \varphi\in W^{1,p}(\Omega) \text{ with } \varphi\leq 0, \text{ for a.a. } t\in(0,T), \qquad \text{with } \chi_t\leq 0,\ \xi\in\partial I_{[0,+\infty)}(\chi), \tag{1.19a}$$

and of the following energy inequality for all $t\in(0,T]$, for $s=0$, and for almost all $0<s\leq t$:

$$\begin{aligned}&\int_s^t\int_\Omega |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r + \int_\Omega (\phi(x,\nabla\chi(t)) + W(\chi(t)))\,\mathrm{d}x\\ &\leq \int_\Omega (\phi(x,\nabla\chi(s)) + W(\chi(s)))\,\mathrm{d}x + \int_s^t\int_\Omega \chi_t\left(-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \vartheta\right)\mathrm{d}x\,\mathrm{d}r.\end{aligned} \tag{1.19b}$$

In Sec. 2.4, several comments and remarks shed light on this weak solvability notion for (1.3). In particular, Proposition 2.14 shows that, if χ is regular enough, (1.19) and the subdifferential inclusion (1.3) are equivalent. In Theorem 4 we state the existence of global-in-time solutions to the weak formulation of system (1.1, 1.18, 1.3) with $\mu>0$, consisting of the (weakly formulated) enthalpy equation, of (1.18) and of (1.19). The proof is again carried out via a time-discretization procedure, combined with Yosida-regularization techniques.

Finally, Theorem 5 focuses on the isothermal case, i.e. with a fixed temperature profile. In this setting, we succeed in proving enhanced regularity for χ, thus solving (1.3) in a stronger sense than (1.19). In the particular case $\phi(x,\nabla\chi)=\frac{1}{p}|\nabla\chi|^p$ the crucial estimate consists in testing (1.3) by $\partial_t(A_p\chi+\beta(\chi))$ (where for simplicity we write β as single-valued). This enables us to estimate separately the terms $\partial I_{(-\infty,0]}(\chi_t)$ (again written as single-valued), $A_p\chi$, and $\beta(\chi)$ in $L^\infty(0,T;L^2(\Omega))$, which is the key step for proving the existence of solutions to the *pointwise* subdifferential inclusion (1.3).

Uniqueness results for the *irreversible* system, even in the isothermal case, do not seem to be at hand, due to the triply nonlinear character of equation (1.3), cf. also Remark 2.18 ahead. Nonetheless, both in the reversible and in the irreversible case, in Thms. 1, 2 and 4 we will prove positivity of the temperature ϑ. In fact, under suitable conditions on the initial temperature, for $\mu>0$ we will also obtain a strictly positive lower bound for ϑ.

For the analysis of the degenerate limit $\delta\downarrow 0$ of (1.1, 1.18, 1.3), we have carefully adapted to the present setting techniques from [9] and [39]. These two papers deal with *complete damage* in the fully rate-independent case, and, respectively, for a system featuring a rate-independent damage flow rule for χ and a displacement equation with viscosity and inertia according to Kelvin-Voigt rheology. In particular, we have extended the results from [39] to the case of a *rate-dependent* equation for χ, also coupled with the temperature equation. Following [9, 39], the key observation is that, for any family $(w_\delta,\mathbf{u}_\delta,\chi_\delta)_\delta$ of solutions to (1.1, 1.18, 1.3) (where w denotes the *enthalpy*), it is possible to deduce for the quantities $\boldsymbol{\mu}_\delta := \sqrt{a(\chi_\delta)+\delta}\,\varepsilon(\partial_t\mathbf{u}_\delta)$ and $\boldsymbol{\eta}_\delta := \sqrt{b(\chi_\delta)+\delta}\,\varepsilon(\mathbf{u}_\delta)$ the estimates

$$\|\boldsymbol{\mu}_\delta\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))},\ \|\boldsymbol{\eta}_\delta\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}\leq C$$

for a positive constant independent of δ. Therefore, there exist $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ such that, up to a subsequence $\boldsymbol{\mu}_\delta \rightharpoonup \boldsymbol{\mu}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ and $\boldsymbol{\eta}_\delta \rightharpoonup^* \boldsymbol{\eta}$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ as $\delta\downarrow 0$. According the terminology of [39], we refer to $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$, respectively, as the viscous and elastic *quasi-stresses.*

In Theorem 6 we will focus on the degenerate limit $\delta\downarrow 0$, confining the discussion to the case where $\rho=0$ and $\mu>0$ (viz. the map $t\mapsto\chi(t,x)$ is nonincreasing for all $x\in\overline{\Omega}$). We refer to Remark 7.5 for a thorough justification of these choices. Passing to the limit as $\delta\downarrow 0$ in (1.18) and exploiting the above convergences for $(\boldsymbol{\mu}_\delta)_\delta$ and $(\boldsymbol{\eta}_\delta)_\delta$ we will prove that there exist a triple $(\mathbf{u},\boldsymbol{\mu},\boldsymbol{\eta})$ solving the *generalized* momentum balance

$$\mathbf{u}_{tt} - \mathrm{div}(\sqrt{a(\chi)}\mathrm{R}_v\boldsymbol{\mu} + \sqrt{b(\chi)}\mathrm{R}_e\boldsymbol{\eta}) = \mathbf{f} \quad \text{in } \Omega\times(0,T), \tag{1.20a}$$

such that the quasi-stresses fulfill

$$\boldsymbol{\mu} = \sqrt{a(\chi)}\,\varepsilon(\mathbf{u}_t),\ \boldsymbol{\eta} = \sqrt{b(\chi)}\,\varepsilon(\mathbf{u}) \text{ a.e. in any open set } A \subset \Omega \times (0,T) \text{ s.t. } A \subset \{\chi > 0\}. \tag{1.20b}$$

In addition to (1.20a)–(1.20b), the notion of weak solution to system (1.1)–(1.3) arising in the limit $\delta \downarrow 0$ consists of the (weak formulation of the) enthalpy equation, of the *one-sided* variational inequality

$$\begin{gathered}\int_0^T \int_\Omega \Big((\chi_t + \gamma(\chi))\,\varphi + \mathbf{d}(x, \nabla\chi) \cdot \nabla\varphi \Big) \leq \int_0^T \int_\Omega \Big(-\frac{1}{2b(\chi)} \boldsymbol{\eta}\, \mathrm{R}_e\, \boldsymbol{\eta} + \vartheta \Big)\, \varphi \, \mathrm{d}x\, \mathrm{d}t \\ \text{for all } \varphi \in L^p(0,T; W^{1,p}(\Omega)) \cap L^\infty(Q) \text{ with } \varphi \geq 0 \text{ and } \mathrm{supp}(\varphi) \subset \{\chi > 0\},\end{gathered} \tag{1.20c}$$

and of a *generalized* total energy inequality, featuring the quasi-stresses $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$. While referring to Remark 7.4 for more comments in this direction, we may observe here that (1.20c) is in fact the integrated version in terms of *quasi-stresses* of the variational inequality (1.19a).

Plan of the paper. In the next Section 2 we introduce the variational formulation for the initial boundary value problem associated to the PDE system (1.1)–(1.3), as well as our main assumptions. Then, we state Theorems 1–5 on the existence/uniqueness of solutions for the reversible and the irreversible *non-degenerating* systems (i.e. $\delta > 0$). The existence Thms. 1, 2, 4, and 5 rely on the time-discretization procedure of Section 3; their proof is carried out by passing to the limit with the time discretization in Sections 4.1, 4.2, 6.1, and 6.2. The continuous dependence Thm. 3 is proved in Section 5. Finally, Section 7 is devoted to the passage to the degenerate limit $\delta \downarrow 0$.
The following table summarizes our results

Results	$\mu = 0$	$\mu = 1$
$\rho = 0,\ \delta > 0$	Theorem 1 (Sec. 2.3): $\exists$	Theorem 4 (Sec. 2.4): $\exists$
$\rho \neq 0,\ \delta > 0$	Theorem 2 (Sec. 2.3): $\exists$	
$\rho = 0,\ \vartheta$ constant, $\delta > 0$	Theorem 3 (Sec. 2.3): uniqueness	Theorem 5 (Sec. 2.4): improved regularity
$\rho \neq 0,\ \vartheta$ constant, $\delta > 0$	Theorem 3 (Sec. 2.3): uniqueness	Theorem 5 (Sec. 2.4): improved regularity
$\rho = 0,\ \delta \downarrow 0$		Theorem 6 (Sec. 7): $\exists$ degenerate case

2 Setup and results for the non-degenerating system

2.1 Notation and preliminaries

Notation 2.1. Throughout the paper, given a Banach space X we shall denote by $\|\cdot\|_X$ its norm, and use the symbol $\langle\cdot,\cdot\rangle_X$ for the duality pairing between X' and X.

Hereafter, we shall suppose that

$$\Omega \subset \mathbb{R}^d, \quad d \in \{2,3\} \ \text{ is a bounded connected domain, with } \mathrm{C}^2\text{-boundary } \partial\Omega.$$

We will identify both $L^2(\Omega)$ and $L^2(\Omega;\mathbb{R}^d)$ with their dual spaces, and denote by $(\cdot,\cdot)$ the scalar product in $\mathbb{R}^d$, by $(\cdot,\cdot)_{L^2(\Omega)}$ both the scalar product in $L^2(\Omega)$, and in $L^2(\Omega;\mathbb{R}^d)$, and by $H^1_0(\Omega;\mathbb{R}^d)$ and $H^2_0(\Omega;\mathbb{R}^d)$ the spaces

$$H^1_0(\Omega;\mathbb{R}^d) := \{\mathbf{v}\in H^1(\Omega;\mathbb{R}^d)\,:\,\mathbf{v}=0 \text{ on } \partial\Omega\},\ \text{endowed with the norm } \|\mathbf{v}\|^2_{H^1_0(\Omega)} := \int_\Omega \varepsilon(\mathbf{v})\colon\varepsilon(\mathbf{v})\,\mathrm{d}x,$$

$$H^2_0(\Omega;\mathbb{R}^d) := \{\mathbf{v}\in H^2(\Omega;\mathbb{R}^d)\,:\,\mathbf{v}=0 \text{ on } \partial\Omega\}.$$

For σ, $p\geq 1$ we will use the notation

$$W^{\sigma,p}_+(\Omega) := \{\zeta\in W^{\sigma,p}(\Omega)\,:\,\zeta(x)\geq 0 \quad \text{for a.a. } x\in\Omega\} \qquad \text{and analogously for } W^{\sigma,p}_-(\Omega). \tag{2.1}$$

We standardly denote by

$$A : H^1(\Omega)\to H^1(\Omega)' \text{ the operator } \langle Au,v\rangle_{H^1(\Omega)} := \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x$$

and, for any $w\in H^1(\Omega)$, by $m(w) := \langle w,1\rangle_{H^1(\Omega)}$ its mean value.

Given a (separable) Banach space X, we will denote by $\mathrm{BV}([0,T];X)$ (by $\mathrm{C}^0_{\mathrm{weak}}([0,T];X)$, respectively), the space of functions from $[0,T]$ with values in X that are defined at every $t\in[0,T]$ and have bounded variation on $[0,T]$ (and are *weakly* continuous on $[0,T]$, resp.)

Finally, throughout the paper we shall denote by the symbols c, c', C, C' various positive constants depending only on known quantities and by v_t (respectively v_{tt}) or (whenever it turns out to be more convenient) $\partial_t v$ (respectively $\partial_{tt}v$) the first (respectively) second partial derivatives with respect to time of a function v.

Preliminaries of mathematical elasticity. In what follows, we shall assume the material to be homogeneous and isotropic, so that the elasticity tensor $\mathrm{R}_e\in\mathbb{R}^{d\times d\times d\times d}$ in equation (1.3) may be represented by

$$\mathrm{R}_e\varepsilon(\mathbf{u}) = \lambda_1\mathrm{tr}(\varepsilon(\mathbf{u}))\mathbf{1} + 2\lambda_2\varepsilon(\mathbf{u}),$$

where $\lambda_1,\lambda_2>0$ are the so-called Lamé constants and $\mathbf{1}$ is the identity tensor. In order to state the variational formulation of the initial-boundary value problem for (1.1)–(1.3), we need to introduce the bilinear forms related to the χ-dependent elliptic operators appearing in (1.2). Hence, given a *non-negative* function $\eta\in L^\infty(\Omega)$, let us consider the continuous bilinear symmetric forms $a_{\mathrm{el}}(\eta\cdot,\cdot)$, $a_{\mathrm{vis}}(\eta\cdot,\cdot)$: $H^1_0(\Omega;\mathbb{R}^d)\times H^1_0(\Omega;\mathbb{R}^d)\to\mathbb{R}$ defined for all $\mathbf{u},\mathbf{v}\in H^1_0(\Omega;\mathbb{R}^d)$ by

$$\begin{aligned} a_{\mathrm{el}}(\eta\mathbf{u},\mathbf{v}) &:= \langle -\mathrm{div}(\eta\mathrm{R}_e\varepsilon(\mathbf{u})),\mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^d)} = \lambda_1\int_\Omega \eta\,\mathrm{div}(\mathbf{u})\,\mathrm{div}(\mathbf{v}) + 2\lambda_2\sum_{i,j=1}^d\int_\Omega \eta\,\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}),\\ a_{\mathrm{vis}}(\eta\mathbf{u},\mathbf{v}) &:= \langle -\mathrm{div}(\eta\mathrm{R}_v\varepsilon(\mathbf{u})),\mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^d)} = \sum_{i,j,k,h=1}^d\int_\Omega \eta\,\ell_{ijkh}\,\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \end{aligned} \tag{2.2}$$

where $(\ell_{ijkh})\in\mathbb{R}^{d\times d\times d\times d}$ is the viscosity tensor R_v. Now, by Korn's inequality (see eg [12, Thm. 6.3-3]), the forms $a_{\mathrm{el}}(\eta\cdot,\cdot)$ and $a_{\mathrm{vis}}(\eta\cdot,\cdot)$ are $H^1_0(\Omega;\mathbb{R}^d)$-elliptic and continuous. Namely, there exist constants C_1, $C_2>0$, only depending on λ_1 and λ_2, such that such that for all $\mathbf{u}$, $\mathbf{v}\in H^1_0(\Omega;\mathbb{R}^d)$

$$a_{\mathrm{el}}(\eta\mathbf{u},\mathbf{u}) \geq \inf_{x\in\Omega}(\eta(x))\,C_1\|\mathbf{u}\|^2_{H^1(\Omega)}, \qquad a_{\mathrm{vis}}(\eta\mathbf{u},\mathbf{u}) \geq \inf_{x\in\Omega}(\eta(x))\,C_1\|\mathbf{u}\|^2_{H^1(\Omega)}, \tag{2.3}$$

$$|a_{\mathrm{el}}(\eta\mathbf{u},\mathbf{v})| + |a_{\mathrm{vis}}(\eta\mathbf{u},\mathbf{v})| \leq C_2\|\eta\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}. \tag{2.4}$$

We shall denote by $\mathcal{E}(\eta\,\cdot) : H^1_0(\Omega;\mathbb{R}^d)\to H^{-1}(\Omega;\mathbb{R}^d)$ and $\mathcal{V}(\eta\,\cdot)$: $H^1_0(\Omega;\mathbb{R}^d)\to H^{-1}(\Omega;\mathbb{R}^d)$ the linear operators associated with $a_{\mathrm{el}}(\eta\cdot,\cdot)$ and $a_{\mathrm{vis}}(\eta\cdot,\cdot)$, respectively, namely

$$\langle\mathcal{E}(\eta\mathbf{v}),\mathbf{w}\rangle_{H^1(\Omega;\mathbb{R}^d)} := a_{\mathrm{el}}(\eta\mathbf{v},\mathbf{w}), \quad \langle\mathcal{V}(\eta\mathbf{v}),\mathbf{w}\rangle_{H^1(\Omega;\mathbb{R}^d)} := a_{\mathrm{vis}}(\eta\mathbf{v},\mathbf{w}) \qquad \text{for all } \mathbf{v},\ \mathbf{w}\in H^1_0(\Omega;\mathbb{R}^d). \tag{2.5}$$

It can be checked via an approximation argument that the following regularity results hold:

$$\text{if } \eta \in L^\infty(\Omega) \text{ and } \mathbf{u} \in H^1_0(\Omega;\mathbb{R}^d), \text{ then } \mathcal{E}(\eta\mathbf{u}), \mathcal{V}(\eta\mathbf{u}) \in H^{-1}(\Omega;\mathbb{R}^d), \tag{2.6a}$$

$$\text{if } \eta \in W^{1,d}(\Omega) \text{ and } \mathbf{u} \in H^2_0(\Omega;\mathbb{R}^d), \text{ then } \mathcal{E}(\eta\mathbf{u}), \mathcal{V}(\eta\mathbf{u}) \in L^2(\Omega;\mathbb{R}^d). \tag{2.6b}$$

Remark 2.2 (The anisotropic inhomogeneous case)**.** In fact, the calculations we will develop extend to the case of an anisotropic and inhomogeneous material, for which the elasticity and viscosity tensors R_e and R_v are of the form $\mathrm{R}_e = (g_{ijkh})$ and $\mathrm{R}_v = (\ell_{ijkh})$, with functions

$$g_{ijkh},\ \ell_{ijkh} \in \mathrm{C}^1(\Omega), \quad i,j,k,h = 1,2,3, \tag{2.7}$$

satisfying the classical symmetry and ellipticity conditions (with the usual summation convention)

$$\begin{aligned} &g_{ijkh} = g_{jikh} = g_{khij}, \qquad \ell_{ijkh} = \ell_{jikh} = \ell_{khij}, \quad i,j,k,h = 1,2,3 \\ &\exists\, C_1 > 0: \quad g_{ijkh}\xi_{ij}\xi_{kh} \geq C_1\xi_{ij}\xi_{ij},\ \ell_{ijkh}\xi_{ij}\xi_{kh} \geq C_1\xi_{ij}\xi_{ij} \quad \text{for all } \xi_{ij} : \xi_{ij} = \xi_{ji}, \quad i,j = 1,2,3. \end{aligned} \tag{2.8}$$

Clearly, (2.8) ensures (2.3), whereas not only does (2.7) imply (2.4), but the C^1-regularity also allows us to perform the third a priori estimate of Section 3.3 rigorously.

In what follows we will use the following elliptic regularity result (see e.g. [12, Thm. 6.3-.6, p. 296], cf. also [42, p. 260]):

$$\exists\, C_3,\, C_4 > 0 \quad \forall\, \mathbf{u} \in H^2_0(\Omega;\mathbb{R}^d): \qquad C_3\|\mathbf{u}\|_{H^2(\Omega)} \leq \|\operatorname{div}(\varepsilon(\mathbf{u}))\|_{L^2(\Omega)} \leq C_4\|\mathbf{u}\|_{H^2(\Omega)}. \tag{2.9}$$

Finally, in the weak formulation of the momentum equation (1.2), besides $\mathcal{V}$ and $\mathcal{E}$ we will also make use of the operator

$$\mathcal{C}_\rho : L^2(\Omega) \to H^{-1}(\Omega;\mathbb{R}^d) \quad \text{defined by} \quad \langle \mathcal{C}_\rho(\vartheta), \mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^d)} := -\rho\int_\Omega \vartheta \operatorname{div}(\mathbf{v})\,\mathrm{d}x. \tag{2.10}$$

Useful inequalities. We recall the celebrated Gagliardo-Nirenberg inequality (cf. [43, p. 125]) in a particular case: for all r, $q \in [1,+\infty]$, and for all $v \in L^q(\Omega)$ such that $\nabla v \in L^r(\Omega)$, there holds

$$\|v\|_{L^s(\Omega)} \leq C_{\mathrm{GN}}\|v\|^\theta_{W^{1,r}(\Omega)}\|v\|^{1-\theta}_{L^q(\Omega)}, \text{ with } \frac{1}{s} = \theta\left(\frac{1}{r} - \frac{1}{d}\right) + (1-\theta)\frac{1}{q}, \quad 0 \leq \theta \leq 1, \tag{2.11}$$

the positive constant C_{GN} depending only on d, r, q, θ. Combining the compact embedding

$$H^2_0(\Omega;\mathbb{R}^d) \Subset W^{1,d^\star-\eta}(\Omega;\mathbb{R}^d), \quad \text{with } d^\star = \begin{cases} \infty & \text{if } d = 2, \\ 6 & \text{if } d = 3, \end{cases} \quad \text{for all } \eta > 0, \tag{2.12}$$

(where for $d = 2$ we mean that $H^2_0(\Omega;\mathbb{R}^d) \Subset W^{1,q}(\Omega;\mathbb{R}^d)$ for all $1 \leq q < \infty$), with [33, Thm. 16.4, p. 102], we have

$$\forall\, \varrho > 0 \ \ \exists\, C_\varrho > 0 \ \ \forall\, \mathbf{u} \in H^2_0(\Omega;\mathbb{R}^d): \ \ \|\varepsilon(\mathbf{u})\|_{L^{d^\star-\eta}(\Omega)} \leq \varrho\|\mathbf{u}\|_{H^2(\Omega)} + C_\varrho\|\mathbf{u}\|_{L^2(\Omega)}. \tag{2.13}$$

We will also make use of the compact Sobolev embedding

$$W^{1,p}(\Omega) \Subset \mathrm{C}^0(\overline{\Omega}) \qquad \text{for } p > d, \text{ with } d \geq 2. \tag{2.14}$$

We conclude with the following Poincaré-type inequality (cf. [26, Lemma 2.2]), with $m(w)$ the mean value of w:

$$\forall\, q > 0 \quad \exists\, C_q > 0 \quad \forall\, w \in H^1(\Omega): \qquad \||w|^q w\|_{H^1(\Omega)} \leq C_q(\|\nabla(|w|^q w)\|_{L^2(\Omega)} + |m(w)|^{q+1}). \tag{2.15}$$

2.2 Assumptions and weak formulations

We enlist below our basic assumptions on the functions c, K, W, $\mathbf{d}$ in system (1.1)–(1.3).
Hypothesis (I). We suppose that

$$\begin{aligned}&\text{the function } \mathsf{c} : [0,+\infty) \to [0,+\infty) \ \text{ is continuous, and}\\ &\exists\, \sigma_1 \geq \sigma > \frac{2d}{d{+}2},\ c_1 \geq c_0 > 0 \ \ \forall\, \vartheta \in [0,+\infty) : \quad c_0(1{+}\vartheta)^{\sigma-1} \leq \mathsf{c}(\vartheta) \leq c_1(1{+}\vartheta)^{\sigma_1-1}\,.\end{aligned} \tag{2.16}$$

Hypothesis (II). We assume that

$$\begin{aligned}&\text{the function } \mathsf{K} : [0,+\infty) \to (0,+\infty) \ \text{ is continuous and}\\ &\exists c_2,\, c_3 > 0 \ \ \forall \vartheta \in [0,+\infty) : \quad c_2\mathsf{c}(\vartheta) \leq \mathsf{K}(\vartheta) \leq c_3(\mathsf{c}(\vartheta) + 1)\,.\end{aligned} \tag{2.17}$$

Hypothesis (III). We require

$$a \in \mathrm{C}^1(\mathbb{R}),\ b \in \mathrm{C}^2(\mathbb{R}) \ \text{ are such that } a(x),\ b(x) \geq 0 \text{ for all } x \in [0,1]. \tag{2.18}$$

Hypothesis (IV). We suppose that the potential W in (1.3) is given by $W = \widehat{\beta} + \widehat{\gamma}$, where

$$\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]\,, \quad \widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R} \ \text{ is proper, l.s.c., convex;} \tag{2.19}$$

$$\widehat{\gamma} \in \mathrm{C}^2(\mathbb{R}). \tag{2.20}$$

Hereafter, we shall denote by $\beta = \partial\widehat{\beta}$ the subdifferential of $\widehat{\beta}$, and set $\gamma := \widehat{\gamma}'$.
Hypothesis (V). We require that there exists

$$\begin{aligned}&\text{a Carathéodory integrand } \phi \,:\, \Omega \times \mathbb{R}^d \to [0,+\infty) \text{ such that for a.a. } x \in \Omega\\ &\text{the map } \phi(x,\cdot) : \mathbb{R}^d \to [0,+\infty) \ \text{ is convex, with } \phi(x,0) = 0, \text{ and in } \mathrm{C}^1(\mathbb{R}^d)\,,\end{aligned} \tag{2.21}$$

and, setting $\mathbf{d} := \nabla_{\boldsymbol{\zeta}}\phi \,:\, \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, the following coercivity and growth conditions hold true:

$$\exists\, p > d, \quad c_4,\, c_5,\, c_6 > 0 \ \text{ for a.a. } x \in \Omega \ \ \forall\, \boldsymbol{\zeta} \in \mathbb{R}^d \ : \ \left\{ \begin{array}{l} \phi(x,\boldsymbol{\zeta}) \geq c_4|\boldsymbol{\zeta}|^p - c_5,\\ |\mathbf{d}(x,\boldsymbol{\zeta})| \leq c_6(1 + |\boldsymbol{\zeta}|^{p-1})\,. \end{array}\right. \tag{2.22}$$

A generalization of the p-Laplace operator. We now consider the realization in $L^2(\Omega)$ of ϕ, i.e.

$$\Phi \,:\, L^2(\Omega) \to [0,+\infty], \quad \Phi(\chi) := \begin{cases} \int_\Omega \phi(x,\nabla\chi(x))\,\mathrm{d}x & \text{if } \phi(\cdot,\nabla\chi(\cdot)) \in L^1(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.23}$$

Relying on [25, Thm. 2.5, p. 22], it is possible to prove that Φ is convex and lower semicontinuous on $L^2(\Omega)$, with domain $D(\Phi) := W^{1,p}(\Omega)$ (due to (2.22)); its subdifferential $\partial\Phi : L^2(\Omega) \rightrightarrows L^2(\Omega)$ is a maximal monotone operator. In (1.3) we will take the elliptic operator

$$\mathcal{B} := \partial\Phi : L^2(\Omega) \rightrightarrows L^2(\Omega). \tag{2.24}$$

Clearly, $\mathcal{B}$ is a generalization of the p-Laplace operator. Note that

$$\mathrm{dom}(\mathcal{B}) := \left\{ v \in W^{1,p}(\Omega) \,:\, \sup_{w \in D(\Phi)\setminus\{0\}} \left| \int_\Omega \mathbf{d}(x,\nabla v(x)) \cdot \nabla w(x)\,dx \right| / \|w\|_{L^2(\Omega)} < +\infty \right\},$$

and (cf. [44, Ex. 2.4])

$$(\mathcal{B}v, w)_{L^2(\Omega)} = \int_\Omega \mathbf{d}(x,\nabla v(x)) \cdot \nabla w(x)\,\mathrm{d}x \quad \text{for all } w \in D(\Phi)\,. \tag{2.25}$$

In order to obtain further regularity and uniqueness results for system (1.1)–(1.3), we will have to assume that either of the following additional hypotheses holds true.

Hypothesis (VI). We require that the function ϕ fulfills the p-coercivity condition

$$\exists\, c_7 > 0 \qquad \forall\, x \in \Omega\ \forall\, \boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d\ : \qquad (\mathbf{d}(x,\boldsymbol{\zeta}) - \mathbf{d}(x,\boldsymbol{\eta}), \boldsymbol{\zeta} - \boldsymbol{\eta}) \geq c_7 |\boldsymbol{\zeta} - \boldsymbol{\eta}|^p, \tag{2.26}$$

and it is Lipschitz with respect to x, viz.

$$\exists\, L > 0 \quad \forall\, x\,, y \in \Omega\ \forall\, \boldsymbol{\zeta} \in \mathbb{R}^d\ : \qquad |\phi(x,\boldsymbol{\zeta}) - \phi(y,\boldsymbol{\zeta})| \leq L|x-y|(1+|\boldsymbol{\zeta}|^p). \tag{2.27}$$

Remark 2.3 (A regularity result)**.** It was proved in [50, Thm. 2, Rmk. 3.5] that, if in addition to Hypothesis (V) the function ϕ fulfills Hypothesis (VI), then

$$\begin{aligned} &\mathrm{dom}(\mathcal{B}) \subset W^{1+\sigma,p}(\Omega) \text{ for all } 0 < \sigma < \frac{1}{p}, \text{ and} \\ &\forall\, 0 < \sigma < \frac{1}{p} \ \ \exists\, C_\sigma > 0\ \forall\, v \in W^{1+\sigma,p}(\Omega)\ :\ \|v\|_{W^{1+\sigma,p}(\Omega)} \leq C_\sigma \|\mathcal{B}(v)\|_{L^2(\Omega)}. \end{aligned} \tag{2.28}$$

Hypothesis (VII). We assume that ϕ complies with (2.27) and with the following *convexity* requirement

$$\exists\, c_8 > 0\ \exists\, \kappa > 0 \qquad \forall\, x \in \Omega\ \forall\, \boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d,\ \ \boldsymbol{\zeta} \neq 0\ : \qquad \mathrm{D}^2_{\boldsymbol{\zeta}} \phi(x,\boldsymbol{\zeta}) \boldsymbol{\eta}\boldsymbol{\eta} \geq c_8 (\kappa + |\boldsymbol{\zeta}|^2) |\boldsymbol{\eta}|^2. \tag{2.29}$$

Remark 2.4. Assumptions (2.27) and (2.29) guarantee the validity of the following inequality (cf. [30] for a proof)

$$\exists\, c_9 > 0 \qquad \forall\, x \in \Omega\ \forall\, \boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d\ : \qquad (\mathbf{d}(x,\boldsymbol{\zeta}) - \mathbf{d}(x,\boldsymbol{\eta}), \boldsymbol{\zeta} - \boldsymbol{\eta}) \geq c_9 (\kappa + |\boldsymbol{\zeta}| + |\boldsymbol{\eta}|)^{p-2} |\boldsymbol{\zeta} - \boldsymbol{\eta}|^2, \tag{2.30}$$

which will play a crucial role in the proof of Thm. 3.

Example 2.5. The two p-Laplacian operators

$$A_p(\chi) := -\operatorname{div}(|\nabla\chi|^{p-1}\nabla\chi), \qquad p > d, \tag{2.31}$$

$$\mathsf{A}_p(\chi) := -\operatorname{div}((1+|\nabla\chi|^2)^{p/2}), \qquad p > d, \tag{2.32}$$

are clearly of the form (2.25), and comply with (2.22) and (2.26)–(2.27) (cf. [14, Ch. I, 4–(iii)]).

Observe that (2.29) is fulfilled by the p-Laplacian operator A_p (2.32), whereas for the *degenerate* operator A_p (2.31), inequality (2.29) holds with $\kappa = 0$.

A *nonlocal* alternative to the p-Laplacian operator. As done in [31], we could replace the p-Laplacian-type operator $\mathcal{B}$ (2.25) in (2.59) with a *linear* operator, with domain compactly embedded in $\mathrm{C}^0(\overline{\Omega})$. More precisely, as in [31] we could choose

$$\mathcal{B} := A_s : H^s(\Omega) \to H^s(\Omega)^* \quad \text{ with } s > \frac{d}{2}. \tag{2.33}$$

In (2.33), $H^s(\Omega)$ denotes the Sobolev-Slobodeckij space $W^{s,2}(\Omega)$, endowed with the inner product

$$(z_1, z_2)_{H^s(\Omega)} := (z_1, z_2)_{L^2(\Omega)} + a_s(z_1, z_2),$$

where

$$a_s(z_1,z_2) := \int_\Omega \int_\Omega \frac{\big(\nabla z_1(x) - \nabla z_1(y)\big) \cdot \big(\nabla z_2(x) - \nabla z_2(y)\big)}{|x-y|^{d+2(s-1)}} \,\mathrm{d}x\,\mathrm{d}y. \tag{2.34}$$

Indeed, since $d \in \{2,3\}$, we may suppose that $s \in (1,2)$. Then, we denote by $A_s : H^s(\Omega) \to H^s(\Omega)^*$ the associated operator, viz.

$$\langle A_s \chi, w \rangle_{H^s(\Omega)} := a_s(\chi, w) \quad \text{for every } \chi,\ w \in H^s(\Omega). \tag{2.35}$$

Observe that, for $s > d/2$ we have $H^s(\Omega) \Subset \mathrm{C}^0(\overline{\Omega})$.

Enthalpy transformation. We now reformulate PDE system (1.1)–(1.3) in terms of the *enthalpy* w, related to the absolute temperature ϑ via

$$w = h(\vartheta) \quad \text{with } h(r) := \int_0^r \mathsf{c}(s)\,\mathrm{d}s. \tag{2.36}$$

It follows from (2.16) that the function h is strictly increasing on $[0, +\infty)$. Thus, we are entitled to define

$$\Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \qquad K(w) := \frac{\mathsf{K}(\Theta(w))}{\mathsf{c}(\Theta(w))}. \tag{2.37}$$

In terms of the enthalpy w, the PDE system (1.1)–(1.3) rewrites as

$$w_t + \chi_t \Theta(w) + \rho\Theta(w)\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(K(w)\nabla w) = g \quad \text{in } \Omega \times (0,T), \tag{2.38}$$

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\Theta(w)\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{2.39}$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \mathrm{div}(\mathbf{d}(x,\nabla\chi)) + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \quad \text{in } \Omega \times (0,T), \tag{2.40}$$

supplemented with the initial and boundary conditions (where $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$)

$$w(0) = w_0, \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}_t(0) = \mathbf{v}_0, \quad \chi(0) = \chi_0 \qquad \text{in } \Omega, \tag{2.41}$$

$$\partial_{\mathbf{n}} w = 0, \qquad \mathbf{u} = 0, \qquad \partial_{\mathbf{n}}\chi = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{2.42}$$

Remark 2.6. The enthalpy transformation (2.36) was proposed in [49], and further developed in [47], in order to deal with PDE systems where a quasilinear internal energy balance analogous to (1.1) is coupled with *rate-independent* processes. The advantage of this change of variables, is that the *nonlinear* term $\mathsf{c}(\vartheta)\vartheta_t$ in (1.1) is replaced by the *linear* contribution w_t in (2.38). We will exploit this fact, when proving the existence of solutions to (an approximation of) system (2.38)–(2.42) by means of a time-discretization scheme.

For later use, let us observe that Hyp. (I) implies

$$\begin{aligned} &\exists\, d_0,\, d_1 > 0 \quad \forall\, w \in [0,+\infty) : \quad d_1(w^{1/\sigma_1} - 1) \leq \Theta(w) \leq d_0(w^{1/\sigma} + 1)\,, \\ &\text{the map } w \mapsto \Theta(w) \quad \text{is Lipschitz continuous.} \end{aligned} \tag{2.43}$$

A straightforward consequence of the first of (2.43) is that for every $s \in (1, \infty)$

$$\exists\, C_s > 0 \quad \forall\, w \in L^1(\Omega) : \quad \|\Theta(w)\|_{L^s(\Omega)} \leq C_s(\|w\|_{L^{s/\sigma}(\Omega)}^{1/\sigma} + 1). \tag{2.44}$$

Moreover, Hypotheses (I) and (II) entail

$$\exists\, \bar{c} > 0 \quad \forall\, w \in \mathbb{R} : \; c_2 \leq K(w) \leq \bar{c}. \tag{2.45}$$

Finally, in order to deal with the case $\rho \neq 0$, we will adopt the following further assumption, which we directly state in terms of the function K instead of K and c, in *replacement* of Hypothesis (II). **Hypothesis (VIII).** We require that the function K defined in (2.37) (where c fulfills Hyp. (I) and $\mathsf{K} : [0,+\infty) \to (0,+\infty)$ is a continuous function) satisfies

$$\exists c_{10} > 0 \; \exists q \geq \frac{d+2}{2d} \quad \forall w \in [0,+\infty) : \quad K(w) = c_{10}\left(w^{2q} + 1\right). \tag{2.46}$$

Indeed, we could slightly weaken (2.46) by prescribing that K is bounded from below and above by two functions behaving like w^{2q}, and we have restricted to (2.46) for simplicity only. Let us stress that, if (2.46) holds, K is no longer bounded from above. The reader may refer to [53] for various examples in which a superquadratic growth of the heat conductivity is imposed.

Problem and Cauchy data. We suppose that bulk force $\mathbf{f}$ and the heat source g fulfill

$$\mathbf{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{2.47}$$
$$g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)'), \tag{2.48}$$

and that the initial data comply with

$$\vartheta_0 \in L^{\sigma_1}(\Omega) \quad \text{whence} \quad w_0 := h(\vartheta_0) \in L^1(\Omega)\,, \tag{2.49}$$
$$\mathbf{u}_0 \in H^2_0(\Omega;\mathbb{R}^d), \quad \mathbf{v}_0 \in H^1_0(\Omega;\mathbb{R}^d)\,, \tag{2.50}$$
$$\chi_0 \in \mathrm{dom}(\mathcal{B}), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega). \tag{2.51}$$

Variational formulation of the non-degenerating system. We now consider the *non-degenerate* version of system (2.38)–(2.42): as already mentioned in the introduction, to rule out the elliptic degeneracy of the momentum equation (2.39), it is indeed sufficient to truncate away from zero only the coefficient $a(\chi)$, cf. (2.58) below. In specifying the variational formulation of the initial-boundary value problem for the *non-degenerate* system, due to the 0-homogeneity of the operator $\partial I_{(-\infty,0]}$ (2.92) we will just distinguish the two cases $\mu = 0$ and $\mu = 1$. We mention in advance that the $L^r(0,T;W^{1,r}(\Omega))$-regularity for w derives from BOCCARDO&GALLOUËT-type estimates [5] on the enthalpy equation, combined with the Gagliardo-Nirenberg inequality (2.11). We refer to the forthcoming Sec. 3.3 and to [49] for all details.

Problem 2.7. *Given $\delta > 0$, $\mu \in \{0,1\}$, find functions*

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*) \quad \textit{for every } 1 \le r < \frac{d+2}{d+1}, \tag{2.52}$$
$$\mathbf{u} \in H^1(0,T;H^2_0(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{2.53}$$
$$\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{2.54}$$

fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \textit{for a.a. } x \in \Omega, \tag{2.55}$$
$$\chi(0,x) = \chi_0(x) \qquad \textit{for a.a. } x \in \Omega, \tag{2.56}$$

the equations

$$\begin{aligned} &\int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t\int_\Omega w\varphi_t \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \chi_t\Theta(w)\varphi\,\mathrm{d}x\,\mathrm{d}s + \rho\int_0^t\int_\Omega \mathrm{div}(\mathbf{u}_t)\Theta(w)\varphi\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_0^t\int_\Omega K(w)\nabla w\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega g\varphi + \int_\Omega w_0\varphi(0)\,\mathrm{d}x \\ &\qquad \textit{for all } \varphi \in \mathscr{F} := \mathrm{C}^0([0,T];W^{1,r'}(\Omega)) \cap W^{1,r'}(0,T;L^{r'}(\Omega)) \textit{ and for all } t \in (0,T], \end{aligned} \tag{2.57}$$

$$\mathbf{u}_{tt} + \mathcal{V}\left((a(\chi)+\delta)\mathbf{u}_t\right) + \mathcal{E}\left(b(\chi)\mathbf{u}\right) + \mathcal{C}_\rho(\Theta(w)) = \mathbf{f} \quad \textit{in } H^{-1}(\Omega;\mathbb{R}^d) \qquad \textit{a.e. in } (0,T), \tag{2.58}$$

and the subdifferential inclusion

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) + \mathcal{B}(\chi) + \beta(\chi) + \gamma(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \;\textit{ in } W^{1,p}(\Omega)^* \;\textit{ a.e. in } (0,T). \tag{2.59}$$

Remark 2.8. Since $w \in \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$, for all $t \in [0,T]$ one has $w(t) \in W^{1,r'}(\Omega)^*$. Combining this with the fact that $w \in L^\infty(0,T;L^1(\Omega))$, we have that $w(t)$ is a Radon measure on Ω for all $t \in [0,T]$, which justifies the notation in the first integral term on the left-hand side of (2.57). Moreover, let us note that the BV-regularity w.r.t. time of the absolute temperature, which is mainly due to the presence of quadratic nonlinearities $\chi_t\Theta(w)$ and $\mathrm{div}(\mathbf{u}_t)\Theta(w)$ in (2.57), is quite natural for this kind of problems (cf., e.g. [5], [17], and [49]).

Remark 2.9. The proof of our results could be carried out with suitable modifications in the case of Neumann boundary conditions on $\mathbf{u}$, as well. We would also be able to handle the case of Neumann conditions on a portion Γ_0 of $\partial\Omega$ and Dirichlet conditions on $\Gamma_1 := \partial\Omega \setminus \Gamma_0$ ($|\Gamma_0|$, $|\Gamma_1| > 0$), provided that the closures of the sets Γ_0 and Γ_1 do not intersect. Indeed, without the latter geometric condition, the elliptic regularity results ensuring the (crucial) $H^2_0(\Omega;\mathbb{R}^d)$-regularity of $\mathbf{u}$ may fail to hold, see [12, Chap. VI, Sec. 6.3].

In what follows, we will refer to system (2.57)–(2.59) with $\mu = 0$ (with $\mu = 1$, respectively), as the (non-degenerating) *reversible full system* (*irreversible full system*, resp). In both cases $\mu = 0$ and $\mu = 1$, we will call *isothermal* the (non-degenerating) system (2.58)–(2.59), where $\Theta(w)$ in (2.59) is replaced by a given temperature profile Θ^*.

2.3 Global existence and uniqueness results for the reversible system

Our first main result states the existence of a solution $(w, \mathbf{u}, \chi)$ to the reversible full system under Hypotheses (I)–(V); under the further Hypothesis (VI) we are able to obtain some enhanced regularity for χ. Its proof will be developed in Section 4 by passing to the limit in the time-discretization scheme set up in Sec. 3.

Theorem 1 (Global existence for the full system, $\mu = 0$, $\rho = 0$)**.** *Let $\mu = 0$, $\rho = 0$, and assume Hypotheses (I)–(V) and conditions* (2.47)–(2.51) *on the data* $\mathbf{f}$, g, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0,. *Then,*

1. *Problem 2.7 admits a solution $(w, \mathbf{u}, \chi)$, such that there exists*

$$\xi \in L^2(0,T;L^2(\Omega)) \text{ with } \xi(x,t) \in \beta(\chi(x,t)) \text{ for a.a. } (x,t) \in \Omega\times(0,T), \text{ fulfilling} \tag{2.60}$$

$$\chi_t + \mathcal{B}(\chi) + \xi + \gamma(\chi) = -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \qquad \text{a.e. in } \Omega\times(0,T). \tag{2.61}$$

Furthermore, $(w, \mathbf{u}, \chi)$ satisfies the total energy equality

$$\begin{aligned}
&\int_\Omega w(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2\,\mathrm{d}x + \int_s^t\int_\Omega |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_s^t a_{\mathrm{vis}}((a(\chi)+\delta)\mathbf{u}_t,\mathbf{u}_t)\,\mathrm{d}r + \frac{1}{2}a_{\mathrm{el}}(b(\chi(t))\mathbf{u}(t),\mathbf{u}(t)) + \Phi(\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x \\
&= \int_\Omega w(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(s)|^2\,\mathrm{d}x + \frac{1}{2}a_{\mathrm{el}}(b(\chi(s))\mathbf{u}(s),\mathbf{u}(s)) + \Phi(\chi(s)) + \int_\Omega W(\chi(s))\,\mathrm{d}x \\
&\qquad + \int_s^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_s^t\int_\Omega g\,\mathrm{d}x\mathrm{d}r \qquad \text{for all } 0\le s\le t\le T.
\end{aligned} \tag{2.62}$$

2. *If, in addition, ϕ complies with Hypothesis (VI), then there holds*

$$\chi \in L^2(0,T;W^{1+\sigma,p}(\Omega)) \quad \text{for all } 0<\sigma<\frac{1}{p}. \tag{2.63}$$

3. *Suppose that*

$$g(x,t)\ge 0 \quad \text{for a.a. } (x,t)\in\Omega\times(0,T). \tag{2.64}$$

Then, $w \ge 0$ a.e. in $\Omega\times(0,T)$, hence

$$\vartheta(x,t) := \Theta(w(x,t)) \ge 0 \quad \text{for a.a. } (x,t)\in\Omega\times(0,T). \tag{2.65}$$

We are going to prove the energy equality (2.62) by testing (2.57) by $\varphi \equiv 1$, (2.58) by $\mathbf{u}_t$, (2.61) by χ_t, adding the resulting relations, integrating in time, and developing the calculations at the end of the proof of Thm. 1 in Sec. 4.1.

We now turn to the case when the thermal expansion coefficient $\rho \neq 0$. As previously mentioned, to prove existence of solutions we need to replace Hyp. (II) with Hyp. (VIII), which has a key role in deriving the enhanced regularity (2.67) for w, cf. Remark 2.10 later on. Observe that (2.67) in particular yields (weak) continuity of the enthalpy variable.

Theorem 2 (Global existence for the full system, $\mu = 0$, $\rho \neq 0$)**.** *Let $\mu = 0$, $\rho \neq 0$, and assume Hypotheses (I) and (III)–(V) and conditions* (2.47)–(2.51) *on the data $\mathbf{f}$, g, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0,. Suppose moreover that Hypothesis (VIII) is satisfied (in place of Hypothesis (II)), and that*

$$w_0 \in L^2(\Omega). \tag{2.66}$$

Then,

1. *Problem 2.7 admits a solution $(w, \mathbf{u}, \chi)$ fulfilling* (2.60)-(2.61)*, such that w has the further regularity*

$$\begin{aligned} w \in L^2(0,T;H^1(\Omega)) \cap L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \\ \cap W^{1,\mathfrak{r}}(0,T;W^{2,\mathfrak{s}}(\Omega)') \quad \text{with } \mathfrak{r} = 1 + \frac{1}{2q+1} \text{ and } \mathfrak{s} = \frac{6q+6}{4q+5} \end{aligned} \tag{2.67}$$

   with q from (2.46)*. Hence, $w \in \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega))$. Furthermore, the weak formulation of the enthalpy equation holds in the form*

$$\begin{aligned} \int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi \,\mathrm{d}x\,\mathrm{d}s + \rho \int_0^t \int_\Omega \mathrm{div}(\mathbf{u}_t)\Theta(w)\varphi\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t \int_\Omega \widehat{K}(w) A\varphi \,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_\Omega g\varphi + \int_\Omega w_0 \varphi(0)\,\mathrm{d}x \end{aligned} \tag{2.68}$$

   for all test functions $\varphi \in \mathscr{F}' := L^{\mathfrak{r}'}(0,T;W^{2,\mathfrak{s}}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$, where $\mathfrak{r}'$ is the conjugate exponent of $\mathfrak{r}$, and $\widehat{K}(w) = \frac{1}{2q+1} w^{2q+1}$ is a primitive of K. Moreover, $(w, \mathbf{u}, \chi)$ complies with the total energy equality (2.62)*.*
2. *If, in addition, ϕ complies with Hypothesis (VI), then the further regularity result* (2.63) *holds true.*
3. *If in addition g complies with* (2.64)*, then* (2.65) *holds.*

Remark 2.10 (Outlook to the enhanced regularity (2.67))**.** Let us justify the additional regularity (2.67) for w, by developing, on a purely *formal* level, enhanced estimates on the enthalpy equation (2.38), based on the stronger Hypothesis (VIII). Indeed, we (formally) choose $\varphi = w$ as a test function for (2.57): re-integrating by parts in time and exploiting (2.46) we obtain for any $t \in (0,T)$:

$$\begin{aligned} &\frac{1}{2}\int_\Omega |w(t)|^2 \,\mathrm{d}x + c_{10} \int_0^t \int_\Omega \left(|w|^{2q} + 1\right) |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\leq \frac{1}{2}\int_\Omega |w(0)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |g||w| \,\mathrm{d}x + \int_0^t \int_\Omega \left(|\chi_t| + |\rho||\,\mathrm{div}(\mathbf{u}_t)|\right) |\Theta(w)||w| \,\mathrm{d}x\,\mathrm{d}s\,. \end{aligned} \tag{2.69}$$

Now, we observe that $\int_\Omega |w|^{2q}|\nabla w|^2 \,\mathrm{d}x = 1/(q+1)^2 \int_\Omega |\nabla(|w|^q w)|^2 \,\mathrm{d}x$ and that, due to the Poincaré inequality (2.15) and to the fact that $w \in L^\infty(0,T;L^1(\Omega))$, there holds

$$\||w|^q w\|^2_{H^1(\Omega)} \leq C(\|\nabla(|w|^q w)\|^2_{L^2(\Omega)} + 1)\,.$$

Therefore, taking into account the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$, for the l.h.s. of (2.69) we have the lower bound

$$\int_0^t \int_\Omega \left(|w|^{2q} + 1\right) |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}s \geq c \int_0^t \left(\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^{2(q+1)}_{L^{6(q+1)}(\Omega)} \right) \mathrm{d}s - C. \tag{2.70}$$

Clearly, relying on (2.48) we can absorb the second term on the r.h.s. of (2.69) into its left-hand side. On the other hand, using the fact that $\ell := \chi_t + \mathrm{div}(\mathbf{u}_t) \in L^2(0,T;L^2(\Omega))$ and taking into account the growth (2.43) of Θ, we can estimate the last summand on the r.h.s. by

$$
\begin{aligned}
d_0 \int_0^t \int_\Omega |\ell|(w^{1+1/\sigma}+1)\,\mathrm{d}x\,\mathrm{d}s &\leq C \int_0^t \int_\Omega |\ell|(w^{q+1}+1)\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \varrho \int_0^t \|w\|^{2(q+1)}_{L^{2(q+1)}(\Omega)}\,\mathrm{d}s + C_\varrho \left(\int_0^t \|\ell\|^2_{L^2(\Omega)} + 1 \right),
\end{aligned}
\tag{2.71}
$$

where the first inequality follows from the fact that $1+1/\sigma < (3d+2)/(2d) \leq q+1$ thanks to (2.16) and (2.46), and $\varrho > 0$ is chosen sufficiently small, in such a way as to absorb $\int_0^t \|w\|^{2(q+1)}_{L^{2(q+1)}(\Omega)}\,\mathrm{d}s$ into the r.h.s. of (2.70). Plugging (2.70) and (2.71) into (2.69) we immediately deduce an estimate for w in the space $w \in L^2(0,T;H^1(\Omega)) \cap L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

An interpolation between $L^2(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ yields (cf. the Gagliardo-Nirenberg inequality (2.11)) that $w \in L^{2(d+2)/d}(Q)$. Therefore, on account of estimate (2.44) for $\Theta(w)$ we conclude that $\Theta(w) \in L^4(Q)$. This ensures that the term $\chi_t\Theta(w) + \rho\Theta(w)\mathrm{div}(\mathbf{u}_t)$ is in $L^{4/3}(Q)$. Furthermore, as a consequence of $w \in L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega))$, we have $w^{2q+1} \in L^{1+1/\epsilon}(0,T;L^{3+3/\epsilon}(\Omega))$, with $\epsilon = 2q+1$. Therefore,

$$
A(\widehat{K}(w)) \text{ is estimated in } L^{\mathfrak{r}}(0,T;W^{2,\mathfrak{s}}(\Omega)'),
$$

with $\mathfrak{r} = 1 + 1/\epsilon = 1 + 1/(2q+1)$ and $\mathfrak{s} = (6q+6)/(4q+5)$ the conjugate exponent of $3 + 3/(2q+1)$. In view of the above observations, a comparison in (2.38) entails an estimate for $w_t \in L^{\mathfrak{r}}(0,T;W^{2,\mathfrak{s}}(\Omega)')$, and we conclude (2.67).

To prove Thm. 2, we will need to combine the time-discretization procedure for system (2.57)–(2.59), with a truncation of the function K in the elliptic operator of (2.57), cf. Problem 3.3 later on. Hence, in order to make the estimates developed in Rmk. 2.10 rigorous, we will have to pass to the limit in two phases, first with the time-step, and then with the truncation parameter, cf. the discussion in Sec. 4.2.

For the *isothermal* reversible system, in both cases $\rho = 0$ and $\rho \neq 0$, we obtain a continuous dependence result, in particular yielding uniqueness of solutions, under the additional convexity property for ϕ in Hypothesis (VII). Indeed, the latter ensures the monotonicity inequality (2.30) for $\mathbf{d}$, which is crucial for the continuous dependence estimate. We also need to restrict to the case in which a is constant.

Theorem 3 (Continuous dependence on the data for the isothermal system, $\mu = 0$, $\rho \in \mathbb{R}$)**.** *Let $\mu = 0$, $\rho \in \mathbb{R}$. Assume that Hypotheses (III)–(V) and (VII) are satisfied, and, in addition, that*

$$
\textit{the function } a \textit{ is constant.} \tag{2.72}
$$

Let $(\mathbf{f}_i, \mathbf{u}_0^i, \mathbf{v}_0^i, \chi_0^i)$, $i = 1,2$, be two sets of data complying with (2.47) *and* (2.50)–(2.51)*, and, accordingly, let $(\mathbf{u}_i, \chi_i)$, $i = 1,2$, be the associated solutions on some $[0,T]$ with fixed temperature profiles $\Theta(w_i) = \bar{\Theta}_i \in L^2(0,T;L^2(\Omega))$. Set $\mathrm{M} := \max_{i=1,2}\left\{ \|\mathbf{u}_i\|_{H^1(0,T;H^2_0(\Omega;\mathbb{R}^d))} + 1 \right\}$. Then there exists a positive constant S_0, depending on M, δ, T, and $|\Omega|$, such that*

$$
\begin{aligned}
&\|\mathbf{u}_1 - \mathbf{u}_2\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^1(0,T;H^1(\Omega;\mathbb{R}^d))} + \|\chi_1 - \chi_2\|_{L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))} \\
&\leq S_0 \Big(\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_{H^1(\Omega;\mathbb{R}^d)} + \|\mathbf{v}_0^1 - \mathbf{v}_0^2\|_{L^2(\Omega;\mathbb{R}^d)} + \|\chi_0^1 - \chi_0^2\|_{L^2(\Omega)} \\
&\qquad + \|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(0,T;H^{-1}(\Omega))} + \|\bar{\Theta}_1 - \bar{\Theta}_2\|_{L^2(0,T;L^2(\Omega))} \Big).
\end{aligned}
\tag{2.73}
$$

In particular, the isothermal reversible system admits a unique solution $(\mathbf{u}, \chi)$.

The proof is postponed to Section 5.

Remark 2.11. Let us notice that, if we consider the s-Laplacian (2.34) instead of the p-Laplacian (2.25), the continuous dependence result stated in Theorem 3 still holds true also without assumption (2.72). In this case, for any two solutions $\mathcal{X}_1$ and $\mathcal{X}_2$ of the isothermal reversible system, (2.73) yields an estimate on $\|\mathcal{X}_1 - \mathcal{X}_2\|_{L^2(0,T;H^s(\Omega))}$. For further details, we refer to Remark 5.1 at the end of Sec. 5.

2.4 Global existence results for the irreversible system

Heuristics for weak solutions. As mentioned in the introduction, the major problem in dealing with the subdifferential inclusion (2.59) in the case $\mu = 1$ is the simultaneous presence of the three nonlinear, maximal monotone operators $\partial I_{(-\infty,0]}$, $\mathcal{B}$, and β, which need to be properly identified when passing to the limit in the time-discretization scheme we are going to set up in Section 3. We now discuss the attached difficulties on a formal level, treating β and $\partial I_{(-\infty,0]}$ as single-valued.

It would be possible to handle $\beta = \partial\widehat{\beta}$ by exploiting the strong-weak closedness (in the sense of graphs) of the (induced operator) $\beta : L^2(0,T;L^2(\Omega)) \rightrightarrows L^2(0,T;L^2(\Omega))$. Nonetheless, a $L^2(0,T;L^2(\Omega))$-estimate of the term $\beta(\mathcal{X})$ in (2.59) cannot be obtained without estimating as well $\mathcal{B}(\mathcal{X})$ (and hence $\partial I_{(-\infty,0]}(\mathcal{X}_t)$ by comparison), in $L^2(0,T;L^2(\Omega))$. To our knowledge, this can be proved by testing (2.59) by $\partial_t(\mathcal{B}(\mathcal{X}) + \beta(\mathcal{X}))$ (cf. also [8]). The related calculations (which we will develop in Sec. 3, on the time-discrete level, for the *isothermal* irreversible system) would involve an integration by parts of the terms on the right-hand side of (2.59). Thus, they would rely on an estimate in $W^{1,1}(0,T;L^2(\Omega))$ of the term $-b'(\mathcal{X})\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w)$. However, presently this enhanced bound for $\Theta(w)$ does not seem to be at hand due to the poor time-regularity of w, cf. (2.52).

That is why, for the temperature-dependent irreversible system we are only able to obtain the existence of solutions $(w, \mathbf{u}, \mathcal{X})$ to a suitable *weak formulation* of (2.59), mutuated from [27], where we also restrict to the particular case in which

$$\widehat{\beta} = I_{[0,+\infty)}. \tag{2.74}$$

Remark 2.12. In the present irreversible context it is sufficient to choose $\widehat{\beta}$ as in (2.74) to enforce the constraint $\mathcal{X} \in [0,1]$ a.e. in $\Omega \times (0,T)$. Indeed, starting from an initial datum $\mathcal{X}_0 \leq 1$ a.e. in Ω we will obtain by irreversibility that $\mathcal{X}(\cdot,t) \leq \mathcal{X}_0 \leq 1$ a.e. in Ω, for almost all $t \in (0,T)$.

The underlying motivation for the weak formulation of (2.59) we will consider is that, due to the 1-homogeneity of $I_{(-\infty,0]}$, it is not difficult to check that (2.59) is equivalent to the system

$$\mathcal{X}_t(x,t) \leq 0 \qquad \text{for a.a. } (x,t) \in \Omega \times (0,T), \tag{2.75a}$$

$$\left\langle \mathcal{X}_t(t) + \mathcal{B}(\mathcal{X}(t)) + \xi(t) + \gamma(\mathcal{X}(t)) + b'(\mathcal{X}(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2} - \Theta(w(t)), \varphi \right\rangle_{W^{1,p}(\Omega)} \geq 0 \quad \text{for all } \varphi \in W^{1,p}_{-}(\Omega) \text{ for a.a. } t \in (0,T), \tag{2.75b}$$

$$\left\langle \mathcal{X}_t(t) + \mathcal{B}(\mathcal{X}(t)) + \xi(t) + \gamma(\mathcal{X}(t)) + b'(\mathcal{X}(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2} - \Theta(w(t)), \mathcal{X}_t(t) \right\rangle_{W^{1,p}(\Omega)} \leq 0 \quad \text{for a.a. } t \in (0,T), \tag{2.75c}$$

with $\xi \in \partial I_{[0,+\infty)}(\mathcal{X})$ a.e. in $\Omega \times (0,T)$ (and $\langle \cdot, \cdot \rangle_{W^{1,p}(\Omega)}$ denoting the duality pairing between $W^{1,p}(\Omega)'$ and $W^{1,p}(\Omega)$, cf. Notation 2.1). In order to see this, it is sufficient to subtract (2.75c) from (2.75b), and use the definition of $\partial I_{[0,+\infty)}$. However, for reasons analogous to those mentioned in the above lines, the proof of (2.75c) is at the moment an open problem. Therefore, following [27], in the forthcoming Definition 2.13 we weakly formulate (2.75) by means of (2.75a), (an integrated version of) (2.75b), and the *energy inequality* (2.78) below, in place of (2.75c).

Definition 2.13 (Weak solution to the (non-degenerating) irreversible full system). *Let* $\mu = 1$. *We call a triple* $(w, \mathbf{u}, \chi)$ *as in* (2.52)–(2.54) *a* weak solution *to Problem 2.7 if, besides fulfilling the weak enthalpy and momentum equations* (2.57)–(2.58)*, it satisfies* $\chi_t(x,t) \leq 0$ *for almost all* $(x,t) \in \Omega \times (0,T)$*, as well as*

$$\int_\Omega \Big(\chi_t(t)\varphi + \mathbf{d}(x, \nabla\chi(t)) \cdot \nabla\varphi + \xi(t)\varphi + \gamma(\chi(t))\varphi + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2}\varphi - \Theta(w(t))\varphi\Big)\,\mathrm{d}x \geq 0$$
$$\text{for all } \varphi \in W^{1,p}_-(\Omega), \quad \text{for a.a. } t \in (0,T), \tag{2.76}$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ *in the following sense:*

$$\xi \in L^1(0,T;L^1(\Omega)) \qquad \text{and} \qquad \langle \xi(t), \varphi - \chi(t)\rangle_{W^{1,p}(\Omega)} \leq 0 \quad \forall\, \varphi \in W^{1,p}_+(\Omega), \text{ for a.a. } t \in (0,T), \tag{2.77}$$

and the energy inequality for all $t \in (0,T]$*, for* $s = 0$*, and for almost all* $0 < s \leq t$*:*

$$\begin{aligned}&\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \Phi(\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x \\ &\qquad \leq \Phi(\chi(s)) + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w)\right)\mathrm{d}x\,\mathrm{d}r.\end{aligned} \tag{2.78}$$

The following result sheds light on the properties of this solution concept. First of all, it states the *total energy inequality* (2.79): from (2.79), we will deduce in Sec. 7 suitable estimates independent of δ, which will allow us to pass to the limit in Problem 2.7 as $\delta \downarrow 0$ for $\mu = 1$. Furthermore, the second part of Proposition 2.14 (whose proof closely follows the argument for [27, Prop. 4.1]) shows that, if χ is regular enough, then (2.75a) and (2.76)–(2.78) are equivalent to (2.59).

Proposition 2.14. *Let* $\mu = 1$. *Then, any weak solution* $(w, \mathbf{u}, \chi)$ *in the sense of Def. 2.13 fulfills the* total energy inequality *for all* $t \in (0,T]$*, for* $s = 0$*, and for almost all* $0 < s \leq t$

$$\begin{aligned}&\int_\Omega w(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2\,\mathrm{d}x + \int_s^t \int_\Omega |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r \\ &\quad + \int_s^t a_{\mathrm{vis}}((a(\chi)+\delta)\mathbf{u}_t, \mathbf{u}_t)\,\mathrm{d}r + \frac{1}{2}a_{\mathrm{el}}(b(\chi(t))\mathbf{u}(t), \mathbf{u}(t)) + \Phi(\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x \\ &\leq \int_\Omega w(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(s)|^2\,\mathrm{d}x + \frac{1}{2}a_{\mathrm{el}}(b(\chi(s))\mathbf{u}(s), \mathbf{u}(s)) + \Phi(\chi(s)) + \int_\Omega W(\chi(s))\,\mathrm{d}x \\ &\qquad + \int_s^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_s^t \int_\Omega g\,\mathrm{d}x\mathrm{d}r\,.\end{aligned} \tag{2.79}$$

Assume now Hypotheses (III)–(V). Let $(w, \mathbf{u}, \chi)$ *be as in* (2.52)–(2.54)*, and suppose in addition that*

$$\begin{aligned}&\mathcal{B}(\chi) \in L^2(0,T;L^2(\Omega)) \text{ and there exists } \xi \in L^2(0,T;L^2(\Omega)) \text{ with} \\ &\xi(x,t) \in \partial I_{[0,+\infty)}(\chi(x,t)) \text{ for a.a. } (x,t) \in \Omega\times(0,T),\end{aligned} \tag{2.80}$$

such that $(w, \mathbf{u}, \chi, \xi)$ *comply with* (2.75a) *and* (2.76)–(2.78)*. Then,*

$$\begin{aligned}&\exists\, \zeta \in L^2(0,T;L^2(\Omega)) \text{ with } \zeta(x,t) \in \partial I_{(-\infty,0]}(\chi_t(x,t)) \text{ for a.a. } (x,t) \in \Omega\times(0,T) \text{ s.t.} \\ &\chi_t + \zeta + \mathcal{B}(\chi) + \xi + \gamma(\chi) = -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \qquad \text{a.e. in } \Omega\times(0,T).\end{aligned} \tag{2.81}$$

Proof. In order to prove (2.79) it is sufficient choose $\varphi \equiv 1$ in (2.57), test (2.58) by $\mathbf{u}_t$, integrate in time, perform the calculations in the proof of Thm. 1, and add the resulting equalities with the energy inequality (2.78). The second part of the statement can be proved considering the energy functional

$$\mathscr{E} : L^2(\Omega) \to (-\infty, +\infty], \qquad \mathscr{E}(\chi) := \Phi(\chi) + \int_\Omega W(\chi)\,\mathrm{d}x. \tag{2.82}$$

It follows from Hypotheses (IV) and (V), as well as the chain rule of [10, Lemma 3.3] that, if χ complies with (2.54) and (2.80), then the map $t \mapsto \mathscr{E}(\chi(t))$ is absolutely continuous on $(0,T)$ and fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\chi(t)) = \int_\Omega (\mathcal{B}(\chi(x,t)) + \xi(x,t) + \gamma(\chi(x,t)))\,\chi_t(x,t)\,\mathrm{d}x \qquad \text{for a.a.}\, t \in (0,T). \tag{2.83}$$

Therefore, differentiating (2.78) in time and using (2.83) we conclude that $(w, \mathbf{u}, \chi, \xi)$ comply with (2.75c), where the duality pairing $\langle \chi_t, \chi_t \rangle_{W^{1,p}(\Omega)}$ is replaced by the scalar product in $(\chi_t, \chi_t)_{L^2(\Omega)}$. Likewise, (2.76) yields (2.75b). Again on account of (2.54) and (2.80), it is not difficult to infer from (2.75) that $-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) - \chi_t - \mathcal{B}(\chi) - \xi - \gamma(\chi) \in \partial I_{(-\infty,0]}(\chi_t)$ a.e. in $\Omega \times (0,T)$ as an element of $L^2(0,T; L^2(\Omega))$, and (2.81) follows. □

Remark 2.15 (Energy inequality (2.79) vs. Energy identity (2.62))**.** As we have already pointed out in the proof of Proposition 2.14, to prove that *any* weak solution $(w, \mathbf{u}, \chi)$ in the sense of Def. 2.13 fulfills in the irreversible case (i.e. with $\mu = 1$) the *total energy inequality* (2.79) it is sufficient to choose $\varphi \equiv 1$ in (2.57), test (2.58) by $\mathbf{u}_t$, integrate in time and add the resulting equalities to (2.78). Instead, the proof of the *total energy equality* (2.62) relies on the fact that, for $\mu = 0$ we are able to obtain the stronger, *pointwise* form (2.61) of the subdifferential inclusion (2.59).

We can now state our existence result in the case $\mu = 1$, with $\rho = 0$. Its proof will be developed in Sec. 6.1 by passing to the limit in the time-discretization scheme devised in Sec. 3. We mention in advance that, in this irreversible setting, in addition the basic assumptions of Hypotheses (I)–(V), we also have to require the p-coercivity condition (2.26). It has a crucial role in proving *strong* convergence of the approximate solutions to χ in $L^p(0,T; W^{1,p}(\Omega))$, which enables us to obtain (2.78). Let us also highlight that, exploiting the additional feature of irreversibility, we prove a slightly more refined (in comparison with (2.65)) positivity result for the temperature ϑ, cf. (2.85) below.

Theorem 4 (Existence of weak solutions for the full system, $\mu = 1$, $\rho = 0$)**.** *Let* $\mu = 1$, $\rho = 0$, *and assume Hypotheses (I)–(V) with* $\widehat{\beta} = I_{[0,+\infty)}$ *as in* (2.74)*, and conditions* (2.47)*–*(2.51) *on the data* $\mathbf{f}$, g, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0*. Suppose moreover that* ϕ *complies with* (2.26)*. Then,*

1. *Problem 2.7 admits a weak solution* $(w, \mathbf{u}, \chi)$ *(cf. Def. 2.13).*
2. *Suppose in addition that*

$$g(x,t) \geq 0 \quad \text{for a.a.}\,(x,t) \in \Omega \times (0,T) \quad \text{and}\ \exists\, \underline{\vartheta}_0 \geq 0 \quad \text{for a.a.}\, x \in \Omega: \quad \vartheta_0(x) > \underline{\vartheta}_0 \geq 0\,. \tag{2.84}$$

Then

$$\vartheta(x,t) := \Theta(w(x,t)) \geq \underline{\vartheta}_0 \geq 0 \quad \text{for a.a.}\,(x,t) \in \Omega \times (0,T). \tag{2.85}$$

Remark 2.16. If $\mathcal{B}$ is given by the s-Laplacian operator A_s (2.33), the Definition 2.13 of weak solution to the irreversible full system is obviously modified: in (2.76) the term $\int_\Omega \mathbf{d}(x, \nabla\chi)\cdot\nabla\varphi$ is replaced by $a_s(\chi, \varphi)$ with $\varphi \in L^2(0,T; W^{s,2}_{-}(\Omega))$, whereas $\Phi(\chi)$ in (2.78) now reads $\frac{1}{2}a_s(\chi,\chi)$. Our existence result Theorem 4 extends to such a case, cf. also the forthcoming Remark 3.12.

In Section 7, we will focus on the irreversible full system (2.57)–(2.59) with $\mathcal{B} = A_s$, and perform an asymptotic analysis of weak solutions (in the sense of Definition 2.13) in the *degenerate* limit $\delta \downarrow 0$.

Remark 2.17. Replacing Hyp. (II) with (VIII) and carefully tailoring the estimates and techniques for the proof of Thm. 2 (cf. Rmk. 2.10) to the irreversible case, we could indeed prove the existence of *weak* (in the sense of Def. 2.13) solutions also for $\mu = 1$ and $\rho \neq 0$. However we expect that, in the latter setting, only the weaker positivity result (2.65) can be proved. Indeed, the estimate yielding the lower bound (2.85) (cf. Step 4 in the proof of Lemma 3.8), cannot be performed on the enthalpy equation due to the presence of term $-\rho\Theta(w)\,\mathrm{div}(\mathbf{u}_t)$.

We finally turn to the irreversible *isothermal* case, and improve the existence result of Theorem 4.

Theorem 5 (Global existence for the isothermal system, $\mu = 1$)**.** *Let $\mu = 1$. In addition to Hypotheses (III)–(V), assume that*

$$b''(x) = 0 \qquad \textit{for all } x \in [0,1], \tag{2.86}$$

and suppose that the data $\mathbf{f}, \mathbf{u}_0$, $\mathbf{v}_0$, χ_0 comply with conditions (2.47) *and* (2.50)–(2.51)*. Suppose in addition ϕ fulfills* (2.26) *and* (2.27)*, that*

$$\mathcal{B}(\chi_0),\ \beta(\chi_0) \in L^2(\Omega), \tag{2.87}$$

and consider a fixed temperature profile

$$\Theta^* \in W^{1,1}(0,T;L^2(\Omega)). \tag{2.88}$$

Then, there exists a quadruple $(\mathbf{u}, \chi, \xi, \zeta)$, fulfilling (2.53)–(2.54)*, $\xi \in \beta(\chi)$ and $\zeta \in \partial I_{(-\infty,0]}(\chi_t)$ a.e. in $\Omega \times (0,T)$, as well as*

$$\chi \in L^\infty(0,T;W^{1+\sigma,p}(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \quad \textit{for every } 0 < \sigma < \frac{1}{p}, \tag{2.89}$$

$$\xi \in L^\infty(0,T;L^2(\Omega)), \tag{2.90}$$

$$\zeta \in L^\infty(0,T;L^2(\Omega)), \tag{2.91}$$

satisfying equations (2.58) *and* (2.81)*, with $\Theta(w)$ replaced by Θ^*.*

Remark 2.18. Uniqueness of solutions for the *irreversible* system, even in the isothermal case, is still an open problem. This is mainly due to the triply nonlinear character of (2.59) (cf. also [13] for non-uniqueness examples for a general doubly nonlinear equation).

A more general dissipation potential. As observed in Remark 6.1 later on, in Thm. 5 we could consider a more general dissipation potential in (1.3). Indeed, in place of subdifferential operator $\partial I_{(-\infty,0]}$, we could allow for a general cyclical monotone operator

$$\begin{gathered} \alpha := \partial\widehat{\alpha} : \mathbb{R} \rightrightarrows \mathbb{R}, \text{ with} \\ \widehat{\alpha} : \mathbb{R} \to \mathbb{R} \text{ convex, lower semicontinuous, with } \mathrm{dom}(\widehat{\alpha}) \subset (-\infty, 0]. \end{gathered} \tag{2.92}$$

3 Time discretization

First, in Section 3.1 we will approximate Problem 2.7 via time discretization. In fact, in the reversible case $\mu = 0$ with $\rho \in \mathbb{R}$, we will set up an *implicit* scheme (cf. Problems 3.2 and 3.3), whereas for the irreversible system $\mu = 1$ with $\rho = 0$, we will employ the *semi-implicit* scheme of Problem 3.4. Moreover, we will tackle separately the discretization of the isothermal irreversible system in Problem 3.6. We refer to Remarks 3.5 and 3.7 for a thorough comparison between the various time-discretization procedures, and more comments. Second, in Sec. 3.2 we will prove existence results for Problems 3.2–3.6. Third, in Sec. 3.3 we will perform suitable a priori estimates.

Notation 3.1. In what follows, also in view of the extension (2.92) mentioned at the end of Sec. 2.4 (cf. Rmk. 6.1), we will use $\widehat{\alpha}$ and α as place-holders for $I_{(-\infty,0]}$ and $\partial I_{(-\infty,0]}$.

3.1 Setup of the time discretization

We consider an equidistant partition of $[0,T]$, with time-step $\tau > 0$ and nodes $t_\tau^k := k\tau$, $k = 0, \dots, K_\tau$. In this framework, we approximate the data $\mathbf{f}$ and g by local means, i.e. setting for all $k = 1, \dots, K_\tau$

$$\mathbf{f}_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \mathbf{f}(s)\,\mathrm{d}s\,, \qquad g_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} g(s)\,\mathrm{d}s\,. \tag{3.1}$$

Problem 3.2 (Time discretization of the full reversible system, $\mu = 0$, $\rho = 0$)**.** Given

$$w_\tau^0 := w_0, \qquad \mathbf{u}_\tau^0 := \mathbf{u}_0, \qquad \mathbf{u}_\tau^{-1} := \mathbf{u}_0 - \tau\mathbf{v}_0, \qquad \chi_\tau^0 := \chi_0, \tag{3.2}$$

find $\{w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k, \xi_\tau^k\}_{k=1}^{K_\tau} \subset H^1(\Omega) \times H_0^2(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega) \times L^2(\Omega)$, with $\xi_\tau^k \in \beta(\chi_\tau^k)$ a.e. in Ω, fulfilling

$$\frac{w_\tau^k - w_\tau^{k-1}}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\Theta(w_\tau^k) + A_{w_\tau^{k-1}}(w_\tau^k) = g_\tau^k \quad \text{in } H^1(\Omega)', \tag{3.3}$$

$$\frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left((a(\chi_\tau^k) + \delta)\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) = \mathbf{f}_\tau^k \quad \text{a.e. in } \Omega, \tag{3.4}$$

$$\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \mathcal{B}(\chi_\tau^k) + \xi_\tau^k + \gamma(\chi_\tau^k) = -b'(\chi_\tau^{k-1})\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathrm{R}_e\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \Theta(w_\tau^k) \quad \text{a.e. in } \Omega, \tag{3.5}$$

where in (3.3) the operator $A_{w_\tau^{k-1}} : H^1(\Omega) \to H^1(\Omega)'$ is defined by

$$\langle A_{w_\tau^{k-1}}(w), v\rangle_{H^1(\Omega)} := \int_\Omega K(w_\tau^{k-1})\nabla w \cdot \nabla v\,\mathrm{d}x \quad \text{for all } w, v \in H^1(\Omega). \tag{3.6}$$

For the full reversible system with $\rho \neq 0$, we work under the stronger Hypothesis (VIII) and thus prescribe a suitable growth on the function K, which is no longer bounded. Therefore, in order to properly deal with the elliptic operator in the enthalpy equation on the time-discrete level, we need to truncate K. We thus introduce the operator

$$\langle A_{w_\tau^{k-1},M}(w), v\rangle_{H^1(\Omega)} := \int_\Omega K_M(w_\tau^{k-1})\nabla w \cdot \nabla v\,\mathrm{d}x \quad \text{for all } w, v \in H^1(\Omega)\,, \tag{3.7}$$

with

$$K_M(r) := \begin{cases} K(-M) & \text{if } r < -M, \\ K(r) & \text{if } |r| \le M, \\ K(M) & \text{if } r > M. \end{cases} \tag{3.8}$$

Observe that

$$K_M(r) \ge c_{10} \qquad \text{for every } r \in \mathbb{R} \tag{3.9}$$

(with c_{10} from (2.46)). Accordingly, we will have to truncate the function Θ, replacing it with

$$\Theta_M(r) := \begin{cases} \Theta(-M) & \text{if } r < -M, \\ \Theta(r) & \text{if } |r| \le M, \\ \Theta(M) & \text{if } r > M. \end{cases} \tag{3.10}$$

Problem 3.3 (Time discretization of the full reversible system, $\mu = 0$, $\rho \neq 0$)**.** *Starting from* $(\mathbf{u}_\tau^0,\ \mathbf{u}_\tau^{-1},\ \chi_\tau^0,\ w_\tau^0)$ *as in* (3.2), *find* $\{w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k, \xi_\tau^k\}_{k=1}^{K_\tau} \subset H^1(\Omega) \times H_0^2(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega) \times L^2(\Omega)$ *with* $\xi_\tau^k \in \beta(\chi_\tau^k)$ *a.e. in* Ω*, fulfilling*

$$\frac{w_\tau^k - w_\tau^{k-1}}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\Theta_M(w_\tau^k) + \rho\,\mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\Theta_M(w_\tau^k) + A_{w_\tau^{k-1},M}(w_\tau^k) = g_\tau^k \quad \text{in } H^1(\Omega)', \tag{3.11}$$

$$\frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left((a(\chi_\tau^k) + \delta)\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) + \mathcal{C}_\rho(\Theta_M(w_\tau^k)) = \mathbf{f}_\tau^k \quad \text{a.e. in } \Omega, \tag{3.12}$$

$$\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \mathcal{B}(\chi_\tau^k) + \xi_\tau^k + \gamma(\chi_\tau^k) = -b'(\chi_\tau^{k-1})\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathrm{R}_e\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \Theta_M(w_\tau^k) \quad \text{a.e. in } \Omega\,. \tag{3.13}$$

We now present the time discretization of the full irreversible system, postponing to Remark 3.5 a detailed comparison between Problem 3.2 and the forthcoming Problem 3.4. Let us only mention in advance that, in the irreversible case we will restrict to the particular choice $\beta = \partial I_{[0,+\infty)}$. Furthermore, in Problem 3.4 instead of the time discretization of (2.59), we will consider the minimum problem (3.19), such that its Euler equation is (2.59) discretized. We resort to this approach in view of the passage to the limit argument as $\tau \to 0$, mutuated from [27], which we will develop in the proof of Thm. 4. Finally, due to technical reasons related to the proof of the *Third a priori estimate* in Sec. 3.3, we will also need to approximate the initial datum w_0 with a sequence

$$(w_{0\tau})_\tau \subset W^{1,\bar{r}}(\Omega) \text{ such that } \sup_{\tau>0} \tau \|\nabla w_{0\tau}\|^{\bar{r}}_{L^{\bar{r}}(\Omega)} \le C, \quad w_{0\tau} \to w_0 \text{ in } L^1(\Omega) \text{ as } \tau \to 0, \tag{3.14}$$

with $\bar{r} = (d+2)/(d+1)$, cf. (2.52). We construct $(w_{0\tau})_\tau$ in such a way that, if ϑ_0 complies with (2.84), then for every $\tau > 0$

$$w_{0\tau}(x) \ge \underline{w}_0 := h(\underline{\vartheta}_0) \ge 0 \qquad \text{for a.a. } x \in \Omega. \tag{3.15}$$

Problem 3.4 (Time discretization of the irreversible *full* system, $\mu = 1$, $\rho = 0$)**.** Starting from the data $(\mathbf{u}_\tau^0, \mathbf{u}_\tau^{-1}, \chi_\tau^0, w_\tau^0)$ as in (3.2), with $w_\tau^0 = w_{0\tau}$ as in (3.14), find $\{w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k, \zeta_\tau^k\}_{k=1}^{K_\tau} \in H^1(\Omega) \times H_0^2(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega) \times L^2(\Omega)$, such that for all $k = 1, \ldots, K_\tau$ there holds

$$\chi_\tau^k \le \chi_\tau^{k-1} \qquad \text{a.e. in } \Omega \text{ and } \qquad \zeta_\tau^k \in \alpha((\chi_\tau^k - \chi_\tau^{k-1})/\tau) \qquad \text{a.e. in } \Omega, \tag{3.16}$$

and $(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)$ fulfill

$$\frac{w_\tau^k - w_\tau^{k-1}}{\tau} + \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\Theta(w_\tau^{k-1}) + A_{w_\tau^{k-1}}(w_\tau^k) = g_\tau^k \quad \text{in } H^1(\Omega)', \tag{3.17}$$

$$\frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left((a(\chi_\tau^k) + \delta)\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) = \mathbf{f}_\tau^k \quad \text{a.e. in } \Omega, \tag{3.18}$$

and

$$\begin{aligned}\chi_\tau^k \in \operatorname{Argmin}_{\chi \in W^{1,p}(\Omega)} \Bigg\{ &\int_\Omega \left(\frac{\tau}{2}\left|\frac{\chi - \chi_\tau^{k-1}}{\tau}\right|^2 + \widehat{\alpha}\left(\frac{\chi - \chi_\tau^{k-1}}{\tau}\right)\right) \mathrm{d}x + \Phi(\chi) + \int_\Omega (\widehat{\beta}(\chi) + \widehat{\gamma}(\chi))\,\mathrm{d}x \\ &+ \int_\Omega \left(b'(\chi_\tau^{k-1})\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathrm{R}_e\varepsilon(\mathbf{u}_\tau^{k-1})}{2} - \Theta(w_\tau^{k-1})\right)\chi\,\mathrm{d}x \Bigg\}.\end{aligned} \tag{3.19}$$

Remark 3.5. The main difference between systems (3.3)–(3.5) and (3.17)–(3.19) consists in the discretization of the coupling term $\Theta(w)$ in the temperature and in the phase parameter equations. Indeed, in the reversible case $\Theta(w)$ on the l.h.s. of (3.3) (and accordingly its coupled term on the r.h.s. of (3.5), which will cancel out in the *First a priori estimate* of Sec. 3.3), is kept *implicit*. Only relying on this we can prove the positivity of the discrete enthalpy w_τ^k, which for system (3.3)–(3.5) would not follow from other considerations. Instead, in the time discretization (3.17)–(3.19) we can allow for an *explicit* coupling term $\Theta(w_\tau^{k-1})$ in (3.17) and in (3.19). Therein, the positivity of the discrete enthalpy will be proved by means of a suitable test of the discrete enthalpy equation, relying on the irreversibility (3.16).

Because of its implicit character, in Lemma 3.9 existence for system (3.3)–(3.5) will be proved by resorting to fixed-point type existence results for elliptic systems featuring pseudo-monotone (cf. e.g. [48, Chap. II]) operators.

Instead, in the semi-implicit scheme of Problem 3.4, equations (3.17), (3.18), and (3.19) are decoupled, hence we will proceed by tackling them separately, solving time-incremental minimization problems. Such a procedure could be useful for the numerical analysis of the problem. That is why, in Sec. 3.2 we will focus on the proof of Lemma 3.8 and develop in detail the calculations for system (3.17)–(3.19), whereas we will only outline the argument for the existence of solutions to (3.3)–(3.5) in Lemma 3.9.

In the time-discretization of the irreversible *isothermal* system, we approximate the given temperature profile Θ^* (cf. (2.88)) by local means as well, i.e.

$$\Theta^{*k}_{\tau} := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \Theta^*(s)\,\mathrm{d}s. \tag{3.20}$$

Contrary to the temperature-dependent irreversible case, we may again address a general maximal monotone $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$. However, in order to perform enhanced estimates on the discrete equation for χ (cf. the *Seventh* and *Eighth a priori estimates* of Sec. 3.3), we will need to replace β with its Yosida regularization $\beta_\tau : \mathbb{R} \to \mathbb{R}$, namely the nondecreasing, Lipschitz continuous derivative of the convex C^1 function $\widehat{\beta}_\tau(x) := \min_{y\in\mathbb{R}}\{|y-x|^2/2\tau + \widehat{\beta}(y)\}$, cf. e.g. [3, 10]. In Problem 3.6 below, we set the regularization parameter equal to the time-step, in view of passing to the limit simultaneously in the time discretization and in the Yosida regularization as $\tau \to 0$. Furthermore, we will have to work with a suitable truncation of the coefficient $a(\chi)$ in (2.58), cf. Remark 3.7 below for further comments.

Problem 3.6 (Time discretization of the irreversible *isothermal* system)**.** Starting from the triple of data $(\mathbf{u}_\tau^0, \mathbf{u}_\tau^{-1}, \chi_\tau^0)$ defined as in (3.2) and considering the discrete approximations $(\Theta^{*k}_{\tau})_{k=1}^{K_\tau}$ of the given temperature profile Θ^*, find $\{\mathbf{u}_\tau^k, \chi_\tau^k, \zeta_\tau^k\}_{k=1}^{K_\tau} \in H_0^2(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega) \times L^2(\Omega)$, such that for all $k = 1,\dots,K_\tau$ there holds

$$\frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} + \mathcal{V}\left(((a(\chi_\tau^k))^+ + \delta)\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right) + \mathcal{E}\left(b((\chi_\tau^k)^+)\mathbf{u}_\tau^k\right) = \mathbf{f}_\tau^k \quad \text{a.e. in } \Omega, \tag{3.21}$$

$$\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \zeta_\tau^k + \mathcal{B}(\chi_\tau^k) + \beta_\tau(\chi_\tau^k) + \gamma(\chi_\tau^k) = -b'((\chi_\tau^{k-1})^+)\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathrm{R}_e\varepsilon(\mathbf{u}_\tau^{k-1})}{2} + \Theta^{*k-1}_{\tau} \quad \text{a.e. in } \Omega, \tag{3.22}$$

$$\zeta_\tau^k \in \alpha\left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right) \quad \text{a.e. in } \Omega. \tag{3.23}$$

Remark 3.7. In Problem 3.6 we need to approximate β by a Lipschitz continuous function β_τ because, only with such a regularization can we test equation (3.22) by the discrete difference $\tau^{-1}((\mathcal{B}(\chi_\tau^k) + \beta_\tau(\chi_\tau^k)) - (\mathcal{B}(\chi_\tau^{k-1}) + \beta_\tau(\chi_\tau^{k-1})))$ (cf. the following Seventh a priori estimate). Hence, we need to take the positive part of a in (3.21) because, replacing β by its Lipschitz regularization β_τ, we are no longer able to enforce the constraint that $\chi_\tau^k \in [0,1]$ a.e. in Ω. Therefore, at the discrete level we loose all positivity information on the coefficient $a(\chi)$. The lack of the constraint $\chi_\tau^k \in [0,1]$ also motivates the truncations $b((\chi_\tau^k)^+)$ in (3.21) and $b'((\chi_\tau^k)^+)$ in (3.22), mainly due to technical reasons (cf. the *First a priori estimate*).

3.2 Existence for the time-discrete problems

First, we prove the existence of solutions to the semi-implicit schemes (3.17)–(3.19) and (3.21)–(3.23).

Lemma 3.8 (Existence for the time-discrete Problems 3.4 and 3.6, $\mu = 1$, $\rho = 0$)**.** *Let $\mu = 1$ and $\rho = 0$. Assume Hypotheses (I)–(V), and* (2.47)–(2.51) *on the data* $\mathbf{f}$, g, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0.

Then, Problems 3.4 and 3.6 admit at least one solution $\{(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k, \zeta_\tau^k)\}_{k=1}^{K_\tau}$ *and* $\{(\mathbf{u}_\tau^k, \chi_\tau^k, \zeta_\tau^k)\}_{k=1}^{K_\tau}$, *resp.*

Furthermore, if (2.84) *holds, then any solution* $\{(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k, \zeta_\tau^k)\}_{k=1}^{K_\tau}$ *of Problem 3.4 fulfills*

$$w_\tau^k(x) \geq \underline{w}_0 = h(\underline{\vartheta}_0) \geq 0 \quad \textit{for a.a. } x \in \Omega. \tag{3.24}$$

Proof. We treat Problems 3.4, and 3.6 in a unified way, and proceed by induction on k. Thus, starting from a quadruple $(\mathbf{u}_\tau^{k-2}, w_\tau^{k-1}, \mathbf{u}_\tau^{k-1}, \chi_\tau^{k-1}) \in H_0^2(\Omega;\mathbb{R}^d) \times H^1(\Omega) \times H_0^2(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega)$, we show that there exist functions $(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k, \zeta_\tau^k)$ and $(\mathbf{u}_\tau^k, \chi_\tau^k, \zeta_\tau^k)$, resp., solving (3.17)–(3.19) for Problem 3.4, and (3.21)–(3.23) for Problem 3.6, resp.

Step 1: discrete equation for χ. In the irreversible isothermal case (i.e. for Problem 3.6), in order to solve (3.22) we start from the approximate equation

$$\begin{aligned}\frac{\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau}{\tau}+\alpha_\varepsilon\left(\frac{\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau}{\tau}\right)+\mathcal{B}(\chi^k_{\tau,\varepsilon})+\beta_\tau(\chi^k_{\tau,\varepsilon})+\gamma(\chi^k_{\tau,\varepsilon})\\ =-b'(\chi^{k-1}_\tau)\frac{\varepsilon(\mathbf{u}^{k-1}_\tau)\mathrm{R}_e\varepsilon(\mathbf{u}^{k-1}_\tau)}{2}+{\Theta^*}^{k-1}_\tau\quad\text{a.e. in }\Omega,\end{aligned}\tag{3.25}$$

where $\varepsilon>0$ and α_ε is the Yosida regularization of the operator α. Clearly, (3.25) is the Euler equation for the minimum problem

$$\min_{\chi\in W^{1,p}(\Omega)}\left\{\tau\int_\Omega\left(\left|\frac{\chi-\chi^{k-1}_\tau}{\tau}\right|^2+\widehat{\alpha}_\varepsilon\left(\frac{\chi-\chi^{k-1}_\tau}{\tau}\right)\right)\mathrm{d}x+\Phi(\chi)+\int_\Omega(\widehat{\beta}_\tau(\chi)+\widehat{\gamma}(\chi))\,\mathrm{d}x+\int_\Omega h^{k-1}_\tau\chi\,\mathrm{d}x\right\},$$

where the function

$$h^{k-1}_\tau:=b'(\chi^{k-1}_\tau)\varepsilon(\mathbf{u}^{k-1}_\tau)\mathrm{R}_e\varepsilon(\mathbf{u}^{k-1}_\tau)/2-{\Theta^*}^{k-1}_\tau\ \text{ is in } L^2(\Omega).\tag{3.26}$$

The latter admits a solution $\chi^k_{\tau,\varepsilon}$ by the direct method of the calculus of variations (also taking into account the fact that $\widehat{\beta}_\tau$ is bounded from below because $\widehat{\beta}$ is). We now want to pass to the limit in (3.25) as $\varepsilon\downarrow 0$. Note that, a comparison in (3.25) and the fact that α_ε is Lipschitz continuous yield that $\mathcal{B}(\chi^k_{\tau,\varepsilon})\in L^2(\Omega)$. Then, following [36, Sec. 3] (to which we refer for all details), we multiply (3.25) firstly by $\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau$, and secondly by $\mathcal{B}(\chi^k_{\tau,\varepsilon})-\mathcal{B}(\chi^{k-1}_\tau)$. To perform the latter estimate, we rely on the Lipschitz continuity of β_τ and γ, as well as on the monotonicity of α_ε, yielding

$$\int_\Omega\alpha_\varepsilon\left(\frac{\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau}{\tau}\right)\left(\mathcal{B}(\chi^k_{\tau,\varepsilon})-\mathcal{B}(\chi^{k-1}_\tau)\right)\mathrm{d}x\geq 0.$$

It follows from these tests that there exists a constant $C>0$, depending on $\tau>0$ but not on $\varepsilon>0$, such that

$$\sup_{\varepsilon>0}(\|\chi^k_{\tau,\varepsilon}\|_{W^{1,p}(\Omega)}+\|\mathcal{B}(\chi^k_{\tau,\varepsilon})\|_{L^2(\Omega)})\leq C.$$

By comparison, $\sup_{\varepsilon>0}\|\alpha_\varepsilon((\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau)/\tau)\|_{L^2(\Omega)}\leq C$. Also in view of the regularity result (2.28), there exist $(\chi^k_\tau,\zeta^k_\tau)\in W^{1+\sigma,p}(\Omega)\times L^2(\Omega)$ for all $0<\sigma<1/p$ such that, up to a subsequence, $(\chi^k_{\tau,\varepsilon})_\varepsilon$ strongly converges in $W^{1,p}(\Omega)$ to χ^k_τ as $\varepsilon\to0$, and $(\alpha_\varepsilon((\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau)/\tau))_\varepsilon$ weakly converges in $L^2(\Omega)$ to ζ^k_τ as $\varepsilon\to0$. Therefore,

$$\limsup_{\varepsilon\to0}\int_\Omega\alpha_\varepsilon\left(\frac{\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau}{\tau}\right)\left(\frac{\chi^k_{\tau,\varepsilon}-\chi^{k-1}_\tau}{\tau}\right)\mathrm{d}x\leq\int_\Omega\zeta^k_\tau\left(\frac{\chi^k_\tau-\chi^{k-1}_\tau}{\tau}\right)\mathrm{d}x,$$

so that $\zeta^k_\tau\in\alpha((\chi^k_\tau-\chi^{k-1}_\tau)/\tau)$ thanks to [3, p. 42]. Thus, passing to the limit as $\varepsilon\to0$ in (3.25) for $\tau>0$ fixed, we conclude that the functions $(\chi^k_\tau,\zeta^k_\tau)$ fulfill (3.22).

Clearly, the direct method of the calculus of variations also yields the existence of a solution to the minimum problem (3.19).

Step 2: discrete equation for u. Next, we solve (3.18), which can be rewritten a.e. in Ω in the form

$$\left(\mathrm{Id}+\tau\mathcal{V}\left((a(\chi^k_\tau)+\delta)\,\cdot\right)+\tau^2\mathcal{E}\left(b(\chi^k_\tau)\,\cdot\right)\right)(\mathbf{u}^k_\tau)=\tau^2\mathbf{f}^k_\tau+\tau\mathcal{V}\left((a(\chi^k_\tau)+\delta)\mathbf{u}^{k-1}_\tau\right)+2\mathbf{u}^{k-1}_\tau-\mathbf{u}^{k-2}_\tau.\tag{3.27}$$

Combining the fact that $\chi^k_\tau\in[0,1]$ a.e. in Ω with (2.18) on a and b and (2.3)–(2.4), we conclude that (the bilinear form associated with) the operator on the left-hand side of the above equation is continuous and coercive. Hence, by Lax-Milgram's theorem, equation (3.27) admits a (unique) solution $\mathbf{u}^k_\tau\in H^1_0(\Omega;\mathbb{R}^d)$. Since the right-hand side of (3.27) is in $L^2(\Omega;\mathbb{R}^d)$, relying on the regularity results of, e.g., [12], we conclude that in fact $\mathbf{u}^k_\tau\in H^2_0(\Omega;\mathbb{R}^d)$. The analysis of (3.21) follows the very same lines.

Step 3: discrete equation for w. Finally, let us consider the functional $\mathcal{G}_\tau^{k-1} : H^1(\Omega) \to \mathbb{R}$

$$\begin{aligned}\mathcal{G}_\tau^{k-1}(w) :=& \frac{1}{2\tau}\int_\Omega |w - w_\tau^{k-1}|^2 \,\mathrm{d}x + \int_\Omega \Theta(w_\tau^{k-1})\left(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right)\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega K(w_\tau^{k-1})|\nabla w|^2 \,\mathrm{d}x - \langle g_\tau^k, w\rangle_{H^1(\Omega)} \,.\end{aligned}$$

Now, $\mathcal{G}_\tau^{k-1}$ is lower semicontinuous w.r.t. the topology of $L^2(\Omega)$. Furthermore, in view of (2.45) and of the Young inequality we have for a fixed $\varrho > 0$

$$\begin{aligned}\mathcal{G}_\tau^{k-1}(w) \geq& \frac{1}{4\tau}\|w\|_{L^2(\Omega)}^2 + \frac{c_2}{2}\|\nabla w\|_{L^2(\Omega)}^2 - \varrho\|w\|_{H^1(\Omega)}^2 \\ &- C_\varrho(\|w_\tau^{k-1}\|_{L^2(\Omega)}^2 + \|g_\tau^k\|_{H^1(\Omega)'}^2 + \|(\chi_\tau^k - \chi_\tau^{k-1})/\tau\|_{L^2(\Omega)}^2).\end{aligned} \tag{3.28}$$

Choosing ϱ sufficiently small, we thus obtain that there exist two positive constants c and C such that

$$\mathcal{G}_\tau^{k-1}(w) \geq c\|w\|_{H^1(\Omega)}^2 - C(1 + \|w_\tau^{k-1}\|_{L^2(\Omega)}^2 + \|g_\tau^k\|_{H^1(\Omega)'}^2 + \|(\chi_\tau^k - \chi_\tau^{k-1})/\tau\|_{L^2(\Omega)}^2 \quad \text{for all } w \in H^1(\Omega)\,.$$

This shows that the sublevels of $\mathcal{G}_\tau^{k-1}$ are bounded in $H^1(\Omega)$. Hence, again by the direct method in the calculus of variations, we conclude that there exists $w_\tau^k \in \mathrm{Argmin}_{w\in H^1(\Omega)}\mathcal{G}_\tau^{k-1}(w)$, and w_τ^k satisfies the associated Euler equation, namely (3.17).

Step 4: positivity. Let us assume in addition that (2.84) holds, and prove (3.24) by induction on k.

Preliminarily, we prove by induction on k that

$$w_\tau^k(x) \geq 0 \qquad \text{for a.a. } x \in \Omega \text{ and for all } k \in \mathbb{N}. \tag{3.29}$$

Clearly (3.29) holds for $k = 0$ thanks to (3.15). It remains to show that, if $w_\tau^{k-1} \geq 0$ a.e. in Ω, then $w_\tau^k \geq 0$ a.e. in Ω. Indeed, let us test (3.17) by $-(w_\tau^k)^-$. Taking into account the definition (2.37) of Θ, we have that $\int_\Omega \Theta(w_\tau^{k-1})(\chi_\tau^k - \chi_\tau^{k-1})(-(w_\tau^k)^-)\,\mathrm{d}x \geq 0$. Combining this with the inequality

$$\frac{1}{\tau}\int_\Omega (w_\tau^k - w_\tau^{k-1})(-(w_\tau^k)^-)\,\mathrm{d}x \geq \frac{1}{2\tau}\int_\Omega (|(w_\tau^k)^-|^2 - |(w_\tau^{k-1})^-|^2)\,\mathrm{d}x, \tag{3.30}$$

and noting that $(w_\tau^{k-1})^- = 0$ a.e. in Ω, also in view of (2.45) we obtain

$$\frac{1}{2\tau}\int_\Omega |(w_\tau^k)^-|^2\,\mathrm{d}x + c_2\int_\Omega |\nabla(w_\tau^k)^-|^2\,\mathrm{d}x \leq -\int_\Omega g_\tau^k (w_\tau^k)^-\,\mathrm{d}x \leq 0,$$

yielding $(w_\tau^k)^- = 0$ a.e. in Ω, whence (3.29).

Now, to prove (3.24), we observe that (3.24) holds for $k = 0$ due to (3.15). Suppose now that $w_\tau^{k-1} \geq \underline{w}_0$ a.e. in Ω: in order to prove that $w_\tau^k \geq \underline{w}_0$ a.e. in Ω, we test (3.17) by $-(w_\tau^k - \underline{w}_0)^-$. With analogous calculations as above we obtain

$$\begin{aligned}\frac{1}{2\tau}\int_\Omega |(w_\tau^k - \underline{w}_0)^-|^2\,\mathrm{d}x + c_2\int_\Omega |\nabla(w_\tau^k - \underline{w}_0)^-|^2\,\mathrm{d}x \leq& -\int_\Omega g_\tau^k (w_\tau^k - \underline{w}_0)^-\,\mathrm{d}x \\ &+ \int_\Omega \Theta(w_\tau^k)(\chi_\tau^{k-1} - \chi_\tau^k)(-(w_\tau^k - \underline{w}_0)^-)\,\mathrm{d}x \leq 0,\end{aligned}$$

where the last inequality is due to the fact that $g_\tau^k \geq 0$ a.e. in Ω, and that $\Theta(w_\tau^k)(\chi_\tau^{k-1} - \chi_\tau^k) \geq 0$ a.e. in Ω by the previously proved (3.29) and the irreversibility constraint. Thus, we conclude (3.24). □

The existence result for Problems 3.2 and 3.3 reads:

Lemma 3.9 (Existence for the time-discrete Problems 3.2, 3.3, $\mu = 0$)**.** *Let $\mu = 0$. Assume Hypotheses (I) and (III)–(V), and* (2.47)–(2.51) *on the data* $\mathbf{f}$, g, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0. *Furthermore,* *1. if $\rho = 0$, assume Hypothesis (II);*

2. if $\rho \neq 0$, assume Hypothesis (VIII) and in addition that $w_0 \in L^2(\Omega)$.

Then, Problems 3.2 and 3.3 admit at least one solution $\{(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k, \xi_\tau^k)\}_{k=1}^{K_\tau}$.

Moreover, if $g \geq 0$ a.e. in $\Omega \times (0,T)$, and $w_0(x) \geq 0$ for a.a. $x \in \Omega$, then any solution $\{(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)\}_{k=1}^{K_\tau}$ of Problem 3.2 and of Problem 3.3 fulfills

$$w_\tau^k(x) \geq 0 \quad \textit{for a.a.}\, x \in \Omega. \tag{3.31}$$

Proof. **Step 1: existence of solutions.** Let us assume first $\rho = 0$. Our argument relies on existence results for elliptic systems from the theory of pseudo-monotone operators which can be found, e.g., in [48, Chap. II]. Indeed, we observe that system (3.3)–(3.5) can be recast as

$$\begin{aligned}
& w_\tau^k + (\chi_\tau^k - \chi_\tau^{k-1})\Theta(w_\tau^k) + \tau A_{w_\tau^{k-1}}(w_\tau^k) = w_\tau^{k-1} + \tau g_\tau^k \quad \text{in } H^1(\Omega)', \\
& \mathbf{u}_\tau^k + \tau \mathcal{V}\left((a(\chi_\tau^k) + \delta)(\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1})\right) + \tau^2 \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right) = 2\mathbf{u}_\tau^{k-1} - \mathbf{u}_\tau^{k-2} + \tau^2 \mathbf{f}_\tau^k \quad \text{a.e. in } \Omega, \\
& \chi_\tau^k + \tau \mathcal{B}(\chi_\tau^k) + \tau\beta(\chi_\tau^k) + \tau\gamma(\chi_\tau^k) - \tau\Theta(w_\tau^k) \ni \chi_\tau^{k-1} - \tau b'(\chi_\tau^{k-1})\frac{\varepsilon(\mathbf{u}_\tau^{k-1})\mathrm{R}_e\varepsilon(\mathbf{u}_\tau^{k-1})}{2} \quad \text{a.e. in } \Omega.
\end{aligned} \tag{3.32}$$

Denoting by $\mathcal{R}_{k-1}$ the operator acting on the unknown $(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)$ and by H_{k-1} the vector of the terms on the r.h.s. of the above equations, we can reformulate system (3.32) in the abstract form

$$\mathcal{R}_{k-1}(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k) = H_{k-1}. \tag{3.33}$$

In fact, mimicking for example the calculations in [47, Lemma 7.4], it can be checked that $\mathcal{R}_{k-1}$ is a pseudo-monotone operator (according to [48, Chap. II, Def. 2.1]) on $H^1(\Omega) \times H_0^1(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega)$, coercive on that space. Therefore, the Leray-Lions type existence result of [48, Chap. II, Thm. 2.6] applies, yielding the existence of a solution $(w_\tau^k, \mathbf{u}_\tau^k, \chi_\tau^k)$ to (3.33). Under the additional Hypothesis (VIII) (which gives (3.9)), an analogous existence proof can be given for Problem 3.3, hence we omit to give the details.

Step 2: non-negativity of w_τ^k. Let us assume in addition that $g \geq 0$ a.e. in $\Omega \times (0,T)$ and $w_0 \geq 0$ a.e. in Ω. Then $g_\tau^k \geq 0$ a.e. in Ω. To prove (3.31), we proceed by induction on k and show that, if $w_\tau^{k-1} \geq 0$ a.e. in Ω, then $w_\tau^k \geq 0$ a.e. in Ω. Indeed, let us test (3.3) by $-(w_\tau^k)^-$. Taking into account the definition (2.37) of Θ, we have that

$$\int_\Omega \Theta(w_\tau^k)\left((\chi_\tau^k - \chi_\tau^{k-1}) + \rho\,\mathrm{div}\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right)(-(w_\tau^k)^-)\,\mathrm{d}x = 0$$

(here we have kept $\rho \in \mathbb{R}$ also to encompass the case with thermal expansion, cf. below). Combining this with the inequality (3.30) and noting that $(w_\tau^{k-1})^- = 0$ a.e. in Ω, also in view of (2.45) we obtain

$$\frac{1}{2\tau}\int_\Omega |(w_\tau^k)^-|^2\,\mathrm{d}x + c_2\int_\Omega |\nabla(w_\tau^k)^-|^2\,\mathrm{d}x \leq -\int_\Omega g_\tau^k (w_\tau^k)^-\,\mathrm{d}x \leq 0,$$

yielding $(w_\tau^k)^- = 0$ a.e. in Ω, whence (3.31).

□

3.3 A priori estimates

Notation and auxiliary results. Hereafter, for a given Banach space X and a K_τ-tuple $(b_\tau^k)_{k=1}^{K_\tau} \subset X$, we shall use the short-hand notation

$$\mathrm{D}_{\tau,k}(b) := \frac{b_\tau^k - b_\tau^{k-1}}{\tau}, \qquad \mathrm{D}^2_{\tau,k}(b) := \mathrm{D}_{\tau,k}(\mathrm{D}_{\tau,k}(b)) = \frac{b_\tau^k - 2b_\tau^{k-1} + b_\tau^{k-2}}{\tau^2}.$$

We recall the well-known *discrete by-part integration* formula

$$\sum_{k=1}^{K_\tau} \tau \mathrm{D}_{\tau,k}(b) v_\tau^k = b_\tau^{K_\tau} v_\tau^{K_\tau} - b_\tau^0 v_\tau^1 - \sum_{k=2}^{K_\tau} \tau b_\tau^{k-1} \mathrm{D}_{\tau,k}(v) \quad \text{for all } \{b_\tau^k\}_{k=1}^{K_\tau},\ \{v_\tau^k\}_{k=1}^{K_\tau} \subset X. \tag{3.34}$$

We consider the left-continuous and right-continuous piecewise constant, and the piecewise linear interpolants of the values $\{b_\tau^k\}_{k=1}^{K_\tau}$, namely the functions

$$\left.\begin{array}{lll} \overline{b}_\tau : (0,T) \to X & \text{defined by} & \overline{b}_\tau(t) := b_\tau^k, \\ \underline{b}_\tau : (0,T) \to X & \text{defined by} & \underline{b}_\tau(t) := b_\tau^{k-1}, \\ b_\tau : (0,T) \to X & \text{defined by} & b_\tau(t) := \frac{t-t_\tau^{k-1}}{\tau} b_\tau^k + \frac{t_\tau^k - t}{\tau} b_\tau^{k-1} \end{array}\right\} \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k].$$

We also introduce the piecewise linear interpolant of the values $\{(b_\tau^k - b_\tau^{k-1})/\tau\}_{k=1}^{K_\tau}$ (namely, the values taken by the -piecewise constant- function b'_τ), viz.

$$\widehat{b}_\tau : (0,T) \to X \qquad \widehat{b}_\tau(t) := \frac{t - t_\tau^{k-1}}{\tau} \frac{b_\tau^k - b_\tau^{k-1}}{\tau} + \frac{t_\tau^k - t}{\tau} \frac{b_\tau^{k-1} - b_\tau^{k-2}}{\tau} \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k].$$

Note that $\widehat{b}'_\tau(t) = \mathrm{D}^2_{\tau,k}(b)$ for $t \in (t_\tau^{k-1}, t_\tau^k]$.

In view of (2.47), (2.48), and (2.88), it is easy to check that the piecewise constant interpolants $(\overline{\mathbf{f}}_\tau)_{k=1}^{K_\tau}$, $(\overline{g}_\tau)_{k=1}^{K_\tau}$, $(\overline{\Theta^*}_\tau)_{k=1}^{K_\tau}$, and $(\Theta^*{}_\tau)_{k=1}^{K_\tau}$ of the values $\mathbf{f}_\tau^k$, g_τ^k (3.1), and ${\Theta^*}_\tau^k$ (3.20), fulfill as $\tau \downarrow 0$

$$\overline{\mathbf{f}}_\tau \to \mathbf{f} \qquad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{3.35a}$$

$$\overline{g}_\tau \to g \qquad \text{in } L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)'), \tag{3.35b}$$

$$\overline{\Theta^*}_\tau \to \Theta^* \qquad \text{in } L^p(0,T;L^2(\Omega)) \text{ for all } 1 \le p < \infty, \tag{3.35c}$$

$$\|\partial_t \Theta^*{}_\tau\|_{L^1(0,T;L^2(\Omega))} \le 2\|\partial_t \Theta^*\|_{L^1(0,T;L^2(\Omega))} \qquad \text{for all } \tau > 0. \tag{3.35d}$$

Finally, we shall denote by $\overline{\mathsf{t}}_\tau$ and by $\underline{\mathsf{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition, i.e. $\overline{\mathsf{t}}_\tau(t) := t_\tau^k$ if $t_\tau^{k-1} < t \le t_\tau^k$ and $\underline{\mathsf{t}}_\tau(t) := t_\tau^{k-1}$ if $t_\tau^{k-1} \le t < t_\tau^k$. Clearly, for every $t \in [0,T]$ we have $\overline{\mathsf{t}}_\tau(t) \downarrow t$ and $\underline{\mathsf{t}}_\tau(t) \uparrow t$ as $\tau \to 0$.

Propositions 3.10 and 3.11 collect in the cases $\rho = 0$ and $\rho \neq 0$ several a priori estimates on the approximate solutions, obtained by interpolation of the discrete solutions to Problems 3.2, 3.4, 3.6, and Problem 3.3, respectively.

Proposition 3.10 ($\mu \in \{0,1\}$, $\rho = 0$)**.** *Let* $\rho = 0$*. Assume Hypotheses (I)–(V) and* (2.47)–(2.51) *on the data* $\mathbf{f}$, g, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0*. Then,*

1. *in the case* $\mu \in \{0,1\}$ *there exist a constant* $S > 0$ *such that for the interpolants of the solutions to*

Problem 3.2 and to Problem 3.4 there holds:

$$\sup_{\tau>0} \|\mathbf{u}_\tau\|_{H^1(0,T;H_0^2(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d))} \leq S, \tag{3.36}$$

$$\sup_{\tau>0} \|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H_0^2(\Omega;\mathbb{R}^d))} \leq S, \tag{3.37}$$

$$\sup_{\tau>0} \|\widehat{\mathbf{u}}_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S, \tag{3.38}$$

$$\sup_{\tau>0} \|\overline{\chi}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq S, \tag{3.39}$$

$$\sup_{\tau>0} \|\chi_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq S, \tag{3.40}$$

$$\sup_{\tau>0} \|\overline{w}_\tau\|_{L^\infty(0,T;L^1(\Omega))} \leq S, \tag{3.41}$$

$$\sup_{\tau>0} \|\overline{w}_\tau\|_{L^r(0,T;W^{1,r}(\Omega))} \leq S \quad \textit{for every } 1 \leq r < \frac{d+2}{d+1}, \tag{3.42}$$

$$\sup_{\tau>0} \|\overline{w}_\tau\|_{\mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)} \leq S, \tag{3.43}$$

$$\sup_{\tau>0} \|\Theta(\overline{w}_\tau)\|_{L^{2+\epsilon}(0,T;L^{2+\epsilon}(\Omega))} \leq S \quad \textit{for any } 0 < \epsilon < \frac{\sigma(d+2)}{d} - 2. \tag{3.44}$$

2. *if $\mu = 0$ in addition there exists $S' > 0$ such that*

$$\sup_{\tau>0} \left(\|\mathcal{B}(\overline{\chi}_\tau)\|_{L^2(0,T;L^2(\Omega))} + \|\overline{\xi}_\tau\|_{L^2(0,T;L^2(\Omega))} \right) \leq S'. \tag{3.45}$$

Moreover, if ϕ also fulfills Hypothesis (VI), then

$$\sup_{\tau>0} \|\overline{\chi}_\tau\|_{L^2(0,T;W^{1+\sigma,p}(\Omega))} \leq S' \quad \textit{for every } 0 < \sigma < \frac{1}{p}. \tag{3.46}$$

3. *in the isothermal case with $\mu = 1$, if $b'' \equiv 0$ (cf.* (2.86)*) and ϕ also fulfills Hypothesis (VI), estimates* (3.36)*–*(3.40) *hold. Moreover, there exists $S'' > 0$ such that for (the interpolants of) the solutions to Problem 3.6*

$$\sup_{\tau>0} \left(\|\mathcal{B}(\overline{\chi}_\tau)\|_{L^\infty(0,T;L^2(\Omega))} + \|\beta_\tau(\overline{\chi}_\tau)\|_{L^\infty(0,T;L^2(\Omega))} \right) \leq S'', \tag{3.47}$$

$$\sup_{\tau>0} \|\overline{\chi}_\tau\|_{L^\infty(0,T;W^{1+\sigma,p}(\Omega))} \leq S'' \quad \textit{for every } 0 < \sigma < \frac{1}{p}, \tag{3.48}$$

$$\sup_{\tau>0} \|\chi_\tau\|_{W^{1,\infty}(0,T;L^2(\Omega))} \leq S'', \tag{3.49}$$

$$\sup_{\tau>0} \|\overline{\zeta}_\tau\|_{L^\infty(0,T;L^2(\Omega))} \leq S''. \tag{3.50}$$

The constants in (3.42), (3.44), *and* (3.46), (3.48) *also depend on the parameters r, ϵ, and σ, respectively.*

Proposition 3.11 ($\mu = 0, \quad \rho \neq 0$)**.** *Let $\mu = 0$ and $\rho \neq 0$. Assume Hypotheses (I), (III)–(V), and Hypothesis (VIII); suppose that the data $\mathbf{f}$, g, ϑ_0, $\mathbf{u}_0$, $\mathbf{v}_0$, χ_0 comply with* (2.47)*–*(2.51)*, and in addition that $w_0 \in L^2(\Omega)$. Then, for the interpolants of the solutions to Problem 3.3 estimates* (3.39)*–*(3.41) *hold with a constant* independent *of M, whereas estimates* (3.36)*–*(3.38)*,* (3.45) *and (under the additional Hypothesis (VI))* (3.46) *hold for a constant* depending *on M. Moreover, there exists a constant $S''' = S'''(M) > 0$ such that*

$$\sup_{\tau>0} \|\overline{w}_\tau\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))} \leq S''', \tag{3.51}$$

$$\sup_{\tau>0} \|w_\tau\|_{H^1(0,T;H^1(\Omega)')} \leq S'''. \tag{3.52}$$

We will treat the proofs of Propositions 3.10 and 3.11 in a unified way, developing a series of a priori estimates.

Proof of Proposition 3.10. Most of the calculations below will be detailed on the discretization scheme (3.3)–(3.5) for the full *reversible* system, and whenever necessary we will outline the differences in comparison with the discrete systems of Problems 3.4 and 3.6. Furthermore, for each estimate we will specify the values of the parameters μ and ρ for which it is valid and, to make the computations more readable, we will illustrate them first on the time-continuous level, i.e. referring to system (2.57)–(2.59). **First a priori estimate for $\mu \in \{0,1\}$, $\rho \in \mathbb{R}$:** *we test* (2.58) *by* $\mathbf{u}_t$ (2.57) *by* 1, (2.59) *by* χ_t, *add them and integrate in time. This is the so-called energy estimate.* We test (3.4) by $\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}$. Note that

$$\tau \int_\Omega \mathrm{D}^2_{\tau,k}(\mathbf{u}) \cdot \mathrm{D}_{\tau,k}(\mathbf{u})\,\mathrm{d}x \geq \frac{1}{2}\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathrm{D}_{\tau,k-1}(\mathbf{u})\|^2_{L^2(\Omega)} \tag{3.53}$$

for all $k = 1, \dots, K_\tau$. Since $\chi_\tau^k \in [0,1]$ a.e. in Ω, thanks to (2.18) we have that $a(\chi_\tau^k) \geq 0$ a.e. in Ω, thus by (2.3) we have

$$\langle \mathcal{V}\left((a(\chi_\tau^k) + \delta)\mathrm{D}_{\tau,k}(\mathbf{u})\right), \mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}\rangle_{H^1(\Omega)} \geq C_1\delta\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^1(\Omega)}. \tag{3.54}$$

On the other hand, using that $\|b(\chi_\tau^k)\|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(0,1)}$ and taking into account (2.4), we find

$$\begin{aligned} |\langle \mathcal{E}\left(b(\chi_\tau^k)\mathbf{u}_\tau^k\right), \mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}\rangle_{H^1(\Omega)}| &\leq C_2\tau\|b\|_{L^\infty(0,1)}\|\mathbf{u}_\tau^k\|_{H^1(\Omega)}\|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^1(\Omega)} \\ &\leq \frac{1}{2}C_1\delta\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^1(\Omega)} + C\tau\|\mathbf{u}_\tau^0\|^2_{H^1(\Omega)} + C_\delta\tau\|\mathbf{u}_\tau^k - \mathbf{u}_\tau^0\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.55}$$

We estimate the latter term by observing that

$$\|\mathbf{u}_\tau^k - \mathbf{u}_\tau^0\|^2_{H^1(\Omega)} = \|\sum_{j=1}^k (\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1})\|^2_{H^1(\Omega)} \leq k\tau^2 \sum_{j=1}^k \|\mathrm{D}_{\tau,j}(\mathbf{u})\|^2_{H^1(\Omega)} \leq T\tau \sum_{j=1}^k \|\mathrm{D}_{\tau,j}(\mathbf{u})\|^2_{H^1(\Omega)}. \tag{3.56}$$

Altogether, collecting (3.53)–(3.56) and summing over the index $k = 1, \dots, K_\tau$, we conclude

$$\begin{aligned} &\frac{1}{2}\|\mathrm{D}_{\tau,K_\tau}(\mathbf{u})\|^2_{L^2(\Omega)} + \frac{1}{2}C_1\delta\sum_{k=1}^{K_\tau}\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^1(\Omega)} + \rho\sum_{k=1}^{K_\tau}\tau\int_\Omega \Theta(w_\tau^k)\,\mathrm{div}(\mathrm{D}_{\tau,k}(\mathbf{u}))\,\mathrm{d}x \\ &\leq \frac{1}{2}\|\mathrm{D}_{\tau,0}(\mathbf{u})\|^2_{L^2(\Omega)} + C\sum_{k=1}^{K_\tau}\tau\left(\sum_{j=1}^k \tau\|\mathrm{D}_{\tau,j}(\mathbf{u})\|^2_{H^1(\Omega)}\right). \end{aligned} \tag{3.57}$$

We multiply (3.5) by $\chi_\tau^k - \chi_\tau^{k-1}$. With standard convexity inequalities, we obtain

$$\begin{aligned} &\tau\|\mathrm{D}_{\tau,k}(\chi)\|^2_{L^2(\Omega)} + \Phi(\chi_\tau^k) + \int_\Omega \widehat{\beta}(\chi_\tau^k)\,\mathrm{d}x + \int_\Omega \gamma(\chi_\tau^k)(\chi_\tau^k - \chi_\tau^{k-1})\,\mathrm{d}x \\ &\leq \Phi(\chi_\tau^{k-1}) + \int_\Omega \widehat{\beta}(\chi_\tau^{k-1})\,\mathrm{d}x + \tau\int_\Omega \mathrm{D}_{\tau,k}(\chi)\left(\Theta(w_\tau^k) - \frac{1}{2}b'(\chi_\tau^{k-1})\varepsilon(\mathbf{u}_\tau^k)\mathrm{R}_e\varepsilon(\mathbf{u}_\tau^k)\right)\mathrm{d}x. \end{aligned} \tag{3.58}$$

We then test (3.3) by τ and add the resulting relation to (3.57) and (3.58), summing over the index $k = 1, \dots, K_\tau$. The terms $\tau\int_\Omega \mathrm{D}_{\tau,k}(\chi)\Theta(w_\tau^k)\,\mathrm{d}x$ and $\rho\tau\int_\Omega \Theta(w_\tau^k)\,\mathrm{div}(\mathrm{D}_{\tau,k}(\mathbf{u}))\,\mathrm{d}x$ cancel out. Furthermore, we note that $|b'(\chi_\tau^{k-1})| \leq C$ a.e. in Ω since $b \in \mathrm{C}^1(\mathbb{R})$ and $0 \leq \chi_\tau^{k-1} \leq 1$ a.e. in Ω, and exploit the Lipschitz continuity of the function γ, which enables us to estimate the last term on the left-hand side of

(3.58). Ultimately, we obtain

$$
\begin{aligned}
&\int_\Omega w_\tau^{K_\tau}\,\mathrm{d}x + \sum_{k=1}^{K_\tau}\tau\|\mathrm{D}_{\tau,k}(\chi)\|^2_{L^2(\Omega)} + \Phi(\chi_\tau^{K_\tau}) + \int_\Omega \widehat{\beta}(\chi_\tau^{K_\tau})\,\mathrm{d}x + \frac{1}{2}\|\mathrm{D}_{\tau,K_\tau}(\mathbf{u})\|^2_{L^2(\Omega)}\\
&\quad + \frac{1}{2}C_1\delta\sum_{k=1}^{K_\tau}\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^1(\Omega)}\\
&\le \int_\Omega w_0\,\mathrm{d}x + \Phi(\chi_\tau^0) + \int_\Omega \widehat{\beta}(\chi_\tau^0)\,\mathrm{d}x + \sum_{k=1}^{K_\tau}\tau\|g_\tau^k\|_{H^1(\Omega)^*}\\
&\quad + \sum_{k=1}^{K_\tau} C\tau(\|\chi_\tau^k\|_{L^2(\Omega)} + \||\varepsilon(\mathbf{u}_\tau^k)|^2\|_{L^2(\Omega)} + 1)\|\mathrm{D}_{\tau,k}(\chi)\|_{L^2(\Omega)}\\
&\le C + \frac{1}{4}\sum_{k=1}^{K_\tau}\tau\|\mathrm{D}_{\tau,k}(\chi)\|^2_{L^2(\Omega)} + C\sum_{k=1}^{K_\tau}\tau\|\varepsilon(\mathbf{u}_\tau^k)\|^4_{L^4(\Omega)} + C\sum_{k=1}^{K_\tau}\tau\left(\sum_{j=1}^{k}\tau\|\mathrm{D}_{\tau,j}(\chi)\|^2_{L^2(\Omega)}\right)\\
&\quad + C\sum_{k=1}^{K_\tau}\tau\left(\sum_{j=1}^{k}\tau\|\mathrm{D}_{\tau,j}(\mathbf{u})\|^2_{H^1(\Omega)}\right),
\end{aligned}
$$

where the last inequality follows from assumptions (2.48)–(2.51) on the data, from the Young inequality, and from estimating $\tau\|\chi_\tau^k\|^2_{L^2(\Omega)} \le 2\tau\|\chi_\tau^0\|^2_{L^2(\Omega)} + 2\tau\|\chi_\tau^k - \chi_\tau^0\|^2_{L^2(\Omega)}$ and dealing with the latter term like in (3.56). Therefore, applying a discrete version of the Gronwall lemma (cf., e.g., [29, Prop. 2.2.1]), we conclude estimates (3.39)–(3.41), as well as estimate

$$
\sup_{\tau>0}\|\mathbf{u}_\tau\|_{H^1(0,T;H_0^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} \le S, \tag{3.59}
$$

which in turn implies

$$
\sup_{\tau>0}\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H_0^1(\Omega;\mathbb{R}^d))} \le S. \tag{3.60}
$$

We can perform this energy estimate on Problem 3.4 as well: calculations (3.53)–(3.57) can be trivially adapted to (3.18), whereas (3.58) derives from choosing in the minimum problem (3.5) χ_τ^{k-1} as a competitor. We again conclude (3.39)–(3.41) as well as (3.59)–(3.60).

In the case of Problem 3.6, (3.22) also features the term ζ_τ^k, whence the additional term $\int_\Omega \zeta_\tau^k \mathrm{D}_{\tau,k}(\chi)\,\mathrm{d}x$ on the left-hand side of (3.58). Since $0 \in \alpha(0)$, by monotonicity the latter term in nonnegative. Taking into account this, replacing $\widehat{\beta}$ with $\widehat{\beta}_\tau$ in (3.58), and observing that the coefficient of $\varepsilon(\mathbf{u}_\tau^k)\mathrm{R}_e\varepsilon(\mathbf{u}_\tau^k)$ on the right-hand side of (3.19) is bounded, we may repeat the same calculations as in the above lines. The coercivity estimate (3.54) goes through because $a(\chi_\tau^k)$, which is no longer guaranteed to be positive, is replaced by $a(\chi_\tau^k)^+$. Furthermore, since $\chi_\tau^k \le \chi_\tau^{k-1} \le \chi_0 \le 1$ a.e. in Ω (due to the irreversibility constraint), we have that $(\chi_\tau^k)^+ \in [0,1]$ a.e. in Ω, thus we may again obtain (3.55).

Second a priori estimate for $\mu \in \{0,1\}$, $\rho = 0$: *following [8] (see also [45]), we test* (2.58) *by* $-\operatorname{div}(\varepsilon(\mathbf{u}_t))$ *and integrate in time.* We test (3.4) by $-\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}))$ and for simplicity and without loss of generality we confine ourselves to the case in which R_v is the identity tensor. This gives rise to the following terms on the left-hand side:

$$-\tau\int_\Omega \mathrm{D}^2_{\tau,k}(\mathbf{u})\cdot \mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x \geq \frac{1}{2}\int_\Omega |\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u}))|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\varepsilon(\mathrm{D}_{\tau,k-1}(\mathbf{u}))|^2\,\mathrm{d}x, \tag{3.61}$$

$$\begin{aligned}-\tau\int_\Omega \mathcal{V}\left((a(\chi^k_\tau)+\delta)\mathrm{D}_{\tau,k}(\mathbf{u})\right)\cdot \mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x &= \tau\int_\Omega (\delta + a(\chi^k_\tau))\,\mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\cdot \mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x\\ &\quad + \tau\int_\Omega \varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u}))\nabla a(\chi^k_\tau)\cdot \mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x\\ &\doteq I_0 + I_1 \geq \delta C_3^2\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^2(\Omega)} + I_1,\end{aligned} \tag{3.62}$$

the latter inequality due to (2.9). Moreover, always on the l.h.s. we have

$$\begin{aligned}-\tau\int_\Omega \mathcal{E}\left(b(\chi^k_\tau)\mathbf{u}^k_\tau\right)\cdot \mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x &= \lambda_1\tau\int_\Omega b(\chi^k_\tau)\Delta(\mathrm{D}_{\tau,k}(\mathbf{u}))\cdot\nabla(\mathrm{div}(\mathbf{u}^k_\tau))\,\mathrm{d}x\\ &\quad + 2\lambda_2\tau\int_\Omega b(\chi^k_\tau)\mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\cdot\mathrm{div}(\varepsilon(\mathbf{u}^k_\tau))\,\mathrm{d}x\\ &\quad + \lambda_1\tau\int_\Omega \mathrm{div}(\mathbf{u}^k_\tau)\nabla b(\chi^k_\tau)\cdot\Delta(\mathrm{D}_{\tau,k}(\mathbf{u}))\,\mathrm{d}x\\ &\quad + 2\lambda_2\tau\int_\Omega \varepsilon(\mathbf{u}^k_\tau)\nabla b(\chi^k_\tau)\cdot\mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x\\ &\doteq I_2+I_3+I_4+I_5,\end{aligned} \tag{3.63}$$

(where Δ stands for the vectorial Laplace operator). On the right-hand side, we have

$$-\tau\int_\Omega \mathbf{f}^k_\tau\cdot\mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x \leq C_\delta\tau\|\mathbf{f}^k_\tau\|^2_{L^2(\Omega)} + \frac{\delta}{8}C_3^2\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^2(\Omega)}, \tag{3.64}$$

where the latter inequality follows from (2.9). We now move the integral terms $I_1,\ldots,I_5$ to the right-hand side. Let us fix $0<\varsigma\leq 3$ such that $p\geq d+\varsigma$ (where p is the exponent in (2.22)). Then,

$$\begin{aligned}|I_1| &\leq \tau\|\mathrm{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\|_{L^2(\Omega)}\|\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u}))\|_{L^{d^\star-\varsigma}(\Omega)}\|\nabla a(\chi^k_\tau)\|_{L^{d+\varsigma}(\Omega)}\\ &\leq \frac{\delta}{4}C_3^2\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^2(\Omega)} + \delta\tau\|\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u}))\|^2_{L^{d^\star-\varsigma}(\Omega)}\|\nabla a(\chi^k_\tau)\|^2_{L^{d+\varsigma}(\Omega)}\\ &\leq \frac{\delta}{4}C_3^2\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^2(\Omega)} + \varrho^2 C\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^2(\Omega)}\|a'\|^2_{L^\infty(-\mathsf{m},\mathsf{m})}\|\nabla\chi^k_\tau\|^2_{L^p(\Omega)}\\ &\qquad + C^2_{\varrho,\delta}C'\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{L^2(\Omega)}\|a'\|^2_{L^\infty(-\mathsf{m},\mathsf{m})}\|\nabla\chi^k_\tau\|^2_{L^p(\Omega)}\\ &\leq \frac{\delta}{2}C_3^2\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^2(\Omega)} + CS^4,\end{aligned} \tag{3.65}$$

where the first and second inequalities respectively follow from the Hölder and Young inequalities, with $d^\star$ as in (2.12), the third one from (2.13), and the last one taking into account estimates (3.59) for $\sup_{k=1,\ldots,K_\tau}\|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{L^2(\Omega)}$, (3.39) for $\sup_{k=1,\ldots,K_\tau}\|\chi^k_\tau\|_{W^{1,p}(\Omega)}$, which in particular yields that $|\chi^k_\tau|\leq\mathsf{m}$ a.e. in $\Omega\times(0,T)$ for some $\mathsf{m}>0$, and from choosing $\varrho\leq C^{-1/2}(\|a'\|_{L^\infty(-\mathsf{m},\mathsf{m})}S)^{-1}$. Furthermore, taking into account that $b(\chi^k_\tau)\in L^\infty(\Omega)$, one easily checks that

$$\begin{aligned}|I_2+I_3| &\leq C\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^2(\Omega)}\|\mathbf{u}^k_\tau\|_{H^2(\Omega)}\\ &\leq \frac{\delta}{8}C_3^2\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|^2_{H^2(\Omega)} + C\|\mathbf{u}^0_\tau\|^2_{H^2(\Omega)} + C\tau\sum_{j=1}^k\tau\|\mathrm{D}_{\tau,j}(\mathbf{u})\|^2_{H^2(\Omega)},\end{aligned} \tag{3.66}$$

where the second inequality follows from the Young inequality, from $\tau\|\mathbf{u}^k_\tau\|^2_{H^2(\Omega)}\leq 2\tau\|\mathbf{u}^0_\tau\|^2_{H^2(\Omega)} + 2\tau\|\mathbf{u}^k_\tau-\mathbf{u}^0_\tau\|^2_{H^2(\Omega)}$, and from estimating the latter term as in (3.56). Analogously, again using that

$\sup_{k=1,\ldots,K_\tau} \|\chi_\tau^k\|_{W^{1,p}(\Omega)} \leq S$ and that $|\chi_\tau^k| \leq \mathsf{m}$ a.e. in Ω, we have

$$
\begin{aligned}
|I_4 + I_5| &\leq \tau \|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^2(\Omega)} \|\nabla b(\chi_\tau^k)\|_{L^3(\Omega)} (\|\operatorname{div}(\mathbf{u}_\tau^k)\|_{L^6(\Omega)} + \|\varepsilon(\mathbf{u}_\tau^k)\|_{L^6(\Omega)}) \\
&\leq C\tau \|b'\|_{L^\infty(-\mathsf{m},\mathsf{m})} \|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^2(\Omega)} \|\chi_\tau^k\|_{W^{1,p}(\Omega)} \|\mathbf{u}_\tau^k\|_{H^2(\Omega)} \\
&\leq \frac{\delta}{8} C_3^2 \tau \|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^2(\Omega)}^2 + CS^2 \|\mathbf{u}_\tau^0\|_{H^2(\Omega)}^2 + CS^2 \tau \sum_{j=1}^{k} \tau \|\mathrm{D}_{\tau,j}(\mathbf{u})\|_{H^2(\Omega)}^2 .
\end{aligned}
\tag{3.67}
$$

Collecting (3.61)–(3.67) and summing over the index $k = 1, \ldots, K_\tau$, we obtain

$$
\begin{aligned}
&\frac{1}{2} \|\varepsilon(\mathrm{D}_{\tau,K_\tau}(\mathbf{u}))\|_{L^2(\Omega)}^2 + \frac{C_3^2 \delta}{8} \sum_{k=1}^{K_\tau} \tau \|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^2(\Omega)}^2 \\
&\qquad \leq C + \frac{1}{2} \|\varepsilon(\mathrm{D}_{\tau,0}(\mathbf{u}))\|_{L^2(\Omega)}^2 + C \sum_{k=1}^{K_\tau} \tau \|\mathbf{f}_\tau^k\|_{L^2(\Omega)}^2 + C \sum_{k=1}^{K_\tau} \tau \sum_{j=1}^{k} \tau \|\mathrm{D}_{\tau,j}(\mathbf{u})\|_{H^2(\Omega)}^2 .
\end{aligned}
$$

Applying the discrete Gronwall Lemma once again, we conclude estimate (3.36), whence (3.37).

It is immediate to check that calculations (3.61)–(3.67) can also be performed on the discrete momentum equation (3.18) in Problem 3.6.

Remark 3.12. The calculations for the *Second a priori estimate* carry over to the case the operator $\mathcal{B}$ is replaced by the s-Laplacian A_s, provided that $s > \frac{d}{2}$. Indeed, this ensures the continuous embedding $W^{s,2}(\Omega) \subset W^{1,\bar{p}}(\Omega)$ for some $\bar{p} > d$, which is crucial in the above calculations, cf. (3.65).

Third a priori estimate for $\mu \in \{0, 1\}$, $\rho = 0$: BOCCARDO&GALLOUËT-*type estimate on* (2.57). As in the proof of [49, Prop. 4.2], we test equation (3.3) by $\Pi(w_\tau^k)$, where

$$
\Pi : [0, +\infty) \to [0, 1] \text{ is defined by } \Pi(w) = 1 - \frac{1}{(1 + w)^{\varsigma}} \quad \text{for some } \varsigma > 0. \tag{3.68}
$$

Note that $\Pi(w_\tau^k)$ is well-defined, since $w_\tau^k \geq 0$ a.e. in Ω, and it belongs to $H^1(\Omega) \cap L^\infty(\Omega)$, as Π is Lipschitz continuous. Such a test function has been first proposed in [16], as a simplification of the technique by BOCCARDO&GALLOUËT [5]. We shall denote by $\widehat{\Pi}$ the primitive of Π such that $\widehat{\Pi}(0) = 0$ (hence $\widehat{\Pi}(w) \geq 0$ for $w \geq 0$). Summing over $k = 1, \ldots, K_\tau$, we obtain

$$
\begin{aligned}
c_2 \varsigma \sum_{k=1}^{K_\tau} \tau \int_\Omega \frac{|\nabla w_\tau^k|^2}{(1 + w_\tau^k)^{\varsigma+1}} \, \mathrm{d}x &\leq \int_\Omega \widehat{\Pi}(w_\tau^{K_\tau}) \, \mathrm{d}x + \sum_{k=1}^{K_\tau} \int_\Omega K(w_\tau^{k-1}) \nabla w_\tau^k \cdot \nabla \Pi(w_\tau^k) \, \mathrm{d}x \\
&\leq \int_\Omega \widehat{\Pi}(w_\tau^0) \, \mathrm{d}x + \sum_{k=1}^{K_\tau} \tau (\|g_\tau^k\|_{L^1(\Omega)} + \|\mathrm{D}_{\tau,k}(\chi) \Theta(w_\tau^k)\|_{L^1(\Omega)}) \|\Pi(w_\tau^k)\|_{L^\infty(\Omega)},
\end{aligned}
$$

where the first inequality follows from $(2.45)_1$, the fact that $\nabla \Pi(w_\tau^k) = \varsigma (\nabla w_\tau^k)/(1 + w_\tau^k)^{\varsigma+1}$, and the second one from the convex analysis inequality $\int_\Omega \Pi(w_\tau^k)(w_\tau^k - w_\tau^{k-1}) \, \mathrm{d}x \geq \int_\Omega (\widehat{\Pi}(w_\tau^k) - \widehat{\Pi}(w_\tau^{k-1})) \, \mathrm{d}x$ and from the fact that, due to assumption (2.49), we have

$$
\int_\Omega \widehat{\Pi}(w_\tau^0) \, \mathrm{d}x \leq C \left(\|w_\tau^0\|_{L^1(\Omega)} + 1 \right) \leq C.
$$

Taking into account that $0 \leq \Pi(w_\tau^k(x)) \leq 1$ for almost all $x \in \Omega$ and all $k = 0, \ldots, K_\tau$, and relying on (2.49) and on (3.35b), we conclude that

$$
\sum_{k=1}^{K_\tau} \tau \int_\Omega \frac{|\nabla w_\tau^k|^2}{(1 + w_\tau^k)^{\varsigma+1}} \, \mathrm{d}x \leq C \sum_{k=1}^{K_\tau} \tau \|\mathrm{D}_{\tau,k}(\chi)\|_{L^2(\Omega)} \|\Theta(w_\tau^k)\|_{L^2(\Omega)} + C'. \tag{3.69}
$$

Now, we argue in the very same way as in [49, Proof of Prop. 4.2]. Combining the Hölder and Gagliardo-Nirenberg inequalities (cf. (2.11)) with the previously proved estimate (3.41) and with (3.69), we see that (cf. [49, Formula (4.35)])

$$\forall\, 1 \le r < \frac{d+2}{d+1}\, \exists\, C_r,\, C_r' > 0\, \forall\, \tau > 0\, :\, \sum_{k=1}^{K_\tau} \tau \|\nabla w_\tau^k\|_{L^r(\Omega)}^r \le C_r \sum_{k=1}^{K_\tau} \tau \|\mathrm{D}_{\tau,k}(\mathcal{X})\|_{L^2(\Omega)} \|\Theta(w_\tau^k)\|_{L^2(\Omega)} + C_r', \tag{3.70}$$

where the restriction on the index r in fact derives from the application of the Gagliardo-Nirenberg inequality (2.11). Next, for a sufficiently small $\epsilon > 0$ such that σ from (2.16) fulfills $\sigma > (2+\epsilon)d/(d+2)$, there holds

$$\begin{aligned} \|\Theta(w_\tau^k)\|_{L^{2+\epsilon}(\Omega)}^{2+\epsilon} \le C(\|w_\tau^k\|_{L^{(2+\epsilon)/\sigma}(\Omega)}^{(2+\epsilon)/\sigma} + 1) &\le C \|w_\tau^k\|_{L^1(\Omega)}^{(2+\epsilon)(1-\theta)/\sigma} \|w_\tau^k\|_{W^{1,r}(\Omega)}^{(2+\epsilon)\theta/\sigma} + C' \\ &\le C S^{(2+\epsilon)(1-\theta)/\sigma} (S + \|\nabla w_\tau^k\|_{L^r(\Omega)})^{(2+\epsilon)\theta/\sigma} + C' \\ &\le \varrho \|\nabla w_\tau^k\|_{L^r(\Omega)}^r + C_\varrho \qquad \text{if } \frac{d(d+2)}{d^2+d+2} < r < \frac{d+2}{d+1} \end{aligned} \tag{3.71}$$

(where we have omitted to indicate the dependence of the constants on ϵ and σ). The first inequality follows from (2.44), the second one from the Gagliardo-Nirenberg inequality (2.11) with $s = 2/\sigma$ and $q = 1$: in fact the constraints

$$\frac{\sigma}{2+\epsilon} > \frac{d}{d+2}, \quad \frac{d(d+2)}{d^2+d+2} < r < \frac{d+2}{d+1} \quad \text{imply} \quad \exists\, \theta \in (0,1)\, :\, \frac{\sigma}{2+\epsilon} = \theta \left(\frac{1}{r} - \frac{1}{d} \right) + 1 - \theta, \tag{3.72}$$

in accord with formula (2.11). Finally, the last inequality in (3.71) is due to the Young inequality, with C_ϱ depending on the constant $\varrho > 0$ to be suitably specified, under the additional condition that $r < (d+2)/(d+1)$ fulfills

$$\frac{(2+\epsilon)\theta}{\sigma} < r.$$

Combining (3.71) with (3.70), we immediately obtain

$$\sum_{k=1}^{K_\tau} \tau \|\nabla w_\tau^k\|_{L^r(\Omega)}^r \le \frac{C_r}{2} \sum_{k=1}^{K_\tau} \tau \|\mathrm{D}_{\tau,k}(\mathcal{X})\|_{L^2(\Omega)}^2 + C\varrho \sum_{k=1}^{K_\tau} \tau \|\nabla w_\tau^k\|_{L^r(\Omega)}^r + C'. \tag{3.73}$$

Hence, we choose $\varrho > 0$ in such a way as to absorb the second term on the right-hand side into the left-hand side. Therefore, on account of (3.40) $\sup_\tau \sum_{k=1}^{K_\tau} \tau \|\nabla w_\tau^k\|_{L^r(\Omega)} \le C$, which yields (3.42) via (3.41) and the Poincaré inequality. Finally, estimate (3.44) ensues from (3.42) and (3.71).

Observe that, when performing this estimate on the semi-implicit equation (3.17), we will obtain on the r.h.s. of (3.73) the term $\sum_{k=1}^{K_\tau} \tau \|\nabla w_\tau^{k-1}\|_{L^r(\Omega)}^r \le \tau \|\nabla w_{0\tau}\|_{L^r(\Omega)}^r + \sum_{k=1}^{K_\tau} \tau \|\nabla w_\tau^k\|_{L^r(\Omega)}^r$, and we can estimate $\tau \|\nabla w_{0\tau}\|_{L^r(\Omega)}^r$ thanks to (3.14).

Fourth a priori estimate for $\mu \in \{0,1\}$, $\rho = 0$: *comparison in* (2.58). It follows from estimates (3.36), (3.37), (3.40), and from the regularity result (2.6b), that

$$\sup_\tau \|\mathcal{V}\left((a(\overline{\mathcal{X}}_\tau) + \delta)\partial_t \mathbf{u}_\tau \right)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))}, \ \sup_\tau \|\mathcal{E}\left(b(\overline{\mathcal{X}}_\tau) \overline{\mathbf{u}}_\tau \right)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le C.$$

Thus, for $\rho = 0$ estimate (3.38) follows from a comparison in (3.4).

The same argument carries over to (3.18) and to (3.21).

Fifth a priori estimate for $\mu \in \{0,1\}$, $\rho = 0$: *comparison in* (2.57). In view of estimates and of (3.35b), a comparison argument in (3.3) yields estimate (3.43). The same for (3.17).

Sixth a priori estimate for $\mu = 0$, $\rho \in \mathbb{R}$: *we test* (2.59) *by* $\mathcal{B}(\mathcal{X}) + \beta(\mathcal{X})$ *and integrate in time.* We test (3.5) by $\tau \left(\mathcal{B}(\mathcal{X}_\tau^k) + \xi_\tau^k \right)$. Arguing as for (3.58) via convexity inequalities and referring to notation

(3.26) for the symbol h_τ^{k-1}, we get

$$\begin{aligned}\Phi(\chi_\tau^k) &+ \int_\Omega \widehat{\beta}(\chi_\tau^k)\,\mathrm{d}x + \tau\|\mathcal{B}(\chi_\tau^k)+\xi_\tau^k\|^2_{L^2(\Omega)}\\
&\leq \Phi(\chi_\tau^{k-1}) + \int_\Omega \widehat{\beta}(\chi_\tau^{k-1})\,\mathrm{d}x + \tau\|\gamma(\chi_\tau^k)+h_\tau^{k-1}\|_{L^2(\Omega)}\|\mathcal{B}(\chi_\tau^k)+\xi_\tau^k\|_{L^2(\Omega)}\\
&\leq \Phi(\chi_\tau^{k-1}) + \int_\Omega \widehat{\beta}(\chi_\tau^{k-1})\,\mathrm{d}x + \frac{1}{2}\tau\|\mathcal{B}(\chi_\tau^k)+\xi_\tau^k\|^2_{L^2(\Omega)} + C\tau(\|h_\tau^{k-1}\|^2_{L^2(\Omega)}+1)\end{aligned}$$

where the last inequality follows from $0 \leq \chi_\tau^k \leq 1$ a.e. in Ω, and the fact that γ is Lipschitz continuous on $[0,1]$. Summing up the above inequality for $k = 1,\ldots,K_\tau$ and taking into account a priori estimates (3.37) and (3.39), we conclude

$$\sup_{\tau>0}\|\mathcal{B}(\overline{\chi}_\tau)+\overline{\xi}_\tau\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{3.74}$$

From this bound, exploiting the monotonicity of β and applying [10, Prop. 2.17], we deduce (3.45). In view of (2.28), from the estimate for $\mathcal{B}(\overline{\chi}_\tau)$ we deduce (3.46).

Seventh a priori estimate for $\mu = 1$, $b'' \equiv 0$, and in the isothermal case: *we test* (2.59) *by* $\partial_t(\mathcal{B}(\chi)+\beta(\chi))$. Since $b'' \equiv 0$, we have that $b'(\chi_\tau^{k-1}) \equiv b$ on Ω. We test (3.22) by $\tau\mathrm{D}_{\tau,k}(\mathcal{B}(\chi)+\beta_\tau(\chi)) = (\mathcal{B}(\chi_\tau^k)+\beta_\tau(\chi_\tau^k) - (\mathcal{B}(\chi_\tau^{k-1})+\beta_\tau(\chi_\tau^{k-1})$. We observe that

$$I_6 := \int_\Omega (\chi_\tau^k - \chi_\tau^{k-1})(\mathcal{B}(\chi_\tau^k)-\mathcal{B}(\chi_\tau^{k-1}))\,\mathrm{d}x = \int_\Omega (\nabla\chi_\tau^k - \nabla\chi_\tau^{k-1})\cdot(\mathbf{d}(x,\nabla\chi_\tau^k)-\mathbf{d}(x,\nabla\chi_\tau^{k-1}))\,\mathrm{d}x \geq 0, \tag{3.75}$$

and, if (2.26) holds, we have in addition

$$I_6 \geq c_7\int_\Omega |\nabla(\chi_\tau^k - \chi_\tau^{k-1})|^p\,\mathrm{d}x = c_7\tau\int_\Omega \tau^{p-1}|\nabla\mathrm{D}_{\tau,k}(\chi)|^p\,\mathrm{d}x \tag{3.76}$$

Moreover, by monotonicity we have

$$\int_\Omega (\chi_\tau^k - \chi_\tau^{k-1})(\beta_\tau(\chi_\tau^k)-\beta_\tau(\chi_\tau^{k-1}))\,\mathrm{d}x \geq 0, \quad \tau\int_\Omega \zeta_\tau^k(\beta_\tau(\chi_\tau^k)-\beta_\tau(\chi_\tau^{k-1}))\,\mathrm{d}x \geq 0. \tag{3.77}$$

Furthermore, always by monotonicity, we get

$$\tau\int_\Omega \zeta_\tau^k(\mathcal{B}(\chi_\tau^k)-\mathcal{B}(\chi_\tau^{k-1}))\,\mathrm{d}x \geq 0.$$

Here, in order to perform a rigorous argument we should approximate the graph α with a Lipschitz continuous function α_ε as we have done in Lemma 3.8. However we prefer not to do it now in order not to overburden the calculations. Combining (3.75) and (3.77) with the inequalities

$$\begin{aligned}&\int_\Omega (\mathcal{B}(\chi_\tau^k)+\beta_\tau(\chi_\tau^k))(\tau\mathrm{D}_{\tau,k}(\mathcal{B}(\chi)+\beta_\tau(\chi)))\,\mathrm{d}x\\
&\geq \frac{1}{2}\int_\Omega |\mathcal{B}(\chi_\tau^k)+\beta_\tau(\chi_\tau^k)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\mathcal{B}(\chi_\tau^{k-1})+\beta_\tau(\chi_\tau^{k-1})|^2\,\mathrm{d}x,\end{aligned}$$

and summing over the index $k = 1,\ldots,K_\tau$, we get (cf. (3.26) for the notation h_τ^{k-1}, with $\Theta(w_\tau^{k-1})$ replaced by ${\Theta^*}_\tau^{k-1}$)

$$\begin{aligned}\frac{1}{2}\int_\Omega |\mathcal{B}(\chi_\tau^{K_\tau})+\beta_\tau(\chi_\tau^{K_\tau})|^2\,\mathrm{d}x \leq &\frac{1}{2}\int_\Omega |\mathcal{B}(\chi_\tau^0)+\beta_\tau(\chi_\tau^0)|^2\,\mathrm{d}x\\
&+ \underbrace{\sum_{k=1}^{K_\tau}\tau\int_\Omega (h_\tau^{k-1}-\gamma(\chi_\tau^k))\mathrm{D}_{\tau,k}(\mathcal{B}(\chi)+\beta_\tau(\chi))\,\mathrm{d}x}_{\doteq I_7}.\end{aligned} \tag{3.78}$$

Clearly, the first term on the right-hand side of (3.78) is bounded thanks to (2.87). Applying the discrete integration by part formula (3.34), we find

$$
\begin{aligned}
I_7 = (\mathcal{B}(\chi_\tau^{K_\tau}) + \beta_\tau(\chi_\tau^{K_\tau}))(h_\tau^{K_\tau-1} + \gamma(\chi_\tau^{K_\tau})) - (\mathcal{B}(\chi_\tau^0) + \beta_\tau(\chi_\tau^0))(h_\tau^0 + \gamma(\chi_\tau^1)) \\
- \sum_{k=2}^{K_\tau} \tau(\mathcal{B}(\chi_\tau^{k-1}) + \beta_\tau(\chi_\tau^{k-1}))(\mathrm{D}_{\tau,k-1}(h) + \mathrm{D}_{\tau,k}(\gamma(\chi)))\,.
\end{aligned}
\tag{3.79}
$$

Now, by the Lipschitz continuity of γ on $[0,1]$, we have $\|\mathrm{D}_{\tau,k}(\gamma(\chi))\|_{L^2(\Omega)} \leq C\|\mathrm{D}_{\tau,k}(\chi)\|_{L^2(\Omega)}$. Furthermore, we find

$$
\begin{aligned}
\|\mathrm{D}_{\tau,k-1}(h)\|_{L^2(\Omega)} &\leq \|\mathrm{D}_{\tau,k-1}(\Theta^*)\|_{L^2(\Omega)} + C\tau|b|\||\varepsilon(\mathbf{u}_\tau^{k-1})|^2 - |\varepsilon(\mathbf{u}_\tau^{k-2})|^2\|_{L^2(\Omega)} \\
&\leq \|\mathrm{D}_{\tau,k-1}(\Theta^*)\|_{L^2(\Omega)} + C'\|\varepsilon(\mathbf{u}_\tau^{k-1}) + \varepsilon(\mathbf{u}_\tau^{k-2})\|_{L^4(\Omega)}^2\|\mathrm{D}_{\tau,k-1}(\mathbf{u})\|_{L^4(\Omega)}^2 \doteq j_\tau^{k-1},
\end{aligned}
$$

where the second inequality also follows from the fact that b' is constant. Collecting (3.78)–(3.79) and the above inequalities, we thus infer

$$
\frac{1}{2}\int_\Omega |\mathcal{B}(\chi_\tau^{K_\tau}) + \beta_\tau(\chi_\tau^{K_\tau})|^2 \,\mathrm{d}x \leq C + \sum_{k=1}^{K_\tau} \tau(\|\mathrm{D}_{\tau,k+1}(\chi)\|_{L^2(\Omega)} + j_\tau^k)\|\mathcal{B}(\chi_\tau^k) + \beta_\tau(\chi_\tau^k)\|_{L^2(\Omega)},
$$

where we set $\mathrm{D}_{\tau,K_\tau+1}(\chi) = 0$. Then, estimate (3.47) ensues via the discrete Gronwall Lemma, taking into account that

$$
\sum_{k=1}^{K_\tau} \tau(\|\mathrm{D}_{\tau,k}(\chi)\|_{L^2(\Omega)} + j_\tau^k) \leq C \tag{3.80}
$$

in view of (3.35d), (3.36), and (3.40). Ultimately, (3.48) follows from (3.47) and the regularity result (2.28).

Eighth a priori estimate for $\mu = 1$ and in the isothermal case: *comparison in* (2.59). From a comparison argument in (3.22), we conclude that $\mathrm{D}_{\tau,k}(\chi) + \zeta_\tau^k$ is estimated in $L^\infty(0,T;L^2(\Omega))$. Then, (3.49) and (3.50) follow from the fact that $\int_\Omega \mathrm{D}_{\tau,k}(\chi)\zeta_\tau^k \,\mathrm{d}x \geq 0$. □

Proof of Proposition 3.11. The a priori bounds (3.39)–(3.41) follow from the calculations developed for the *First a priori estimate*, which also yields (3.59) and (3.60) for a constant independent of $M > 0$. The BOCCARDO&GALLOUËT-type *Third estimate* is replaced by the following

Ninth a priori estimate for $\mu = 0$, $\rho \neq 0$: *test* (2.57) *by* w. We test (3.11) by τw_τ^k. Summing over $k = 1, \ldots, K_\tau$ and recalling (3.7)–(3.10) we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |w_\tau^{K_\tau}|^2 \,\mathrm{d}x + c_{10}\sum_{k=1}^{K_\tau}\tau\int_\Omega |\nabla w_\tau^k|^2 \,\mathrm{d}x \\
&\leq \int_\Omega |w_\tau^0|^2 \,\mathrm{d}x + \sum_{k=1}^{K_\tau}\left(\tau\int_\Omega g_\tau^k w_\tau^k \,\mathrm{d}x + \tau\int_\Omega (|\mathrm{D}_{\tau,k}(\chi)| + |\rho||\operatorname{div}(\mathrm{D}_{\tau,k}(\mathbf{u}))|)\,|\Theta_M(w_\tau^k)||w_\tau^k| \,\mathrm{d}x\right) \\
&\leq \int_\Omega |w_0|^2 \,\mathrm{d}x + \nu\sum_{k=1}^{K_\tau}\tau\|w_\tau^k\|_{H^1(\Omega)}^2 + C_\nu\sum_{k=1}^{K_\tau}\tau\left(\|g_\tau^k\|_{H^1(\Omega)'}^2 + \|\mathrm{D}_{\tau,k}(\chi)\|_{L^2(\Omega)}^2 + \rho^2\|\operatorname{div}(\mathrm{D}_{\tau,k}(\mathbf{u}))\|_{L^2(\Omega)}^2\right),
\end{aligned}
\tag{3.81}
$$

for a suitably small constant $\nu > 0$, where we have used that the terms $\Theta_M(w_\tau^k)$ are uniformly bounded (by a constant depending on $M > 0$). Hence, estimate (3.51) follows from using that $w_0 \in L^2(\Omega)$, taking into account the previously proved bounds (3.40) and (3.59), and applying the discrete Gronwall Lemma. Estimate (3.52) then ensues from a comparison in (3.11), in view of the previously proved estimates.

Finally, relying on (3.51), we are able to perform the analogue of the *Second a priori estimate* on the momentum equation in the case $\rho \neq 0$ as well, as the following calculations show.

Tenth a priori estimate for $\mu = 0$, $\rho \neq 0$: *test* (2.58) *by* $-\operatorname{div}(\varepsilon(\mathbf{u}_t))$ *and integrate in time.* We test (3.12) by $-\operatorname{div}(\varepsilon(\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}))$. Every term can be dealt with like in the *Second estimate*, in addition we need to estimate the term

$$\left|\tau\rho\int_\Omega \nabla(\Theta_M(w_\tau^k))\operatorname{div}(\varepsilon(\mathrm{D}_{\tau,k}(\mathbf{u})))\,\mathrm{d}x\right| \leq C_\nu\rho\tau\|\nabla(\Theta_M(w_\tau^k))\|_{L^2(\Omega)}^2 + \nu\tau\|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^2(\Omega)}^2 .$$

Choosing ν sufficiently small in such a way as to absorb $\|\mathrm{D}_{\tau,k}(\mathbf{u})\|_{H^2(\Omega)}^2$ into (3.62), and estimating $\|\nabla(\Theta_M(w_\tau^k))\|_{L^2(\Omega)}^2$ via (3.51) (observe that Θ_M is Lipschitz continuous), we re-obtain (3.36)–(3.37), for a constant *depending* on M. Moreover, estimate (3.38) ensues from a comparison in (3.12).

Finally, estimates (3.45) and (3.46) can be obtained by repeating on equation (3.13) the very same calculations developed for the *Sixth estimate*: again, we get bounds depending on the truncation parameter M. $\square$

Remark 3.13. A close perusal of the proof of Proposition 3.10, and in particular of the calculations performed in the Second and Fourth a priori estimates, reveals that in fact estimates (3.39)–(3.43) hold for constants *independent* of $\delta > 0$ in both cases $\mu = 0$ and $\mu = 1$. This will play a key role in the proof of Theorem 6.

We conclude this section by mentioning in advance, for the reader's convenience, that the relevant estimates

1. for the proof of Thm. 1 are the *First, Second, Third, Fourth, Fifth*, and *Sixth a priori estimate*;
2. for the proof of Thm. 2 are the *First, Fourth, Sixth, Ninth*, and *Tenth a priori estimate*;
3. for the proof of Thm. 4 are the *First, Second, Third, Fourth*, and *Fifth a priori estimate*;
4. for the proof of Thm. 5 are the *First, Second, Fourth, Seventh*, and the *Eighth a priori estimate.*

4 Proofs of Theorems 1, 2, and 3

4.1 Proof of Theorem 1

Preliminarily, we rewrite equations (3.3)–(3.5) in terms of the interpolants $\overline{w}_\tau$, $\underline{w}_\tau$, $\overline{\mathbf{u}}_\tau$, $\underline{\mathbf{u}}_\tau$, $\mathbf{u}_\tau$, $\widehat{\mathbf{u}}_\tau$, $\overline{\chi}_\tau$, $\underline{\chi}_\tau$, χ_τ, and $\overline{\xi}_\tau$, namely

$$\begin{aligned} -\int_0^{\bar{\mathrm{t}}_\tau(t)}\int_\Omega \overline{w}_\tau\varphi_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^{\bar{\mathrm{t}}_\tau(t)}\int_\Omega \partial_t\chi_\tau\Theta(\overline{w}_\tau)\varphi\,\mathrm{d}x\,\mathrm{d}s + \int_0^{\bar{\mathrm{t}}_\tau(t)}\int_\Omega K(\underline{w}_\tau)\nabla\overline{w}_\tau\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s \\ = \int_0^{\bar{\mathrm{t}}_\tau(t)}\int_\Omega \overline{g}_\tau\varphi\,\mathrm{d}x\,\mathrm{d}s - \int_\Omega \overline{w}_\tau(t)\varphi(t)\,\mathrm{d}x + \int_\Omega w_0\varphi(0)\,\mathrm{d}x \qquad \text{for all } \varphi\in\mathscr{F},\ t\in[0,T], \end{aligned} \tag{4.1}$$

$$\partial_t\widehat{\mathbf{u}}_\tau(t) + \mathcal{V}\left((a(\overline{\chi}_\tau(t))+\delta)\partial_t\mathbf{u}_\tau(t)\right) + \mathcal{E}\left(b(\overline{\chi}_\tau(t))\overline{\mathbf{u}}_\tau(t)\right) = \overline{\mathbf{f}}_\tau(t) \quad \text{a.e. in } \Omega,\ \text{for a.a.}\, t\in(0,T), \tag{4.2}$$

$$\begin{aligned} \partial_t\chi_\tau(t) + \mathcal{B}(\overline{\chi}_\tau(t)) + \overline{\xi}_\tau(t) + \gamma(\overline{\chi}_\tau(t)) = -b'(\underline{\chi}_\tau(t))\frac{\varepsilon(\underline{\mathbf{u}}_\tau(t))\mathrm{R}_e\varepsilon(\underline{\mathbf{u}}_\tau(t))}{2} + \Theta(\overline{w}_\tau(t)) \\ \text{a.e. in } \Omega,\ \text{for a.a.}\, t\in(0,T), \end{aligned} \tag{4.3}$$

where for later use in (4.1) we have already integrated by parts in time, and $\mathscr{F}$ is as in (2.57). In what follows we will take the limit of (4.1)–(4.3) as $\tau\downarrow 0$ by means of compactness arguments, combined with techniques from maximal monotone operator theory.

Step 1: *compactness.* First of all, we observe that due to estimates (3.36) and (3.38), there holds

$$\begin{aligned} \|\mathbf{u}_\tau - \overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H_0^2(\Omega;\mathbb{R}^d))} \leq \tau^{1/2}\|\partial_t\mathbf{u}_\tau\|_{L^2(0,T;H_0^2(\Omega;\mathbb{R}^d))} \leq S\tau^{1/2}, \\ \|\widehat{\mathbf{u}}_\tau - \partial_t\mathbf{u}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq \tau^{1/2}\|\partial_t\widehat{\mathbf{u}}_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S\tau^{1/2}. \end{aligned} \tag{4.4}$$

Therefore, (3.36)–(3.38), joint with (4.4) and well-known weak and strong compactness results (cf. [51]), yield that there exist a vanishing sequence of time-steps (τ_k) and $\mathbf{u}$ as in (2.53) such that as $k \to \infty$

$$\begin{aligned} &\mathbf{u}_{\tau_k} \rightharpoonup^* \mathbf{u} && \text{in } H^1(0,T;H_0^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d)), \\ &\mathbf{u}_{\tau_k},\, \overline{\mathbf{u}}_{\tau_k},\, \underline{\mathbf{u}}_{\tau_k} \to \mathbf{u} && \text{in } L^\infty(0,T;H^{2-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \\ &\partial_t \widehat{\mathbf{u}}_{\tau_k} \rightharpoonup \partial_{tt}\mathbf{u} && \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \\ &\partial_t \mathbf{u}_{\tau_k} \to \partial_t \mathbf{u} && \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)). \end{aligned} \tag{4.5}$$

A stability estimate analogous to the first of (4.4), the a priori bounds (3.39), (3.40) and (3.45) and the previously mentioned compactness arguments, imply that there exist $\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $\xi,\, \lambda \in L^2(0,T;L^2(\Omega))$ such that, along a not relabeled subsequence there hold as $k \to \infty$

$$\begin{aligned} &\chi_{\tau_k} \rightharpoonup^* \chi && \text{in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ &\chi_{\tau_k},\, \overline{\chi}_{\tau_k},\, \underline{\chi}_{\tau_k} \to \chi && \text{in } L^\infty(0,T;W^{1-\epsilon,p}(\Omega)) \text{ for all } \epsilon \in (0,1], \end{aligned} \tag{4.6}$$

as well as

$$\overline{\xi}_{\tau_k} \rightharpoonup \xi \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{4.7}$$

$$\mathcal{B}(\overline{\chi}_{\tau_k}) \rightharpoonup \lambda \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{4.8}$$

Furthermore, if in addition ϕ complies with (2.26) and (2.27), then, due to (3.46) we also have the enhanced regularity (2.63), and the strong convergence

$$\chi_{\tau_k},\, \overline{\chi}_{\tau_k},\, \underline{\chi}_{\tau_k} \to \chi \qquad \text{in } L^s(0,T;W^{1,p}(\Omega)) \text{ for all } 1 \le s < \infty. \tag{4.9}$$

As for $(\overline{w}_\tau)_\tau$, estimates (3.41)–(3.43) and a generalization of the Aubin-Lions theorem to the case of time derivatives as measures (see e.g. [48, Chap. 7, Cor. 7.9]) yield that there exists w as in (2.52) such that, up to the extraction of a further subsequence, as $k \to \infty$ there hold

$$\begin{aligned} &\overline{w}_{\tau_k},\, \underline{w}_{\tau_k} \rightharpoonup w && \text{in } L^r(0,T;W^{1,r}(\Omega)), \\ &\overline{w}_{\tau_k},\, \underline{w}_{\tau_k} \to w && \text{in } L^r(0,T;W^{1-\epsilon,r}(\Omega)) \cap L^s(0,T;L^1(\Omega)) \text{ for all } \epsilon \in (0,1] \text{ and } 1 \le s < \infty. \end{aligned} \tag{4.10}$$

Furthermore, by an infinite-dimensional version of Helly's selection principle (cf. e.g. [4]) we have $\overline{w}_{\tau_k}(t) \rightharpoonup w(t)$ in $W^{1,r'}(\Omega)^*$ for all $t \in [0,T]$. Taking into account the a priori bound (3.41) of $(\overline{w}_{\tau_k}(t))_{\tau_k}$ in $L^1(\Omega)$, we then conclude that

$$\overline{w}_{\tau_k}(t) \rightharpoonup w(t) \qquad \text{in } \mathrm{M}(\Omega) \text{ for all } t \in [0,T]. \tag{4.11}$$

Clearly, the second of (4.10) implies that $\overline{w}_{\tau_k} \to w$ a.e. in $\Omega \times (0,T)$, hence by the continuity of Θ we also have $\Theta(\overline{w}_{\tau_k}) \to \Theta(w)$ a.e. in $\Omega \times (0,T)$. Moreover, estimate (3.44) guarantees that $(\Theta(\overline{w}_{\tau_k}))_{\tau_k}$ is uniformly integrable in $L^2(0,T;L^2(\Omega))$. Therefore, thanks e.g. to [15, Thm. III.3.6], we conclude that

$$\Theta(\overline{w}_{\tau_k}) \to \Theta(w) \qquad \text{in } L^2(0,T;L^2(\Omega)). \tag{4.12}$$

Step 2: passage to the limit in (4.1)–(4.3). It follows from (4.6) and (4.12) that

$$\partial_t \chi_{\tau_k} \Theta(\overline{w}_{\tau_k}) \rightharpoonup \chi_t \Theta(w) \qquad \text{in } L^1(0,T;L^1(\Omega)). \tag{4.13}$$

Moreover, (4.10) and (2.17) easily yield that $(K(\underline{w}_{\tau_k})\nabla \overline{w}_{\tau_k})_{\tau_k}$ is bounded in $L^r(0,T;L^r(\Omega))$. Since

$$K(\underline{w}_{\tau_k}) \to K(w) \qquad \text{in } L^s(0,T;L^s(\Omega)) \text{ for every } s \in [1,\infty) \tag{4.14}$$

taking into account (4.10), we can pass to the limit as $k \to \infty$ in the third integral in the first line of (4.1). Convergences (4.10)–(4.11), (4.13)–(4.14), as well as (3.35b) for $(\overline{g}_{\tau_k})_{\tau_k}$, allow us to take the limit of (4.1) as $\tau_k \downarrow 0$. Hence we conclude that (w, χ) comply with (2.57).

As for the passage to the limit in (4.2), we observe that (4.6), the compact embedding (2.14), and (2.18) imply that $a(\overline{\chi}_{\tau_k}) \to a(\chi)$ and $b(\overline{\chi}_{\tau_k}) \to b(\chi)$ in $L^\infty(0,T;L^\infty(\Omega))$. Therefore, by the (4.5) we immediately conclude that $\mathcal{V}\left((a(\overline{\chi}_{\tau_k}(t)) + \delta)\partial_t \mathbf{u}_{\tau_k}(t)\right) \to \mathcal{V}\left((a(\chi(t)) + \delta)\partial_t \mathbf{u}(t)\right)$ and $\mathcal{E}\left(b(\overline{\chi}_{\tau_k}(t))\overline{\mathbf{u}}_{\tau_k}(t)\right) \to \mathcal{E}\left(b(\chi(t))\mathbf{u}(t)\right)$ in $L^2(\Omega)$ for almost all $t \in (0,T)$. Also relying on the third of (4.5) and on (3.35a) for $(\overline{\mathbf{f}}_{\tau_k})_k$, we find that $(\mathbf{u}, \chi)$ fulfill (2.58).

Finally, combining (4.6) with (4.7)–(4.8) and taking into account the strong-weak closedness of the graphs of the operators $\partial\beta$, $\mathcal{B} : L^2(\Omega) \rightrightarrows L^2(\Omega)$, we immediately conclude that (4.7)–(4.8) hold with

$$\xi(x,t) \in \beta(\chi(x,t)) \text{ and } \lambda(x,t) = \mathcal{B}(\chi(x,t)) \qquad \text{for a.a. } (x,t) \in \Omega \times (0,T).$$

We also observe that (4.5), (4.6), and (2.18) yield

$$b'(\underline{\chi}_{\tau_k}(t))\frac{\varepsilon(\underline{\mathbf{u}}_{\tau_k}(t))\mathrm{R}_e\varepsilon(\underline{\mathbf{u}}_{\tau_k}(t))}{2} \to b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2} \qquad \text{in } L^2(0,T;L^2(\Omega)). \tag{4.15}$$

Therefore, relying on (4.6) and the Lipschitz continuity (2.20) of γ on bounded intervals, we easily pass to the limit in (4.3) and infer that (χ, ξ) fulfill (2.60)–(2.61).

Step 3*: proof of total energy equality* (2.62)*.* We choose $\varphi \equiv 1$ in (2.57), test (2.58) by $\mathbf{u}_t$ and integrate on time, and test (2.61) by χ_t and integrate on time, then add the resulting relations. Some terms cancel out, and to conclude (2.62) we use the chain rule formula (2.83) for the functional $\mathscr{E}(\chi) := \Phi(\chi) + \int_\Omega W(\chi)\,\mathrm{d}x$, as well as the fact that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}a_{\mathrm{el}}(b(\chi)\mathbf{u},\mathbf{u})\right) &= \frac{1}{2}a_{\mathrm{el}}(b'(\chi)\chi_t\mathbf{u},\mathbf{u}) + a_{\mathrm{el}}(b(\chi)\mathbf{u},\mathbf{u}_t)\\ &= \frac{1}{2}\int_\Omega b'(\chi)\chi_t\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2}\,\mathrm{d}x + \left\langle \mathcal{E}\left(b(\chi)\mathbf{u}\right), \mathbf{u}_t\right\rangle_{H^1(\Omega;\mathbb{R}^d)} \qquad \text{for a.a. } t \in (0,T).\end{aligned}$$

□

4.2 Proof of Theorem 2

Outline. First of all, relying on the a priori estimates of Prop. 3.11, we pass to the limit as $\tau \to 0$ in the time-discretization scheme (3.11)–(3.13): we thus obtain the existence of a triple $(w_M, \mathbf{u}_M, \chi_M)$ solving the truncated version of system (2.57)–(2.59).

Secondly, we perform the passage to the limit as the truncation parameter M tends to $+\infty$. In this step, we need to obtain for the functions $(w_M)_M$ a bound in the spaces specified in (2.67), *independent* of the parameter M. In this direction, the key estimate consists in testing (4.17) below by (a truncation of) $w_M \in H^1(\Omega)$, which is now an admissible test function for (4.17): it is indeed in view of performing this test, that we need to keep the two passages to the limit as $\tau \to 0$ and as $M \to \infty$ distinct. In order to carry out the calculations related to such an estimate, we need to carefully tailor to the present truncated setting the *formal* computations outlined in Remark 2.10.

Step 1*: passage to the limit as* $\tau \to 0$*, for* $M > 0$ *fixed, in* (3.11)–(3.13)*.* The argument follows the very same lines as the one in the proof of Thm. 1: it is even easier, due to the truncations of the functions K and Θ. Therefore, we omit the details. Let us just observe that passing to the limit as $\tau \to 0$ in (3.11) leads to a solution

$$w_M \in L^2(0,T;H^1(\Omega)) \cap \mathrm{C}^0([0,T];L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)') \tag{4.16}$$

of the *truncated* enthalpy equation

$$\langle w_t, \varphi\rangle_{H^1(\Omega)} + \int_\Omega \chi_t\Theta_M(w)\varphi\,\mathrm{d}x + \rho\int_\Omega \mathrm{div}(\mathbf{u}_t)\Theta_M(w)\varphi\,\mathrm{d}x + \int_\Omega K_M(w)\nabla w\nabla\varphi\,\mathrm{d}x = \langle g, \varphi\rangle_{H^1(\Omega)} \tag{4.17}$$

for all $\varphi \in H^1(\Omega)$. Indeed, regularity (4.16) follows from estimates (3.51)–(3.52), and in turn it allows for the *stronger* formulation (4.17) (in comparison with (2.57)) of the enthalpy equation. Therefore, for every $M > 0$ the (Cauchy problem for the) truncated version of system (2.57)–(2.59), consisting of (4.17) and of (2.58)–(2.59) with Θ replaced by Θ_M, admits a solution $(w_M, \mathbf{u}_M, \chi_M)$, with the regularity (4.16) for w_M and (2.53)–(2.54) for $(\mathbf{u}_M, \chi_M)$, further fulfilling (2.60)–(2.61) (with Θ_M in place of Θ).
Step 2: passage to the limit as $M \to \infty$. Let $(w_M, \mathbf{u}_M, \chi_M)_M$ be the family of solutions constructed in the previous step. Since estimates (3.39)–(3.41) and (3.59)–(3.60) hold with a constant *independent* of M, we conclude by lower semicontinuity that

$$\begin{aligned}\exists\, C > 0 \quad \forall\, M > 0 \, : \quad & \|w_M\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}_M\|_{H^1(0,T;H^1_0(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} \\ & + \|\chi_M\|_{L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))} \leq C\,.\end{aligned} \tag{4.18}$$

Next, introduce the truncation operator

$$\mathcal{T}_M(r) = \begin{cases} -M & \text{if } r < -M, \\ r & \text{if } |r| \leq M, \\ M & \text{if } r > M, \end{cases}$$

and the sets

$$\begin{cases} \mathscr{A}_M := \{(x,t) \in \Omega \times (0,T) \, : \, |w_M(x,t)| \leq M\}, & \mathscr{A}_M^t := \{x \in \Omega \, : (x,t) \in \mathscr{A}_M\} \\ \mathscr{O}_M := \{(x,t) \in \Omega \times (0,T) \, : \, |w_M(x,t)| > M\}, & \mathscr{O}_M^t := \{x \in \Omega \, : (x,t) \in \mathscr{O}_M\}\,. \end{cases} \tag{4.19}$$

Hence, we test (4.17) by $\mathcal{T}_M(w_M)$ and integrate on $(0,t)$, $t \in (0,T)$: observing that

$$\left.\begin{array}{l} K_M(w_M)\nabla w_M \cdot \nabla(\mathcal{T}_M(w_M)) = K(\mathcal{T}_M(w_M))|\nabla(\mathcal{T}_M(w_M))|^2 \\ \Theta_M(w_M) = \Theta(\mathcal{T}_M(w_M)) \end{array}\right\} \quad \text{a.e. in } \Omega \times (0,T), \tag{4.20}$$

we thus obtain

$$\begin{aligned} & \frac{1}{2}\int_\Omega |\mathcal{T}_M(w_M(t))|^2 \, \mathrm{d}x + \int_0^t \int_\Omega K(\mathcal{T}_M(w_M))|\nabla(\mathcal{T}_M(w_M))|^2 \, \mathrm{d}x \, \mathrm{d}s \\ & \leq \frac{1}{2}\int_\Omega |\mathcal{T}_M(w_M(0))|^2 \, \mathrm{d}x + \int_0^t \int_\Omega |g||\mathcal{T}_M(w_M)| \, \mathrm{d}x + \int_0^t \int_\Omega |\ell_M||\Theta(\mathcal{T}_M(w_M))||\mathcal{T}_M(w_M)| \, \mathrm{d}x \, \mathrm{d}s\,, \end{aligned} \tag{4.21}$$

where we have used the place-holder $\ell_M := \partial_t \chi_M + \rho \, \mathrm{div}(\partial_t \mathbf{u}_M)$. Now, arguing in the very same way as in Rmk. 2.10 we observe that

$$\int_0^t \int_\Omega K(\mathcal{T}_M(w_M))|\nabla(\mathcal{T}_M(w_M))|^2 \, \mathrm{d}x \geq c \int_0^t \left(\|\nabla \mathcal{T}_M(w_M)\|^2_{L^2(\Omega)} + \|\mathcal{T}_M(w_M)\|^{2(q+1)}_{L^{6(q+1)}(\Omega)} \right) \mathrm{d}s - C \tag{4.22}$$

for positive constants c and C *independent* of M. Let us now focus on the on the r.h.s. of (4.21): note that $\|\mathcal{T}_M(w_M(0))\|^2_{L^2(\Omega)} \leq \|w_0\|^2_{L^2(\Omega)}$, whereas the second integral term can be estimated thanks to (2.48) on g. Taking into account the growth (2.43) of Θ and (2.46), the third integral can be estimated by

$$C \int_0^t \int_\Omega |\ell_M|(|\mathcal{T}_M(w_M)|^{q+1} + 1) \, \mathrm{d}x \, \mathrm{d}s \leq \varrho \int_0^t \int_\Omega |\mathcal{T}_M(w_M)|^{2(q+1)} \, \mathrm{d}s + C_\varrho, \tag{4.23}$$

where we have used estimates (4.18). Choosing $\varrho > 0$ sufficiently small, we can absorb the integral term on the r.h.s. of (4.23) into the r.h.s. of (4.22). As in Rmk. 2.10, we thus conclude that

$$\exists\, C > 0 \; \forall\, M > 0 \, : \; \|\mathcal{T}_M(w_M)\|_{L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega))} \leq C\,. \tag{4.24}$$

We now use (4.24) in order to infer an analogous estimate for the family $(w_M)_M$. To do so, we preliminarily observe that from the bound for $\|\mathcal{T}_M(w_M)\|_{L^\infty(0,T;L^2(\Omega))}$ we infer

$$C \geq \int_\Omega |\mathcal{T}_M(w_M)(t)|^2 \, \mathrm{d}x \geq \int_{\mathscr{O}_M^t} M^2 \, \mathrm{d}x = M^2 |\mathscr{O}_M^t|. \tag{4.25}$$

Therefore, upon testing (4.17) by w_M, integrating in time, and repeating the same calculations as above (also relying on (3.9)), we end up with

$$\begin{aligned} &\frac{1}{2}\int_\Omega |w_M(t)|^2\,\mathrm{d}x + c_{10}\int_0^t \|\nabla w_M\|_{L^2(\Omega)}^2\,\mathrm{d}s + c\int_0^t \|w_M\|_{L^{6(q+1)}(\mathscr{A}_M^s)}^{2(q+1)}\,\mathrm{d}s \\ &\leq C + \frac{1}{2}\int_\Omega |w_0|^2\,\mathrm{d}x + \varrho_1\int_0^t \|w_M\|_{H^1(\Omega)}^2\,\mathrm{d}s + C_{\varrho_1}\int_0^t \|g\|_{H^1(\Omega)'}^2\,\mathrm{d}s + I_8 \end{aligned} \tag{4.26}$$

for some $\varrho_1 > 0$ to be specified later, and we estimate

$$\begin{aligned} I_8 &= \int_0^t\int_{\mathscr{A}_M^s} |\ell_M||\Theta_M(w_M)||w_M|\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathscr{O}_M^s} |\ell_M||\Theta_M(w_M)||w_M|\,\mathrm{d}x\,\mathrm{d}s \\ &\leq C_{\varrho_2}\left(\int_0^t \|\ell_M\|_{L^2(\Omega)}^2 + 1\right) + \varrho_2\int_0^t \|w_M\|_{L^{2(q+1)}(\mathscr{A}_M^s)}^{2(q+1)}\,\mathrm{d}s \\ &\quad + C_{\varrho_3}\int_0^t \|\ell_M\|_{L^2(\Omega)}^2\|\Theta_M(w_M)\|_{L^3(\mathscr{O}_M^s)}^2\,\mathrm{d}s + \varrho_3\int_0^t \|w_M\|_{H^1(\mathscr{O}_M^s)}^2\,\mathrm{d}s \end{aligned} \tag{4.27}$$

where we have argued along the same lines as in Rmk. 2.10. Now, observe that for almost all $t \in (0,T)$

$$\|\Theta_M(w_M)\|_{L^3(\mathscr{O}_M^t)}^2 \leq C(|\Theta(M)| + |\Theta(-M)|)^2|\mathscr{O}_M^t|^{2/3} \leq C(|M|^{2/\sigma}+1)|\mathscr{O}_M^t|^{2/3} \leq C'\frac{|M|^{2/\sigma}+1}{M^{4/3}},$$

where the second inequality is due to the growth (2.43) of Θ, and the last one to (4.25). Observe that $2/\sigma - 4/3 < 0$ for $d = 3$ thanks to (2.16), therefore $(|M|^{2/\sigma}+1)/M^{4/3} \to 0$ as $M \to \infty$. For $d = 2$, in (4.27) taking into account the Sobolev embedding $H^1(\mathscr{O}_M^s) \subset L^s(\mathscr{O}_M^s)$ for every $s \in [1,\infty)$, we can replace $\|\Theta_M(w_M)\|_{L^3(\mathscr{O}_M^s)}^2$ with $\|\Theta_M(w_M)\|_{L^{2+\epsilon}(\mathscr{O}_M^s)}^2$ for any $\epsilon > 0$, and tune ϵ in such a way that the latter term will again converge to 0 as $M \to \infty$. Therefore the third term on the r.h.s. of (4.27) is bounded. It remains to choose ϱ_2 in such a way as to absorb the term $\int_0^t \|w_M\|_{L^{2(q+1)}(\mathscr{A}_M^s)}^{2(q+1)}\,\mathrm{d}s$ into the l.h.s. of (4.26), and ϱ_1, ϱ_3 so that $\varrho_1 + \varrho_3$ is sufficiently small. Also applying the Gronwall Lemma, we conclude that

$$\exists\, C > 0\ \forall\, M > 0 \ :\ \|w_M\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))} \leq C\,. \tag{4.28}$$

With easy calculations we also find

$$\exists\, C > 0\ \forall\, M > 0 : \|w_M\|_{L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega))} + \|\partial_t w_M\|_{L^{\mathfrak{r}}(0,T;W^{2,\mathfrak{s}}(\Omega)')} \leq C, \tag{4.29}$$

with $\mathfrak{r}$ and $\mathfrak{s}$ as in (2.67), the estimate for $\partial_t w_M$ following from a comparison in (4.17) (cf. Remark 2.10).

We are now in the position to obtain the further estimates

$$\begin{aligned} \exists\, C > 0 \quad \forall\, M > 0 : \quad &\|\mathbf{u}_M\|_{H^1(0,T;H_0^2(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d))\cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))} \\ &+\|\mathcal{B}(\chi_M)\|_{L^2(0,T;L^2(\Omega))} + \|\xi_M\|_{L^2(0,T;L^2(\Omega))} \leq C\,, \end{aligned} \tag{4.30}$$

where ξ_M is the selection in $\beta(\chi_M)$ fulfilling (2.61). Indeed, (4.30) can be proved relying on the previously obtained (4.18) and (4.28)–(4.29), by performing on the truncated version of system (2.57)–(2.59), the time-continuous analogues of the *Sixth* and *Tenth a priori estimates* (cf. Sec. 3.3).

Hence, we can carry out the passage to the limit argument as $M \to \infty$. Relying on the bounds (4.18) and (4.30) and on the compactness tools already exploited in the proof of Thm. 1, we find that there exist $(w, \mathbf{u}, \chi)$ and a (not relabeled) subsequence of $(w_M, \mathbf{u}_M, \chi_M)_M$ such that (the time-continuous analogues of) convergences (4.5)–(4.9) hold as $M \to \infty$. Furthermore, in view of estimates (4.28)–(4.29), the compactness result [51, Corollary 4] guarantees that convergences (4.10) improve to

$$\begin{aligned} &w_M \rightharpoonup^* w \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap W^{1,\mathfrak{r}}(0,T;W^{2,\mathfrak{s}}(\Omega)'), \\ &w_M \to w \quad \text{in } L^2(0,T;W^{1-\epsilon,2}(\Omega)) \cap L^\sigma(0,T;L^2(\Omega)) \cap \mathrm{C}^0([0,T];X) \\ &\qquad\qquad \text{for all } \epsilon \in (0,1],\ 1 \leq \sigma < \infty, \text{ and every space } X \text{ s.t. } L^2(\Omega) \Subset X \subset W^{2,\mathfrak{s}}(\Omega)' \end{aligned} \tag{4.31}$$

with $\mathfrak{r}$ and $\mathfrak{s}$ as in (2.67). Relying on (4.31) and on the Lipschitz continuity of Θ, it is not difficult to infer that $\nabla\Theta_M(w_M) \rightharpoonup \nabla\Theta(w)$ in $L^2(0,T;L^2(\Omega))$ as $k \to \infty$. Furthermore, combining (4.31) and the last of (4.5) for the sequence $(\chi_M)_M$ we also have that $\mathrm{div}(\partial_t \mathbf{u}_M)\Theta_M(w_M) \rightharpoonup \mathrm{div}(\mathbf{u}_t)\Theta(w)$ and $\Theta_M(w_M)\partial_t\chi_M \rightharpoonup \Theta(w)\chi_t$ in $L^{4/3}(Q)$. Therefore, we are able to pass to the limit in (4.17) (with test functions as in the statement of Thm. 2), and in the corresponding equations for $\mathbf{u}$ and χ in the case $\rho \neq 0$, which concludes the proof. □

5 Proof of Theorem 3

Let $(\mathbf{u}_i, \chi_i)$, $i = 1, 2$, be two solution pairs like in the statement of Theorem 3 and set $(\mathbf{u}, \chi) := (\mathbf{u}_1 - \mathbf{u}_2, \chi_1 - \chi_2,)$. Taking into account that a is constant (cf. (2.72)), hence $a(\chi_i) \equiv \bar{a} \geq 0$ for $i = 1, 2$ (cf. (2.72)), it is immediate to check that $(\mathbf{u}, \chi)$ fulfill a.e. in $\Omega \times (0,T)$

$$\mathbf{u}_{tt} + \mathcal{E}(b(\chi_1)\mathbf{u}) + \mathcal{E}((b(\chi_1) - b(\chi_2))\mathbf{u}_2) + \mathcal{V}((\bar{a}+\delta)\mathbf{u}_t) + \mathcal{C}_\rho(\Theta_1^* - \Theta_2^*) = \mathbf{f}_1 - \mathbf{f}_2\,, \tag{5.1}$$

$$\begin{aligned} &\chi_t + \mathcal{B}\chi_1 - \mathcal{B}\chi_2 + \beta(\chi_1) - \beta(\chi_2) + \gamma(\chi_1) - \gamma(\chi_2) \\ &\ni -b'(\chi_1)\left(\frac{\varepsilon(\mathbf{u}_1)\mathrm{R}_e\varepsilon(\mathbf{u}_1)}{2} - \frac{\varepsilon(\mathbf{u}_2)\mathrm{R}_e\varepsilon(\mathbf{u}_2)}{2}\right) - (b'(\chi_1) - b'(\chi_2))\frac{\varepsilon(\mathbf{u}_2)\mathrm{R}_e\varepsilon(\mathbf{u}_2)}{2} + \Theta_1^* - \Theta_2^*\,. \end{aligned} \tag{5.2}$$

Now, we test (5.1) by $\mathbf{u}_t$ and integrate in time. Recalling (2.3), it is not difficult to infer

$$\frac{1}{2}\|\mathbf{u}_t(t)\|^2_{L^2(\Omega)} + \delta\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s \leq \frac{1}{2}\|\mathbf{v}_0^1 - \mathbf{v}_0^2\|^2_{L^2(\Omega)} + \int_0^t \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^{-1}(\Omega)}\,\|\mathbf{u}_t\|_{H^1(\Omega)}\,\mathrm{d}s + I_9 + I_{10} + I_{11}, \tag{5.3}$$

where we have

$$\begin{aligned} I_9 = -\int_0^t \langle \mathcal{E}(b(\chi_1)\mathbf{u}), \mathbf{u}_t\rangle_{H^1(\Omega;\mathbb{R}^d)}\,\mathrm{d}s &\leq C\int_0^t \|b(\chi_1)\|_{L^\infty(\Omega)}\,\|\mathbf{u}\|_{H^1(\Omega)}\,\|\mathbf{u}_t\|_{H^1(\Omega)}\,\mathrm{d}s \\ &\leq \frac{\delta}{4}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s + C\int_0^t \|\mathbf{u}\|^2_{H^1(\Omega)}\,\mathrm{d}s\,, \end{aligned} \tag{5.4}$$

whereas, the Lipschitz continuity of b on bounded intervals (cf. (2.18)) and the Hölder inequality yield

$$\begin{aligned} I_{10} &= \int_0^t \langle \mathcal{E}((b(\chi_1) - b(\chi_2))\mathbf{u}_2), \mathbf{u}_t\rangle_{H^1(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \\ &\leq C\int_0^t \|\mathbf{u}_2\|_{W^{1,6}(\Omega)}\|\chi\|_{L^3(\Omega)}\|\mathbf{u}_t\|_{H^1(\Omega)}\,\mathrm{d}s \\ &\leq \frac{\delta}{4}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s + C\|\mathbf{u}_2\|^2_{L^\infty(0,T;H^2(\Omega))}\int_0^t \|\chi\|^2_{L^3(\Omega)}\,\mathrm{d}s \\ &\leq \frac{\delta}{4}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s + \nu\int_0^t \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^t \|\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s\,, \end{aligned} \tag{5.5}$$

where in the last inequality we have exploited the embeddings $H^1(\Omega) \Subset L^3(\Omega) \subset L^2(\Omega)$ and [33, Thm. 16.4, p. 102], with $\nu > 0$ a suitable constant to be chosen later and the constant C also depending on $\|\mathbf{u}_2\|^2_{L^\infty(0,T;H^2(\Omega))}$. Moreover, we get

$$\begin{aligned} I_{11} =& \rho\int_0^t\int_\Omega (\Theta_1^* - \Theta_2^*)\,\mathrm{div}(\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}s \\ &\leq \frac{\delta}{4}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s + \rho^2 C_\delta\int_0^t \|\Theta_1^* - \Theta_2^*\|^2_{L^2(\Omega)}\,\mathrm{d}s\,. \end{aligned}$$

Noting that $\|\mathbf{u}(t)\|^2_{H^1(\Omega)} \le 2\|\mathbf{u}^1_0 - \mathbf{u}^2_0\|^2_{H^1(\Omega)} + 2t\int_0^t \|\mathbf{u}_t(r)\|^2_{H^1(\Omega)}\,\mathrm{d}r$, we obtain from (5.3)–(5.5) that

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}_t(t)\|^2_{L^2(\Omega)} + \frac{\delta}{4}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s \\
&\le \frac{1}{2}\|\mathbf{v}^1_0 - \mathbf{v}^2_0\|^2_{L^2(\Omega)} + C\|\mathbf{f}_1 - \mathbf{f}_2\|^2_{L^2(0,T;H^{-1}(\Omega))} + \frac{\delta}{8}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s + C\|\mathbf{u}^1_0 - \mathbf{u}^2_0\|^2_{H^1(\Omega)} \\
&\quad + C\int_0^t \left(\int_0^s \|\mathbf{u}_t(r)\|^2_{H^1(\Omega)}\,\mathrm{d}r\right)\mathrm{d}s + \nu\int_0^t \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^t \|\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\quad + C\rho^2\int_0^t \|\Theta^*_1 - \Theta^*_2\|^2_{L^2(\Omega)}\,\mathrm{d}s.
\end{aligned}$$

Next, we test (5.2) by χ integrate the resulting equation in time. With elementary computations, also taking into account the Lipschitz continuity of γ (2.20), the monotonicity of β (2.19), and the crucial inequality (2.30), we get

$$\frac{1}{2}\|\chi(t)\|^2_{L^2(\Omega)} + c_9\kappa\int_0^t \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s \le \frac{1}{2}\|\chi^1_0 - \chi^2_0\|^2_{L^2(\Omega;\mathbb{R}^d)} + C\int_0^t \|\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s + I_{12} + I_{13}\,, \tag{5.6}$$

$$\begin{aligned}
I_{12} &:= -\int_0^t\int_\Omega b'(\chi_1)\left(\frac{\varepsilon(\mathbf{u}_1)\mathrm{R}_e\varepsilon(\mathbf{u}_1)}{2} - \frac{\varepsilon(\mathbf{u}_2)\mathrm{R}_e\varepsilon(\mathbf{u}_2)}{2}\right)\chi\,\mathrm{d}x\,\mathrm{d}s, \\
I_{13} &:= -\int_0^t\int_\Omega \left((b'(\chi_1) - b'(\chi_2))\frac{\varepsilon(\mathbf{u}_2)\mathrm{R}_e\varepsilon(\mathbf{u}_2)}{2}\chi + (\Theta^*_1 - \Theta^*_2)\chi\right)\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}$$

Using (2.14) and the fact that $b'(\chi_1) \in L^\infty(\Omega)$ we get

$$\begin{aligned}
|I_{12}| &\le \int_0^t\int_\Omega |b'(\chi_1)|\left(\frac{|(\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2))\mathrm{R}_e(\varepsilon(\mathbf{u}_1) + \varepsilon(\mathbf{u}_2))|}{2}\right)|\chi|\,\mathrm{d}x\,\mathrm{d}s \\
&\le C\int_0^t \|\mathbf{u}\|_{H^1(\Omega)}(\|\mathbf{u}_1\|_{W^{1,6}(\Omega)} + \|\mathbf{u}_2\|_{W^{1,6}(\Omega)})\|\chi\|_{L^3(\Omega)}\,\mathrm{d}s \\
&\le \nu\int_0^t \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^t \|\chi\|^2_{L^2(\Omega)} + C\int_0^t\left(\|\mathbf{u}^1_0 - \mathbf{u}^2_0\|^2_{H^1(\Omega)}\,\mathrm{d}s + \int_0^s \|\partial_t\mathbf{u}\|^2_{H^1(\Omega)}\right)\mathrm{d}s\,,
\end{aligned} \tag{5.7}$$

where the last inequality is obtained arguing as for (5.5). Again exploiting the Lipschitz continuity of b' on bounded intervals and the bound for $\mathbf{u}_2$ in $L^\infty(0,T;H^2_0(\Omega;\mathbb{R}^d))$, we get

$$\begin{aligned}
|I_{13}| &\le C\int_0^t\int_\Omega |\chi|^2|\varepsilon(\mathbf{u}_2)|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \|\Theta^*_1 - \Theta^*_2\|_{L^2(\Omega)}\|\chi\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\le \int_0^t \|\chi\|_{L^6(\Omega)}\|\chi\|_{L^2(\Omega)}\|\mathbf{u}_2\|^2_{W^{1,6}(\Omega)}\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\Theta^*_1 - \Theta^*_2\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\le \nu\int_0^t\int_\Omega \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\Theta^*_1 - \Theta^*_2\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^t \|\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s\,,
\end{aligned} \tag{5.8}$$

where the last estimate also follows from the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$ and the Young inequality. Collecting now (5.6)–(5.8), we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|\chi(t)\|^2_{L^2(\Omega)}\,\mathrm{d}s + c_9\kappa\int_0^t \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\le \frac{1}{2}\|\chi^1_0 - \chi^2_0\|^2_{L^2(\Omega)} + 2\nu\int_0^t \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\qquad + C\int_0^t \|\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^t\left(\|\mathbf{u}^1_0 - \mathbf{u}^2_0\|^2_{H^1(\Omega)} + \int_0^s \|\partial_t\mathbf{u}\|^2_{H^1(\Omega)}\,\mathrm{d}r + \int_0^t \|\Theta^*_1 - \Theta^*_2\|^2_{L^2(\Omega)}\right)\mathrm{d}s\,.
\end{aligned} \tag{5.9}$$

Summing up (5.6) and (5.9) and choosing $\nu \le c_9\kappa/6$, we conclude

$$
\begin{aligned}
&\frac{1}{2}\|\mathbf{u}_t(t)\|^2_{L^2(\Omega)} + \frac{\delta}{8}\int_0^t \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s + \frac{1}{2}\|\chi(t)\|^2_{L^2(\Omega)} + \frac{c_9\kappa}{6}\int_0^t \|\nabla\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\le C\Big(\|\chi_0^1-\chi_0^2\|^2_{L^2(\Omega)} + \|\mathbf{u}_0^1-\mathbf{u}_0^2\|^2_{H^1(\Omega)} + \|\mathbf{v}_0^1-\mathbf{v}_0^2\|^2_{L^2(\Omega)} + \|\mathbf{f}_1-\mathbf{f}_2\|^2_{L^2(0,T;H^{-1}(\Omega))} \\
&\quad + \int_0^t \|\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_0^t\int_0^s \|\mathbf{u}_t\|^2_{H^1(\Omega)}\,\mathrm{d}r\,\mathrm{d}s + \int_0^t \|\Theta_1^*-\Theta_2^*\|^2_{L^2(\Omega)}\,\mathrm{d}s\Big)\,.
\end{aligned}
$$

The application of the standard Gronwall lemma gives immediately the desired continuous dependence estimate (2.73). □

Remark 5.1. If we replace the p-Laplacian (2.25) with the linear s-Laplacian (2.34) in the equation for χ, the continuous dependence estimate of Theorem 3 can be performed without assuming a to be constant (cf. (2.72)). Indeed, in this case we would be able to deal with the additional term $\int_0^t \langle \mathcal{V}((a(\chi_1)-a(\chi_2))\partial_t\mathbf{u}_2, \partial_t\mathbf{u}\rangle_{H^1(\Omega;\mathbb{R}^d)}\,\mathrm{d}s$, which results from subtracting the equations fulfilled by solution pairs $(\mathbf{u}_i,\chi_i)$, $i=1,2$. It would be possible to estimate it by means of the $H^s(\Omega)$-norm of $\chi=\chi_1-\chi_2$, which would pop in on the left-hand side of (5.6).

6 Proofs of Theorems 4 and 5

6.1 Proof of Theorem 4

Step 0: approximate equations. Equations (3.17) and (3.18) can be rephrased in terms of the interpolants $\overline{w}_\tau$, $\underline{w}_\tau$, $\overline{\mathbf{u}}_\tau$, $\underline{\mathbf{u}}_\tau$, $\widehat{\mathbf{u}}_\tau$, $\overline{\chi}_\tau$, χ_τ, in a way analogous to (4.1)–(4.2).

Furthermore, taking into account that $\alpha = \partial I_{(-\infty,0]}$ and that $\beta = \partial I_{[0,+\infty)}$ we observe that the minimum problem (3.19) yields for every $k=1,\ldots,K_\tau$ that

$$
\frac{\tau}{2}\int_\Omega\left(\left|\frac{\eta-\chi_\tau^{k-1}}{\tau}\right|^2 - \left|\frac{\chi_\tau^{k}-\chi_\tau^{k-1}}{\tau}\right|^2\right)\mathrm{d}x + \Phi(\eta)-\Phi(\chi_\tau^k) + \int_\Omega\left(\widehat{\gamma}(\eta)-\widehat{\gamma}(\chi_\tau^k) + h_\tau^{k-1}(\eta-\chi_\tau^k)\right)\mathrm{d}x \ge 0
$$
$$
\text{for all } \eta\in W^{1,p}(\Omega) \text{ with } 0\le\eta \text{ and } \eta\le\chi_\tau^{k-1} \text{ a.e. in } \Omega\,,
$$

(recall the short-hand notation (3.26) for h_τ^{k-1}). Writing necessary optimality conditions for the above minimum problem, we infer

$$
\begin{aligned}
&\int_\Omega\Big(\partial_t\chi_\tau(t)(\eta-\overline{\chi}_\tau(t)) + \mathbf{d}(x,\nabla\overline{\chi}_\tau(t))\cdot\nabla(\eta-\overline{\chi}_\tau(t)) + \gamma(\overline{\chi}_\tau(t))(\eta-\overline{\chi}_\tau(t)) + \underline{h}_\tau(t)(\eta-\overline{\chi}_\tau(t))\Big)\,\mathrm{d}x \ge 0 \\
&\quad\text{for all } t\in[0,T] \text{ and all } \eta\in W^{1,p}(\Omega) \text{ with } 0\le\eta \text{ and } \eta\le\underline{\chi}_\tau(t) \text{ a.e. in } \Omega,
\end{aligned}
\tag{6.1}
$$

where we have used the short-hand notation (cf. (3.26))

$$
\underline{h}_\tau(t) := b'(\underline{\chi}_\tau(t))\frac{\varepsilon(\underline{\mathbf{u}}_\tau(t))\mathrm{R}_e\varepsilon(\underline{\mathbf{u}}_\tau(t))}{2} - \Theta(\underline{w}_\tau(t)). \tag{6.2}
$$

Letting $\eta = \nu\varphi + \overline{\chi}_\tau(t)$ in (6.1) and dividing the resulting inequality by $\nu>0$, we deduce that

$$
\begin{aligned}
&\int_\Omega\Big(\partial_t\chi_\tau(t)\varphi + \mathbf{d}(x,\nabla\overline{\chi}_\tau(t))\cdot\nabla\varphi + \gamma(\overline{\chi}_\tau(t))\varphi + \underline{h}_\tau(t)\varphi\Big)\,\mathrm{d}x \ge 0 \quad\text{for all } t\in[0,T] \\
&\text{and all } \varphi\in W^{1,p}(\Omega) \text{ s.t. there exists } \nu>0 \text{ with } 0\le\nu\varphi+\overline{\chi}_\tau(t)\le\underline{\chi}_\tau(t) \text{ a.e. in } \Omega.
\end{aligned}
\tag{6.3}
$$

Choosing $\varphi = -\partial_t\chi_\tau(t)$ (observe that it complies with the constraint above, upon taking $\nu=\tau$), we therefore obtain

$$\begin{aligned}&\int_\Omega \big(|\partial_t\chi_\tau(t)|^2 + \mathbf{d}(x,\nabla\overline{\chi}_\tau(t))\cdot\nabla(\partial_t\chi_\tau(t)) + \gamma(\chi_\tau(t))\partial_t\chi_\tau(t)\big)\,\mathrm{d}x\\ &\qquad\le -\int_\Omega \underline{h}_\tau(t)\partial_t\chi_\tau(t)\,\mathrm{d}x + \int_\Omega (\gamma(\chi_\tau(t)) - \gamma(\overline{\chi}_\tau(t)))\partial_t\chi_\tau(t)\,\mathrm{d}x\,.\end{aligned} \tag{6.4}$$

Therefore, upon summing (6.4) over the index k we deduce the *discrete version* of the energy inequality (2.78) for all $0\le s\le t\le T$, viz.

$$\begin{aligned}&\int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)}\int_\Omega |\partial_t\chi_\tau|^2\,\mathrm{d}x\,\mathrm{d}r + \Phi(\overline{\chi}_\tau(\bar{\mathfrak{t}}_\tau(t))) + \int_\Omega W(\overline{\chi}_\tau(\bar{\mathfrak{t}}_\tau(t)))\,\mathrm{d}x\\ &\le \Phi(\overline{\chi}_\tau(\bar{\mathfrak{t}}_\tau(s))) + \int_\Omega W(\overline{\chi}_\tau(\bar{\mathfrak{t}}_\tau(s)))\,\mathrm{d}x + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)}\int_\Omega \partial_t\chi_\tau\left(-b'(\underline{\chi}_\tau)\frac{\varepsilon(\underline{\mathbf{u}}_\tau)\mathrm{R}_\mathrm{e}\varepsilon(\underline{\mathbf{u}}_\tau)}{2} + \Theta(\underline{w}_\tau)\right)\mathrm{d}x\,\mathrm{d}r\\ &\qquad + C\tau^{1/2}\|\partial_t\chi_\tau\|^2_{L^2(0,T;L^2(\Omega))}\,,\end{aligned} \tag{6.5}$$

where we have estimated the last term on the right-hand side of (6.4) using that $\|\gamma(\chi_\tau(t))-\gamma(\overline{\chi}_\tau(t))\|_{L^2(0,T;L^2(\Omega))}\le C\tau^{1/2}\|\partial_t\chi_\tau\|_{L^2(0,T;L^2(\Omega))}$, thanks to the Lipschitz continuity of γ.

Step 1*: compactness.* In view of the a priori estimates from Proposition 3.10, we infer that there exist a vanishing subsequence $(\tau_k)_k$ and limit functions $(w,\mathbf{u},\chi)$ such that convergences (4.5), (4.6), (4.10)–(4.12) hold true as $k\to\infty$. Observe that (4.6) in particular yields that $\chi\ge 0$ and $\chi_t\le 0$ a.e. in $\Omega\times(0,T)$. Arguing as in the proof of [27, Lemma 5.11], we now prove that

$$\overline{\chi}_{\tau_k}\to\chi \qquad \text{in } L^p(0,T;W^{1,p}(\Omega))\,. \tag{6.6}$$

Indeed, [27, Lemma 5.2] gives a sequence $(\varphi_{\tau_k})_k\subset L^p(0,T;W^{1,p}_+(\Omega))\cap L^\infty(Q)$ of test functions for (6.3), fulfilling

$$\varphi_{\tau_k}\to\chi \text{ in } L^p(0,T;W^{1,p}(\Omega)),\qquad 0\le\varphi_{\tau_k}\le\underline{\chi}_{\tau_k} \text{ a.e. in } \Omega\times(0,T). \tag{6.7}$$

Observe that the first of (6.7) and convergences (4.6) yield in particular

$$\overline{\chi}_{\tau_k}\to\chi \qquad \text{in } L^2(0,T;L^2(\Omega)). \tag{6.8}$$

We have

$$\begin{aligned}&c_7\int_0^T\int_\Omega |\nabla\overline{\chi}_{\tau_k}-\nabla\chi|^p\,\mathrm{d}x\,\mathrm{d}s\\ &\le \int_0^T\int_\Omega \big(\mathbf{d}(x,\nabla\overline{\chi}_{\tau_k}) - \mathbf{d}(x,\nabla\chi)\big)\cdot\nabla(\overline{\chi}_{\tau_k}-\chi)\,\mathrm{d}x\,\mathrm{d}s\\ &\le \int_0^T\int_\Omega \mathbf{d}(x,\nabla\overline{\chi}_{\tau_k})\cdot\nabla(\overline{\chi}_{\tau_k}-\varphi_{\tau_k})\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_\Omega \mathbf{d}(x,\nabla\overline{\chi}_{\tau_k})\cdot\nabla(\varphi_{\tau_k}-\chi)\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad\qquad - \int_0^T\int_\Omega \mathbf{d}(x,\nabla\chi)\cdot\nabla(\overline{\chi}_{\tau_k}-\chi)\,\mathrm{d}x\,\mathrm{d}s \doteq I_{14}+I_{15}+I_{16}\end{aligned} \tag{6.9}$$

where the first inequality follows from (2.26) and the second one from elementary algebraic manipulations. Now, choosing $\varphi := \varphi_{\tau_k}-\overline{\chi}_{\tau_k}$ in (6.3) (which we are allowed to do thanks to (6.7)) and integrating in time, we obtain

$$I_{14} = \int_0^T\int_\Omega \big(\partial_t\chi_{\tau_k} + \gamma(\overline{\chi}_{\tau_k}) + \underline{h}_{\tau_k}\big)(\varphi_{\tau_k}-\overline{\chi}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}s \le C\|\varphi_{\tau_k}-\overline{\chi}_{\tau_k}\|_{L^2(0,T;L^2(\Omega))}\to 0 \text{ as } k\to\infty,$$

due to the bounds (3.37) and (3.40), and to (6.7) and (6.8). We also have

$$|I_{15}|\le \|\mathbf{d}(x,\nabla\overline{\chi}_{\tau_k})\|_{L^{p'}(0,T;L^{p'}(\Omega))}\|\nabla(\varphi_{\tau_k}-\chi)\|_{L^p(0,T;L^p(\Omega))}\le C\|\nabla(\varphi_{\tau_k}-\chi)\|_{L^p(0,T;L^p(\Omega))}\to 0,$$

where the second inequality follows from (2.22) and (3.40), and the last passage is due to (6.7). Taking into account that $\overline{\chi}_{\tau_k} \rightharpoonup \chi$ in $L^p(0,T;W^{1,p}(\Omega))$ by (4.9), we also prove that $I_{16} \to 0$ as $k \to \infty$. In this way, from (6.9) we conclude (6.6). Observe that, (6.6) combined with the bound (3.40) then yields (4.9).

Step 2: passage to the limit. Arguing in the very same way as for the proof of Thm. 1, it is possible to prove that $(w, \mathbf{u}, \chi)$ solve equations (2.57) and (2.58). It now remains to prove the variational inequality (2.76), together with (2.77), and the energy inequality (2.78). As for the latter, it is sufficient to pass to the limit as $k \to \infty$ in (6.5). For this, we use convergences (4.5), (4.6), (4.10)–(4.12), (4.13), (4.15), as well as (6.6), which in particular yields

$$\Phi(\overline{\chi}_\tau(\bar{\mathrm{t}}_\tau(s))) \to \Phi(\chi(s)) \qquad \text{for a.a. } s \in (0,T).$$

Clearly, the last term on the right-hand side of (6.5) tends to zero. Since the argument for (2.76)–(2.77) is perfectly analogous to the one developed in the proof of [27, Thm. 4.4], we refer the reader to [27] for all details and here just outline its main steps. Passing to the limit in (6.3) as $\tau_k \downarrow 0$ with suitable test functions from [27, Lemma 5.2], we prove that for almost all $t \in (0,T)$

$$\int_\Omega \Big(\chi_t(t)\tilde{\varphi} + \mathbf{d}(x, \nabla\chi(t)) \cdot \nabla\tilde{\varphi} + \gamma(\chi(t))\tilde{\varphi} + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2}\tilde{\varphi} - \Theta(w(t))\tilde{\varphi} \Big)\,\mathrm{d}x \geq 0$$
$$\text{for all } \tilde{\varphi} \in W^{1,p}_-(\Omega) \text{ with } \{\tilde{\varphi} = 0\} \supset \{\chi(t) = 0\}.$$

From this, arguing as in the proof of [27, Thm. 4.4] we deduce that for almost all $t \in (0,T)$

$$\begin{aligned} &\int_\Omega \Big(\chi_t(t)\varphi + \mathbf{d}(x, \nabla\chi(t)) \cdot \nabla\varphi + \gamma(\chi(t))\varphi + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2}\varphi - \Theta(w(t))\varphi \Big)\,\mathrm{d}x \\ &\geq \int_{\{\chi(t)=0\}} \left(\gamma(\chi(t)) + b'(\chi(t))\frac{\varepsilon(\mathbf{u}(t))\mathrm{R}_e\varepsilon(\mathbf{u}(t))}{2} - \Theta(w(t)) \right)^+ \varphi\,\mathrm{d}x \quad \text{for all } \varphi \in W^{1,p}_-(\Omega). \end{aligned} \tag{6.10}$$

Therefore, we take

$$\begin{aligned} \xi(x,t) := -\mathfrak{I}_{\{\chi=0\}}(x,t) \left(\gamma(\chi(x,t)) + b'(\chi(x,t))\frac{\varepsilon(\mathbf{u}(x,t))\mathrm{R}_e\varepsilon(\mathbf{u}(x,t))}{2} - \Theta(w(x,t)) \right)^+ \\ \text{for a.a. } (x,t) \in \Omega \times (0,T), \end{aligned} \tag{6.11}$$

$\mathfrak{I}_{\{\chi=0\}}$ denoting the characteristic function of the set $\{\chi = 0\}$. From (6.10) we deduce that, with this ξ inequality (2.76) holds. Moreover, it is immediate to check that ξ also complies with (2.77).

Step 3: strict positivity (2.85) of the temperature. Suppose that (2.84) holds: the discrete strict positivity (3.24) and convergences (4.10) yield that, in the limit, $w(x,t) \geq \underline{w}_0(x) = \Theta^{-1}(\underline{\vartheta}_0(x))$ for almost all $(x,t) \in \Omega \times (0,T)$. Therefore, (2.85) ensues. □

6.2 Proof of Theorem 5

Step 1: compactness. For the interpolants $\overline{\mathbf{u}}_\tau$, $\underline{\mathbf{u}}_\tau$, $\mathbf{u}_\tau$, $\widehat{\mathbf{u}}_\tau$, $\overline{\chi}_\tau$, χ_τ, $\overline{\xi}_\tau$ of the solutions $(\mathbf{u}^k_\tau, \chi^k_\tau, \zeta^k_\tau)_{k=1}^{K_\tau}$ of the discrete Problem 3.6, estimates (3.36)–(3.40) and (3.47)–(3.50) hold. Therefore, standard strong and weak compactness results yield that there exist $(\mathbf{u}, \chi)$ fulfilling (2.53)–(2.54) and a subsequence $\tau_k \downarrow 0$ such that convergences (4.5) and (4.6) hold. Moreover, estimates (3.47)–(3.50) also imply that χ has the enhanced regularity (2.89), and that

$$\begin{aligned} &\chi_{\tau_k} \rightharpoonup^* \chi && \text{in } L^\infty(0,T;W^{1+\sigma,p}(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \text{ for all } 0 < \sigma < \tfrac{1}{p}, \\ &\chi_{\tau_k}, \overline{\chi}_{\tau_k}, \underline{\chi}_{\tau_k} \to \chi && \text{in } L^\infty(0,T;W^{1,p}(\Omega)). \end{aligned} \tag{6.12}$$

Furthermore, there exist ζ and ξ such that, possibly along a further subsequence,

$$\overline{\zeta}_{\tau_k} \rightharpoonup^* \zeta \qquad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{6.13}$$

$$\beta_{\tau_k}(\overline{\chi}_{\tau_k}) \rightharpoonup^* \xi \qquad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{6.14}$$

Step 2: passage to the limit. Relying on convergences (4.5), (4.6), and (3.35a), we take the limit of the discrete momentum equation (3.18). As for (3.22), we observe that, thanks to estimate (3.47) and the second of (6.12), there holds

$$\mathcal{B}(\overline{\chi}_{\tau_k}) \to \mathcal{B}(\chi) \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and strongly in } L^\infty(0,T;W^{1,p}(\Omega)^*). \tag{6.15}$$

Therefore, also taking into account (4.15) we pass to the limit in (3.22) and conclude $(\mathbf{u},\chi,\xi,\zeta)$ fulfill (2.81), with $\Theta(w)$ replaced by Θ^*. Furthermore, combining (4.7) with (6.14) we have

$$\limsup_{k\to\infty} \int_0^T \int_\Omega \beta_{\tau_k}(\overline{\chi}_{\tau_k})\overline{\chi}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}s \le \int_0^T \int_\Omega \xi\chi\,\mathrm{d}x\,\mathrm{d}s.$$

Thanks to [10, Prop. 2.5, p. 27] we conclude that $\xi \in \beta(\chi)$ a.e. in $\Omega\times(0,T)$. Finally, testing equation (3.22) by $\partial_t\chi_{\tau_k}$ and integrating in time, with calculations analogous to (3.58) we find for all $t\in[0,T]$

$$\begin{aligned}
&\limsup_{k\to\infty} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \partial_t\chi_{\tau_k}\overline{\zeta}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}s \\
&\le -\liminf_{k\to\infty} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\partial_t\chi_{\tau_k}|^2\,\mathrm{d}x\,\mathrm{d}s - \liminf_{k\to\infty} \Phi(\overline{\chi}_{\tau_k}(t)) - \liminf_{k\to\infty} \int_\Omega \widehat{\beta}_{\tau_k}(\overline{\chi}_{\tau_k}(t)) \\
&\qquad - \liminf_{k\to\infty} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \gamma(\overline{\chi}_{\tau_k})\partial_t\chi_{\tau_k}\,\mathrm{d}x\,\mathrm{d}s - \liminf_{k\to\infty} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \underline{h}_{\tau_k}\partial_t\chi_{\tau_k}\,\mathrm{d}x\,\mathrm{d}s \\
&\le -\int_0^t \int_\Omega |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}s - \Phi(\chi(t)) - \int_\Omega \widehat{\beta}(\chi(t)) - \int_0^t \int_\Omega \gamma(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad - \int_0^t \int_\Omega h\partial_t\chi\,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_\Omega \chi_t\zeta\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

(here $\underline{h}_\tau$ is as in (6.2), with $\underline{\Theta^*}_\tau$ in place of $\Theta(\underline{w}_\tau)$, and $h := b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} - \Theta^*$), where the last inequality is due to (4.6) and (6.12), the Mosco-convergence of $(\widehat{\beta}_{\tau_k})_{\tau_k}$ to $\widehat{\beta}$, the Lipschitz continuity of γ, and (4.15). The last identity follows from equation (2.81). The aforementioned tokens of maximal monotone operator theory allow us to deduce from the fact that

$$\limsup_{k\to\infty} \int_0^t \int_\Omega \partial_t\chi_{\tau_k}\overline{\zeta}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_\Omega \chi_t\zeta\,\mathrm{d}x\,\mathrm{d}s,$$

that $\zeta \in \alpha(\chi_t)$ a.e. in $\Omega\times(0,T)$, which concludes the proof. □

Remark 6.1. Indeed, in the proof of Theorem 5 the fact that $\alpha = \partial I_{(-\infty,0]}$ has never been specifically used, therefore Thm. 5 extends to a maximal monotone operator α as in (2.92), observing that, up to perturbing $\widehat{\alpha}$ with an affine function, it is not restrictive to suppose that

$$0 \in \alpha(0) \text{ and } \widehat{\alpha}(0) = 0, \quad \text{whence } \ \widehat{\alpha}(x) \ge 0 \text{ for all } x \in \mathbb{R}.$$

7 Analysis of the degenerating system

We now address the passage to the *degenerate* limit $\delta \downarrow 0$ in the full system (2.57)–(2.59). For technical reasons which will be clarified in Remark 7.5 later on, we focus on the *irreversible* case $\mu = 1$, and neglect the thermal expansion term in the momentum equation, i.e. take $\rho = 0$. Furthermore, we confine the discussion to the case, in which, for $\delta > 0$ the coefficients of *both* the elliptic operators in (2.58) are truncated, cf. Remark 7.2 below. In particular, we will take the functions a and b of the form

$$a(\chi) = \chi, \quad b(\chi) = \chi, \text{ and replace both coefficients by } \chi + \delta. \tag{7.1}$$

Remark 7.1. The choice $a(\chi) = 1 - \chi$ and $b(\chi) = \chi$ in (2.58) and the truncation of both coefficients would lead to the momentum equation

$$\mathbf{u}_{tt} + \mathcal{V}\left((1 - \chi + \delta)\mathbf{u}_t\right) + \mathcal{E}\left((\chi + \delta)\mathbf{u}\right) = \mathbf{f} \quad \text{in } H^{-1}(\Omega; \mathbb{R}^d), \quad \text{a.e. in } (0, T) \tag{7.2}$$

for which the asymptotic analysis $\delta \downarrow 0$ would be less meaningful in the case of an *irreversible* evolution for χ. For, starting from an initial datum $\chi_0 \in W^{1,p}(\Omega)$ with $\max_{x \in \overline{\Omega}} \chi_0(x) < 1$, we would have $1 - \chi(x, t) \geq 1 - \max_{x \in \overline{\Omega}} \chi_0(x) > 0$ for all $(x, t) \in \overline{\Omega} \times (0, T]$. Hence, the limit $\delta \downarrow 0$ would not lead to elliptic degeneracy in (7.2).

Observe that the ensuing discussion can be suitably adjusted to the choice $a(\chi) = \chi$, $b(\chi) = 1 - \chi$, which is meaningful for phase transition models.

Remark 7.2. It seems to us that *both* the coefficients a and b need to be truncated when taking the degenerate limit in the momentum equation. Indeed, on the one hand the truncation of a allows us to deal with the *main part* of the elliptic operator in (2.58). On the other hand, in order to pass to the limit in the quadratic term on the right-hand side of (2.59), we will also need to truncate b, cf. (7.35) later on.

Theorem 4 guarantees that for every $\delta > 0$ there exists a triple $(w_\delta, \mathbf{u}_\delta, \chi_\delta)$ as in (2.52)–(2.54) fulfilling the enthalpy equation (2.57) with $\rho = 0$, the momentum equation

$$\partial_{tt}\mathbf{u}_\delta - \operatorname{div}((\chi + \delta)\mathrm{R}_v \varepsilon(\partial_t \mathbf{u}_\delta)) - \operatorname{div}((\chi + \delta)\mathrm{R}_e \varepsilon(\mathbf{u}_\delta)) = \mathbf{f} \quad \text{in } H^{-1}(\Omega; \mathbb{R}^d), \text{ a.e. in } (0, T), \tag{7.3}$$

(where for later convenience we have dropped the operator notation (2.5)), as well as

$$\partial_t \chi_\delta(x, t) \leq 0 \quad \text{for a.a. } (x, t) \in \Omega \times (0, T), \tag{7.4}$$

$$\begin{aligned} &\int_\Omega \Big(\left(\partial_t \chi_\delta(t) + \xi_\delta(t) + \gamma(\chi_\delta(t))\right) \varphi + \mathbf{d}(x, \chi_\delta(t)) \cdot \nabla \varphi \Big) \,\mathrm{d}x \leq \int_\Omega \left(-\frac{\varepsilon(\mathbf{u}_\delta(t)) \mathrm{R}_e \varepsilon(\mathbf{u}_\delta(t))}{2} + \Theta(w_\delta(t)) \right) \varphi \,\mathrm{d}x \\ &\qquad\qquad \text{for all } \varphi \in W^{1,p}_+(\Omega), \quad \text{for a.a. } t \in (0, T), \end{aligned} \tag{7.5}$$

$$\text{with } \xi_\delta(x, t) = -\mathcal{I}_{\{\chi_\delta = 0\}}(x, t) \left(\gamma(\chi_\delta(x, t)) + \frac{\varepsilon(\mathbf{u}_\delta(x, t)) \mathrm{R}_e \varepsilon(\mathbf{u}_\delta(x, t))}{2} - \Theta(w_\delta(x, t)) \right)^+ \tag{7.6}$$

for almost all $(x, t) \in \Omega \times (0, T)$, (changing sign in (2.76) and recalling (6.11)), and the energy inequality (2.78). As observed in Section 2.4, the family $(w_\delta, \mathbf{u}_\delta, \chi_\delta)_\delta$ then fulfills for all $t \in (0, T]$ the *energy inequality*

$$\begin{aligned} &\int_\Omega w_\delta(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\partial_t \chi_\delta|^2 \,\mathrm{d}x \,\mathrm{d}r + \int_0^t a_{\mathrm{vis}}((\chi_\delta + \delta)\partial_t \mathbf{u}_\delta, \partial_t \mathbf{u}_\delta) \,\mathrm{d}r \\ &\quad + \frac{1}{2} a_{\mathrm{el}}((\chi_\delta(t) + \delta)\mathbf{u}_\delta(t), \mathbf{u}_\delta(t)) + \Phi(\chi_\delta(t)) + \int_\Omega W(\chi_\delta(t)) \,\mathrm{d}x \\ &\leq \int_\Omega w_0 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2 \,\mathrm{d}x + \frac{1}{2} a_{\mathrm{el}}((\chi_0 + \delta)\mathbf{u}_0, \mathbf{u}_0) + \Phi(\chi_0) + \int_\Omega W(\chi_0) \,\mathrm{d}x \\ &\qquad + \int_0^t \int_\Omega \mathbf{f} \cdot \partial_t \mathbf{u}_\delta \,\mathrm{d}x \mathrm{d}r + \int_0^t \int_\Omega g \,\mathrm{d}x. \end{aligned} \tag{7.7}$$

First of all, following [39], in Prop. 7.3 below we deduce from equations (2.57), (7.3), and from (7.7) some a priori estimates for the family $(w_\delta, \mathbf{u}_\delta, \chi_\delta)_\delta$, *independent* of $\delta > 0$.

Let us mention in advance that estimate (7.11) for $(w_\delta)_\delta$ holds true only for the solutions $(w_\delta, \mathbf{u}_\delta, \chi_\delta)_\delta$ obtained through the time-discretization procedure of Section 3.3. Such solutions shall be referred to as *approximable*. Indeed, on the one hand, Remark 3.13 ensures that the *discrete* estimates (3.39)–(3.43) are valid with constants independent on δ: hence they are inherited by the approximable solutions $(w_\delta)_\delta$, yielding estimate (7.11) below. On the other hand, the calculations developed for the Fourth a priori estimate in Sec. 3.3 suggest that, in order to prove (7.11) for *all* weak solutions $(w_\delta)_\delta$ to (2.57), it would be necessary to test (2.57) by $\varphi = \Pi(w_\delta)$ with Π as in (3.68). This is not an admissible choice due to the poor regularity of w_δ. Since we do not dispose of a uniqueness result for the *irreversible full* system, we cannot conclude (7.11) for *all* weak solutions (in the sense of Def. 2.13) $(w_\delta)_\delta$, and therefore we will restrict to *approximable* solutions.

As it will be clear from the proof of Prop. 7.3, estimates (7.8)–(7.10) instead hold for *all* weak solutions $(\mathbf{u}_\delta, \chi_\delta)$.

Proposition 7.3. *Assume Hypotheses (I), (II), and (IV) with* $\widehat{\beta} = I_{[0,+\infty)}$*, conditions* (2.47)–(2.51) *on the data* $\mathbf{f}$*,* g, ϑ_0*,* $\mathbf{u}_0$*,* $\mathbf{v}_0$*,* χ_0*, and suppose that* a*,* b *are given by* (7.1)*. Then, there exists a constant* $\overline{S} > 0$ *such that for all* $\delta > 0$ *and for all* $(w_\delta, \mathbf{u}_\delta, \chi_\delta)_\delta$ (approximable) weak *solutions to the irreversible full system, the following estimates hold*

$$\|w_\delta\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_t \mathbf{u}_\delta\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} + \|\chi_\delta\|_{L^\infty(0,T;W^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega))} + \|W(\chi_\delta)\|_{L^\infty(0,T;L^1(\Omega))} \leq \overline{S}, \tag{7.8}$$

$$\|\sqrt{\chi+\delta}\,\mathrm{R}_v\,\varepsilon(\partial_t \mathbf{u}_\delta)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} + \|\sqrt{\chi+\delta}\,\mathrm{R}_e\,\varepsilon(\mathbf{u}_\delta)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \leq \overline{S}, \tag{7.9}$$

$$\|\partial_{tt}\mathbf{u}_\delta\|_{L^2(0,T;H^{-1}(\Omega;\mathbb{R}^d))} \leq \overline{S}, \tag{7.10}$$

$$\|w_\delta\|_{L^r(0,T;W^{1,r}(\Omega))\cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)} \leq \overline{S}. \tag{7.11}$$

Proof. Estimates (7.8)–(7.9) are straightforward consequences of the energy inequality (7.7), taking into account that $\int_\Omega W(\chi_\delta(t))\,\mathrm{d}x \geq -C$ for a constant independent of $t \in [0,T]$, estimating

$$\int_0^t \int_\Omega \mathbf{f} \cdot \partial_t \mathbf{u}_\delta \,\mathrm{d}x\mathrm{d}r \leq \frac{1}{2}\int_0^t \int_\Omega |\mathbf{f}|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t \int_\Omega |\partial_t \mathbf{u}_\delta|^2\,\mathrm{d}x\mathrm{d}r,$$

and applying the Gronwall Lemma. Hence, (7.10) follows from a comparison in (7.3), in view of (2.6a). Finally, (7.11) can be proved by observing that the discrete estimates (3.42)–(3.43) are in fact *independent* of the parameter $\delta > 0$, hence they carry over to the *approximable* solutions $(w_\delta)_\delta$. □

As pointed out in [39] (see also [9]), estimates (7.9) suggest that for the analysis $\delta \downarrow 0$ it is meaningful to work with the quantities

$$\boldsymbol{\mu}_\delta := \sqrt{\chi_\delta+\delta}\,\varepsilon(\partial_t \mathbf{u}_\delta), \qquad \boldsymbol{\eta}_\delta := \sqrt{\chi_\delta+\delta}\,\varepsilon(\mathbf{u}_\delta) \tag{7.12}$$

in terms of which (7.3) rewrites as

$$\partial_{tt}\mathbf{u}_\delta - \mathrm{div}(\sqrt{\chi_\delta+\delta}\,\mathrm{R}_v\,\boldsymbol{\mu}_\delta) - \mathrm{div}(\sqrt{\chi_\delta+\delta}\,\mathrm{R}_e\,\boldsymbol{\eta}_\delta) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \text{ a.e. in } (0,T). \tag{7.13}$$

For later purposes, we also observe that, in the setting of (7.1) and with notation (7.12), the total energy inequality (2.79) for the triple $(w_\delta, \mathbf{u}_\delta, \chi_\delta)$ can be reformulated as

$$\begin{aligned} &\int_\Omega w_\delta(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2\,\mathrm{d}x + \int_s^t \int_\Omega |\partial_t \chi_\delta|^2\,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}\int_s^t \int_\Omega \boldsymbol{\mu}_\delta(r)\,\mathrm{R}_v\,\boldsymbol{\mu}_\delta(r)\,\mathrm{d}x\,\mathrm{d}r \\ &\quad + \frac{1}{2}\int_\Omega \boldsymbol{\eta}_\delta(t)\,\mathrm{R}_e\,\boldsymbol{\eta}_\delta(t)\,\mathrm{d}x + \Phi(\chi_\delta(t)) + \int_\Omega W(\chi_\delta(t))\,\mathrm{d}x \\ &\leq \int_\Omega w_\delta(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(s)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \boldsymbol{\eta}_\delta(s)\,\mathrm{R}_e\,\boldsymbol{\eta}_\delta(s)\,\mathrm{d}x + \Phi(\chi_\delta(s)) \\ &\qquad + \int_\Omega W(\chi_\delta(s))\,\mathrm{d}x + \int_s^t \int_\Omega \mathbf{f} \cdot \partial_t \mathbf{u}_\delta\,\mathrm{d}x\mathrm{d}r + \int_s^t \int_\Omega g\,\mathrm{d}x\mathrm{d}r\,. \end{aligned} \tag{7.14}$$

The following result shows that the limit $\delta \downarrow 0$ preserves the structure (7.13) of the momentum equation, as well as the enthalpy equation (2.57). The weak formulation (2.76)–(2.78) of the equation for χ is generalized by (7.19)–(7.20), cf. Rmk. 7.4.

Theorem 6. *Assume Hypotheses (I), (II), (IV) with* $\widehat{\beta} = I_{[0,+\infty)}$*, and (V) with* ϕ *fulfilling* (2.26)*. Assume conditions* (2.47)*–*(2.51) *on the data* $\mathbf{f}$*,* g, ϑ_0*,* $\mathbf{u}_0$*,* $\mathbf{v}_0$*,* χ_0*, and suppose that* a*,* b *are given by* (7.1)*. Then, there exist* w *as in* (2.52)*, and*

$$\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}(\Omega;\mathbb{R}^d)), \ \boldsymbol{\mu} \in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \tag{7.15}$$

$$\begin{aligned} &\boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \\ &\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \chi(x,t) \geq 0, \quad \chi_t(x,t) \leq 0 \qquad \text{for a.a. } (x,t) \in \Omega \times (0,T), \end{aligned} \tag{7.16}$$

such that

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t), \ \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}) \ \text{a.e. in any open set } A \subset \Omega \times (0,T) \text{ s.t. } \chi > 0 \text{ a.e. in } A, \tag{7.17}$$

fulfilling the weak enthalpy equation (2.57) *with* $\rho = 0$*, the* weak *momentum equation*

$$\partial_{tt}\mathbf{u} - \mathrm{div}(\sqrt{\chi}\,\mathrm{R}_v\,\boldsymbol{\mu}) - \mathrm{div}(\sqrt{\chi}\mathrm{R}_e\,\boldsymbol{\eta})) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \text{ a.e. in } (0,T), \tag{7.18}$$

as well as

$$\begin{aligned} &\int_0^T \int_\Omega \Big((\partial_t\chi + \gamma(\chi))\,\varphi + \mathbf{d}(x,\nabla\chi)\cdot\nabla\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \int_\Omega \Big(-\frac{1}{2\chi}\boldsymbol{\eta}\,\mathrm{R}_e\,\boldsymbol{\eta} + \Theta(w) \Big)\,\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \text{for all } \varphi \in L^p(0,T;W^{1,p}_+(\Omega)) \cap L^\infty(Q) \text{ with } \mathrm{supp}(\varphi) \subset \{\chi > 0\}, \end{aligned} \tag{7.19}$$

and the total energy inequality *for almost all* $t \in (0,T]$

$$\begin{aligned} &\mathcal{H}(t) + \int_0^t \int_\Omega |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}\int_0^t \int_\Omega \boldsymbol{\mu}(r)\,\mathrm{R}_v\,\boldsymbol{\mu}(r)\,\mathrm{d}x\,\mathrm{d}r \\ &\leq \int_\Omega w_0\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2}a_{\mathrm{el}}(\chi_0\mathbf{u}_0,\mathbf{u}_0) + \Phi(\chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x \\ &\qquad + \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_0^t \int_\Omega g\,\mathrm{d}x\mathrm{d}r \end{aligned} \tag{7.20}$$

$$\begin{aligned} &\text{with } \mathcal{H}(t) \geq \int_\Omega w(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t\mathbf{u}(t)|^2\,\mathrm{d}x + \Phi(\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x + \mathcal{J}(t), \\ &\qquad\qquad \text{where } \mathcal{J}(t) := \frac{1}{2}\liminf_{\delta_k\downarrow 0} \int_\Omega \boldsymbol{\eta}_{\delta_k}(t)\,\mathrm{R}_e\,\boldsymbol{\eta}_{\delta_k}(t)\,\mathrm{d}x, \end{aligned} \tag{7.21}$$

(with $(\boldsymbol{\eta}_{\delta_k})$ *a suitable subsequence of* $(\boldsymbol{\eta}_\delta)$ *from* (7.12)*), and for all* $0 \leq t_1 \leq t_2 \leq T$ *there holds*

$$\int_{t_1}^{t_2} \mathcal{H}(r)\,\mathrm{d}r \geq \int_{t_1}^{t_2} \left(\int_\Omega w(r)(\mathrm{d}x) + \Phi(\chi(r)) + \int_\Omega \left(\frac{1}{2}|\partial_t\mathbf{u}(r)|^2 + W(\chi(r)) + \frac{1}{2}\boldsymbol{\eta}(r)\,\mathrm{R}_e\,\boldsymbol{\eta}(r) \right)\mathrm{d}x \right)\mathrm{d}r\,. \tag{7.22}$$

Remark 7.4. Let us briefly compare the concept of weak solution (to the *degenerating* irreversible full system (2.57)–(2.59)) arising from (7.17)–(7.20), with the notion of weak solution (to the *non-degenerating* irreversible full system (2.57)–(2.59)) given in Definition 2.13, in the case in which $a(\chi) = b(\chi) = \chi$. Suppose that the functions $(\mathbf{u},\chi)$ in (7.15) and (7.16) have further regularity properties (2.53)–(2.54), and that $\chi > 0$ a.e. in $\Omega \times (0,T)$. Then, (7.17) holds a.e. in $\Omega \times (0,T)$, hence it is immediate to realize that (7.19) coincides with (2.76). Furthermore, subtracting from (7.20) the weak enthalpy equation (2.57) tested by 1, we obtain a generalized form of the energy inequality (2.78) for almost all $t \in (0,T]$ and for $s = 0$.

Proof. It follows from estimates (7.8)–(7.11) and the same compactness arguments as in the proofs of Thms. 1 and 4 that there exist a vanishing sequence $\delta_k \downarrow 0$ and functions w as in (2.52) and $(\mathbf{u}, \chi, \boldsymbol{\mu}, \boldsymbol{\eta})$ as in (7.15)–(7.16) such that as $k \to \infty$

$$w_{\delta_k} \to w \text{ in } L^r(0,T;W^{1-\epsilon,r}(\Omega)) \cap L^s(0,T;L^1(\Omega)) \text{ for all } \epsilon \in (0,1] \text{ and all } 1 \le s < \infty, \tag{7.23}$$

$$\mathbf{u}_{\delta_k} \rightharpoonup^* \mathbf{u} \text{ in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}(\Omega;\mathbb{R}^d)), \tag{7.24}$$

$$\boldsymbol{\mu}_{\delta_k} \rightharpoonup \boldsymbol{\mu} \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \tag{7.25}$$

$$\boldsymbol{\eta}_{\delta_k} \rightharpoonup^* \boldsymbol{\eta} \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \tag{7.26}$$

$$\chi_{\delta_k} \rightharpoonup^* \chi \text{ in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{7.27}$$

$$\chi_{\delta_k} \to \chi \text{ in } \mathrm{C}^0([0,T];\mathrm{C}^0(\overline{\Omega})), \tag{7.28}$$

the latter convergence due to the compactness results in [51] and the compact embedding $W^{1,p}(\Omega) \Subset \mathrm{C}^0(\overline{\Omega})$. Observe that (7.24) and (7.27) respectively yield

$$\partial_t \mathbf{u}_{\delta_k} \to \mathbf{u}_t \text{ in } \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{R}^d)), \qquad \chi_{\delta_k} \to \chi \text{ in } \mathrm{C}^0_{\mathrm{weak}}([0,T];W^{1,p}(\Omega)). \tag{7.29}$$

From (7.23), exploiting (2.43) we deduce that

$$\Theta(w_{\delta_k}) \to \Theta(w) \quad \text{ in } L^2(0,T;L^2(\Omega)). \tag{7.30}$$

Thus, we are in the position of passing to the limit as $\delta_k \downarrow 0$ in (2.57) for the functions $(w_{\delta_k}, \chi_{\delta_k})$, and conclude (2.57) for (w, χ).

Exploiting (7.28) and the fact that $t \mapsto \chi_\delta(x,t)$ is nonincreasing for all $x \in \overline{\Omega}$, with the very same argument as in the proof of [39, Prop. 4.3] it is possible to prove that $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ have the form (7.17). In order to do that we can use the boundedness of $\varepsilon(\mathbf{u}_{\delta_k})$ and of $\varepsilon(\partial_t \mathbf{u}_{\delta_k})$ in $L^2(K;\mathbb{R}^{d\times d})$ for any compact cylinder K of the form $K_0 \times [0,t]$ on which $\chi > 0$. Notice that on these cylinders $\chi \ge \bar{\delta} > 0$ for some $\bar{\delta} > 0$. Hence, exploiting convergence (7.28), we infer that there exists $\delta_0 > 0$ such that, for any $0 < \delta_k \le \delta_0$, we have $\chi_{\delta_k}(x,t) + \delta_k \ge \bar{\delta}$ for all $x \in K_0$. Thus also $\chi_{\delta_k}(x,s) + \delta_k \ge \bar{\delta}$ for all $(x,s) \in K = K_0 \times [0,t]$ because $t \mapsto \chi_\delta(x,t)$ is nonincreasing for all $x \in \overline{\Omega}$. Then we can identify at the limit $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ and cover A in (7.17) by cylinders of the form K above. Hence, relying on (7.24)–(7.28) it is immediate to pass to the limit in (7.13) and conclude (7.18).

Next, we prove that

$$\chi_{\delta_k} \to \chi \qquad \text{in } L^p(0,T;W^{1,p}(\Omega))\,. \tag{7.31}$$

For this, we repeat the arguments from Step 2 in the proof of Thm. 4, based on [27, Lemma 5.11]. Namely, we apply [27, Lemma 5.2], which gives a sequence $(\varphi_{\delta_k})_k \subset L^p(0,T;W^{1,p}_+(\Omega)) \cap L^\infty(Q)$, such that $\varphi_{\delta_k} \to \chi$ in $L^p(0,T;W^{1,p}(\Omega))$ and $0 \le \varphi_{\delta_k} \le \chi_{\delta_k}$ a.e. in $\Omega \times (0,T)$. Relying on assumption (2.26), with the same calculations as in (6.9) we then have

$$\begin{aligned} &c_7 \int_0^T \int_\Omega |\nabla \chi_{\delta_k} - \nabla \chi|^p \,\mathrm{d}x\,\mathrm{d}s \\ &\le \int_0^T \int_\Omega \mathbf{d}(x, \nabla \chi_{\delta_k}) \cdot \nabla(\chi_{\delta_k} - \varphi_{\delta_k}) \,\mathrm{d}x\,\mathrm{d}s + \int_0^T \int_\Omega \mathbf{d}(x, \nabla \chi_{\delta_k}) \cdot \nabla(\varphi_{\delta_k} - \chi) \,\mathrm{d}x\,\mathrm{d}s \\ &\qquad - \int_0^T \int_\Omega \mathbf{d}(x, \nabla \chi) \cdot \nabla(\chi_{\delta_k} - \chi) \,\mathrm{d}x\,\mathrm{d}s \doteq I_{17} + I_{18} + I_{19}. \end{aligned} \tag{7.32}$$

Now, choosing $\widetilde{\varphi}_{\delta_k} := \varphi_{\delta_k} - \chi_{\delta_k}$ as a test function for (7.5) and integrating in time, we obtain

$$I_{17} = \int_0^T \int_\Omega \left(\partial_t \chi_{\delta_k} + \xi_{\delta_k} + \gamma(\chi_{\delta_k}) + \frac{\varepsilon(\mathbf{u}_{\delta_k}) \mathrm{R}_e \varepsilon(\mathbf{u}_{\delta_k})}{2} - \Theta(w_{\delta_k}) \right) (\varphi_{\delta_k} - \chi_{\delta_k}) \,\mathrm{d}x\,\mathrm{d}s \doteq I^a_{17} + I^b_{17} + I^c_{17},$$

where

$$I_{17}^a = \int_0^T \int_\Omega (\partial_t \chi_{\delta_k} + \gamma(\chi_{\delta_k}) - \Theta(w_{\delta_k}))\,(\varphi_{\delta_k} - \chi_{\delta_k})\,\mathrm{d}x\,\mathrm{d}s \le C\|\varphi_{\delta_k} - \chi_{\delta_k}\|_{L^2(0,T;L^2(\Omega))} \to 0 \text{ as } k \to \infty,$$

$$I_{17}^b = \int_0^T \langle \xi_{\delta_k}, \varphi_{\delta_k} - \chi_{\delta_k} \rangle_{W^{1,p}(\Omega)}\,\mathrm{d}s \le 0$$

$$I_{17}^c = \int_0^T \int_\Omega \frac{\varepsilon(\mathbf{u}_{\delta_k})\mathrm{R}_e\varepsilon(\mathbf{u}_{\delta_k})}{2}\,(\varphi_{\delta_k} - \chi_{\delta_k})\,\mathrm{d}x\,\mathrm{d}s \le 0$$

the second inequality due to (2.77), and the third one to the fact that $\varphi_{\delta_k} \le \chi_{\delta_k}$ a.e. in $\Omega \times (0,T)$. Calculations completely analogous to the ones developed in the proof of Thm. 4 yield that I_{18}, $I_{19} \to 0$ as $k \to \infty$. In this way, from (7.32) we conclude (7.31).

We are now in the position to pass to the limit in (the time-integrated version of) (7.5) and conclude (7.19). To this aim, we observe that, for any fixed test function φ as in (7.19), $\mathrm{supp}(\varphi)$ is a compact subset of $\overline{\Omega} \times [0,T]$. Hence there exists $\underline{\chi} > 0$ such that $\chi(x,t) \ge \underline{\chi} > 0$ for all $(x,t) \in \mathrm{supp}(\varphi)$, and, by (7.28), there exists $\bar{k} \in \mathbb{N}$ such that for $k \ge \bar{k}$

$$\chi_{\delta_k}(x,t) \ge \frac{1}{2}\underline{\chi} > 0 \quad \text{for all } (x,t) \in \mathrm{supp}(\varphi). \tag{7.33}$$

Therefore,

$$\lim_{k\to\infty} \int_0^T \int_\Omega \xi_{\delta_k} \varphi\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{7.34}$$

since $\mathrm{supp}(\xi_\delta) \subset \{\chi_\delta = 0\}$ by (7.6). Also exploiting (7.31), we succeed in taking the limit of the left-hand side of (7.5). As for the right-hand side, we use (7.30) and argue in the following way

$$\begin{aligned} \limsup_{\delta_k\to 0} \left(-\int_0^T \int_\Omega \frac{\varepsilon(\mathbf{u}_{\delta_k})\mathrm{R}_e\varepsilon(\mathbf{u}_{\delta_k})}{2}\varphi\,\mathrm{d}x\,\mathrm{d}t \right) &= -\liminf_{\delta_k\to 0} \int_0^T \int_\Omega \frac{1}{2(\chi_{\delta_k} + \delta_k)} \boldsymbol{\eta}_{\delta_k}\,\mathrm{R}_e\,\boldsymbol{\eta}_{\delta_k}\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &\le -\int_0^T \int_\Omega \frac{1}{2\chi}\boldsymbol{\eta}\,\mathrm{R}_e\,\boldsymbol{\eta}\varphi\,\mathrm{d}x\,\mathrm{d}t\,, \end{aligned} \tag{7.35}$$

where we have used that, thanks to (7.28) and (7.33), $\frac{1}{2(\chi_{\delta_k}+\delta_k)} \to \frac{1}{2\chi}$ uniformly on $\mathrm{supp}(\varphi)$, thus the last inequality e.g. follows from the lower semicontinuity result of [2].

Finally, (7.20) follows from taking the limit as $\delta_k \to 0$ of the total energy inequality (7.14), written on the interval $(0,t)$ for *any* $t \in (0,T]$. Observe that, by (7.14) (cf. also the arguments in the proofs of [39, Prop. 4.3]), the map

$$t \mapsto \mathcal{H}_\delta(t) := \int_\Omega w_\delta(t)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \boldsymbol{\eta}_\delta(t)\,\mathrm{R}_e\,\boldsymbol{\eta}_\delta(t)\,\mathrm{d}x + \Phi(\chi_\delta(t)) + \int_\Omega W(\chi_\delta(t))\,\mathrm{d}x$$

has (uniformly) bounded variation. Therefore, by Helly's theorem up to a subsequence there exists $\mathcal{H}$ such that $\mathcal{H}_{\delta_k}(t) \to \mathcal{H}(t)$ for all $t \in [0,T]$. To identify $\mathcal{H}$, we take the $\liminf$ as $\delta_k \to 0$ of the first, second, fourth, and fifth term in $\mathcal{H}_\delta(t)$, exploiting convergences (7.23), (7.27), (7.28), (7.29), as well as (7.31), and relying on lower semicontinuity arguments. Therefore we conclude that (7.21) holds. Finally, inequality (7.22) follows combining the following facts: on the one hand, since $(\mathcal{H}_\delta)_\delta \subset L^\infty(0,T)$ is uniformly bounded due to estimates (7.8)–(7.11), the dominated convergence theorem ensures

$$\int_{t_1}^{t_2} \mathcal{H}_{\delta_k}(r)\,\mathrm{d}r \to \int_{t_1}^{t_2} \mathcal{H}(r)\,\mathrm{d}r \text{ as } k \to \infty \quad \text{for all } 0 \le t_1 \le t_2 \le T.$$

On the other hand, on account convergences (7.23)–(7.28), by weak lower semicontinuity arguments we have that $\liminf_{k\to\infty} \int_{t_1}^{t_2} \mathcal{H}_{\delta_k}(r)\,\mathrm{d}r$ is greater or equal than the right-hand side of (7.22). This concludes the proof. □

Remark 7.5. As it is clear from the above lines, the proof of Thm. 6 strongly relies on the following properties:

1. the compact embedding of $W^{1,p}(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$;
2. the fact that $t \mapsto \chi_\delta(t,x)$ is nonincreasing for all $x \in \overline{\Omega}$, which follows from the irreversibility constraint.

These are the reasons why we have restricted the analysis of the degenerate limit to the irreversible system. Within this setting, we further need to assume $\rho = 0$. Indeed, because of the lack of estimates on $\mathrm{div}(\mathbf{u}_t)$ for $\delta \downarrow 0$, we would not be able to the limit in the term $\rho\,\mathrm{div}(\mathbf{u}_t)\Theta(w)$ in (2.57) as $\delta \downarrow 0$.

We also point out that, seemingly, the total energy inequality (7.20) cannot be improved to an inequality holding on any subinterval $(s,t) \subset (0,T)$. Indeed, for the sequence $(\boldsymbol{\eta}_{\delta_k})_k$ only the weak convergence (7.26) is available, which does not allow us to take the limit of the right-hand side of (7.14) but for $s = 0$.

Finally, observe that the proof of Thm. 6 simplifies if the operator $\mathcal{B}$ is given by the nonlocal s-Laplacian operator A_s. In this case, in order to pass to the limit in (7.5) it is no longer necessary to prove the strong convergence (7.31) for $(\chi_{\delta_k})_k$. In fact, the term $\mathbf{d}(x,\chi_{\delta_k}) \cdot \nabla\varphi$ in (7.5) is replaced by $a_s(\chi_{\delta_k},\varphi)$, which can be dealt with by weak convergence arguments due to the linearity of the operator A_s.

Acknowledgments. The authors would like to thank all the referees for their careful reading of the paper, and in particular one of them for a very useful suggestion on how to improve Thm. 6. They are also grateful to Christiane Kraus and Christian Heinemann for fruitful discussions on some topics related to this paper.

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