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Existence and Approximation Results for General Rate-independent Problems via a Variable Time-step Discretization Scheme

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Abstract. In this note, we prove an existence and approximation result for a class of *state-dependent* rate-independent problems (which have already been investigated in [6]), by passing to the limit in a time-discretization scheme with suitably constructed variable-time steps.

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1. Introduction

In this paper, we present an existence and approximation result for the Cauchy problem for the doubly nonlinear evolution equation:

$$\partial\Psi(z(t),\dot{z}(t)) + \partial\mathcal{E}(t,z(t)) \ni 0 \quad \text{in } Z, \quad t \in (0,T). \tag{1.1}$$

Here, Z is a (separable) reflexive Banach space, and the symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between Z' and Z. Throughout the paper, we will always assume the two proper functionals $\mathcal{E} : [0,T]\times Z \to \mathbb{R}$ and $\Psi : Z \times Z \to [0,+\infty)$ to fulfil

$$\begin{aligned} \mathcal{E}(t,\cdot) : Z \to \mathbb{R} \quad &\text{is convex and l.s.c.} \quad \text{for a.e. } t \in (0,T), \\ \mathcal{E}(\cdot,z) : [0,T] \to \mathbb{R} \quad &\text{is differentiable} \quad \forall\, z \in Z, \end{aligned} \tag{1.2}$$

$$\Psi(z,\cdot) : Z \to [0,+\infty) \text{ is convex and 1-positively homogeneous } \forall z \in Z, \tag{1.3}$$

and the symbol ∂ in (1.1) denotes the subdifferential of both functionals w.r.t. their second variables, i.e., for $t \in [0,T]$ and $z, v \in Z$ we have

$$\begin{aligned} \xi \in \partial\mathcal{E}(t,z) \;&\Leftrightarrow\; \mathcal{E}(t,\hat{z}) - \mathcal{E}(t,z) \geq \langle \xi, \hat{z} - z\rangle \quad \forall \hat{z} \in Z, \\ \omega \in \partial\Psi(z,v) \;&\Leftrightarrow\; \Psi(z,\hat{v}) - \Psi(z,v) \geq \langle \omega, \hat{v} - v\rangle \quad \forall \hat{v} \in Z. \end{aligned}$$

Let us point out that our 1-homogeneity assumption on $\Psi(z,\cdot)$ entails that a solution to (1.1) remains so if the time variable is rescaled.

As a matter of fact, (1.1) models *rate-independent* processes, occurring in plasticity, phase transformations in elastic solids, dry friction on surfaces, and several other fields of continuum mechanics (see the survey [4]). In this framework, z is referred to as the *state variable* of the process, while the functionals $\mathcal{E}$ and Ψ are respectively related to the *potential energy* and to the *dissipation*. Rate-independence means that such processes are insensitive to changes in the time-scales, which is also connected to the fact that they may display a hysteresis behavior.

The energetic formulation. A new modelling approach for rate-independent problems has been recently proposed in the seminal papers [5, 8, 7], however dealing with the simpler case of a *state-independent* functional Ψ (i.e., $\Psi(z, v) = \Psi(v)$ for all $(z, v) \in Z \times Z$). Such an approach stems from the fact that, in the applied problems arising in, e.g., plasticity, the energy functional $\mathcal{E}$ cannot be expected to be convex or smooth w.r.t. the state variable. Hence, the subdifferential formulation (1.1) appears to be too restrictive. On the other hand, the functional $t \mapsto \mathcal{E}(t, \cdot)$, which takes into account the external loadings, can be assumed smooth. In this spirit, in the paper [6] the following *energetic formulation* of (1.1) has been considered:

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, \hat{z}) + \Psi(z(t), \hat{z}-z(t)) \quad \forall \hat{z} \in Z, \tag{S}$$

$$\mathcal{E}(t, z(t)) + \int_0^t \Psi(z(\tau), \dot{z}(\tau))\, \mathrm{d}\tau = \mathcal{E}(0, z(0)) + \int_0^t \partial_t \mathcal{E}(\tau, z(\tau))\, \mathrm{d}\tau. \tag{E}$$

Note that (S) is a *stability condition*, which states that switching from the state z to the state $\hat{z}$ enforces a release of potential energy $\mathcal{E}(t, z(t)) - \mathcal{E}(t, \hat{z})$, smaller than the dissipated energy $\Psi(z(t), \hat{z}-z(t))$. On the other hand, (E) is an energy balance. Let us stress that the formulation (S)-(E) does not involve the derivative of $\mathcal{E}$ w.r.t. the variable z, but only the assumedly smooth power of the external forces $\partial_t \mathcal{E}$. Actually, it has been shown in [6] that, under suitable conditions on $\mathcal{E}$ and Ψ (see Sec. 3 later on), the two formulations (1.1) and (S)-(E) are equivalent. Moreover, an existence and approximation result (cf. [6, Thm. 4.7]) has been proved for (S)-(E) by passing to the limit in a suitable time-discretization scheme, which we briefly introduce.

Approximation via time-discretization. To introduce this approximation scheme, let us fix a partition of the interval $(0, T)$

$$\mathcal{P}_\tau := \left\{ t_\tau^0 = 0 < t_\tau^1 < \cdots < t_\tau^{N-1} < t_\tau^N = T \right\}, \quad \tau := \max_{j=1,\ldots,N} \{ t_\tau^j - t_\tau^{j-1} \}, \tag{1.4}$$

and let us introduce the following *time incremental problem*, associated with the *time-continuous* Problem (1.1): *given* $z_\tau^0 := z_0$*, find* $z_\tau^1, \ldots, z_\tau^N \in Z$ *such that*

$$z_\tau^k \in \operatorname{argmin}\{ \mathcal{E}(t_\tau^k, z) + \Psi(z_\tau^{k-1}, z - z_\tau^{k-1}) \mid z \in Z \} \quad \text{for } k = 1, \ldots, N. \tag{\textbf{IP}}$$

It can be shown (cf. Section 3) that, under suitable assumptions on Ψ and $\mathcal{E}$, for any $k = 1, \ldots, N$ (**IP**) has a *unique* solution $\{z_\tau^k\}$, fulfilling the *subdifferential* *inclusion*

$$\partial\Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) + \partial\mathcal{E}(t_\tau^k, z_\tau^k) \ni 0 \quad \forall k = 1, \ldots, N, \tag{1.5}$$

the *stability condition*

$$\mathcal{E}(t_\tau^k, z_\tau^k) \le \mathcal{E}(t_\tau^k, \hat{z}) + \Psi(z_\tau^{k-1}, \hat{z} - z_\tau^k) \quad \forall \hat{z} \in Z, \tag{1.6}$$

as well as the *energy inequality*

$$\mathcal{E}(t_\tau^k, z_\tau^k) + (t_\tau^k - t_\tau^{k-1})\Psi\left(z_\tau^{k-1}, \frac{z_\tau^k - z_\tau^{k-1}}{t_\tau^k - t_\tau^{k-1}}\right) \le \mathcal{E}(t_\tau^{k-1}, z_\tau^{k-1}) + \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t \mathcal{E}(r, z_\tau^{k-1}) \mathrm{d}r. \tag{1.7}$$

Indeed, (1.5), (1.6) and (1.7) are nothing but the *discrete* versions of (1.1), (S) and (E). In this setting, the approximate solutions of (1.1) are then constructed as suitable interpolants of the discrete solutions $\{z_\tau^k\}_{k=1}^N$. Yet, in [6] the passage to the limit procedure is performed *only* in the case of *uniform* time-step partitions.

Our main result. Nonetheless, it has been suggested in [6, Rem. 4.10] that it should also be possible to pass to the limit in an approximation scheme constructed with a *variable* time-step partition of $[0, T]$, somehow "adjusted" to the data of the problem, cf. (3.19) later on. In the what follows, we will indeed prove this conjecture, thus obtaining our main existence and approximation result, Theorem 3.8. The proof of this result relies on a (version of) the compactness result for Young measures (which we briefly present in Sec. 2), and on a *chain rule* argument. However, let us point out that the problem of the convergence of a *generic* variable time-step approximation scheme for (S)-(E) remains open.

Error estimates and uniqueness. As for the analysis of the rate of convergence of the approximate solutions, this issue is actually tightly connected to the problem of uniqueness of the solutions to (1.1). In fact, the strong convergence of the approximation scheme in the *state-independent* case has been achieved in [7], by means of the same *energetic estimates* exploited for proving a result of continuous dependence on the initial data. However, the latter type of result is definitely harder to prove in the state-dependent case, due to the quasivariational character of the problem. Uniqueness and continuous dependence theorems for (1.1) have indeed been obtained in [6] under more restrictive assumptions on Ψ and $\mathcal{E}$ than the ones for existence, by carefully combining the *energetic* method of [7] with the *convex analysis* arguments of [2]. Up to now, it has turned out to be arduous to adapt this complex uniqueness proof for proving the strong convergence of the approximate solutions, even in the case of a *uniform* time-step partition.

2. The fundamental theorem for Young measures for the weak topology

Notation. Let X be a separable reflexive Banach space. We denote by $\mathcal{B}(X)$ the Borel σ-algebra of X, while $\mathcal{L}$ is the σ-algebra of the Lebesgue measurable subsets of $(0, T)$ ($|\cdot|$ stands for the Lebesgue measure on $(0, T)$), and $\mathcal{L} \otimes \mathcal{B}(X)$ is the product σ-algebra on $(0,T)\times X$. We say that a $\mathcal{L}\otimes\mathcal{B}(X)$-measurable functional $h:(0,T)\times X\to(-\infty,+\infty]$ is a *weakly normal integrand* if

$$v\mapsto h_t(v)=h(t,v) \text{ is sequentially weakly l.s.c. for a.e. } t\in(0,T). \tag{2.1}$$

We also recall that a sequence $\{u_n\}\subset L^1(0,T;X)$ is *uniformly integrable* if

$$\forall\,\varepsilon>0\ \exists\,\delta>0 \qquad \forall\,J\subset(0,T)\ |J|\le\delta \qquad \Rightarrow \sup_{n\in\mathbb{N}}\int_J\|u_n(t)\|\,dt\le\varepsilon. \tag{2.2}$$

Definition 2.1 (Young measures). A *Young measure* (or *parametrized measure*) in X is a family $\boldsymbol{\nu}:=\{\nu_t\}_{t\in(0,T)}$ of Borel probability measures on X such that

$$t\in(0,T)\mapsto\nu_t(B)\quad\text{is}\quad\mathcal{L}\text{-measurable}\quad\forall B\in\mathcal{B}(X). \tag{2.3}$$

We denote by $\mathcal{Y}(0,T;X)$ the set of all Young measures.

The following result, which is a version of the so-called fundamental compactness result for Young measures (see, e.g., [1, Thm.1]) for the *weak topology*, has been proved in [10] (cf. Thm. 3.2 therein).

Theorem 2.2 (The fundamental theorem for weak topologies). *Let $\{v_n\}_{n\in\mathbb{N}}$ be a* bounded *sequence in $L^p(0,T;X)$, for some $p\ge1$; if $p=1$, let $\{v_n\}$ also be uniformly integrable. Then, there exists a subsequence $k\mapsto v_{n_k}$ and a Young measure $\boldsymbol{\nu}=\{\nu_t\}_{t\in(0,T)}\in\mathcal{Y}(0,T;X)$ such that for a.e. $t\in(0,T)$*

$$\nu_t \textit{ is concentrated on the set } L(t):=\bigcap_{p=1}^{\infty}\overline{\{v_{n_k}(t):k\ge p\}}^{w} \tag{2.4}$$

of the weak limit points *of $\{v_{n_k}(t)\}$, and*

$$\liminf_{k\to\infty}\int_0^T h(t,v_{n_k}(t))\,\mathrm{d}t\ge\int_0^T\left(\int_X h(t,\xi)\,\mathrm{d}\nu_t(\xi)\right)\mathrm{d}t \tag{2.5}$$

for every weakly normal *integrand h such that $\{h^-(\cdot,v_{n_k}(\cdot))\}$ is uniformly integrable (h^- denoting the negative part of h). As a consequence, setting $v(t):=\int_X\xi\,\mathrm{d}\nu_t(\xi)$, we have*

$$v_{n_k}\rightharpoonup v \textit{ in } L^p(0,T;X) \textit{ if } p<\infty,\quad v_{n_k}\overset{*}{\rightharpoonup}v \textit{ in } L^\infty(0,T;X). \tag{2.6}$$

Remark 2.3. Note that our L^p-boundedness assumption (together with the uniform integrability for $p=1$) yields by itself that, up to a subsequence, $\{v_n\}$ weakly converges in $L^p(0,T;X)$ (for $p=1$, this is guaranteed by the *Dunford-Pettis criterion* for L^1 vector-valued functions, see, e.g., [3, Thm. IV.2.1]). Hence, (2.6) (which is a straightforward consequence of the general lower semicontinuity inequality (2.5)) has rather to be interpreted as an identification of the weak-L^p limit in terms of a suitable Young measure $\boldsymbol{\nu}$. Such a measure actually retains some information on the *pointwise* (weak) limiting behavior of the sequence, see (2.4).

3. The main result

First of all, let us enlist all the assumptions on the functionals $\mathcal{E}$ and Ψ which will come into play in the proof of our main theorem (i.e., Thm. 3.8) and of the related intermediate results.

Statement of the assumptions

Besides (1.2), we assume that $\mathcal{E}$ complies with

$$\begin{aligned} &\partial_t\mathcal{E}(\cdot,z):[0,T]\to\mathbb{R} \quad \text{is measurable } \forall z\in Z, \text{ and} \\ &\exists C_0>0\ \exists\lambda_0\in L^1(0,T;[0,\infty))\ \forall z\in Z:\ |\partial_t\mathcal{E}(t,z)|\le\lambda_0(t)(\mathcal{E}(t,z)+C_0); \end{aligned} \tag{3.1}$$

$$\begin{aligned} \exists\,\lambda_1\in L^1(0,T;[0,\infty)) \quad \text{for a.e.}\,t\in(0,T), \quad \forall z,\hat z\in Z: \\ |\partial_t\mathcal{E}(t,z)-\partial_t\mathcal{E}(t,\hat z)|\le\lambda_1(t)\|z-\hat z\|; \end{aligned} \tag{3.2}$$

$$\text{for a.e.}\,t\in(0,T) \text{ the map } z\mapsto\partial_t\mathcal{E}(t,z) \text{ is weakly continuous on } Z; \tag{3.3}$$

$$\partial\mathcal{E}\subset[0,T]\times Z\times Z' \text{ is closed in the strong-weak-weak topology.} \tag{3.4}$$

Finally, we require $z\mapsto\mathcal{E}(t,z)$ to be uniformly convex in the z variable, with a modulus of convexity κ independent of $t\in[0,T]$, i.e., (setting $z_\theta:=(1-\theta)z_0+\theta z_1$):

$$\begin{aligned} &\exists\kappa>0 \quad \forall z_0,\ z_1\in Z, \quad \forall t\in[0,T], \quad \forall\theta\in[0,1]: \\ &\mathcal{E}(t,z_\theta)\le(1-\theta)\mathcal{E}(t,z_0)+\theta\mathcal{E}(t,z_1)-\frac{\kappa}{2}\theta(1-\theta)\|z_0-z_1\|^2. \end{aligned} \tag{3.5}$$

Remark 3.1. Indeed, (3.1) ensures that $\mathcal{E}$ is bounded from below and absolutely continuous in time (see also [4, Sect. 3]): namely, $\forall\,t,s\in[0,T]$ and $\forall\,z\in Z$ we have

$$\mathcal{E}(t,z)\ge -C_0, \quad \text{and} \quad \mathcal{E}(t,z)+C_0\le(\mathcal{E}(s,z)+C_0)\exp\left(\Big|\int_s^t\lambda_0(\tau)\,\mathrm{d}\tau\Big|\right). \tag{3.6}$$

Remark 3.2. Under the assumptions (1.2), (3.1) and (3.2) on $\mathcal{E}$, the following chain rule for the subdifferential $\partial\mathcal{E}$ of $\mathcal{E}$ holds (for the proof, see [6, Prop. 2.6]): *for any curve $z\in W^{1,1}(0,T;Z)$ such that there exists a selection g with*

$$g(t)\in\partial\mathcal{E}(t,z(t)) \quad \text{for a.e. } t\in(0,T) \quad \text{and} \quad g\in L^\infty(0,T;Z'), \tag{3.7}$$

then the map $t\mapsto\mathcal{E}(t,z(t))$ is absolutely continuous on $(0,T)$, and for every measurable selection $\zeta(t)\in\partial\mathcal{E}(t,z(t))$ we have the identity

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,z(t))=\langle\zeta(t),\dot z(t)\rangle+\partial_t\mathcal{E}(t,z(t)) \quad \text{for a.e. } t\in(0,T). \tag{3.8}$$

As for the dissipation functional Ψ, in addition to (1.3) we impose:

$$\exists\,C_\Psi>0\ \forall(z,v)\in Z\times Z:\ \Psi(z,v)\le C_\Psi\|v\|; \tag{3.9}$$

$$\begin{aligned} \exists\,\psi^*>0 \quad \forall\,z,\hat z,v,\in Z: \quad |\Psi(z,v)-\Psi(\hat z,v)|\le\psi^*\|v\|\|z-\hat z\|, \\ \text{with} \quad \psi^*<\kappa; \end{aligned} \tag{3.10}$$

$$\Psi: Z\times Z\to[0,\infty) \text{ is sequentially weakly lower semicontinuous;} \tag{3.11}$$

$$\forall v\in Z:\ \Psi(\cdot,v):Z\to[0,\infty) \text{ is sequentially weakly continuous.} \tag{3.12}$$

We refer to [6, Sec. 4.2] for a non trivial example of functionals Ψ and $\mathcal{E}$ complying with all the above assumptions.

Remark 3.3. Loosely speaking, the requirement $\psi^* < \kappa$ in (3.10) means that the variations of Ψ with respect to z are weak enough so that the uniform convexity of $\mathcal{E}$ is able to compensate for them. In fact, a simple example (see [6, Sec. 3]) shows that, when $\psi^* \geq \kappa$, the Cauchy problem for (1.1) might not possess absolutely continuous solutions.

Remark 3.4. It is possible to show (cf. [6, Lemma 4.1]) that, under the present conditions on Ψ, (3.12) is equivalent to the fact that the map $z \mapsto \partial\Psi(z,0)$ has a sequentially closed graph in the weak-weak topology of $Z \times Z'$.

Remark 3.5. Assumption (1.3) has an interesting *geometrical* interpretation: i.e., that for every $z \in Z$ there exists a non-empty, closed and convex set $C(z) \subset Z'$, with $\Psi(z,v) := \sup\{ \langle \sigma, v\rangle \mid \sigma \in C(z) \}$ for all $v \in Z$ (i.e., for all $z \in Z$ $\Psi(z,\cdot)$ is the *support function* of the set $C(z)$). Standard convex analysis results (see [9]) ensure that

$$\partial\Psi(z,v) = \operatorname{argmax}\{ \langle \sigma, v\rangle \mid \sigma \in C(z) \} \subset C(z) = \partial\Psi(z,0) \quad \forall\, v, z \in Z.$$

Besides, it is easy to check that, due to (3.9), for all $z \in Z$

$$C(z) \subset B'_{C_\Psi}(0),$$

$B'_{C_\Psi}(0)$ being the ball of Z' centered at 0, with radius C_Ψ.

Remark 3.6. It has been verified (see [6, Prop. 2.7]) that, under the present assumptions, for a.e. $t \in [0,T]$ the stability condition (S) is equivalent to

$$\partial\Psi(z(t),0) + \partial\mathcal{E}(t,z(t)) \ni 0. \tag{3.13}$$

Approximation

Let us a fix a partition $\mathcal{P}_\tau$ (1.4) of $[0,T]$, and consider the discrete time incremental problem (**IP**), with an initial datum $z_0 \in Z$. In [6] (cf. Lemma 4.4 and Cor. 4.5 therein), it has been proved that, under (some of) the assumptions (1.2)–(1.3) and (3.1)–(3.4), problem (**IP**) admits a unique solution, complying with (1.5)–(1.7), as well as with the following *variational inequality*: $\forall \hat{z} \in Z$

$$\frac{\kappa}{2}\|z_\tau^k - \hat{z}\|^2 \leq \mathcal{E}(t_\tau^k, \hat{z}) - \mathcal{E}(t_\tau^k, z_\tau^k) + \Psi(z_\tau^{k-1}, \hat{z} - z_\tau^{k-1}) - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}). \tag{3.14}$$

Furthermore, note that (1.5) may be rephrased in this way: for all $k = 1, \ldots, N$ there exist $\xi_k \in \partial\mathcal{E}(t_\tau^k, z_\tau^k)$, $\omega_k \in \partial\Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1})$ fulfilling $\xi_k + \omega_k = 0$.

We can now introduce the piecewise constant interpolants $\overline{Z}_\tau$, $\underline{Z}_\tau : [0,T] \to Z$ and the piecewise linear interpolant $\widehat{Z}_\tau : [0,T] \to Z$ of the discrete solutions $\{z_\tau^k\}_{k=0}^N$ of Problem (**IP**), defined by

$$\overline{Z}_\tau(t) := z_\tau^k \text{ for } t \in (t_\tau^{k-1}, t_\tau^k], \quad \underline{Z}_\tau(t) := z_\tau^{k-1} \text{ for } t \in [t_\tau^{k-1}, t_\tau^k),$$
$$\widehat{Z}_\tau(t) = \tfrac{t - t_\tau^{k-1}}{t_\tau^k - t_\tau^{k-1}} z_\tau^k + \tfrac{t_\tau^k - t}{t_\tau^k - t_\tau^{k-1}} z_\tau^{k-1}, \quad \text{for } t \in [t_\tau^{k-1}, t_\tau^k].$$

Analogously, we introduce the (left-continuous piecewise constant) interpolants $\bar{\xi}_\tau$ and $\bar{\omega}_\tau$ of $\{\xi_k\}_{k=1}^N$ and $\{\omega_k\}_{k=1}^N$. Also, let $\bar{\mathsf{t}}_\tau : [0,T] \to [0,T]$ be defined by $\bar{\mathsf{t}}_\tau(0) := 0$ and $\bar{\mathsf{t}}_\tau(t) := t_\tau^k$ for $t \in (t_\tau^{k-1}, t_\tau^k]$. Of course, for every $t \in [0,T]$ we have $\bar{\mathsf{t}}_\tau(t) \downarrow t$ as $\tau \searrow 0$.

Hence, on account of Remark 3.5, (1.5) yields for every $t \in [0,T]$

$$\overline{\xi}_\tau(t) \in \partial\mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{Z}_\tau(t)), \quad \overline{\omega}_\tau(t) \in \partial\Psi(\underline{Z}_\tau(t), 0), \quad \overline{\omega}_\tau(t) + \overline{\xi}_\tau(t) = 0. \tag{3.15}$$

On the other hand, summing up (1.7) on each subinterval of the partition $\mathcal{P}_\tau$, we end up with the energy estimate

$$\int_{\mathsf{s}}^{\mathsf{t}} \Psi(\underline{Z}_\tau(r), \widehat{Z}'_\tau(r))\,\mathrm{d}r + \mathcal{E}(\mathsf{t}, \overline{Z}_\tau(\mathsf{t})) \le \mathcal{E}(\mathsf{s}, \overline{Z}_\tau(\mathsf{s})) + \int_{\mathsf{s}}^{\mathsf{t}} \partial_t\mathcal{E}(r, \underline{Z}_\tau(r))\,\mathrm{d}r \tag{3.16}$$

for every pair of nodes $\mathsf{s}, \mathsf{t} \in \mathcal{P}_\tau$, with $\mathsf{s} < \mathsf{t}$.

A priori estimates

We will always denote by the symbol C any positive constant occurring in the following estimates, without specifying the quantities C may depend on.

Claim 1. *There exists a positive constant C such that for all $\tau > 0$*

$$\max\left\{|\mathcal{E}(t, \overline{Z}_\tau(t))|, |\mathcal{E}(t, \underline{Z}_\tau(t))|\right\} \le C \quad \forall\, t \in [0,T]. \tag{3.17}$$

Indeed, using that $\overline{Z}_\tau(\mathsf{t}) = \underline{Z}_\tau(\mathsf{t})$ for every $\mathsf{t} \in \mathcal{P}_\tau$, taking $\mathsf{s} = 0$ and $\mathsf{t} = \bar{\mathsf{t}}_\tau(t)$ in (3.16) we end up with

$$\mathcal{E}(\bar{\mathsf{t}}_\tau(t), \underline{Z}_\tau(t)) \le \mathcal{E}(0, z_0) + \int_0^{\bar{\mathsf{t}}_\tau(t)} \partial_t\mathcal{E}(r, \underline{Z}_\tau(r))\,\mathrm{d}r.$$

Hence, by (3.1) we have

$$\mathcal{E}(\bar{\mathsf{t}}_\tau(t), \underline{Z}_\tau(\mathsf{t})) \le \mathcal{E}(0, z_0) + \textstyle\int_0^{\bar{\mathsf{t}}_\tau(t)} \lambda_0(r)\,(\mathcal{E}(r, \underline{Z}_\tau(r)) + C_0)\ \mathrm{d}r.$$

Then, by the Gronwall Lemma and the first of (3.6) we conclude

$$0 \le \mathcal{E}(\bar{\mathsf{t}}_\tau(t), \underline{Z}_\tau(\mathsf{t})) + C_0 \le (\mathcal{E}(0, z_0) + C_0)\exp(\textstyle\int_0^{\bar{\mathsf{t}}_\tau(t)} \lambda_0(r)\ \mathrm{d}r),$$

so that the sequence $\{\mathcal{E}(\bar{\mathsf{t}}_\tau(\cdot), \underline{Z}_\tau(\cdot))\}$ is bounded in $L^\infty(0,T)$. Exploiting the second of (3.6), we conclude a L^∞-bound for the sequence $\{\mathcal{E}(\cdot, \underline{Z}_\tau(\cdot))\}$. We argue likewise for $\{\mathcal{E}(\cdot, \overline{Z}_\tau(\cdot))\}$, and (3.17) is proved.

Claim 2. *There exists a positive constant C such that for all $\tau > 0$*

$$\|\overline{Z}_\tau\|_{L^\infty(0,T;Z)},\ \|\underline{Z}_\tau\|_{L^\infty(0,T;Z)},\ \|\widehat{Z}_\tau\|_{L^\infty(0,T;Z)} \le C. \tag{3.18}$$

This estimate is a straightforward consequence of (3.17) and of the uniform convexity assumption (3.5) on $\mathcal{E}$.

Claim 3. *The sequences $\{\bar{\xi}_\tau\}$ and $\{\bar{\omega}_\tau\}$ are bounded in $L^\infty(0,T;Z')$.*
This is due to (3.15) and to (1.3), (3.9): indeed, in view of Remark 3.5 we have

$$\bar{\omega}_\tau(t) \in \partial\Psi(\underline{Z}_\tau(t), 0) \subset B'_{C_\Psi}(0) \quad \forall\, t \in [0,T].$$

Claim 4. *Set* $\Lambda_* := \int_0^T \lambda_1(s)\,\mathrm{d}s$, *and, for every* $N \geq 1$, *let* $\mathcal{P}_{\tau_N}$ *be the partition:*

$$0 = t_\tau^0 < \cdots \leq t_\tau^k \leq \cdots < t_\tau^N = T, \quad \int_0^{t_\tau^k} \lambda_1(s)\,ds = k\Lambda_*/N, \quad k = 1, \ldots, N. \tag{3.19}$$

Then, the following estimates hold:

$$\|\widehat{Z}_{\tau_N} - \overline{Z}_{\tau_N}\|_{L^\infty(0,T;Z)} \leq \|\overline{Z}_{\tau_N} - \underline{Z}_{\tau_N}\|_{L^\infty(0,T;Z)} \leq \frac{\Lambda_*}{(\kappa - \psi^*)N} \quad \forall\, N \geq 1. \tag{3.20}$$

The first inequality in (3.20) ensues from trivial calculations. As for the second one, we first choose $\hat{z} := z_\tau^{k-1}$ in (3.14), thus obtaining

$$\tfrac{\kappa}{2}\|z_\tau^k - z_\tau^{k-1}\|^2 \leq \mathcal{E}(t_\tau^k, z_\tau^{k-1}) - \mathcal{E}(t_\tau^k, z_\tau^k) - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}). \tag{3.21}$$

Secondly, we write (3.14) at the $(k-1)$th step and plug in $\hat{z} := z_\tau^k$. Adding the resulting inequality and (3.21), and exploiting (3.1)–(3.2), (3.5), (3.9)–(3.10) (see [6, Prop. 4.7] for details), we get

$$\begin{aligned}
&\kappa\|z_\tau^k - z_\tau^{k-1}\|^2 \\
&\leq \mathcal{E}(t_\tau^k, z_\tau^{k-1}) - \mathcal{E}(t_\tau^k, z_\tau^k) + \mathcal{E}(t_\tau^{k-1}, z_\tau^k) - \mathcal{E}(t_\tau^{k-1}, z_\tau^{k-1}) \\
&\quad + \Psi(z_\tau^{k-2}, z_\tau^k - z_\tau^{k-1}) - \Psi(z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) \\
&\leq \int_{t_\tau^{k-1}}^{t_\tau^k} (\partial_t\mathcal{E}(\tau, z_\tau^{k-1}) - \partial_t\mathcal{E}(\tau, z_\tau^k))\,d\tau + \psi^*\|z_\tau^{k-1} - z_\tau^{k-2}\|\|z_\tau^k - z_\tau^{k-1}\| \\
&\leq \|z_\tau^k - z_\tau^{k-1}\| \int_{t_\tau^{k-1}}^{t_\tau^k} \lambda_1(\tau)\,d\tau + \psi^*\|z_\tau^{k-1} - z_\tau^{k-2}\|\|z_\tau^k - z_\tau^{k-1}\|.
\end{aligned}$$

On the other hand, by (3.19) we have $\int_{t_\tau^{k-1}}^{t_\tau^k} \lambda_1(\tau)\,d\tau = \Lambda_*/N$. Thus, dividing both sides of the above inequality by $\kappa\delta_k$, with $\delta_k := \|z_\tau^k - z_\tau^{k-1}\|$ for $k = 1, \ldots, N$ and $\delta_0 := 0$, we end up with the recurrence relation

$$\delta_k \leq \frac{\Lambda_*}{\kappa N} + \frac{\psi^*}{\kappa}\delta_{k-1} \quad \forall k = 1, \ldots, N,$$

and the second inequality in (3.20) easily follows.

Claim 5. *Assume* (3.19). *Then, the sequence* $\{\widehat{Z}'_{\tau_N}\}_N \subset L^1(0,T;Z)$ *is bounded and uniformly integrable.*

As a consequence of (3.20), we have for all $k = 1, \ldots, N$

$$\int_{t_\tau^{k-1}}^{t_\tau^k} \|\widehat{Z}'_{\tau_N}(s)\|\,\mathrm{d}s \leq \frac{\Lambda_*}{(\kappa - \psi^*)N} = \frac{1}{\kappa - \psi^*}\int_{t_\tau^{k-1}}^{t_\tau^k} \lambda_1(s)\,\mathrm{d}s, \tag{3.22}$$

and the first assertion is proved upon summing on the subintervals of the partition. To check the uniform integrability, we preliminarily note that, for any fixed $N \geq 1$, and for any $s, t \in [0,T]$, with $t_{h-1} < s < t_h \le t_{j-1} \le t < t_j$, there holds

$$\begin{aligned}\int_s^t \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r &\le \int_{t_{h-1}}^{t_h} \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r + \int_{t_h}^{t_{j-1}} \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r + \int_{t_{j-1}}^{t_j} \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r \\ &\le \frac{2\Lambda_*}{(\kappa - \psi^*)N} + \int_{t_h}^{t_{j-1}} \lambda_1(r) \,\mathrm{d}r \le \frac{2\Lambda_*}{(\kappa - \psi^*)N} + \int_s^t \lambda_1(r) \,\mathrm{d}r,\end{aligned}$$

in view of (3.22). Then, let us fix $\varepsilon > 0$, and hence $N_* > 1$ such that, for $N > N_*$, $2\Lambda_*/((\kappa - \psi^*)N) < \varepsilon/4$. Let us also pick $\delta_* > 0$ such that

$$s, t \in [0,T],\ |s-t| \le \delta_* \Longrightarrow \begin{cases} \int_s^t \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r \le \frac{\varepsilon}{4} & \text{for } 1 \le N \le N_*, \\ \int_s^t \lambda_1(r) \,\mathrm{d}r \le \frac{\varepsilon}{4}. \end{cases}$$

Combining the two estimates above, we finally end up with

$$\begin{aligned}|t-s| \le \delta_* \ \Rightarrow\ \sup_N \int_s^t \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r &\le \sup_{N \le N_*} \int_s^t \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r \\ &\quad + \sup_{N > N_*} \int_s^t \|\widehat{Z}'_{\tau_N}(r)\| \,\mathrm{d}r \le \frac{3\varepsilon}{4}.\end{aligned}$$

Remark 3.7. Let us point out that Claims 1–3 hold for *any* variable time-step partition $\mathcal{P}_\tau$, whereas we have been able to prove Claims 4 and 5 just for the partition $\mathcal{P}_{\tau_N}$ defined by (3.19), which is in fact tailored to the function λ_1, cf. (3.2). On the other hand, it can be proved that, if $\lambda_1 \in L^\infty(0,T)$ and if *uniform* time-step partitions are considered, then a stronger version of (3.20) holds, namely

$$\|\widehat{Z}'_\tau\|_{L^\infty(0,T;Z)} \le \frac{\|\lambda_1\|_{L^\infty(0,T)}}{\kappa - \psi^*} \quad \text{for } \tau > 0,$$

see [6, Prop. 4.8]

Our existence and approximation result

Theorem 3.8. *Assume* (1.2)–(1.3)*,* (3.1)–(3.5)*,* (3.9)–(3.12)*. Let $z_0 \in Z$ fulfil the stability condition*

$$\mathcal{E}(0, z_0) \le \mathcal{E}(0, \hat{z}) + \Psi(z_0, \hat{z} - z_0) \quad \forall \hat{z} \in Z. \tag{3.23}$$

Then, the sequence of partitions $\{\mathcal{P}_{\tau_N}\}_N$ of $[0,T]$ defined by (3.19) *admits a subsequence* (*not relabeled*)*, for which there exists a curve $z \in W^{1,1}(0,T;Z)$ such that $z(0) = z_0$, the following convergences hold as $N \nearrow \infty$:*

$$\forall t \in [0,T]: \quad \widehat{Z}_{\tau_N}(t),\ \overline{Z}_{\tau_N}(t),\ \underline{Z}_{\tau_N}(t) \rightharpoonup z(t) \quad \text{in } Z, \tag{3.24}$$

$$\widehat{Z}'_{\tau_N} \rightharpoonup \dot{z} \quad \text{in } L^1(0,T;Z). \tag{3.25}$$

Moreover, z fulfils the energetic formulation (S)–(E) *of* (1.1) *for all $t \in [0,T]$.*

Proof. It follows from Claim 5 that the sequence $\{\widehat{Z}_{\tau_N}\} \subset \mathrm{C}^0([0,\mathrm{T}];\mathrm{Z})$ is equicontinuous. Hence, the estimate (3.18), combined with the Ascoli-Arzelà compactness theorem in the framework of the weak topology of (the reflexive space) Z, ensures that there exists a limit curve $z \in L^\infty(0,T;Z)$ such that, along a subsequence, the convergence for $\widehat{Z}_{\tau_N}$ in (3.24) holds. We trivially have $z(0) = z_0$. Thanks to (3.20), we also deduce the convergences for $\overline{Z}_{\tau_N}$ and $\underline{Z}_{\tau_N}$. Besides, owing to Claim 5 and the Dunford-Pettis criterion [3, Thm. IV.2.1], we conclude that $z \in W^{1,1}(0,T;Z)$ and that (3.25) holds up to a further extraction.

Proof of (S). In view of Theorem 2.2 and of Claim 3, up to a further extraction the sequences $\{\bar{\xi}_{\tau_N}\}$ and $\{\bar{\omega}_{\tau_N}\}$ have two limiting Young measures $\{\zeta_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;Z')$ and, respectively, $\{\mu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;Z')$ such that for a.e. $t \in (0,T)$ the measure ζ_t (μ_t, respectively) is concentrated on the set $L_\xi(t)$ ($L_\omega(t)$, resp.) of the weak limit points of the sequence $\{\overline{\xi}_{\tau_N}(t)\}$ ($\{\overline{\omega}_{\tau_N}(t)\}$, resp.). Further,

$$\begin{array}{llll} \bar{\xi}_{\tau_N} \overset{*}{\rightharpoonup} \xi & \text{in } L^\infty(0,T;Z'), & \text{with } \xi(t) := \int_{Z'} v \, d\zeta_t(v) & \text{for a.e. } t \in (0,T), \\ \bar{\omega}_{\tau_N} \overset{*}{\rightharpoonup} \omega & \text{in } L^\infty(0,T;Z'), & \text{with } \omega(t) := \int_{Z'} v \, d\mu_t(v) & \text{for a.e. } t \in (0,T). \end{array}$$

Passing to the limit in (3.15), we have

$$\xi(t) + \omega(t) = 0 \quad \text{for a.e. } t \in (0,T). \tag{3.26}$$

On the other hand, thanks to (3.4), (3.11)–(3.12), Remark 3.4, (3.15) and (3.24), we readily conclude that

$$L_\xi(t) \subset \partial\mathcal{E}(t, z(t)), \quad L_{\omega(t)} \subset \partial\Psi(z(t), 0) \quad \text{for a.e. } t \in (0,T).$$

Thus, a convexity argument yields that $\xi(t) \in \partial\mathcal{E}(t, z(t))$ and $\omega(t) \in \partial\Psi(z(t), 0)$ for a.e. $t \in (0,T)$. Combining this with (3.26), we end up with (3.13). Therefore, by Remark 3.6, (S) holds for a.e. $t \in (0,T)$, hence for all $t \in [0,T]$ due to (3.23) and by a continuity argument. By the way, let us stress that the above computations yield that there exists $\xi \in L^\infty(0,T;Z')$ fulfilling $\xi(t) \in \partial\mathcal{E}(t, z(t)) \cap (-\partial\Psi(z(t), 0))$ for a.e. $t \in (0,T)$. Recalling that $\Psi(z(t), 0) = 0$, we conclude that

$$\Psi(z(r), \dot{z}(r)) + \langle \xi(r), \dot{z}(r) \rangle \geq 0 \quad \text{for a.e. } r \in (0,t), \tag{3.27}$$

Proof of (E). First, we establish the one-sided estimate

$$\int_0^t \Psi(z(r), \dot{z}(r)) \, \mathrm{d}r + \mathcal{E}(t, z(t)) \leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(r, z(r)) \, \mathrm{d}r \tag{3.28}$$

by passing to the limit as $N \nearrow \infty$ in the discrete energy inequality

$$\begin{aligned} \int_0^{\bar{\mathfrak{t}}_{\tau_N}(t)} \Psi(\underline{Z}_{\tau_N}(r), \widehat{Z}'_{\tau_N}(r)) \, \mathrm{d}r + \mathcal{E}(\bar{\mathfrak{t}}_{\tau_N}(t), \overline{Z}_{\tau_N}(\bar{\mathfrak{t}}_{\tau_N}(t))) \\ \leq \mathcal{E}(0, z_0) + \int_0^{\bar{\mathfrak{t}}_{\tau_N}(t)} \partial_t \mathcal{E}(r, \underline{Z}_{\tau_N}(r)) \, \mathrm{d}r \end{aligned} \tag{3.29}$$

for all $t \in [0,T]$. Owing to the assumptions (3.1)–(3.3), it is easy to pass to the limit in the terms involving $\mathcal{E}$ and $\partial_t\mathcal{E}$, see the proof of [6, Thm. 4.6] for further details. In fact, we end up with

$$\partial_t \mathcal{E}(\cdot, \underline{Z}_{\tau_N}(\cdot)) \to \partial_t \mathcal{E}(\cdot, z(\cdot)) \quad \text{in } L^1(0,T), \tag{3.30}$$

$$\liminf_{N\to\infty} (\mathcal{E}(\bar{\mathfrak{t}}_{\tau_N}(t), \overline{Z}_\tau(\bar{\mathfrak{t}}_{\tau_N}(t)) \geq \mathcal{E}(t, z(t))), \tag{3.31}$$

In order to pass to the limit in the first integral term on left-hand side of (3.29), we remark that, by Claims 2 and 5, the sequence $\{(\underline{Z}_{\tau_N}, \widehat{Z}'_{\tau_N})\}$ is bounded in

$L^1(0,T;Z\times Z)$. Thus, applying Theorem 2.2 in the space $X := Z\times Z$, we conclude that, up to a subsequence, $(\underline{Z}_{\tau_N}, \widehat{Z}'_{\tau_N})$ generates a limiting Young measure $\{\nu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;Z\times Z)$. Recalling that Ψ is a weakly normal integrand (cf. Section 2) on the space $(0,T)\times Z\times Z$, we thus obtain

$$\liminf_{N\to\infty}\int_0^{\bar{\mathrm{t}}_{\tau_N}(t)} \Psi(\underline{Z}_\tau(r), \widehat{Z}'_{\tau_N}(r))\,\mathrm{d}r \geq \int_0^t \left(\int_{Z\times Z} \Psi(z,v)\,\mathrm{d}\nu_r(z,v)\right)\mathrm{d}r. \tag{3.32}$$

On the other hand, in view of (3.24), (3.25), (2.4) and (2.6), for a.e. $t\in(0,T)$ we have $\nu_t = \delta_{z(t)}\otimes\sigma_t$, with $\{\sigma_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;Z)$ and $\dot{z}(t) = \int_Z v\,\mathrm{d}\sigma_t(v)$ for a.e. $t\in(0,T)$. Therefore, also by the Jensen inequality we conclude

$$\begin{aligned}\int_0^t \left(\int_{Z\times Z} \Psi(z,v)\,\mathrm{d}\nu_r(z,v)\right)\mathrm{d}r &= \int_0^t \left(\int_Z \Psi(z(r),v)\,\mathrm{d}\sigma_r(v)\right)\mathrm{d}r \\ &\geq \int_0^t \Psi(z(r),\dot{z}(r))\,\mathrm{d}r. \end{aligned}\tag{3.33}$$

Collecting (3.30), (3.31), (3.32) and (3.33), we conclude (3.28). To obtain the opposite inequality, we combine (3.27) with the chain rule formula (3.8) (see Remark 3.2), applied to the L^∞-selection ξ of $\partial\mathcal{E}(\cdot,z(\cdot))$ previously retrieved. Therefore, we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,z(t)) + \Psi(z(t),\dot{z}(t)) \geq \partial_t\mathcal{E}(t,z(t)) \text{ for a.e. } t\in(0,T).$$

Integration of this inequality leads to the opposite estimate in (3.28), and (E) ensues. □

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