# From Visco-Energetic to Energetic and Balanced Viscosity solutions of rate-independent systems

Riccarda Rossi and Giuseppe Savaré

*Dedicated to Gianni Gilardi on the occasion of his 70th birthday*

**Abstract** This paper focuses on weak solvability concepts for rate-independent systems in a metric setting. *Visco-Energetic* solutions have been recently obtained by passing to the time-continuous limit in a time-incremental scheme, akin to that for Energetic solutions, but perturbed by a 'viscous' correction term, as in the case of Balanced Viscosity solutions. However, for Visco-Energetic solutions this viscous correction is tuned by a *fixed* parameter $\mu$. The resulting solution notion is characterized by a stability condition and an energy balance analogous to those for Energetic solutions, but, in addition, it provides a fine description of the system behavior at jumps as Balanced Viscosity solutions do. Visco-Energetic evolution can be thus thought as 'in-between' Energetic and Balanced Viscosity evolution. Here we aim to formalize this intermediate character of Visco-Energetic solutions by studying their singular limits as $\mu \downarrow 0$ and $\mu \uparrow \infty$. We shall prove convergence to Energetic solutions in the former case, and to Balanced Viscosity solutions in the latter situation.

## 1 Introduction

A large class of *rate-independent systems* are driven by

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- a time-dependent energy functional $\mathscr{E} : [0,T] \times X \to (-\infty,\infty]$, with $[0,T]$ the time span during which the system is observed, and $X$ the space of the states of the system,
- a (positive) dissipation functional $\mathscr{D} : X \times X \to [0,\infty)$, keeping track of the energy dissipated by the curve $u : [0,T] \to X$ describing the evolution of the system, that satisfies suitable structural properties peculiar of rate-independence.

When $X$ is a (separable) Banach space, a natural class of dissipations is provided by translation invariant functionals of the form $\mathscr{D}(u_1,u_2) := \Psi(u_2 - u_1)$, where $\Psi : X \to [0,\infty)$ is a (convex, lower semicontinuous) dissipation potential, positively homogeneous of degree 1, namely $\Psi(\lambda v) = \lambda\Psi(v)$ for all $\lambda \geq 0$ and $v \in X$. The evolution of the rate-independent system is governed by the doubly nonlinear differential inclusion

$$\partial\Psi(u'(t)) + \partial_u\mathscr{E}(t,u(t)) \ni 0 \quad \text{in } X^* \quad \text{for a.a. } t \in (0,T), \tag{1.1}$$

where $\partial\Psi : X \rightrightarrows X^*$ is the subdifferential in the sense of convex analysis, while $\partial_u\mathscr{E} : [0,T] \times X \rightrightarrows X^*$ is a suitable notion of subdifferential of $\mathscr{E}$ w.r.t. the variable $u$. As it will be apparent from the forthcoming discussion, in general (1.1) is only formally written.

More generally, throughout this paper we shall assume that the dissipation $\mathscr{D}$ is induced by a distance $\mathsf{d}$ on the space $X$, such that

$$(X,\mathsf{d}) \text{ is a } \textit{complete} \text{ metric space.} \tag{X}$$

We will henceforth denote a (metric) rate-independent system by $(X,\mathscr{E},\mathsf{d})$.

Rate-independent evolution occurs in manifold problems in physics and engineering, cf. [9, 10] for a survey. In addition to its wide range of applicability, over the last two decades the analysis of rate-independent systems has attracted considerable interest due to its intrinsic mathematical challenges: first and foremost, the quest of a proper solvability concept for the system $(X,\mathscr{E},\mathsf{d})$. In fact, since the dissipation potential has linear growth at infinity, one can in general expect only BV-time regularity for the curve $u$ (unless the energy functional is uniformly convex). Thus $u$ may have jumps as a function of time. Therefore, the pointwise derivative $u'$ in the subdifferential inclusion (1.1) in the Banach setting, and the metric derivative $|u'|$ in the general metric setup (X), need not be defined. This calls for a suitable weak formulation of rate-independent evolution, also able to satisfactorily capture the behavior of the system in the jump regime.

In what follows we illustrate the three solution concepts this paper is concerned with, referring to Sections 2 and 3 for more details and precise statements.

## 1.1 Energetic, Balanced Viscosity, and Visco-Energetic solutions

The pioneering papers [16, 17] advanced the by now classical concept of (Global) Energetic solution to the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ (cf. also the notion of 'quasistatic evolution' in the realm of crack propagation, dating back to [4]), which can be in fact given in a more general topological setting [8]. It is a curve $u : [0,T] \to X$ with bounded variation, complying for every $t \in [0,T]$ with

- the global stability condition

$$\mathscr{E}(t,u(t)) \le \mathscr{E}(t,v) + \mathsf{d}(u(t),v) \quad \text{for every } v \in X, \tag{$\mathrm{S_d}$}$$

- the energy balance

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d}}(u,[0,t]) = \mathscr{E}(0,u(0)) + \int_0^t \partial_t \mathscr{E}(s,u(s))\,\mathrm{d}s. \tag{$\mathrm{E_d}$}$$

Here, $\mathrm{Var}_{\mathsf{d}}(u,[0,t])$ denotes the (pointwise) total variation of the curve $u$ induced by the metric $\mathsf{d}$, which is related to 'energy dissipation': in fact, ($\mathrm{E_d}$) balances the stored energy at the process time $t$ and the energy dissipated up to $t$ with the initial energy and the work of the external loadings, encoded in the second integral on the right-hand side. Existence results for Energetic solutions may be proved by resorting to a well understood time discretization procedure. Indeed, for every fixed partition $\mathscr{T}_\tau := \{t_\tau^0 = 0 < t_\tau^1 < \ldots < t_\tau^{N-1} < t_\tau^N = T\}$ of the interval $[0,T]$, with fineness $\tau := \max_{i=1,\ldots,N}(t_\tau^i - t_\tau^{i-1})$, discrete solutions $(U_n^\tau)_{n=1}^N$ are constructed as solutions of the time-incremental minimization scheme

$$\min_{U \in X} \left( \mathscr{E}(t_\tau^n, U) + \mathsf{d}(U_\tau^{n-1}, U) \right). \tag{$\mathrm{IM}_\tau$}$$

Under suitable conditions it can be shown that, for every null sequence $(\tau_k)_k$, up to a subsequence the piecewise constant interpolants $(\overline{U}_{\tau_k})_k$ of the discrete solutions converge to an Energetic solution. While widely applied, the Energetic concept has also been criticized on the grounds that the global stability condition ($\mathrm{S_d}$) is too strong a requirement, when dealing with nonconvex energies. To avoid violating it, the system may in fact have to change instantaneously in a very drastic way, jumping into very far-apart energetic configurations, possibly 'too early'. In this connection, we refer to the discussions from [6, Ex. 6.3], [11, Ex. 6.1], as well as to [21], providing a characterization of Energetic solutions to one-dimensional rate-independent systems (i.e., with $X = \mathbb{R}$), driven by a fairly broad class of nonconvex energies. In [21], the input-output relation associated with the Energetic concept is shown to be related to the so-called *Maxwell rule* for hysteresis processes [22]. These features are also reflected in the jump conditions satisfied by an Energetic solution $u$ at every jump point $t \in \mathrm{J}_u$ ($u(t-)$, $u(t+)$ denoting the left/right limits of $u$ at $t$ and $\mathrm{J}_u$ its jump set), namely

$$\begin{aligned} \mathsf{d}(u(t-),u(t)) &= \mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t)), \\ \mathsf{d}(u(t),u(t+)) &= \mathscr{E}(t,u(t)) - \mathscr{E}(t,u(t+)), \end{aligned} \tag{1.2}$$

which show the influence of the global energy landscape of $\mathscr{E}$.

The global stability condition ($\mathrm{S}_\mathsf{d}$) in fact stems from the global minimization problem ($\mathrm{IM}_\tau$), whereas a scheme based on *local* minimization would be preferable, cf. [3] for a first discussion of this in the realm of crack propagation, and [5] in the frame of abstract (finite-dimensional) rate-independent systems. This localization can be achieved by perturbing the variational scheme ($\mathrm{IM}_\tau$) with a term, modulated by a viscosity parameter $\varepsilon$, which penalizes the squared distance from the previous step $U_\tau^{n-1}$. One is thus led to consider the time-incremental minimization

$$\min_{U\in X}\left(\mathscr{E}(t_\tau^n,U)+\mathsf{d}(U_\tau^{n-1},U)+\frac{\varepsilon}{2\tau}\mathsf{d}^2(U_\tau^{n-1},U)\right), \tag{$\mathrm{IM}_{\varepsilon,\tau}$}$$

which may be considered as a *viscous* approximation of ($\mathrm{IM}_\tau$). For fixed $\varepsilon > 0$, the limit passage as $\tau \downarrow 0$ in ($\mathrm{IM}_{\varepsilon,\tau}$) leads to solutions (of the metric formulation) of the Generalized Gradient System $(X,\mathscr{E},\mathsf{d},\psi_\varepsilon)$, where the dissipation function $\psi_\varepsilon : [0,\infty) \to [0,\infty)$ is given by

$$\psi_\varepsilon(r) = r + \frac{\varepsilon}{2}r^2 = \frac{1}{\varepsilon}\psi(\varepsilon r) \qquad \text{with} \quad \psi(r) = r + \frac{1}{2}r^2. \tag{1.3}$$

We refer to [20] for existence results for gradient systems in metric spaces, driven by dissipation potentials with superlinear growth at infinity like $\psi_\varepsilon$. In turn, it has been shown in [11] (cf. also [13]) that, under suitable conditions on the energy functional, time-continuous solutions (to the metric formulation) of $(X,\mathscr{E},\mathsf{d},\psi_\varepsilon)$ converge as $\varepsilon \downarrow 0$, up to reparamerization, to a <u>*Balanced Viscosity*</u> (BV) solution of the rate-independent system $(X,\mathscr{E},\mathsf{d})$. The latter is a curve $u \in \mathrm{BV}([0,T];X)$ satisfying

- the *local* stability condition

$$|\mathrm{D}\mathscr{E}|(t,u(t)) \leq 1 \quad \text{for every } t \in [0,T]\setminus \mathrm{J}_u, \tag{$\mathrm{S}_{\mathsf{d},\mathrm{loc}}$}$$

- the energy balance

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[0,t]) = \mathscr{E}(0,u(0)) + \int_0^t \partial_t \mathscr{E}(s,u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T]. \tag{$\mathrm{E}_{\mathsf{d},\mathsf{v}}$}$$

Here, $|\mathrm{D}\mathscr{E}| : [0,T]\times X \to [0,\infty]$ is the *metric slope* of the energy functional $\mathscr{E}$, namely

$$|\mathrm{D}\mathscr{E}|(t,u) := \limsup_{v\to u} \frac{(\mathscr{E}(t,u)-\mathscr{E}(t,v))_+}{\mathsf{d}(u,v)}, \tag{1.4}$$

and $\mathrm{Var}_{\mathsf{d},\mathsf{v}}$ is a suitably *augmented* notion of total variation, fulfilling $\mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[a,b]) \geq \mathrm{Var}_\mathsf{d}(u,[a,b])$ for all $[a,b] \subset [0,T]$, which measures the energy dissipated along the jump, at a point $t \in \mathrm{J}_u$, by means of the cost

$$\mathsf{v}(t,u(t-),u(t+)) := \inf\Big\{ \int_{r_0}^{r_1} |\vartheta'|(r)\,(|\mathrm{D}\mathscr{E}|(t,\vartheta(r))\vee 1)\,\mathrm{d}r : \\ \vartheta \in \mathrm{AC}([r_0,r_1];X),\ \vartheta(r_0)=u(t-),\ \vartheta(r_1)=u(t+)\Big\} \tag{1.5}$$

that is reminiscent of the viscous approximation ($\mathrm{IM}_{\varepsilon,\tau}$). Indeed, it is possible to show (cf. (1.6) ahead) that every BV solution to $(X,\mathscr{E},\mathsf{d})$ complies with the jump conditions

$$\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t+)) = \mathsf{v}(t,u(t-),u(t+)) \\ = \int_{r_0}^{r_1} |\vartheta'|(r)\,(|\mathrm{D}\mathscr{E}|(t,\vartheta(r))\vee 1)\,\mathrm{d}r \tag{1.6}$$

at every jump point $t \in \mathrm{J}_u$, with $\vartheta$ an optimal jump transition between $u(t-)$ and $u(t+)$. Any optimal transition can be decomposed into an (at most) countable collection of *sliding* transitions, evolving in the rate-independent mode, and *viscous* transitions, i.e. (metric) solutions of the Generalized Gradient System $(X,\mathscr{E},\mathsf{d},\psi)$ with the superlinear $\psi$ from (1.3), and where the time variable in the energy functional is frozen at the jump time $t$. Therefore, BV solutions account for the onset of viscous behavior at jumps of the system, which can be in fact interpreted as fast transitions (possibly) governed by viscosity. The characterization in the one-dimensional case, with a nonconvex driving energy, from [21] reveals that the input-output relation underlying BV solutions follows the *delay rule* [22], as they tend to jump 'as late as possible'.

A notable feature of BV solutions is that they can be directly obtained as limits of the discrete solutions arising from the perturbed scheme ($\mathrm{IM}_{\varepsilon,\tau}$), when the parameters $\varepsilon$ and $\tau$ *jointly* tend to zero with convergence rates such that

$$\lim_{\varepsilon,\tau\downarrow 0} \frac{\varepsilon}{\tau} = +\infty; \tag{1.7}$$

the argument developed in [12, 15] in the Banach setting can be in fact easily extended to the metric framework, cf. the discussion in Sec. 3.1. This remarkable property has somehow inspired the approach in [18]. There, a new notion of rate-independent evolution has been obtained in the time-continuous limit, as $\tau \downarrow 0$, of the perturbed time-incremental minimization scheme

$$\min_{U\in X}\Big(\mathscr{E}(t_\tau^n,U) + \mathsf{d}(U_\tau^{n-1},U) + \frac{\mu}{2}\mathsf{d}^2(U_\tau^{n-1},U)\Big) \quad \text{with } \mu>0 \textit{ fixed} \tag{$\mathrm{IM}_\mu$}$$

as a parameter. The analysis carried out in [18] in fact covers a more general, topological setting, akin to that of [8], with a general viscous correction $\delta : X\times X \to [0,\infty)$ compatible, in a suitable sense, with the metric d: a particular case is in fact $\delta(u,v) = \frac{\mu}{2}\mathsf{d}^2(u,v)$ as in ($\mathrm{IM}_\mu$). In the simplified metric setting of (X), under the same conditions ensuring the existence of Energetic solutions it is possible to show that the (piecewise constant interpolants of the) discrete solutions arising from ($\mathrm{IM}_\mu$) converge, as $\tau\downarrow 0$ and $\mu>0$ is fixed, to a *($\mu$-)Visco-Energetic* solution to the rate-independent system $(X,\mathscr{E},\mathsf{d})$. In what follows, we will simply speak of *Visco-Energetic* (VE) solutions, and often highlight their dependence on the parameter $\mu$ in the acronym $\mathrm{VE}_\mu$. A $\mathrm{VE}_\mu$ solution is a curve $u \in \mathrm{BV}([0,T];X)$ complying with the

- 'perturbed', still global, stability condition

$$\mathscr{E}(t,u(t)) \le \mathscr{E}(t,v) + \mathsf{d}(u(t),v) + \frac{\mu}{2}\mathsf{d}^2(u(t),v) \quad \text{for every } v \in X \text{ and for every } t \in [0,T]\setminus \mathrm{J}_u, \tag{$\mathrm{S_D}$}$$

- the energy balance

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{c}}(u,[0,t]) = \mathscr{E}(0,u(0)) + \int_0^t \partial_t\mathscr{E}(s,u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T]. \tag{$\mathrm{E_{d,c}}$}$$

Here, $\mathrm{Var}_{\mathsf{d},\mathsf{c}}$ is an alternative augmented total variation functional, again estimating the total variation induced by $\mathsf{d}$, but featuring a different notion of jump dissipation cost. In analogy with (1.5), the visco-energetic cost $\mathsf{c}$ (we shall often write $\mathsf{c}_\mu$ to highlight its dependence on the parameter $\mu$, and accordingly write ($\mathrm{E}_{\mathsf{d},\mathsf{c}_\mu}$)), is still obtained by minimizing a suitable transition cost $\mathrm{Trc}_{\mathrm{VE}}$ over a class of continuous, but not necessarily absolutely continuous, curves $\vartheta : E \to X$, with $E$ an arbitrary compact subset of $\mathbb{R}$ having a possibly more complicated structure than that of an interval. The transition cost $\mathrm{Trc}_{\mathrm{VE}}$ evaluates (1) the $\mathsf{d}$-total variation $\mathrm{Var}_{\mathsf{d}}(\vartheta,E)$ of $\vartheta$ over $E$; (2) a quantity related to the "gaps" of the set $E$; (3) a quantity measuring the violation of the (global) stability condition ($\mathrm{S_D}$) along the jump transition $\vartheta$, cf. [18] and Sec. 2.2 ahead for all details and precise formulae. In this context as well, it can be proved (cf. [18, Prop. 3.8]) that any VE solution $u$ satisfies at its jump points $t \in \mathrm{J}_u$ the jump conditions

$$\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t+)) = \mathsf{c}(t,u(t-),u(t+)) = \mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E) \tag{1.8}$$

with $\vartheta : E \to X$ an optimal transition curve between $u(t-)$ and $u(t+)$. Furthermore, any optimal transition can be decomposed into an (at most countable) collection of *sliding transitions*, parameterized by a continuous variable and fulfilling the stability condition ($\mathrm{S_D}$), and *pure jump transitions*, defined on discrete subsets of $E$, along which the stability ($\mathrm{S_D}$) may be violated. A notable property of VE solutions is that, if an optimal jump transition $\vartheta : E \to X$ at a jump point $t$ does not comply with the stability condition ($\mathrm{S_D}$) at some $s \in E$, then $s$ is isolated and, denoting by $s_- := \max(E \cap (-\infty,s))$, there holds

$$\vartheta(s) \in \mathrm{Argmin}_{y\in X}\left\{\mathscr{E}(t,y) + \mathsf{d}(\vartheta(s_-),y) + \frac{\mu}{2}\mathsf{d}^2(\vartheta(s_-),y)\right\}.$$

A complete characterization of VE solutions to one-dimensional rate-independent systems has been recently provided in [19], showing that their behavior strongly depends on the parameter $\mu$. When $\mu = 0$, VE solutions coincide with Energetic solutions and therefore they satisfy the *Maxwell rule*. For a sufficiently 'strong' viscous correction, i.e. with $\mu$ above a certain threshold depending on the (nonconvex) driving energy, VE solutions exhibit a behavior akin to that of BV solutions, and follow the *delay rule*. With a 'weak' correction, VE solutions have an intermediate character between Energetic and BV solutions.

## *1.2 Main results*

In this paper, we aim to gain further insight into this in-between quality of VE solutions and into the role of the tuning parameter $\mu$, revealed by the analysis in [19], in a more general context. To this end, we shall study the singular limits of $\mathrm{VE}_\mu$ solutions to the (metric) rate-independent system $(X,\mathscr{E},\mathsf{d})$ as $\mu \downarrow 0$ and $\mu \uparrow \infty$.

With **Theorem 1** we will show that, any sequence $(u_n)_n$ of $\mathrm{VE}_{\mu_n}$ solutions corresponding to a null sequence $\mu_n \downarrow 0$ converges, up to a subsequence, to an Energetic solution of $(X,\mathscr{E},\mathsf{d})$. **Theorem 2** will address the behavior of a sequence $(u_n)_n$ of $\mathrm{VE}_{\mu_n}$ solutions with parameters $\mu_n \uparrow \infty$. In this case, in accordance with condition (1.7), we expect to obtain BV solutions. We will prove indeed that, up to a subsequence, as $\mu_n \uparrow \infty$ $\mathrm{VE}_{\mu_n}$ solutions converge to a BV solution of $(X,\mathscr{E},\mathsf{d})$.

While referring to Sections 4 and 5 for further comments and all details, let us mention here that the proof of Thm. 2 is quite challenging. In fact, it involves passing from the transitions that describe the jump behavior of a sequence of $\mathrm{VE}_{\mu_n}$ solutions, and that are given by a collection of 'sliding pieces' and discrete trajectories, to the jump transitions for BV solutions, that are instead *absolutely continuous* curves. This can be achieved by means of a careful reparameterization technique, combined with a delicate compactness argument for transition curves in *varying* domains.

#### Plan of the paper

In Section 2 we collect some preliminary results, set up the basic assumptions on the energy functional $\mathscr{E}$, and give the precise definitions of Energetic, Balanced Viscosity, and Visco-Energetic solutions to the rate-independent system $(X,\mathscr{E},\mathsf{d})$. In Section 3 we recapitulate the existence results for the three solution concepts, and state our own Theorems 1 and 2, whose proof is developed throughout Sections 4 and 5, also resorting to some auxiliary results stated and proved in the Appendix.

# 2 Preliminary results and overview of the solution concepts for rate-independent systems

We start by fixing some notation: for a given arbitrary $E \subset \mathbb{R}$, we shall denote by

$$\begin{aligned}&\mathfrak{P}_f(E) \text{ the collection of all finite subsets of } E,\\ &E^- := \inf E, \quad E^+ := \sup E\,.\end{aligned} \tag{2.9}$$

**Kuratowski convergence of sets**

In view of the compactness argument developed in Section 5 ahead, we provide a minimal aside on the notion of *Kuratowski* convergence of sets, confining the discussion to *closed* sets, and referring to [2] for all details. We say that a sequence $(C_n)_n$ of closed subsets of $X$ converge in the sense of Kuratowski to a closed set $C$, if

$$\mathrm{Li}_{n\to\infty}C_n = \mathrm{Ls}_{n\to\infty}C_n = C, \tag{2.10}$$

where

$$\mathrm{Li}_{n\to\infty}C_n := \{x \in X \,:\, \exists x_n \in C_n \text{ such that } x_n \to x\}, \tag{2.11a}$$

$$\begin{aligned}\mathrm{Ls}_{n\to\infty}C_n := \{x \in X \,:\, &\exists\, j \mapsto n_j \text{ increasing and}\\ &x_{n_j} \in C_{n_j} \text{ such that } x_{n_j} \to x\}.\end{aligned} \tag{2.11b}$$

If all the closed sets $C_n$ are contained in a compact set $K$, then Kuratowski convergence coincides with the convergence induced by the Hausdorff distance [2, Prop. 4.4.14]. That is why, the *Blaschke* Theorem (cf., e.g., [2, Thm. 4.4.15]) is applicable, ensuring that, if $K \subset X$ is a fixed compact set, then every sequence of closed sets $(C_n)_n \subset K$ admits a subsequence converging in the Kuratowski sense to a closed set $C \subset K$. If the sets $C_n$ are connected, then $C$ is also connected.

## *2.1 Preliminaries on functions of bounded variation and absolutely continuous functions*

Let us first recall some preliminary definitions and properties related to functions of bounded variation with values in the metric space $(X,\mathsf{d})$. The *pointwise* total variation $\mathrm{Var}_{\mathsf{d}}(u,E)$ of a function $u : E \to X$ is defined by

$$\begin{aligned}&\mathrm{Var}_{\mathsf{d}}(u,E) :=\\ &\sup\left\{\sum_{j=1}^{M} \mathsf{d}(u(t_{j-1}),u(t_j)) \,:\, t_0 < t_1 < \ldots < t_M,\ \{t_j\}_{j=0}^{M} \in \mathfrak{P}_f(E)\right\},\end{aligned} \tag{2.12}$$

with $\mathrm{Var}_{\mathsf{d}}(u,\emptyset) := 0$. We define the space of functions with bounded variation via

$$\mathrm{BV}_{\mathsf{d}}(E;X) := \{u : E \to X \,:\, \mathrm{Var}_{\mathsf{d}}(u,E) < \infty\}\,.$$

For every $u \in \mathrm{BV}_{\mathsf{d}}(E;X)$ we may introduce the function

$$V_u : [E^-, E^+] \to [0, \infty) \quad \text{given by} \quad V_u(t) := \mathrm{Var}_{\mathsf{d}}(u, E \cap [E^-, t])\,. \tag{2.13}$$

Observe that $V_u$ is monotone nondecreasing and satisfies

$$\mathsf{d}(u(t_0), u(t_1)) \le \mathrm{Var}_{\mathsf{d}}(u, [t_0, t_1]) = V_u(t_1) - V_u(t_0) \quad \text{for all } t_0, t_1 \in E \text{ with } t_0 \le t_1\,.$$

Since the metric space $(X, \mathsf{d})$ is complete, every function $u \in \mathrm{BV}_{\mathsf{d}}(E;X)$ is *regulated*, i.e. at every $t \in E$ the left and right limits $u(t-)$ and $u(t+)$ exist (with obvious adjustments at $E^-$ and $E^+$). We recall that $u$ only has jump discontinuities, and that its (at most) *countable* jump set $\mathrm{J}_u$ coincides with the jump set of $V_u$.

We will also consider the distributional derivative $\nu_u$ of the function $V_u$ and recall that the Borel measure $\nu_u$ can be decomposed into the sum

$$\nu_u = \nu_u^{\mathrm{d}} + \nu_u^{\mathrm{J}} \tag{2.14}$$

with $\nu_u^{\mathrm{d}}$ the diffuse part of $\nu_u$ (i.e. the sum of its absolutely continuous and Cantor parts), fulfilling $\nu_u^{\mathrm{d}}(\{t\}) = 0$ for every $t \in [E^-, E^+]$, and $\nu_u^{\mathrm{J}}$ its jump part, concentrated on the set $\mathrm{J}_u$, so that

$$\nu_u^{\mathrm{J}}(\{t\}) = \mathsf{d}(u(t-), u(t)) + \mathsf{d}(u(t)), u(t+)) \quad \text{for every } t \in \mathrm{J}_u.$$

Therefore we have

$$\mathrm{Var}_{\mathsf{d}}(u, [t_0, t_1]) = \nu_u^{\mathrm{d}}([t_0, t_1]) + \mathrm{Jmp}_{\mathsf{d}}(u; [t_0, t_1]) \tag{2.15}$$

for every interval $[t_0, t_1] \subset E$, with the jump contribution

$$\begin{aligned} \mathrm{Jmp}_{\mathsf{d}}(u; [t_0, t_1]) :=& \mathsf{d}(u(t_0)), u(t_0+)) + \mathsf{d}(u(t_1-)), u(t_1)) \\ &+ \sum_{t \in \mathrm{J}_u \cap (t_0, t_1)} \left( \mathsf{d}(u(t-), u(t)) + \mathsf{d}(u(t), u(t+)) \right)\,. \end{aligned} \tag{2.16}$$

In the definition of Balanced Viscosity and Visco-Energetic solutions, there will come into play an alternative notion of total variation for a curve $u \in \mathrm{BV}([0,T];X)$, which will reflect the energetic behavior of the (Balanced Viscosity/Visco-Energetic) solution at jump points. It will be obtained by suitably modifying the jump contribution to the total variation induced by $\mathsf{d}$, cf. (2.15), in terms of a (general) *cost function* $\mathsf{e} : [0,T] \times X \times X \to [0,\infty]$, with $\mathsf{e} \ge \mathsf{d}$, that shall measure the energy dissipated along a jump. Thus, hereafter we will refer to $\mathsf{e}$ as *jump dissipation cost*. As particular cases of $\mathsf{e}$, we will consider

- the *viscous (jump dissipation) cost* $\mathsf{v}$, cf. (2.29) ahead, in the case of Balanced Viscosity solutions;
- the *visco-energetic (jump dissipation) cost* $\mathsf{c}$, cf. (2.36) ahead, in the case of Visco-Energetic solutions.

With the jump dissipation cost $\mathsf{e}$ we associate the *incremental cost*

$$\Delta_{\mathsf{e}} : [0,T] \times X \times X \to [0,\infty], \quad \Delta_{\mathsf{e}}(t, u_-, u_+) := \mathsf{e}(t, u_-, u_+) - \mathsf{d}(u_-, u_+) \tag{2.17}$$

for all $t \in [0,T]$ and $u_-, u_+ \in X$, where the notation $u_-, u_+$ is suggestive of the fact that, in the definition of the total variation functional induced by $\mathsf{e}$, the incremental cost will be evaluated at the left and right limits $u(t-)$ and $u(t+)$ at a jump point of a curve $u$. We will also use the notation

$$\Delta_{\mathsf{e}}(t,u_-,u,u_+) := \Delta_{\mathsf{e}}(t,u_-,u) + \Delta_{\mathsf{e}}(t,u,u_+)\,.$$

We are now in a position to introduce the *augmented total variation* functional induced by $\mathsf{e}$.

**Definition 1.** Given a (jump dissipation) cost function $\mathsf{e}$ and the associated incremental cost $\Delta_{\mathsf{e}}$, and given a curve $u \in \mathrm{BV}([0,T];X)$, we define the *incremental jump variation* of $u$ on a sub-interval $[t_0,t_1] \subset [0,T]$ by

$$\begin{aligned}\mathrm{Jmp}_{\Delta_{\mathsf{e}}}(u;[t_0,t_1]) &:= \Delta_{\mathsf{e}}(t_0,u(t_0)),u(t_0+))\\ &\quad + \Delta_{\mathsf{e}}(t_1,u(t_1-)),u(t_1)) + \sum_{t\in \mathrm{J}_u\cap(t_0,t_1)} \Delta_{\mathsf{e}}(t,u(t-),u(t),u(t+))\,. \end{aligned} \tag{2.18}$$

This induces the *augmented total variation* functional

$$\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u,[t_0,t_1]) := \mathrm{Var}_{\mathsf{d}}(u,[t_0,t_1]) + \mathrm{Jmp}_{\Delta_{\mathsf{e}}}(u;[t_0,t_1]) \tag{2.19}$$

along any sub-interval $[t_0,t_1] \subset [0,T]$.

Since we have subtracted from the $\mathsf{e}$-jump contribution the $\mathsf{d}$-distance of the jump end-points, cf. (2.17), the $\mathsf{d}$-jump contribution (2.16) to $\mathrm{Var}_{\mathsf{d}}$ cancels out, and in fact only the *diffuse* contribution $\nu_u^{\mathsf{d}}([t_0,t_1])$ remains. In fact, one could rewrite $\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u,[t_0,t_1])$ as

$$\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u,[t_0,t_1]) = \nu_u^{\mathsf{d}}([t_0,t_1]) + \mathrm{Jmp}_{\mathsf{e}}(u;[t_0,t_1]), \tag{2.20}$$

with $\mathrm{Jmp}_{\mathsf{e}}(u;[t_0,t_1])$ defined by (2.18) with the "whole" cost $\mathsf{e}$ in place of its incremental version $\Delta_{\mathsf{e}}$, i.e.

$$\begin{aligned}\mathrm{Jmp}_{\mathsf{e}}(u;[t_0,t_1]) :=& \mathsf{e}(t_0,u(t_0),u(t_0+)) + \mathsf{e}(t_1,u(t_1-),u(t_1))\\ &+ \sum_{t\in \mathrm{J}_u\cap(t_0,t_1)} \left(\mathsf{e}(t,u(t-),u(t)) + \mathsf{e}(t,u(t),u(t+))\right)\,. \end{aligned} \tag{2.21}$$

Clearly, $\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u,[t_0,t_1]) \geq \mathrm{Var}_{\mathsf{d}}(u,[t_0,t_1])$, and they coincide if $\mathsf{e} = \mathsf{d}$, or when $\mathrm{J}_u = \emptyset$. Moreover, as observed in [18], although it need not be induced by a distance on $X$, $\mathrm{Var}_{\mathsf{d},\mathsf{e}}$ still enjoys the additivity property

$$\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u,[a,c]) = \mathrm{Var}_{\mathsf{d},\mathsf{e}}(u,[a,b]) + \mathrm{Var}_{\mathsf{d},\mathsf{e}}(u,[b,c]) \qquad \text{for all } 0 \leq a \leq b \leq c \leq T.$$

Finally, we recall that a curve $u : [0,T] \to X$ is absolutely continuous (and write $u \in \mathrm{AC}([0,T];X)$) if there exists $m \in L^1(0,T)$ such that

$$\mathsf{d}(u(s),u(t)) \leq \int_s^t m(r)\,\mathrm{d}r \quad \text{for all } 0 \leq s \leq t \leq T. \tag{2.22}$$

For every $u \in \mathrm{AC}([0,T];X)$, the limit

$$|u'|(t) = \lim_{s\to t} \frac{\mathsf{d}(u(s),u(t))}{|t-s|} \quad \text{exists for a.a. } t \in (0,T), \tag{2.23}$$

cf. [1, Sec. 1.1]. We will refer to it as the *metric derivative* of $u$ at $t$. The map $t \mapsto |u'|(t)$ belongs to $L^1(0,T)$ and it is minimal within the class of functions $m \in L^1(0,T)$ fulfilling (2.22).

## 2.2 Energetic, Balanced Viscosity, and Visco-Energetic solutions at a glance

We now give a quick overview of the notions of rate-independent evolution this paper is concerned with. We aim to somehow motivate the various solution concepts and in addition highlight both the common points, and the differences, in their structure.

Underlying the upcoming definitions, there will be the following basic conditions on the energy functional $\mathscr{E}$. Let us mention in advance that we in fact allow for a possibly nonsmooth time-dependence $t \mapsto \mathscr{E}(t,u)$. However, in what follows for simplicity we will confine our analysis to the case in which the domain of $\mathscr{E}(t,\cdot)$ in fact coincides with $X$ for every $t \in [0,T]$, referring to [18, Rmk. 2.7] for a discussion of the more general case in which $\mathrm{dom}(\mathscr{E}(t,\cdot))$ is a proper subset of $X$ (still independent of the time variable).

#### Basic assumptions on the energy

Throughout the paper, we will require that $\mathscr{E}$ complies with two basic properties, involving the perturbed energy functional $\mathscr{F} : [0,T] \times X \to \mathbb{R}$

$$\mathscr{F}(t,u) := \mathscr{E}(t,u) + \mathsf{d}(x_o,u) \quad \text{with } x_o \text{ a given reference point in } X \tag{2.24}$$

and its sublevel sets $S_C := \{(t,u) \in [0,T] \times X \,:\, \mathscr{F}(t,u) \leq C\}$. Namely,

**Lower semicontinuity and compactness:** for all $C \in \mathbb{R}$

$$\mathscr{E} \text{ is lower semicontinuous on } S_C \text{ and the sets } S_C \text{ are compact in } [0,T] \times X; \tag{E$_1$}$$

**Power control:** there exists a map $\mathscr{P} : [0,T] \times X \to \mathbb{R}$ fufilling

$$\begin{aligned} &\liminf_{s\uparrow t} \frac{\mathscr{E}(t,u) - \mathscr{E}(s,u)}{t-s} \geq \mathscr{P}(t,u) \geq \limsup_{s\downarrow t} \frac{\mathscr{E}(s,u) - \mathscr{E}(t,u)}{s-t} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (t,u) \in [0,T] \times X, \\ &\exists C_P > 0 \ \forall (t,u) \in [0,T] \times X : \quad |\mathscr{P}(t,u)| \leq C_P \mathscr{F}(t,u). \end{aligned} \tag{E$_2$}$$

We may understand the power functional $\mathscr{P}$ as a sort of "time superdifferential" of the energy functional, surrogating its partial time derivative in the case where the functional $t \mapsto \mathscr{E}(t,u)$ is not differentiable at every point of $[0,T] \times X$. This for instance occurs for reduced energies having the form $\mathscr{E}(t,u) = \min_{\varphi \in \Phi} \mathscr{I}(t,\varphi,u)$ and such that the set of minimizers does not reduce to a singleton, as considered, e.g., in [7, 14, 13, 18]. By repeating the very same arguments as in [18], we may deduce from ($E_1$) & ($E_2$) that

$$\begin{aligned} &\text{the function } t \mapsto \mathscr{E}(t,u) \text{ is Lipschitz continuous for every } u \in X, \text{ with} \\ &\mathscr{P}(t,u) = \partial_t \mathscr{E}(t,u) \qquad \text{for almost all } t \in [0,T] \text{ and for all } u \in X. \end{aligned} \tag{2.25}$$

Therefore,

$$\mathscr{E}(t,u) = \mathscr{E}(s,u) + \int_s^t \mathscr{P}(r,u)\,\mathrm{d}r \qquad \text{for every } [s,t] \subset [0,T]. \tag{2.26}$$

Combining this with the power control estimate in ($E_2$) and exploiting the Gronwall Lemma, we conclude that

$$\mathscr{F}(t,u) \le \mathscr{F}(s,u) \exp\left(C_P|t-s|\right) \quad \text{for all } s, t \in [0,T]. \tag{2.27}$$

That is why, it is significant (and notationally convenient) to work with the functional $\mathscr{F}_0(u) := \mathscr{F}(0,u)$, which controls $\mathscr{F}(t,u)$, and thus the power functional $\mathscr{P}(t,u)$, at all $t \in [0,T]$.

We are now in a position to give the concept of **Energetic** solution, dating back to [16, 17], cf. also [9].

**Definition 2 (Energetic solution).** A curve $u \in \mathrm{BV}([0,T];X)$ is an Energetic solution of the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ if it satisfies for every $t \in [0,T]$

- the global stability condition

$$\mathscr{E}(t,u(t)) \le \mathscr{E}(t,v) + \mathsf{d}(u(t),v) \quad \text{for every } v \in X, \tag{$S_\mathsf{d}$}$$

- the energy balance

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_\mathsf{d}(u,[0,t]) = \mathscr{E}(0,u(0)) + \int_0^t \mathscr{P}(s,u(s))\,\mathrm{d}s. \tag{$E_\mathsf{d}$}$$

For later use, we introduce the d-stable set

$$\mathscr{S}_\mathsf{d} := \{(t,u) \in [0,T] \times X \,:\, \mathscr{E}(t,u) \le \mathscr{E}(t,v) + \mathsf{d}(u,v) \text{ for all } v \in X\},$$

with its time-dependent sections $\mathscr{S}_\mathsf{d}(t) := \{u \in X \,:\, (t,u) \in \mathscr{S}_\mathsf{d}\}$. We postpone to Section 3.1 a discussion on the existence of Energetic solutions.

As already mentioned in the Introduction, **Balanced Viscosity** solutions arise in the time-continuous limit of the time-incremental scheme ($IM_{\varepsilon,\tau}$), when the parameters $\varepsilon$ and $\tau$ both tend to zero with $\frac{\varepsilon}{\tau} \uparrow \infty$ cf. (1.7). They fulfill the *local version* of the stability condition ($S_\mathsf{d}$), involving the *metric slope* of the energy functional

$\mathscr{E}$, cf. (1.4). The "viscous" character of the approximation that underlies condition (1.7), is also reflected in the *viscous jump dissipation cost*. Indeed, at fixed process time $t \in [0,T]$, $\mathsf{v}(t,u_-,u_+)$ is obtained by minimizing the *transition cost*

$$\mathrm{Trc}_{\mathrm{BV}}(t,\vartheta,[r_0,r_1]) := \int_{r_0}^{r_1} |\vartheta'|(r)\,(|\mathrm{D}\mathscr{E}|(t,\vartheta(r)) \vee 1)\,\mathrm{d}r \tag{2.28}$$

over all *absolutely continuous* curves $\vartheta$ on an interval $[r_0,r_1]$, connecting the two points $u_-$ and $u_+$, where we recall that $|\vartheta'|$ is the (almost everywhere defined) *metric derivative* of the curve $\vartheta$. Namely,

$$\begin{aligned}\mathsf{v}(t,u_-,u_+) := \inf\Big\{&\mathrm{Trc}_{\mathrm{BV}}(t,\vartheta,[r_0,r_1]) : \\ &\vartheta \in \mathrm{AC}([r_0,r_1];X),\ \vartheta(r_0)=u_-,\ \vartheta(r_1)=u_+\Big\}.\end{aligned} \tag{2.29}$$

We can then introduce the incremental cost $\Delta_{\mathsf{v}}$ (2.17) and the jump variation $\mathrm{Jmp}_{\Delta_{\mathsf{v}}}$ (2.18) associated with $\mathsf{v}$, and thus arrive at the induced augmented total variation $\mathrm{Var}_{\mathsf{d},\mathsf{v}}$ (2.19), which enters into the energy balance involved in the Balanced Viscosity concept.

**Definition 3 (Balanced Viscosity solution).** A curve $u \in \mathrm{BV}([0,T];X)$ is a Balanced Viscosity (BV) solution of the rate-independent system $(X,\mathscr{E},\mathsf{d})$ if it satisfies

- the local stability condition

$$|\mathrm{D}\mathscr{E}|(t,u(t)) \le 1 \quad \text{for every } t \in [0,T]\setminus \mathrm{J}_u, \tag{$\mathrm{S}_{\mathsf{d},\mathrm{loc}}$}$$

- the energy balance

$$\begin{aligned}&\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[0,t]) \\ &= \mathscr{E}(0,u(0)) + \int_0^t \mathscr{P}(s,u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T].\end{aligned} \tag{$\mathrm{E}_{\mathsf{d},\mathsf{v}}$}$$

The notion of **Visco-Energetic** solution features a modified concept of stability which also involves the viscous correction $\delta(u,v) = \frac{\mu}{2}\mathsf{d}^2(u,v)$. We then define the functional

$$\mathsf{D}(u,v) := \mathsf{d}(u,v) + \delta(u,v) = \mathsf{d}(u,v) + \frac{\mu}{2}\mathsf{d}^2(u,v) \tag{2.30}$$

and we say that a point $(t,x) \in [0,T]\times X$ is $\mathsf{D}$-stable if

$$\mathscr{E}(t,x) \le \mathscr{E}(t,y) + \mathsf{D}(x,y) = \mathscr{E}(t,y) + \mathsf{d}(x,y) + \frac{\mu}{2}\mathsf{d}^2(x,y) \qquad \text{for all } y \in X. \tag{2.31}$$

We denote by $\mathscr{S}_{\mathsf{D}}$ the collection of all $\mathsf{D}$-stable points, and by $\mathscr{S}_{\mathsf{D}}(t)$ its section at time $t \in [0,T]$. We also introduce the *residual stability function* $\mathscr{R} : [0,T]\times X \to \mathbb{R}$ given by

$$
\begin{aligned}
\mathscr{R}(t,x) :&= \sup_{y\in X}\{\mathscr{E}(t,x)-\mathscr{E}(t,y)-\mathsf{D}(x,y)\}\\
&= \mathscr{E}(t,x)-\inf_{y\in X}\{\mathscr{E}(t,y)+\mathsf{D}(x,y)\}
\end{aligned}
\tag{2.32}
$$

(for simplicity, we choose to neglect the $\mu$-dependence of the functionals $\mathsf{D}$ and $\mathscr{R}$ in their notation). Observe that

$$
\begin{aligned}
&\mathscr{R}(t,x) \geq 0 \quad \text{for all } (t,x) \in [0,T]\times X \quad \text{with}\\
&\mathscr{R}(t,x) = 0 \text{ if and only if } (t,x) \in \mathscr{S}_{\mathsf{D}}\,,
\end{aligned}
\tag{2.33}
$$

so that $\mathscr{R}$ may be interpreted as "measuring the failure" of the stability condition at a given point $(t,x) \in [0,T]\times X$. It can be straightforwardly checked that, under the basic lower semicontinuity assumption ($\mathrm{E}_1$) on $\mathscr{E}$, the functional $\mathscr{R}$ is lower semicontinuous on $[0,T]\times X$.

We now have all the ingredients to define the jump-dissipation cost for Visco-Energetic solutions. In the same way as for Balanced Viscosity solutions, such a cost is obtained by minimizing a suitable transition cost over a class of curves connecting the two end-points of the jump. However, such curves, while still continuous, need not be absolutely continuous. Further, they are in general defined on a *compact* subset $E \subset \mathbb{R}$ that may have a more complicated structure than that of an interval. To describe it, we introduce

$$
\text{the collection } \mathfrak{h}(E) \text{ of the connected components of the set } [E^-,E^+]\setminus E, \tag{2.34}
$$

where we recall that $E^- = \inf E$ and $E^+ = \sup E$. Since $[E^-,E^+]\setminus E$ is an open set, $\mathfrak{h}(E)$ consists of at most countably many open intervals, which we will often refer to as the "holes" of $E$. Hence, the transition cost at the basis of the concept of Visco-Energetic solution evaluates (1) the $\mathsf{d}$-total variation of a continuous curve defined on a set $E$, (2) the sum, over all the holes of $E$, of a quantity related to the gaps (3) the measure of "how much" the curve $\vartheta$ fails to comply with the $\mathsf{D}$-stability condition (2.31) at the points in $E\setminus\{E^+\}$.

**Definition 4.** Let $E$ be a compact subset of $\mathbb{R}$ and $\vartheta \in \mathrm{C}(E;X)$. For every $t \in [0,T]$ we define the *transition cost function*

$$
\mathrm{Trc}_{\mathsf{VE}}(t,\vartheta,E) := \mathrm{Var}_{\mathsf{d}}(\vartheta,E) + \mathrm{GapVar}_{\mathsf{d}}(\vartheta,E) + \sum_{s\in E\setminus E^+} \mathscr{R}(t,\vartheta(s))\,, \tag{2.35}
$$

with

1. $\mathrm{Var}_{\mathsf{d}}(\vartheta,E)$ from (2.12);
2. $\mathrm{GapVar}_{\mathsf{d}}(\vartheta,E) := \sum_{I\in\mathfrak{h}(E)} \frac{\mu}{2}\mathsf{d}^2(\vartheta(I^-),\vartheta(I^+))$;
3. the (possibly infinite) sum

$$
\sum_{s\in E\setminus E^+} \mathscr{R}(t,\vartheta(s)) := \begin{cases} \sup\{\sum_{s\in P}\mathscr{R}(t,\vartheta(s))\,:\, P\in\mathfrak{P}_f(E)\} & \text{if } E\setminus E^+ \neq \emptyset,\\ 0 & \text{otherwise} \end{cases}
$$

(recall that $\mathfrak{P}_f(E)$ denotes the collection of all finite subsets of $E$).

Along with [18], we observe that, for every fixed $t \in [0,T]$ and $\vartheta \in \mathrm{C}(E;X)$, the transition cost fulfills the additivity property

$$\mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E\cap[a,c]) = \mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E\cap[a,b]) + \mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E\cap[b,c])$$

for all $a<b<c$. We are now in a position to define the associated *visco-energetic jump dissipation cost* $\mathsf{c} : [0,T]\times X\times X \to [0,\infty]$ via

$$\begin{aligned} \mathsf{c}(t,u_-,u_+) := \inf\{ &\mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E) : \\ &E \Subset \mathbb{R},\ \vartheta \in \mathrm{C}(E;X),\ \vartheta(E^-)=u_-,\ \vartheta(E^+)=u_+\}, \end{aligned} \tag{2.36}$$

whence the incremental dissipation cost $\Delta_{\mathsf{c}}$ according to (2.17), the jump variation $\mathrm{Jmp}_{\Delta_{\mathsf{c}}}$ as in (2.18), and the augmented total variation $\mathrm{Var}_{\mathsf{d},\mathsf{c}}$ as in (2.19).

We can now give the following

**Definition 5 (Visco-Energetic solution).** A curve $u \in \mathrm{BV}([0,T];X)$ is a Visco-Energetic (VE) solution of the rate-independent system $(X,\mathscr{E},\mathsf{d})$ if it satisfies

- the $\mathsf{D}$-stability condition

$$\begin{aligned} \mathscr{E}(t,u(t)) \le \mathscr{E}(t,v) + \mathsf{d}(u(t),v) + \frac{\mu}{2}\mathsf{d}^2(u(t),v) \\ \text{for every } v\in X \text{ and for every } t\in[0,T]\setminus \mathrm{J}_u, \end{aligned} \tag{$\mathrm{S}_{\mathsf{D}}$}$$

- the energy balance

$$\begin{aligned} &\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{c}}(u,[0,t]) \\ &= \mathscr{E}(0,u(0)) + \int_0^t \mathscr{P}(s,u(s))\,\mathrm{d}s \quad \text{for all } t\in[0,T]. \end{aligned} \tag{$\mathrm{E}_{\mathsf{d},\mathsf{c}}$}$$

## 3 Main results

Prior to stating our own results on the singular limits of VE solutions in Section 3.2, in Sec. 3.1 below we recall the known existence results for Energetic, BV, and VE solutions. Under the same conditions ensuring the existence for the two former solution concepts, we will prove our convergence statements for $\mathrm{VE}_\mu$ solutions in the limits $\mu \downarrow 0$ and $\mu \uparrow \infty$, respectively.

### *3.1 A survey on existence results*

In what follows, in addition to the basic conditions ($\mathrm{E}_1$) and ($\mathrm{E}_2$), we will introduce further assumptions on the energy functional $\mathscr{E}$ that will be at the core of the upcoming existence results for Energetic (Thm. 1), BV (Thm. 2), and VE (Thm. 3) solutions. We will also illustrate the main ideas underlying their proofs.

**Energetic solutions.**

For the existence of Energetic solutions in the metric setting of (X) we refer to [8, Thm. 4.5], cf. also [9] and [10, Sec. 2.1]. In accordance with these results, in addition to the coercivity ($E_1$) and the power control ($E_2$), we require that

**Upper semicontinuity of the power:** $\mathscr{P} : [0,T] \times X \to \mathbb{R}$ satisfies the *conditional upper semicontinuity* condition

$$
\begin{aligned}
&((t_n,u_n) \to (t,u) \text{ in } [0,T]\times X,\ \mathscr{E}(t_n,u_n) \to \mathscr{E}(t,u)) \\
&\Longrightarrow \limsup_{n\to\infty} \mathscr{P}(t_n,u_n) \le \mathscr{P}(t,u)\,.
\end{aligned}
\tag{$E_3$}
$$

We thus have

**Theorem 1.** *Let $\mathscr{E} : [0,T]\times X \to \mathbb{R}$ comply with* ($E_1$), ($E_2$) *and* ($E_3$). *Then, for every initial datum $u_0$ stable at $t = 0$, i.e. $u_0 \in \mathscr{S}_{\mathsf{d}}(0)$, there exists at least one Energetic solution to the rate-independent system $(X,\mathscr{E},\mathsf{d})$ with $u(0) = u_0$.*

The *proof* is based on a (by now standard in the frame of rate-independent systems) time-discretization procedure, with the discrete solutions constructed by recursively solving the time-incremental minimization scheme ($IM_\tau$). Their (piecewise constant) interpolants are shown to comply with the discrete versions of the stability condition ($S_{\mathsf{d}}$) and of the upper energy estimate in ($E_{\mathsf{d}}$), whence all a priori estimates stem, also based on the power control ($E_2$). With a Helly-type compactness result, crucially relying on ($E_1$), we thus infer that the approximate solutions pointwise converge to a curve $u \in \mathrm{BV}([0,T];X)$. The continuity (cf. (2.25)) and lower semicontinuity properties

$$
\begin{aligned}
&t_n \to t \ \Rightarrow\ \mathscr{E}(t_n,y) \to \mathscr{E}(t,y) \quad \text{for all } y \in X, \\
&(t_n \to t,\ u_n \to u) \ \Rightarrow\ \liminf_{n\to\infty} \mathscr{E}(t_n,u_n) \ge \mathscr{E}(t,u)
\end{aligned}
\tag{3.37}
$$

ensure the closedness of the stable set $\mathscr{S}_{\mathsf{d}}$, which allows us to pass to the limit in the discrete stability condition and conclude that $u$ complies with ($S_{\mathsf{d}}$). Lower semicontinuity arguments, joint with ($E_3$), lead to the limit passage in the discrete upper energy estimate, so that $u$ complies with the upper energy estimate $\le$ of ($E_{\mathsf{d}}$). The lower energy estimate $\ge$ can be then deduced from the stability condition either via a Riemann-sum argument, formalized in, e.g., [10, Prop, 2.1.23], or by applying [18, Lemma 6.2].

**Balanced Viscosity solutions.**

Along the footsteps of [13, Thm. 4.2], for the existence of Balanced Viscosity solutions, in addition to ($E_1$) and ($E_2$), we again need to impose the (conditional) upper semicontinuity of the power functional and, *in addition*, the lower semicontinuity of the slope along sequences *with bounded energy and slope*. These requirements are subsumed by the following condition:

**Upper semicontinuity of the power, lower semicontinuity of the slope:**]$\mathscr{E} : [0,T]\times X \to \mathbb{R}$ and $\mathscr{P} : [0,T]\times X \to \mathbb{R}$ satisfy

$$\left( (t_n,u_n) \to (t,u) \text{ in } [0,T]\times X,\ \sup_{n\in\mathbb{N}} \mathscr{F}_0(u_n) < \infty,\ \sup_{n\in\mathbb{N}} |\mathrm{D}\mathscr{E}|(t_n,u_n) < \infty \right) \\ \Longrightarrow \begin{cases} \liminf_{n\to\infty} |\mathrm{D}\mathscr{E}|(t_n,u_n) \geq |\mathrm{D}\mathscr{E}|(t,u), \\ \limsup_{n\to\infty} \mathscr{P}(t_n,u_n) \leq \mathscr{P}(t,u)\,. \end{cases} \tag{$E_3'$}$$

The last, key condition underlying the existence of Balanced Viscosity solutions is that $\mathscr{E}$ complies with the

**Chain-rule inequality:**]for every curve $u \in \mathrm{AC}([0,T];X)$ the function $t \mapsto \mathscr{E}(t,u(t))$ is absolutely continuous on $[0,T]$, and there holds

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t,u(t)) + \mathscr{P}(t,u(t)) \leq |u'|(t)|\mathrm{D}\mathscr{E}|(t,u(t)) \qquad \text{for a.a. } t \in (0,T)\,. \tag{$E_4$}$$

Under these conditions, the following existence result was proved in [13].

**Theorem 2.** *Let* $\mathscr{E} : [0,T]\times X \to \mathbb{R}$ *comply with* ($E_1$), ($E_2$), ($E_3'$)*, and* ($E_4$)*. Then, for every* $u_0 \in X$ *there exists at least one Balanced Viscosity solution to the rate-independent system* $(X,\mathscr{E},\mathsf{d})$ *with* $u(0) = u_0$.

As mentioned in the Introduction, in the *proof* of [13, Thm. 4.2] (cf. also [11]), BV solutions arise by taking the vanishing-viscosity limit, as $\varepsilon \downarrow 0$, of the time-continuous solutions of the Gradient Systems $(X,\mathscr{E},\mathsf{d},\psi_\varepsilon)$ with $\psi_\varepsilon$ from (1.3). Nonetheless, exploiting the arguments from [12, 15] in the Banach setting, the vanishing-viscosity analysis developed in [13] could be easily adapted to the direct limit passage in the time-discretization scheme ($IM_{\varepsilon,\tau}$). In fact, the lower semicontinuity of the slope from ($E_3'$) serves to the purpose of passing to the limit in the dissipation term in the discrete energy-dissipation inequality arising from the scheme ($IM_{\varepsilon,\tau}$). This leads to the total variation term $\mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[0,t])$ in the energy balance ($E_{\mathsf{d},\mathsf{v}}$). Instead, the upper semicontinuity of the power allows us to take the limit in the power term of the discrete energy inequality. In this way, it is possible to conclude that any limit curve $u \in \mathrm{BV}([0,T];X)$ of the discrete solutions complies with the local stability condition ($S_{\mathsf{d},\mathrm{loc}}$) and with the upper energy estimate

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[0,t]) \leq \mathscr{E}(0,u(0)) + \int_0^t \mathscr{P}(s,u(s))\,\mathrm{d}s\,. \tag{$E_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$}$$

Unlike the case of Energetic solutions, where the validity of global stability condition ($\mathrm{S_d}$) was sufficient to conclude the lower energy estimate for ($\mathrm{E_d}$), ($\mathrm{S_{d,loc}}$) is not strong enough to lead to the converse inequality of ($\mathrm{E^{ineq}_{d,v}}$). This is instead ensured by a chain-rule argument based on ($\mathrm{E_4}$), cf. [12, Prop. 4.2, Thm. 4.3].

Finally, let us mention that, under the very assumptions for the existence Thm. 2, trivially adapting the argument for [15, Thm. 3.15] it can be shown that a curve $u \in \mathrm{BV}([0,T];X)$ is a BV solution to the rate-independent system $(X,\mathscr{E},\mathsf{d})$ if and only if it satisfies ($\mathrm{S_{d,loc}}$), the localized energy inequality

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d}}(u,[s,t]) \leq \mathscr{E}(s,u(s)) + \int_s^t \mathscr{P}(r,u(r))\,\mathrm{d}r \tag{3.38}$$

for all $0 \leq s \leq t \leq T$, and the jump conditions

$$\begin{aligned} &\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t)) = \mathsf{v}(t,u(t-),u(t)), \\ &\mathscr{E}(t,u(t)) - \mathscr{E}(t,u(t+)) = \mathsf{v}(t,u(t),u(t+)), \\ &\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t+)) = \mathsf{v}(t,u(t-),u(t+))\,. \end{aligned} \tag{3.39}$$

**Visco-Energetic solutions.**

As already hinted, Visco-Energetic solutions were introduced in [18] within a more complex topological setting, featuring an *asymmetric* distance and a topology $\sigma$, involved in the coercivity condition on the energy functional. It turns out that, in the present metric setting where $\sigma$ is the topology induced by $\mathsf{d}$, ($\mathrm{E_1}$), ($\mathrm{E_2}$) and ($\mathrm{E_3'}$) coincide with the conditions required on the energy functional $\mathscr{E}$ within [18, Assumption $< A >$, Sec. 2.2]. Furthermore, the particular choice $\delta(u,v) = \frac{\mu}{2}\mathsf{d}^2(u,v)$ for the viscous correction ensures the validity of [18, Assumption $< B >$, Sec. 3.1]. In particular, condition [18, $< B.3 >$, Sec. 3.1] is fulfilled, namely $\mathsf{D}$-stability implies local $\mathsf{d}$-stability, as it can be straightforwardly checked. Finally, thanks to the lower semicontinuity of the residual functional $\mathscr{R}$ from (2.32), also [18, Assumption $< C >$, Sec. 3.3] is fulfilled. Therefore, [18, Thm. 3.9] applies, ensuring the convergence of the time-incremental scheme ($\mathrm{IM}_\mu$), with $\mu > 0$ fixed, to a Visco-Energetic solution. In particular, we have the following existence result, under the *same* conditions on the energy functional as in the existence Thm. 1 for Energetic solutions.

**Theorem 3.** *Let $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ comply with* ($\mathrm{E_1}$)*,* ($\mathrm{E_2}$) *and* ($\mathrm{E_3}$)*. Then, for every $\mu > 0$ and every initial datum $u_0 \in X$ there exists at least one* $\mathrm{VE}_\mu$ *solution to the rate-independent system $(X,\mathscr{E},\mathsf{d})$ with $u(0) = u_0$.*

The outline of the existence argument is the same as for Energetic solutions, though the technical difficulties attached to the single steps are peculiar of the Visco-Energetic case. The $\mathsf{D}$-stability condition ($\mathrm{S_D}$) and the upper energy estimate in ($\mathrm{E_{d,c}}$) are derived by passing to the limit in their discrete versions, valid for the discrete solutions to the time-incremental scheme ($\mathrm{IM}_\mu$). As shown in [18, Thm. 6.5], the lower energy estimate can then be derived from ($S_D$) by applying [18, Lemma 6.2].

Under the same conditions as for the existence Thm. 3, we have the following 'stability' result for VE solutions with respect to convergence of the parameters $\mu_n$ to some *strictly positive* $\mu$.

**Proposition 1.** *Let* $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ *comply with* ($E_1$)*,* ($E_2$) *and* ($E_3$)*. Let* $(\mu_n) \subset$ *fulfill*

$$\mu_n \to \mu > 0 \qquad \text{as } n \to \infty.$$

*Let* $(u_n^0)_n, u_0 \subset X$ *fulfill*

$$u_n^0 \to u_0 \quad \text{and} \quad \mathscr{E}(0,u_n^0) \to \mathscr{E}(0,u_0) \text{ as } n \to \infty. \tag{3.40}$$

*Then, there exist a subsequence* $(u_{n_k})_k$ *and a curve* $u \in \mathrm{BV}([0,T];X)$ *such that* $u(0) = u_0$*,*

$$u_{n_k}(t) \to u(t) \;\text{ and }\; \mathscr{E}(t,u_{n_k}(t)) \to \mathscr{E}(t,u(t)) \quad \text{for every } t \in [0,T], \tag{3.41}$$

*and u is a* $\mathrm{VE}_\mu$ *solution to the rate-independent system* $(X,\mathscr{E},\mathsf{d})$.

We will outline the *proof* of Proposition 1 at the end of Sec. 5.1.

We conclude this section by recalling that, VE solutions as well can be characterized in terms of suitable jump conditions. Namely, it was proved in [18, Prop. 3.8] that a curve $u \in \mathrm{BV}([0,T];X)$ is a VE solution to the rate-independent system $(X,\mathscr{E},\mathsf{d})$ if and only if it satisfies ($S_D$), the energy-dissipation inequality (3.38), and the jump conditions

$$\begin{aligned}
&\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t)) = \mathsf{c}(t,u(t-),u(t)),\\
&\mathscr{E}(t,u(t)) - \mathscr{E}(t,u(t+)) = \mathsf{c}(t,u(t),u(t+)),\\
&\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t+)) = \mathsf{c}(t,u(t-),u(t+))\,.
\end{aligned} \tag{3.42}$$

## 3.2 Main results: Singular limits of Visco-Energetic solutions

We now consider a sequence $(\mu_n)_n \subset (0,\infty)$, either converging to 0, or diverging to $\infty$. Accordingly, let $(u_n^0)_n \subset X$ be a sequence of initial data for the rate-independent system $(X,\mathscr{E},\mathsf{d})$. Under conditions ($E_1$), ($E_2$) and ($E_3$), there exists a corresponding sequence of Visco-Energetic solutions $(u_n)_n \subset \mathrm{BV}([0,T];X)$ to the rate-independent system $(X,\mathscr{E},\mathsf{d})$, arising from the viscous corrections $\delta_n(u,v) = \frac{\mu_n}{2}\mathsf{d}^2(u,v)$ and satisfying the initial condition $u_n(0) = u_n^0$.

Our first result addresses the behavior of the sequence $(u_n)_n$ in the case $\mu_n \downarrow 0$, under the *sole* conditions ($E_1$), ($E_2$) and ($E_3$) guaranteeing the existence of Visco-Energetic and Energetic solutions, cf. Theorems 1 and 3.

**Theorem 1 (Convergence to Energetic solutions as** $\mu \downarrow 0$**)** *Let* $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ *comply with* ($E_1$)*,* ($E_2$) *and* ($E_3$)*. Let* $(u_n^0)_n, u_0 \subset X$ *fulfill* (3.40) *and suppose* *that* $u_0 \in \mathscr{S}_{\mathsf{d}}(0)$. *Let* $(\mu_n)_n \subset (0,\infty)$ *be a null sequence, and, correspondingly, let* $(u_n)_n \subset \mathrm{BV}([0,T];X)$ *be a sequence of* $\mathrm{VE}_{\mu_n}$ *solutions to the rate-independent system* $(X,\mathscr{E},\mathsf{d})$ *fulfilling* $u_n(0) = u_n^0$.

*Then, there exist a subsequence* $(u_{n_k})_k$ *and a curve* $u \in \mathrm{BV}([0,T];X)$ *such that* $u(0) = u_0$*, convergences* (3.41) *hold, and and u is an Energetic solution to* $(X,\mathscr{E},\mathsf{d})$.

We will prove the convergence (along a subsequence) of a sequence of $\mathrm{VE}_{\mu_n}$ solutions, as $\mu_n \uparrow \infty$, to a Balanced Viscosity solution, under the same conditions as in the existence Theorem 2 for Balanced Viscosity solutions. Hence we need to strengthen ($\mathrm{E}_3$) with ($\mathrm{E}_3'$), and require the chain-rule inequality ($\mathrm{E}_4$) as well.

**Theorem 2 (Convergence to Balanced Viscosity solutions as $\mu \uparrow \infty$)** *Let* $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ *comply with* ($\mathrm{E}_1$)*,* ($\mathrm{E}_2$)*,* ($\mathrm{E}_3'$)*, and* ($\mathrm{E}_4$)*. Let* $(u_n^0)_n, u_0 \subset X$ *fulfill* (3.40)*. Let* $(\mu_n)_n \subset (0,\infty)$ *be a diverging sequence, and, correspondingly, let* $(u_n)_n \subset \mathrm{BV}([0,T];X)$ *be a sequence of* $\mathrm{VE}_{\mu_n}$ *solutions to the rate-independent system* $(X,\mathscr{E},\mathsf{d})$ *fulfilling* $u_n(0) = u_n^0$.

*Then, there exist a subsequence* $(u_{n_k})_k$ *and a curve* $u \in \mathrm{BV}([0,T];X)$ *such that* $u(0) = u_0$*, convergences* (3.41) *hold, and u is an Balanced Viscosity solution to* $(X,\mathscr{E},\mathsf{d})$.

Both proofs will be carried out throughout Sections 4 and 5.

## 4 Proofs of Theorems 1 and 2

### A preliminary compactness result.

We start with a Helly-type compactness result for a sequence of $\mathrm{VE}_{\mu_n}$ solutions, associated with parameters $(\mu_n)_n$, which applies both to the limit $\mu_n \downarrow 0$, and to the limit $\mu_n \uparrow \infty$, under the basic conditions ($\mathrm{E}_1$) and ($\mathrm{E}_2$) on $\mathscr{E}$. The key starting observation is that, since

$$\mathrm{Var}_{\mathsf{d},\mathsf{c}_\mu}(u,[0,t]) \geq \mathrm{Var}_{\mathsf{d}}(u,[0,t]) \quad \text{for every } u \in \mathrm{BV}([0,T];X) \text{ and every } \mu > 0, \tag{4.43}$$

every VE solution complies with the upper energy estimate of the energy balance ($\mathrm{E}_{\mathsf{d}}$), cf. (4.44) below, where the (either vanishing or blowing up) parameters $\mu_n$ no longer feature. From this energy estimate there stem all the a priori estimates and compactness properties common to the two singular limits $\mu_n \downarrow 0$ and $\mu_n \uparrow \infty$.

**Proposition 2 (A priori estimates and compactness).** *Let* $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ *comply with* ($\mathrm{E}_1$) *and* ($\mathrm{E}_2$)*. Consider a sequence* $(u_n)_n \subset \mathrm{BV}([0,T];X)$ *of curves starting from initial data* $(u_0^n)_n \subset X$ *converging to some* $u_0 \in X$ *as in* (3.40)*. Suppose that the curves* $u_n$ *fulfill for every* $n \in \mathbb{N}$ *the upper energy estimate*

$$\mathscr{E}(t,u_n(t)) + \mathrm{Var}_{\mathsf{d}}(u_n,[0,t]) \leq \mathscr{E}(0,u_0^n) + \int_0^t \mathscr{P}(s,u_n(s))\,\mathrm{d}s \tag{4.44}$$

*for all* $t \in [0,T]$. *Set* $V_n := V_{u_n}$ *(cf. (2.13)).*

*Then,*

$$\exists C > 0 \; \forall n \in \mathbb{N} : \qquad \sup_{t\in[0,T]} \mathscr{F}_0(u_n(t)) + V_n(T) \le C\,. \tag{4.45}$$

*Furthermore, there exist a subsequence* $k \mapsto n_k$ *and functions* $u \in \mathrm{BV}([0,T];X)$*,* $\mathsf{E}, \mathsf{V} \in \mathrm{BV}([0,T])$*, and* $\mathsf{P} \in L^\infty(0,T)$*, such that*

$$u_{n_k}(t) \to u(t) \qquad \text{for all } t \in [0,T], \tag{4.46a}$$

$$\mathscr{E}(t, u_{n_k}(t)) \to \mathsf{E}(t) \qquad \text{for all } t \in (0,T], \tag{4.46b}$$

$$V_{n_k}(t) \to \mathsf{V}(t) \qquad \text{for all } t \in (0,T], \tag{4.46c}$$

$$\mathscr{P}(t, u_{n_k}(t)) \rightharpoonup^* \mathsf{P} \qquad \text{in } L^\infty(0,T), \tag{4.46d}$$

*so that* $u(0) = u_0$ *and there hold*

$$\mathsf{d}(u(s), u(t)) \le \mathsf{V}(t) - \mathsf{V}(s) \qquad \text{for all } 0 \le s \le t \le T, \tag{4.47a}$$

$$\mathsf{E}(t) \ge \mathscr{E}(t, u(t)) \qquad \text{for all } t \in (0,T], \text{ with } \mathsf{E}(0) = \mathscr{E}(0, u_0). \tag{4.47b}$$

*Furthermore, for every* $t \in \mathrm{J}_u$ *there exist two sequences* $\alpha_k \uparrow t$ *and* $\beta_k \downarrow t$ *such that*

$$u_{n_k}(\alpha_k) \to u(t-) \quad \text{and} \quad u_{n_k}(\beta_k) \to u(t+)\,. \tag{4.48}$$

*Finally, the functions* $(u, \mathsf{E}, \mathsf{V}, \mathsf{P})$ *comply with*

$$\mathsf{E}(t) + \mathsf{V}(t) = \mathsf{E}(s) + \mathsf{V}(s) + \int_s^t \mathsf{P}(r)\,\mathrm{d}r \quad \text{for all } 0 \le s \le t \le T\,. \tag{4.49}$$

The *proof* follows by trivially adapting the argument for [18, Thm. 7.2]. Let us only mention that estimate (4.45) derives from (4.44), where the integral term on the right-hand side involving the power functional is estimated by resorting to the power control ($E_2$). As for (4.48), it can be shown by suitably adapting the Helly-type compactness argument yielding (4.46a).

In the next Secs. 4.1 and 4.2, we will carry out the proof of Theorem 1 and, respectively, outline the argument for Theorem 2. In fact, in Section 5 we will develop the proof of the main technical lower semicontinuity result underlying the limit passage as $\mu_n \uparrow \infty$ in the Visco-Energetic energy balance ($E_{\mathsf{d},\mathsf{c}_{\mu_n}}$) and leading to the upper energy estimate ($E_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$).

## 4.1 Proof Theorem 1

We apply Proposition 2 and deduce that there exist a subsequence $(u_{n_k})_k$ of $\mathrm{VE}_{\mu_{n_k}}$ solutions, and a curve $u \in \mathrm{BV}([0,T];X)$, such that (4.46), (4.47), and (4.49) hold. In what follows, for simplicity we shall denote the sequence of curves $(u_{n_k})_k$ by $(u_k)_k$

and accordingly write $\mu_k$ in place of $\mu_{n_k}$. We split the argument for proving that the limiting curve $u$ is an Energetic solution in some steps.

***Claim* 1*: there holds***

$$\mathsf{E}(t) = \mathscr{E}(t,u(t)), \quad \limsup_{k\to\infty} \mathscr{P}(t,u_k(t)) \leq \mathscr{P}(t,u(t)) \\ \text{for all } t \in [0,T] \setminus \widetilde{\mathrm{J}} \text{ with } \widetilde{\mathrm{J}} := \cap_{m\in\mathbb{N}} \cup_{k\geq m} \mathrm{J}_{u_k}, \tag{4.50}$$

*i.e., the countable set* $\widetilde{\mathrm{J}}$ *is the* $\limsup$ *of the sets* $(\mathrm{J}_{u_k})_k$*. As a result,*

$$\mathsf{P}(t) \leq \mathscr{P}(t,u(t)) \quad \text{for a.a. } t \in (0,T). \tag{4.51}$$

To prove (4.50) at a fixed $t \in [0,T] \setminus \widetilde{\mathrm{J}}$, we observe that, since $t \in [0,T] \setminus \mathrm{J}_{u_k}$ for every $k \geq m$ and $m \in \mathbb{N}$ a given index (only) depending on $t$, the stability condition for all $y \in X$ and for all $k \geq m$

$$\mathscr{E}(t,u_k(t)) \leq \mathscr{E}(t,y) + \mathsf{d}(u_k(t),y) + \frac{\mu_k}{2}\mathsf{d}^2(u_k(t),y) \tag{4.52}$$

holds. We choose $y = u(t)$ in (4.52) and thus deduce that $\limsup_{k\to\infty} \mathscr{E}(t,u_k(t)) \leq \mathscr{E}(t,u(t))$. Hence, we conclude the energy convergence

$$\mathscr{E}(t,u_k(t)) \to \mathscr{E}(t,u(t)) \qquad \text{for all } t \in [0,T] \setminus \widetilde{\mathrm{J}}, \tag{4.53}$$

whence the first of (4.50). The $\limsup$ inequality for the power term in (4.50) follows from ($\mathrm{E}_3$). Then, since the set $\widetilde{\mathrm{J}}$ is negligible, we have for every $t \in (0,T)$ and $r \in (0,(T-t)\wedge t)$

$$\int_{t-r}^{t+r} \mathsf{P}(s)\,\mathrm{d}s \leq \limsup_{k\to\infty} \int_{t-r}^{t+r} \mathscr{P}(s,u_k(s))\,\mathrm{d}s \leq \int_{t-r}^{t+r} \mathscr{P}(s,u(s))\,\mathrm{d}s, \tag{4.54}$$

where the second inequality follows from the second of (4.50) and the Fatou Lemma, taking into account that $\sup_{t\in[0,T]} \mathscr{P}(t,u_k(t)) \leq C_P \sup_{t\in[0,T]} \mathscr{F}(t,u_k(t)) \leq C$ by virtue of ($\mathrm{E}_2$), (2.27), and estimate (4.45). Therefore, (4.51) ensues upon dividing (4.54) by $r$ and taking the limit as $r \downarrow 0$.

***Claim* 2*: the curve u complies with***

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d}}(u,[s,t]) \leq \mathscr{E}(s,u(s)) + \int_s^t \mathscr{P}(r,u(r))\,\mathrm{d}r \\ \text{for all } t \in (0,T],\ s \in (0,t) \setminus \widetilde{\mathrm{J}}, \text{ and } s = 0. \tag{4.55}$$

The upper energy estimate (4.55) ensues from (4.49), taking into account (4.47), (4.50), and (4.51).

***Claim* 3*:***

$$u(t) \in \mathscr{S}_{\mathsf{d}}(t) \quad \text{for every } t \in [0,T] \setminus \widetilde{\mathrm{J}}. \tag{4.56}$$

It follows from passing to the limit as $k \to \infty$ in the stability condition (4.52).

***Claim* 4*:***

$$
\begin{aligned}
&u(t-), u(t+) \in \mathscr{S}_{\mathsf{d}}(t) \quad \text{for every } t \in (0,T),\\
&u(0+) \in \mathscr{S}_{\mathsf{d}}(0), \qquad u(T-) \in \mathscr{S}_{\mathsf{d}}(T).
\end{aligned}
\tag{4.57}
$$

Let us only prove the assertion at $t \in (0,T)$ and for $u(t+)$: since the latter right limit exists, we have that $u(t+) = \lim_{s\downarrow t,\, s\in(t,T)\setminus\widetilde{\mathrm{J}}} u(s)$. Therefore, $u(t+) \in \mathscr{S}_{\mathsf{d}}(t)$ follows from the previously obtained (4.56), combined with the closedness of the stable set $\mathscr{S}_{\mathsf{d}}$, cf. (3.37).

***Claim*** *5:*

$$
u(t) \in \mathscr{S}_{\mathsf{d}}(t) \quad \text{for every } t \in (0,T] \cap \widetilde{\mathrm{J}}. \tag{4.58}
$$

*Therefore, u complies with the stability condition* ($\mathrm{S}_{\mathsf{d}}$)*.*

We consider the upper energy estimate (4.55) written on the interval $[s,t]$, for every $s \in (0,t) \setminus \widetilde{\mathrm{J}}$, and then take the limit of the right-hand side as $s \uparrow t$. We use that $u(t-) = \lim_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} u(s)$, and that

$$
\limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \mathscr{E}(s,u(s)) \le \mathscr{E}(t,u(t-)). \tag{4.59}
$$

This follows from applying the stability condition $u(s) \in \mathscr{S}_{\mathsf{d}}(s)$, which holds at all $s \in (0,t) \setminus \widetilde{\mathrm{J}}$, with competitor $y = u(t-)$. Therefore $\mathscr{E}(s,u(s)) \le \mathscr{E}(s,u(t-)) + \mathsf{d}(u(s),u(t-))$, which yields

$$
\limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \mathscr{E}(s,u(s)) \le \limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \mathscr{E}(s,u(t-)). \tag{4.60}
$$

In turn,

$$
\begin{aligned}
&\limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \left(\mathscr{E}(s,u(t-)) - \mathscr{E}(t,u(t-))\right) \mathrm{d}t\\
&\overset{(1)}{\le} \limsup_{s\uparrow t} \int_s^t |\mathscr{P}(r,u(t-))|\,\mathrm{d}r\\
&\overset{(2)}{\le} C \limsup_{s\uparrow t} (t-s) = 0
\end{aligned}
\tag{4.61}
$$

with (1) due to (2.26) and (2) to the power-control estimate

$$
|\mathscr{P}(r,u(t-))| \le C\mathscr{F}_0(u(t-)) \ \le C\,. \tag{4.62}
$$

In (4.62) the first inequality ensues from ($\mathrm{E}_2$) and (2.27), while the second one from the lower semicontinuity of $u \mapsto \mathscr{F}_0(u)$, which gives $\mathscr{F}_0(u(t-)) \le \liminf_{s\uparrow t} \mathscr{F}_0(u(s))$ $\le C$ thanks to the energy bound $\sup_{t\in[0,T]} \mathscr{F}_0(u(t)) \le C$, deriving from estimate (4.45) by the lower semicontinuity of $\mathscr{F}_0$. Combining (4.60) with (4.61) we thus conclude (4.59). We also observe that

$$
\liminf_{s\uparrow t} \mathrm{Var}_{\mathsf{d}}(u,[s,t]) \ge \mathsf{d}(u(t-),u(t))\,. \tag{4.63}
$$

On account of (4.59) and (4.63), from (4.55) we deduce the jump estimate

$$\mathscr{E}(t,u(t)) + \mathsf{d}(u(t-),u(t)) \leq \mathscr{E}(t,u(t-)) \qquad \text{for every } t \in (0,T] \cap \widetilde{\mathrm{J}}. \tag{4.64}$$

We combine this with the previously obtained stability condition (4.57) to conclude (4.58).

***Claim*** 6***:*** *the curve u complies with the lower energy estimate*

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d}}(u,[0,t]) \geq \mathscr{E}(0,u(0)) + \int_0^t \mathscr{P}(r,u(r))\,\mathrm{d}r \tag{4.65}$$

*for all* $t \in [0,T]$, *and thus with the energy balance* ($\mathrm{E}_{\mathsf{d}}$).
We either apply [10, Prop. 2.1.23] or [18, Lemma 6.2, Thm. 6.5], to conclude (4.65) from the previously obtained ($\mathrm{S}_{\mathsf{d}}$).

***Claim*** 7***:*** *the convergence of the energies* $\mathscr{E}(t,u_k(t)) \to \mathscr{E}(t,u(t))$ *holds at every* $t \in [0,T]$.

It follows from (4.46b) and (4.47b) that $\liminf_{k\to\infty} \mathscr{E}(t,u_k(t)) \geq \mathscr{E}(t,u(t))$ for every $t \in [0,T]$. To prove the converse inequality for the lim sup, we resort to a by now classical argument based on the comparison of the energy balances ($\mathrm{E}_{\mathsf{d}}$) and ($\mathrm{E}_{\mathsf{d,c}}$). Indeed, we have

$$\begin{aligned}
&\limsup_{k\to\infty} \mathscr{E}(t,u_k(t)) \\
&\overset{(1)}{\leq} \limsup_{k\to\infty} \mathscr{E}(0,u_k^0) + \limsup_{k\to\infty} \int_0^t \mathscr{P}(r,u_k(r))\,\mathrm{d}r - \liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_{\mu_k}}(u_k,[0,t]) \\
&\overset{(2)}{\leq} \mathscr{E}(0,u_0) + \int_0^t \mathscr{P}(r,u(r))\,\mathrm{d}r - \mathrm{Var}_{\mathsf{d}}(u,[0,t]) \overset{(3)}{=} \mathscr{E}(t,u(t))\,,
\end{aligned}$$

with (1) due to ($\mathrm{E}_{\mathsf{d,c}}$), (2) following from the assumed convergence of the initial data (3.40), from (4.46d) combined with (4.51), and from (4.43) and, finally, (3) due to the just obtained energy balance ($\mathrm{E}_{\mathsf{d}}$).

This concludes the proof of Theorem 1. ∎

## 4.2 Proof Theorem 2

Proposition 2 ensures that any sequence $(u_n)_n$ of VE solutions, corresponding to parameters $\mu_n \to \infty$, admits a subsequence $(u_{n_k})_k$ converging to a curve $u \in \mathrm{BV}([0,T];X)$ in the sense of (4.46) and (4.47); as in the proof of Thm. 1, hereafter we will write $u_k$, $\mu_k$, and $\mathsf{c}_k$ in place of $u_{n_k}$, $\mu_{n_k}$, and $\mathsf{c}_{\mu_k}$, respectively. Thanks to the chain rule from condition ($\mathrm{E}_4$), in order to prove that $u$ is a BV solution it is sufficient to verify the local stability ($\mathrm{S}_{\mathsf{d},\mathrm{loc}}$) and the upper energy estimate ($\mathrm{E}_{\mathsf{d,v}}^{\mathrm{ineq}}$), cf. [12, Prop. 4.2, Thm. 4.3]. The convergence of the energies $\mathscr{E}(t,u_k(t)) \to \mathscr{E}(t,u(t))$ holds at every $t \in [0,T]$ will then follow from comparing the energy balances ($\mathrm{E}_{\mathsf{d,c}}$) and ($\mathrm{E}_{\mathsf{d,v}}$), similarly as in Claim 7 of the proof of Thm. 1.

▷ **The local stability condition** ($\mathrm{S}_{\mathsf{d},\mathrm{loc}}$)**.**

As in the proof of Theorem 1, we introduce the set $\widetilde{\mathrm{J}} := \cap_{m\in\mathbb{N}} \cup_{k\geq m} \mathrm{J}_{u_k}$. Since D-stability implies local stability, we have that for every $t \in [0,T]\setminus\widetilde{\mathrm{J}}$ there holds

$$|\mathrm{D}\mathscr{E}|(t,u_k(t)) \leq 1 \quad \text{for all } k \geq m, \tag{4.66}$$

with $m \in \mathbb{N}$ depending on $t$. Taking into account the energy bound (4.45) as well, we are in a position to exploit the lower semicontinuity property ensured by ($\mathrm{E}_3'$). Taking the $\liminf_{k\to\infty}$ of (4.66), we thus deduce that

$$|\mathrm{D}\mathscr{E}|(t,u(t)) \leq 1 \quad \text{for all } t \in [0,T]\setminus\widetilde{\mathrm{J}}. \tag{4.67}$$

We also conclude that

$$|\mathrm{D}\mathscr{E}|(t,u(t-)),\ |\mathrm{D}\mathscr{E}|(t,u(t+)) \leq 1 \quad \text{for all } t \in (0,T), \tag{4.68}$$

and analogously for $|\mathrm{D}\mathscr{E}|(0,u(0+))$ and $|\mathrm{D}\mathscr{E}|(T,u(T-))$, by arguing in the very same way as for **Claim** 4 in the proof of Theorem 2. Clearly, we then have the local stability condition at all points in $[0,T]\setminus\mathrm{J}_u$.

▷ **The upper energy estimate** ($\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$)**.**

Combining the energy bound (4.45) and the slope estimate (4.66) with convergence (4.46a) and resorting to ($\mathrm{E}_3'$), we conclude that $\limsup_{k\to\infty}\mathscr{P}(t,u_k(t)) \leq \mathscr{P}(t,u(t))$ for all $t\in[0,T]\setminus\widetilde{\mathrm{J}}$. Therefore, the very same argument as for **Claim** 1 in the proof of Theorem 2 yields that $\mathsf{P}(t) \leq \mathscr{P}(t,u(t))$ for almost all $t\in(0,T)$. All in all, taking the $\liminf_{k\to\infty}$ in ($\mathrm{E}_{\mathsf{d},\mathsf{c}_{\mu_k}}$) and exploiting the initial data convergence (3.40), the previously obtained (4.47b), and the above estimate for $\mathsf{P}$, we infer that

$$\mathscr{E}(T,u(T)) + \liminf_{k\to\infty}\mathrm{Var}_{\mathsf{d},\mathsf{c}_{\mu_k}}(u_k,[0,T]) \leq \mathscr{E}(0,u(0)) + \int_0^T \mathscr{P}(r,u(r))\,\mathrm{d}r\,.$$

In order to conclude ($\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$), it thus remains to show that

$$\liminf_{k\to\infty}\mathrm{Var}_{\mathsf{d},\mathsf{c}_{\mu_k}}(u_k,[0,T]) \geq \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[0,T])\,.$$

This will be guaranteed by the upcoming result, whose proof will be developed throughout Section 5. ∎

**Theorem 4.** *Let* $\mathscr{E}:[0,T]\times X\to\mathbb{R}$ *comply with* ($\mathrm{E}_1$), ($\mathrm{E}_2$), *and* ($\mathrm{E}_3'$). *Let* $\mu_k\uparrow\infty$ *and* $(u_k)_k$, $u\in\mathrm{BV}([0,T];X)$ *fulfill*

$$\exists C_F > 0 \; \forall k \in \mathbb{N} : \sup_{t\in[0,T]} \mathscr{F}_0(u_k(t)) \le C_F , \tag{4.69a}$$

$$u_k(t) \to u(t) \quad \textit{for every } t \in [0,T], \tag{4.69b}$$

$$\begin{aligned} &\forall t \in \mathrm{J}_u \; \exists (\alpha_k)_k, (\beta_k)_k \subset [0,T] \textit{ with} \\ &\alpha_k \uparrow t, \; \beta_k \downarrow t \textit{ and } u_k(\alpha_k) \to u(t-), \; u_k(\beta_k) \to u(t+). \end{aligned} \tag{4.69c}$$

*Then,*

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b]) \ge \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[a,b]) \qquad \textit{for all } [a,b] \subset [0,T]. \tag{4.70}$$

## 5 Proof of Theorem 4

Let us mention in advance the argument for proving the lower semicontinuity inequality (4.70) follows the same steps, outlined below, as those for the lower semicontinuity result [15, Prop. 7.3] in the context of the limit passage from 'viscous' gradient systems to BV solutions. Nevertheless, we have to cope with the (nontrivial) technical issues peculiar of the fact that the kind of transitions describing the system behavior at jumps changes upon passing from VE to BV solutions. This problem will be addressed in the proof of Proposition 3 ahead.

**Outline of the proof of Theorem 4.**

Up to the extraction of a (not relabeled) subsequence and modifying the constant $C_F$ from (4.69a), we may suppose that

$$\sup_k \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b]) \le C_F , \tag{5.71}$$

too. We introduce a sequence of non-negative and bounded Borel measures $\eta_k$ by defining them on intervals via

$$\eta_k([a,b]) := \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b]) \quad \text{for all } [a,b] \subset [0,T].$$

In view of (5.71), we have that, up to a further extraction, there exists a Borel measure $\eta$ such that $\eta_k \rightharpoonup^* \eta$ in duality with $\mathrm{C}([0,T])$. Observe that, by (4.43), we have

$$\eta([a,b]) \ge \limsup_{k\to\infty} \eta_k([a,b]) \ge \limsup_{k\to\infty} \mathrm{Var}_{\mathsf{d}}(u_k,[a,b]) \ge \mathrm{Var}_{\mathsf{d}}(u,[a,b]) \ge \nu_u^{\mathrm{d}}([a,b]),$$

with $\nu_u^{\mathrm{d}}$ the diffuse measure associated with $u$ via (2.14). Therefore we obtain

$$\eta \ge \nu_u^{\mathrm{d}} . \tag{5.72}$$

We now exploit Proposition 3 ahead to conclude that, for every $t \in \mathrm{J}_u$ and any two sequences $\alpha_k \uparrow t$ and $\beta_k \downarrow t$ fulfilling (4.69c), there holds

$$\eta(\{t\}) \geq \limsup_{k\to\infty} \eta_k([\alpha_k,\beta_k]) \geq \liminf_{k\to\infty} \eta_k([\alpha_k,\beta_k]) \geq \mathsf{v}(t,u(t-),u(t+)). \quad (5.73)$$

Analogously, we can prove that

$$\begin{aligned} &\limsup_{k\to\infty} \eta_k([\alpha_k,t]) \geq \mathsf{v}(t,u(t-),u(t)), \\ &\limsup_{k\to\infty} \eta_k([t,\beta_k]) \geq \mathsf{v}(t,u(t),u(t+)). \end{aligned} \quad (5.74)$$

Arguing in the very same way as in the proof of [15, Prop. 7.3], we combine (5.72), (5.73), and (5.74) with the representation

$$\begin{aligned} &\mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[a,b]) \\ &= \nu_u^{\mathrm{d}}([a,b]) + \mathrm{Jmp}_{\mathsf{v}}(u;[a,b]) \\ &= \nu_u^{\mathrm{d}}([a,b]) + \mathsf{v}(a,u(a),u(a+)) + \mathsf{v}(b,u(b-),u(b)) \\ &\qquad + \sum_{t\in \mathrm{J}_u\cap(a,b)} (\mathsf{v}(t,u(t-),u(t))+\mathsf{v}(t,u(t),u(t+))), \end{aligned}$$

cf. (2.20), to conclude the desired lower semicontinuity inequality (4.70). ∎

The proof of the upcoming result is developed throughout Section 5.1.

**Proposition 3.** *Let* $\mathscr{E} : [0,T]\times X \to \mathbb{R}$ *comply with* ($\mathrm{E}_1$)*,* ($\mathrm{E}_2$)*, and* ($\mathrm{E}_3'$)*. Let* $\mu_k \uparrow \infty$ *and* $(u_k)_k, u \in \mathrm{BV}([0,T];X)$ *fulfill* (4.69) *and* (5.71)*. For every* $t \in \mathrm{J}_u$*, pick two sequences* $(\alpha_k)_k$*,* $(\beta_k)_k$ *converging to* $t$ *and fulfilling* (4.69c)*. Then,*

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[\alpha_k,\beta_k]) \geq \mathsf{v}(t,u(t-),u(t+)). \quad (5.75)$$

## *5.1 Proof of Proposition 3*

We split the argument in some steps, some of which in turn rely on some technical results proved in the Appendix.

**Step 1: reparameterization.**

The curve $u_k$ has at most countably many jump points $(t_m^k)_{m\in M_k}$ between the points $\alpha_k$ and $\beta_k$. We now suitably reparameterize both the continuous pieces of the trajectory $u_k$, as well as the optimal transitions $\vartheta_j^k$ connecting the left and right limits $u_k(t_j^k-)$ and $u_k(t_j^k+)$ at a jump point $t_j^k$. We will then glue all of them together to obtain a sequence of curves $(\mathsf{u}_k)_k$, defined on compact sets $(C_k)_k$, which shall enjoy suitable estimates (cf. Step 2), allowing for a refined compactness argument both for the curves $\mathfrak{u}_k$ and for the sets $C_k$.

We set

$$\mathfrak{m}_k := \beta_k - \alpha_k + \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) + \sum_{m \in M_k} 2^{-m}$$

and define the rescaling function $\mathfrak{s}_k : [\alpha_k, \beta_k] \to [0, \mathfrak{m}_k]$ by

$$\mathfrak{s}_k(t) := t - \alpha_k + \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, t]) + \sum_{\{m \in M_k : t_m^k \le t\}} 2^{-m}.$$

Observe that $\mathfrak{s}_k$ is strictly increasing, with jump set $\mathrm{J}_{\mathfrak{s}_k} = (t_m^k)_{m \in M_k}$. We introduce the notation

$$I_m^k := (\mathfrak{s}_k(t_m^k-), \mathfrak{s}_k(t_m^k+)), \quad I_k := \cup_{m \in M_k} I_m^k, \quad \Lambda_k := [\mathfrak{s}_k(\alpha_k), \mathfrak{s}_k(\beta_k)].$$

On $\Lambda_k \setminus I_k$ the inverse $\mathfrak{t}_k : \Lambda_k \setminus I_k \to [\alpha_k, \beta_k]$ of $\mathfrak{s}_k$ is well defined and Lipschitz continuous. We set

$$\mathfrak{u}_k(s) := (u_k \circ \mathfrak{t}_k)(s) \qquad \text{for all } s \in \Lambda_k \setminus I_k. \tag{5.76}$$

The curve $\mathfrak{u}_k$ is also Lipschitz, and satisfies

$$\mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(\mathfrak{u}_k, [s_0, s_1]) \le (s_1 - s_0) \quad \text{for all } [s_0, s_1] \subset \Lambda_k \setminus I_k. \tag{5.77}$$

We check (5.77) in the case in which $s_0 = \mathfrak{s}_k(t_0)$ and $s_1 = \mathfrak{s}_k(t_1)$, with $t_0 < t_1$ belonging to the same connected component of $[\alpha_k, \beta_k] \setminus (t_m^k)_{m \in M_k}$ (the other case is completely analogous). Then, we observe that

$$s_1 - s_0 = \mathfrak{s}_k(t_1) - \mathfrak{s}_k(t_0) = t_1 - t_0 + \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [t_0, t_1]) \ge \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(\mathfrak{u}_k, [s_0, s_1]).$$

We now recall [18, Thm. 3.14], ensuring that at every jump point $t_m^k$ there exists an optimal transition $\vartheta_m^k$ that is continuous on a compact set $E_m^k$, *tight* (i.e. it fulfills $\vartheta_m^k(J^-) \ne \vartheta_m^k(J^+)$ for every "hole" $J \in \mathfrak{h}(E_m^k)$), and such that

$$\begin{aligned}
&u(t_m^k-) = \vartheta_m^k((E_m^k)^-), \qquad u(t_m^k+) = \vartheta_m^k((E_m^k)^+), \qquad u(t_m^k) \in \vartheta_m^k(E_m^k),\\
&\mathscr{E}(t_m^k, u(t_m^k-)) - \mathscr{E}(t_m^k, u(t_m^k+)) = \mathsf{c}(t_m^k, u(t_m^k-), u(t_m^k+))\\
&= \mathrm{Trc}_{\mathrm{VE}}(t_m^k, \vartheta_m^k, E_m^k)\\
&= \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \sum_{r \in E_m^k \setminus (E_m^k)^+} \mathscr{R}(t_m^k, \vartheta_m^k(r)).
\end{aligned} \tag{5.78}$$

We adapt the calculations from [18, Lemma 5.1] and define the rescaling function $\sigma_m^k$ on $E_m^k$ by

$$\begin{aligned}
\sigma_m^k(t) := &\frac{1}{2^m} \frac{t - (E_m^k)^-}{(E_m^k)^+ - (E_m^k)^-} + \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k \cap [(E_m^k)^-, t])\\
&+ \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k \cap [(E_m^k)^-, t]) + \sum_{r \in [(E_m^k)^-, t] \setminus (E_m^k)^+} \mathscr{R}(t_m^k, \vartheta_m^k(r)) + \mathfrak{s}_k(t_m^k-)
\end{aligned}$$

for all $t \in E_m^k$. It can be checked that $\sigma_m^k$ is continuous and strictly increasing, with image a compact set $S_m^k \subset I_m^k$ such that

$$
\begin{aligned}
&(S_m^k)^- = \sigma_m^k((E_m^k)^-) = \mathfrak{s}_k(t_m^k-) \quad \text{ and} \\
&(S_m^k)^+ = \sigma_m^k((E_m^k)^+) \\
&= \frac{1}{2^m} + \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \sum_{r \in E_m^k \setminus (E_m^k)^+} \mathscr{R}(t_m^k, \vartheta_m^k(r)) + \mathfrak{s}_k(t_m^k-) \\
&= \mathfrak{s}_k(t_m^k+) \,.
\end{aligned}
$$

The inverse function $\tau_m^k : S_m^k \to E_m^k$ is Lipschitz continuous.

We then introduce the set

$$
C_k := (\Lambda_k \backslash I_k) \cup (\cup_{m \in M_k} S_m^k) \,.
$$

It is not difficult to check that $C_k$ is a closed subset of $\Lambda_k$. We extend the functions $\mathfrak{t}_k$ and $\mathfrak{u}_k$, so far defined on $\Lambda_k \setminus I_k$, only, to the set $C_k$ by setting

$$
\mathfrak{t}_k(s) \equiv t_m^k \quad \text{and} \quad \mathfrak{u}_k(s) := \vartheta_m^k(\tau_m^k(s)) \qquad \text{whenever } s \in S_m^k \text{ for some } m \in M_k.
$$

Since $u(t_m^k-) = \vartheta_m^k((E_m^k)^-)$ and $u(t_m^k+) = \vartheta_m^k((E_m^k)^+)$, we have that the extended curve $\mathfrak{u}_k \in \mathrm{C}(C_k;X)$. Furthermore, $\mathfrak{u}_k \in \mathrm{BV}(C_k;X)$: indeed,

$$
\begin{aligned}
\mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, S_m^k) = \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k), \qquad \mathrm{GapVar}_{\mathsf{d}}(\mathfrak{u}_k, S_m^k) = \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k), \\
\sum_{s \in S_m^k \setminus \{(S_m^k)^+\}} \mathscr{R}(t_m^k, \mathfrak{u}_k(s)) = \sum_{r \in E_m^k \setminus \{(E_m^k)^+\}} \mathscr{R}(t_m^k, \vartheta_m^k(r)),
\end{aligned}
\tag{5.79a}
$$

as well as

$$
\mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, S_m^k \cap [s_0, s_1]) \leq (s_1 - s_0) \qquad \text{for all } s_0, s_1 \in S_m^k \text{ with } s_0 < s_1 \,. \tag{5.79b}
$$

**Step 2: a priori estimates**

It follows from (5.71) and from the fact that $(\beta_k - \alpha_k) \downarrow 0$, that

$$
C_k^+ = \mathfrak{m}_k \leq \beta_k - \alpha_k + \mathrm{Var}_{\mathsf{d},\mathsf{c}}(u_k, [\alpha_k, \beta_k]) + 2 \leq 2C_F \tag{5.80}
$$

(up to modifying the constant $C_F$). Moreover, in view of (5.71), (5.77), and (5.79b) we have

$$
\sup_{k \in \mathbb{N}} \mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, C_k) \leq C, \tag{5.81a}
$$

$$
\mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, C_k \cap [s_0, s_1]) \leq (s_1 - s_0) \quad \text{for all } s_0, s_1 \in C_k, \ s_0 < s_1, \text{ and } k \in \mathbb{N}. \tag{5.81b}
$$

Finally, we remark that

$$\sup_{k\in\mathbb{N}}\sup_{s\in C_k}\mathscr{F}_0(\mathfrak{u}_k(s))\leq C_F. \tag{5.81c}$$

Indeed, we have that

$$\sup_{s\in\Lambda_k\setminus I_k}\mathscr{F}_0(\mathfrak{u}_k(s))=\sup_{t\in[\alpha_k,\beta_k]\setminus(t_m^k)_{m\in M_k}}\mathscr{F}_0(u_k(t))\leq C_F$$

in view of (4.69). Furthermore, it follows from [18, Thm. 3.16] that for all $r\in E_m^k$ there holds

$$\begin{aligned}\mathscr{E}(t_m^k,\vartheta_m^k(r))+\mathsf{d}(\vartheta_m^k(r),\vartheta_m^k((E_m^k)^-))&\leq\mathscr{E}(t_m^k,\vartheta_m^k(r))+\mathrm{Var}_{\mathsf{d}}(\vartheta_m^k,E_m^k\cap[(E_m^k)^-,r])\\&\leq\mathscr{E}(t_m^k,\vartheta_m^k((E_m^k)^-))=\mathscr{E}(t_m^k,u_k(t_m^k-)).\end{aligned}$$

Therefore,

$$\sup_{s\in S_m^k}\mathscr{F}_0(\mathfrak{u}_k(s))=\sup_{r\in E_m^k}\mathscr{F}_0(\vartheta_m^k(r))\leq\mathscr{F}_0(u_k(t_m^k-))\leq C_F.$$

All in all, we conclude (5.81c).

**Step 3: compactness**

By virtue of estimates (5.81), we are in a position to apply the compactness result [18, Thm. 5.4] and conclude that there exist a (not relabeled) subsequence, a compact set $C\subset[0,2C_F]$, and a function $\mathfrak{u}\in\mathrm{BV}(C;X)$ such that, as $k\to\infty$, there hold

1. $C_k\to C$ à la Kuratowski;
2. $\mathrm{graph}(\mathfrak{u})\subset\mathrm{Li}_{k\to\infty}\mathrm{graph}(\mathfrak{u}_k)$;
3. whenever $(s_k)_k\in C_k$ converge to $s\in C$, then $\mathfrak{u}_k(s_k)\to\mathfrak{u}(s)$;
4. $\mathfrak{u}_k((C_k)^\pm)\to\mathfrak{u}(C^\pm)$.

Therefore, $\mathfrak{u}(C^-)=u(t-)$, and $\mathfrak{u}(C^+)=u(t+)$. Furthermore, it follows from (5.81b) that the curve $\mathfrak{u}$ is Lipschitz on $C$. Finally, for later use let us point out that, since the functions $\mathfrak{t}_k$ take values in the intervals $[\alpha_k,\beta_k]$ shrinking to the singleton $\{t\}$, there holds

$$\lim_{k\to\infty}\sup_{s\in C_k}|\mathfrak{t}_k(s)-t|=0. \tag{5.82}$$

**Step 4: connectedness of $C$**

Observe that, since the sets $C_k$ are not, in general, connected, we cannot immediately deduce that $C$ is connected. We will however show that,

$$\forall I\in\mathfrak{h}(C)\text{ there holds }\mathfrak{u}(I^-)=\mathfrak{u}(I^+)=:\mathfrak{u}_I. \tag{5.83}$$

In view of this, we may extend $\mathfrak{u}$ to the whole interval $[0,C^+]$ by defining

$$\mathfrak{u}(s) := \mathfrak{u}_I \qquad \text{for all } s \in I \quad \text{for all } I \in \mathfrak{h}(C).$$

Hereafter, we will replace $C$ by $[0, C^+]$. We will split the proof of (5.83) in two claims.

***Claim*** 1***:*** *for every $I \in \mathfrak{h}(C)$ there exist $J_k$ such that*

$$J_k \in \mathfrak{h}(C_k) \text{ and } \lim_{k\to\infty} J_k^- = I^-, \quad \lim_{k\to\infty} J_k^+ = I^+. \tag{5.84}$$

This follows by repeating the very same arguments as in the proof of [18, Thm. 5.3].

***Claim*** 2***:*** *there holds* $\mathfrak{u}(I^-) = \mathfrak{u}(I^+)$. In view of the compactness property (3) from Step 3, there holds $\mathfrak{u}_k(J_k^\pm) \to \mathfrak{u}(I^\pm)$. Therefore,

$$\begin{aligned} \mathsf{d}(\mathfrak{u}(I^-), \mathfrak{u}(I^+)) &= \lim_{k\to\infty} \mathsf{d}(\mathfrak{u}_k(J_k^-), \mathfrak{u}_k(J_k^+)) \\ &\le \limsup_{k\to\infty} \frac{1}{2\mu_k^{1/2}} \left(\mu_k \mathsf{d}^2(\mathfrak{u}_k(J_k^-), \mathfrak{u}_k(J_k^+)) + 1\right) \\ &\le \limsup_{k\to\infty} \frac{1}{\mu_k^{1/2}} \left(\mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) + 1\right) = 0, \end{aligned}$$

where we have used Young's equality and estimate (5.71).

## Step 5: estimate of the transition cost and conclusion of the proof

With Steps 3 and 4 we have shown that the Lipschitz continuous curve $\mathfrak{u}$ is defined on the interval $[0, C^+]$ and connects the left and right limits $u(t-)$ and $u(t+)$. We now aim to prove that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) \ge \mathrm{Trc}_{\mathrm{BV}}(t, \mathfrak{u}, [0, C^+]) \ge \mathsf{v}(t, u(t-), u(t+)), \tag{5.85}$$

which will lead to (5.75).

Indeed, it follows from Lemma 1 that

$$\begin{aligned} \mathrm{Trc}_{\mathrm{BV}}(t, \mathfrak{u}, [0, C^+]) &= \int_0^{C^+} |\mathfrak{u}'|(s) \left(|\mathrm{D}\mathscr{E}|(t, \mathfrak{u}(s)) \vee 1\right) \mathrm{d}s \\ &= \sup \Big\{ \sum_{i=1}^N \mathsf{d}(\mathfrak{u}(\sigma_{i-1}), \mathfrak{u}(\sigma_i)) \inf_{\sigma\in[\sigma_{i-1},\sigma_i]} \left(|\mathrm{D}\mathscr{E}|(t, \mathfrak{u}(\sigma)) \vee 1\right) : \\ &\qquad (\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0, C^+]) \Big\}. \end{aligned} \tag{5.86}$$

Therefore, in what follows we will prove that

$$
\begin{aligned}
&\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[\alpha_k,\beta_k]) \\
&\geq \sum_{i=1}^{N} \mathsf{d}(\mathfrak{u}(\sigma_{i-1}),\mathfrak{u}(\sigma_i)) \inf_{\sigma\in[\sigma_{i-1},\sigma_i]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma))\vee 1)
\end{aligned} \tag{5.87}
$$

for every $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0,C^+])$.

Let us consider a given partition $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0,C^+])$ and fix an index $j \in \{1,\dots,N\}$. Preliminarily, we observe that, by the compactness property (1) in Step 3, there exist sequences $(\sigma_{j-1}^k)_k$, $(\sigma_j^k)_k \subset C_k$ such that, as $k\to\infty$, there holds

$$
\sigma_{j-1}^k \to \sigma_{j-1},\ \sigma_j^k \to \sigma_j \quad \text{and} \quad \mathfrak{u}_k(\sigma_{j-1}^k) \to \mathfrak{u}(\sigma_{j-1}),\ \mathfrak{u}_k(\sigma_j^k) \to \mathfrak{u}(\sigma_j) \tag{5.88}
$$

where the second convergence follows from the compactness property (3). We now distinguish two cases

1. $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma))\vee 1) = 1$;
2. $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} |\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) > 1$.

Clearly, the second case is equivalent to $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma))\vee 1) > 1$.
**Case (1):** In view of (5.88), we have

$$
\begin{aligned}
&\mathsf{d}(\mathfrak{u}(\sigma_{j-1}),\mathfrak{u}(\sigma_j)) \inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma))\vee 1) \\
&= \lim_{k\to\infty} \mathsf{d}(\mathfrak{u}_k(\sigma_{j-1}^k),\mathfrak{u}_k(\sigma_j^k)).
\end{aligned} \tag{5.89}
$$

**Case (2):** We have that $|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) > \delta > 1$ for all $\sigma\in[\sigma_{j-1},\sigma_j]$. First of all, we observe that

$$
\exists\,\bar\delta \in (1,\delta) \quad \exists\,\bar k \in \mathbb{N} \quad \inf_{k\geq \bar k} \inf_{\sigma\in[\sigma_{j-1}^k,\sigma_j^k]\cap C_k} |\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) \geq \bar\delta. \tag{5.90}
$$

To show this, we argue by contradiction and suppose that there exists a (not relabeled) subsequence along which $\inf_{\sigma\in[\sigma_{j-1}^k,\sigma_j^k]\cap C_k} |\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) \leq 1$. Since for every $k\in\mathbb{N}$ the inf on the compact set $[\sigma_{j-1}^k,\sigma_j^k]\cap C_k$ is attained by lower semicontinuity of the map $\sigma\mapsto |\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma))$, we deduce that there exists a sequence $(\tilde\sigma_k)_k$ with $|\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\tilde\sigma_k),\mathfrak{u}_k(\tilde\sigma_k)) \leq 1$, converging up to a subsequence to some $\tilde\sigma\in[\sigma_{j-1},\sigma_j]$. Now, $\mathfrak{t}_k(\tilde\sigma_k)\to t$ by (5.82) and $\mathfrak{u}_k(\tilde\sigma_k)\to\mathfrak{u}(\tilde\sigma)$ by the compactness property (3) from Step 3. Hence, using the lower semicontinuity of $|\mathrm{D}\mathscr{E}|$ granted by ($\mathrm{E}_3'$) we conclude that $|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\tilde\sigma)) \leq 1$, in contradiction with the assumption that $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} |\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) > 1$.

Observe that (5.90) implies that $\mathscr{R}(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) > 0$ for all $\sigma\in[\sigma_{j-1}^k,\sigma_j^k]\cap C_k$ and all $k\geq\bar k$. We now deduce the *uniform* positivity property

$$
\exists\, r>0 \qquad \inf_{k\geq\bar k} \inf_{\sigma\in[\sigma_{j-1}^k,\sigma_j^k]\cap C_k} \mathscr{R}(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) \geq r. \tag{5.91}
$$

Indeed, as for (5.90) we proceed by contradiction: if (5.91) did not hold, there would exist a sequence $(\tilde{\sigma}_k)_k$ with $\mathscr{R}(\mathsf{t}_k(\tilde{\sigma}_k), \mathsf{u}_k(\tilde{\sigma}_k)) \to 0$, converging to some $\tilde{\sigma} \in [\sigma_{j-1}, \sigma_j]$ that would fulfill $\mathscr{R}(t, \mathsf{u}(\tilde{\sigma})) = 0$ by the lower semicontinuity of $\mathscr{R}$. Now, by property (2.33), $\mathscr{R}(t, \mathsf{u}(\tilde{\sigma})) = 0$ would imply that $(t, \mathsf{u}(\tilde{\sigma}))$ belongs to the stable set $\mathscr{S}_{\mathsf{D}}$. In turn, the $\mathsf{D}$-stability condition (2.31) would imply that $|\mathrm{D}\mathscr{E}|(t, \mathsf{u}(\tilde{\sigma})) \leq 1$, against the standing assumption that $\inf_{\sigma \in [\sigma_{j-1}, \sigma_j]} |\mathrm{D}\mathscr{E}|(t, \mathsf{u}(\sigma)) > 1$.

Now, (5.91) entails that $\mathsf{t}_k(\sigma) \in (t_m^k)_{m \in M_k}$ for all $\sigma \in [\sigma_{j-1}^k, \sigma_j^k] \cap C_k =: \mathscr{L}_k$. But then, it is not difficult to realize that the function $\mathsf{t}_k$ must be constant on $\mathscr{L}_k$. Namely, there exists $m_k \in M_k$ such that $\mathsf{t}_k(\sigma) \equiv t_{m_k}^k$ for all $\sigma \in \mathscr{L}_k$. It was observed in [18, Rmk. 3.15] that the set $C_k^{\mathscr{R}} := \{s \in S_{m_k}^k \setminus \{(S_{m_k}^k)^+\} : \mathscr{R}(t_{m_k}^k, \mathsf{u}_k(s)) > 0\}$ is discrete. Trivially adapting the argument from [18, Rmk. 3.15], from (5.91) we in fact conclude that for all $k \geq \bar{k}$ the set $\mathscr{L}_k \subset C_k^{\mathscr{R}}$ consists of finitely many points $(r_\ell^k)_{\ell=1}^{L_k}$, and that the cardinality $L_k$ of the sets $\mathscr{L}_k$ is uniformly bounded with respect to $k$, i.e.

$$\sup_{k \geq \bar{k}} L_k \leq C < \infty. \tag{5.92}$$

Furthermore, notice that $r_\ell^k$ is the extremum of a hole of $C_k$ for every $\ell = 1, \ldots, L_k$. The compactness statement from Step 3 (cf. again [18, Thm. 5.4]) applies, yielding that, up to a subsequence,

1. the sets $(\mathscr{L}_k)_k$ converge in the sense of Kuratowski to a finite, thanks to (5.92), set $\mathscr{L} = (r_l)_{l=1}^L \subset [\sigma_{j-1}, \sigma_j]$, such that $\sigma_{j-1}$, $\sigma_j \in \mathscr{L}$.
2. for every $r_l \in \mathscr{L}$ there exists a sequence $(r_{\ell_k}^k(l))_k$, with $r_{\ell_k}^k(l) \in \mathscr{L}_k$ for every $k \in \mathbb{N}$, such that $\mathsf{u}_k(r_{\ell_k}^k(l)) \to \mathsf{u}(r_l)$. From now on, we will use the simplified notation $r_k(l)$ in place of $r_{\ell_k}^k(l)$;
3. whenever $r_{\ell_n}^{k_n} \in \overline{\mathscr{L}}_{k_n}$ converge to some $r_l \in \mathscr{L}$ as $n \to \infty$, then $\mathsf{u}_{k_n}(r_{\ell_n}^{k_n}) \to \mathsf{u}(r_l)$.

We now estimate $\mathsf{d}(\mathsf{u}(\sigma_{j-1}), \mathsf{u}(\sigma_j)) \inf_{\sigma \in [\sigma_{j-1}, \sigma_j]} (|\mathrm{D}\mathscr{E}|(t, \mathsf{u}(\sigma)) \vee 1)$ by interpolating between the points $\sigma_{j-1}$ and $\sigma_j$ the points $\mathscr{L} = (r_l)_{l=1}^L$. Thus we have

$$
\begin{aligned}
&\mathsf{d}(\mathfrak{u}(\sigma_{j-1}),\mathfrak{u}(\sigma_j)) \inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma))\vee 1)\\
&\le \mathsf{d}(\mathfrak{u}(\sigma_{j-1}),\mathfrak{u}(\sigma_j)) + \mathsf{d}(\mathfrak{u}(\sigma_{j-1}),\mathfrak{u}(\sigma_j)) \inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma))-1)\\
&\le \mathsf{d}(\mathfrak{u}(\sigma_{j-1}),\mathfrak{u}(\sigma_j)) + \sum_{l=1}^{L} \mathsf{d}(\mathfrak{u}(r_{l-1}),\mathfrak{u}(r_l))\,(|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(r_l))-1)\\
&\overset{(1)}{\le} \liminf_{k\to\infty} \mathsf{d}(\mathfrak{u}_k(\sigma_{j-1}^k),\mathfrak{u}_k(\sigma_j^k))\\
&\quad + \sum_{l=1}^{L} \liminf_{k\to\infty} \mathsf{d}(\mathfrak{u}_k(r_k(l-1)),\mathfrak{u}_k(r_k(l)))\sqrt{2\mu_k \mathscr{R}(t_{m_k}^k,\mathfrak{u}_k(r_k(l)))}\\
&\overset{(2)}{\le} \liminf_{k\to\infty} \mathsf{d}(\mathfrak{u}_k(\sigma_{j-1}^k),\mathfrak{u}_k(\sigma_j^k)) + \liminf_{k\to\infty} \sum_{l=1}^{L} \frac{\mu_k}{2}\mathsf{d}^2(\mathfrak{u}_k(r_k(l-1)),\mathfrak{u}_k(r_k(l)))\\
&\qquad + \liminf_{k\to\infty} \sum_{l=1}^{L} \mathscr{R}(t_{m_k}^k,\mathfrak{u}_k(r_k(l)))\,.
\end{aligned}
\tag{5.93}
$$

For (1), we have used that for every $l = 1,\ldots,L$ there exists a sequence $(r_k(l))_k$ fulfilling the aforementioned convergence property (2), and applied the forthcoming Lemma 3 with the choice $\psi(r) = r + \frac{1}{2}r^2$ (cf. (1.3)), so that $\psi^*(S) = \frac{1}{2}((S-1)_+)^2$, with $\tau_k := \mu_k^{-1}$, with $t_k := t_{m_k}^k \to t$ as $k \to \infty$, and with $u_k := \mathfrak{u}_k(r_k(l)) \to \mathfrak{u}(r_l)$. We then conclude that (cf. (6.102) ahead for the definition of the generalized Moreau-Yosida approximation $\mathscr{Y}^{\psi}_{\mu_k^{-1}}(\mathscr{E})$)

$$
\begin{aligned}
&(|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(r_l))-1) = (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(r_l))-1)_+\\
&\le \liminf_{k\to\infty} \sqrt{2\mu_k\left(\mathscr{E}(t_{m_k}^k,\mathfrak{u}_k(r_k(l))) - \mathscr{Y}^{\psi}_{\mu_k^{-1}}(\mathscr{E})(t_{m_k}^k,\mathfrak{u}_k(r_k(l)))\right)}\\
&= \liminf_{k\to\infty} \sqrt{2\mu_k \mathscr{R}(t,\mathfrak{u}_k(r_k(l)))} \qquad \text{for all } l = 1,\ldots,L.
\end{aligned}
\tag{5.94}
$$

Finally, for (2) in (5.93) we have applied Young's inequality.

Observe that the term multiplied by $\mu_k$ featuring on the right-hand side of (5.93) involves points that are extrema of holes in $C_k$. Therefore, it is estimated by $\mathrm{GapVar}_{\mathsf{d}}(\mathfrak{u}_k,C_k)$, whereas the third term is bounded by $\sum_{s\in S_{m_k}^k\setminus\{(S_{m_k}^k)^+\}} \mathscr{R}(t_{m_k}^k,\mathfrak{u}_k(s))$. Combining (5.89), and (5.93), and summing over all the points of $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0,C^+])$, we conclude the desired (5.87). This finishes the proof of Theorem 4. ∎

We conclude this section by giving the

**Outline of the proof of Proposition 1.**

The argument borrows some ideas both from the proof of Theorem 1, and of Theorem 2. Let us briefly sketch its steps.

$\triangleright$ **Compactness:** We again apply Prop. 2 and deduce the existence of a subsequence $(u_{n_k})_k$ converging to some $u \in \mathrm{BV}([0,T];X)$ in the sense of (4.46) and (4.47); hereafter we will again use the short-hands $u_k$, $\mu_k$, and $\mathsf{c}_k$ in place of $u_{n_k}$, $\mu_{n_k}$, and $\mathsf{c}_{\mu_k}$, respectively. We will use the notation

$$\mathsf{D}_{\mu_k}(u,v) := \mathsf{d}(u,v) + \tfrac{\mu_k}{2}\mathsf{d}^2(u,v), \qquad \mathsf{D}_{\mu}(u,v) := \mathsf{d}(u,v) + \tfrac{\mu}{2}\mathsf{d}^2(u,v),$$

and write $\mathrm{GapVar}_{\mathsf{d}}^{\mu_k}$, $\mathrm{GapVar}_{\mathsf{d}}^{\mu}$, $\mathscr{R}^{\mu_k}$, $\mathscr{R}^{\mu}$.

$\triangleright$ **The $\mathsf{D}_\mu$-stability condition:** As in *Claim* 1 within the proof of Thm. 1, we introduce the set $\widetilde{J} = \cap_{m\in\mathbb{N}} \cup_{k\geq m} \mathrm{J}_{u_k}$. First, we prove that the limit curve $u$ fulfills the stability condition $(\mathrm{S}_{\mathsf{D}_\mu})$ at every $t \in [0,T]\setminus\widetilde{J}$ by passing to the limit as $k\to\infty$ in the $\mathsf{D}_{\mu_k}$-stability condition for the curves $u_k$, holding on $[0,T]\setminus \mathrm{J}_{u_k}$. Secondly, we deduce the validity of the $\mathsf{D}_\mu$-stability condition at every $t\in[0,T]\setminus \mathrm{J}_u$ by density argument, similarly as in the proof of Thm. 1, *Claim* 4. Here we exploit the closedness of the $\mathsf{D}_\mu$-stable set $\mathscr{S}_{\mathsf{D}_\mu}$, which is in turn ensured by the lower semicontinuity of $\mathscr{R}^{\mu}$.

$\triangleright$ **The upper energy estimate $\leq$ in $(\mathrm{E}_{\mathsf{d},\mathsf{c}_\mu})$:** We show that for all $t\in[0,T]$

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{c}_\mu}(u,[0,t]) \leq \mathscr{E}(0,u(0)) + \int_0^t \mathscr{P}(s,u(s))\,\mathrm{d}s \tag{5.95}$$

by taking the $\liminf_{k\to\infty}$ in the analogous upper energy estimate for the curves $(u_k)_k$. Let us only comment on the proof of the key lower semicontinuity inequality

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b]) \geq \mathrm{Var}_{\mathsf{d},\mathsf{c}_\mu}(u,[a,b]) \qquad \text{for all } [a,b]\subset[0,T], \tag{5.96}$$

since for dealing with the other terms in (5.95) we repeat the very same arguments as in the proofs of Thms. 1 and 2.

First of all, we may suppose that the sequence $(u_k)_k$ complies with the conditions (4.69) of Thm. 4. Along the footsteps of the proof of Thm. 4, we introduce the Borel measures $\eta_k([a,b]) := \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b])$ and show that, up to a subsequence, they converge to a measure $\eta \geq \nu_u^{\mathsf{d}}$. It then remains to deduce that $\eta(\{t\}) \geq \mathsf{c}(t,u(t-),u(t+)) \quad$ for all $t\in \mathrm{J}_u$, as well as the analogue of (5.74), to conclude (5.96). With this aim we adapt the proof of Proposition 3 to show that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[\alpha_k,\beta_k]) \geq \mathsf{c}(t,u(t-),u(t+))$$

at every point $t\in \mathrm{J}_u$, and for every pair of sequences $(\alpha_k)_k$, $(\beta_k)_k$ converging to $t$ and fulfilling (4.69c). Hence, we reparameterize the curves $u_k$ in the very same way as in Step 1 of the proof of Prop. 3. By virtue of the a priori estimates from Step 2, the compactness arguments in Step 3 yield the existence of a Lipschitz continuous limit curve $\mathfrak{u} : C \to X$, with $C \Subset [0,\infty)$ and $\mathfrak{u}(C^-) = u(t-)$, $\mathfrak{u}(C^+) = u(t+)$. Here, we can no longer replace $C$ with the interval $[0,C^+]$ as in the proof of Prop. 3, but we can still observe property (5.84), based on [18, Thm. 5.3]. We now show that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[\alpha_k,\beta_k]) \geq \mathrm{Trc}_{\mathrm{VE}}(t,\mathfrak{u},C) \geq \mathsf{c}(t,u(t-),u(t+))\,. \tag{5.97}$$

The liminf-inequality for the $\mathrm{Var}_{\mathsf{d}}$ contribution to $\mathrm{Var}_{\mathsf{d},\mathsf{c}_k}$ easily follows from the aforementioned compactness arguments. For the $\mathrm{GapVar}_{\mathsf{d}}^{\mu_k}$-contribution (which depends on the parameter $\mu_k$ via the viscous correction $\frac{\mu_k}{2}\mathsf{d}^2$), it is essential to use property (5.84). For the $\mathscr{R}^{\mu_k}$ contribution, we can adapt the arguments from the discussion of Case (2) in Step 5 of the proof of Prop. 3, also exploiting the liminf-estimate

$$(t_k \to t,\ x_k \to x) \ \Rightarrow\ \liminf_{k\to\infty} \mathscr{R}^{\mu_k}(t_k,x_k) \geq \mathscr{R}^{\mu}(t,x).$$

This concludes the proof of (5.96).

$\triangleright$ **The lower energy estimate $\geq$ in ($\mathrm{E}_{\mathsf{d},\mathsf{c}_\mu}$):** It follows from [18, Thm. 6.5]. Again, the energy convergence $\mathscr{E}(t,u_k(t)) \to \mathscr{E}(t,u(t))$ for every $t\in[0,T]$ follows from the limit passage in the energy balance. ∎

## 6 Auxiliary results

We start with the proof of the representation formula (5.86) for the transition cost $\mathrm{Trc}_{\mathrm{BV}}(t,\mathfrak{u},[0,C^+])$. In the upcoming statement, we replace the functional $u \mapsto |\mathrm{D}\mathscr{E}|(t,u)\vee 1$ by a general

$$g : X \to \mathbb{R} \qquad \text{positive and lower semicontinuous.}$$

**Lemma 1.** *Let $v \in \mathrm{AC}([a,b];X)$. Then, there holds*

$$\begin{aligned}&\int_a^b |v'|(s)g(v(s))\,\mathrm{d}s\\ &= \sup\left\{\sum_{i=1}^N \mathsf{d}(v(\sigma_{i-1}),v(\sigma_i)) \inf_{\sigma\in[\sigma_{i-1},\sigma_i]} g(v(\sigma))\ :\ (\sigma_i)_{i=1}^N \in \mathfrak{P}_f([a,b])\right\}\\ &=: \mathsf{S}\,.\end{aligned} \tag{6.98}$$

*In particular, the map $s \mapsto |v'|(s)g(v(s))$ is integrable on $[a,b]$ if and only if $\mathsf{S}<\infty$.*

*Proof.* Let us fix $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([a,b])$. Observe that

$$\begin{aligned}\mathsf{d}(v(\sigma_{i-1}),v(\sigma_i)) \inf_{\tilde\sigma\in[\sigma_{i-1},\sigma_i]} g(v(\tilde\sigma)) &\overset{(1)}{\leq} \int_{\sigma_{i-1}}^{\sigma_i} |v'|(\sigma) \inf_{\tilde\sigma\in[\sigma_{i-1},\sigma_i]} g(v(\tilde\sigma))\,\mathrm{d}\sigma\\ &\leq \int_{\sigma_{i-1}}^{\sigma_i} |v'|(\sigma)g(v(\sigma))\,\mathrm{d}\sigma\end{aligned}$$

with (1) due to (2.22). Therefore, upon summing up over the index $i=1,\ldots,N$ and using that $(\sigma_i)_{i=1}^N$ is arbitrary, we conclude

$$\int_a^b |v'|(s) g(v(s))\,\mathrm{d}s \geq \mathsf{S}\,.$$

As for the converse inequality, we now consider a partition $a = \sigma_1 < \ldots < \sigma_i < \ldots = \sigma_N = b$ with fineness $\tau := \max_{i=1,\ldots,N}(\sigma_i - \sigma_{i-1})$ and introduce the functions

$$\underline{\sigma}_\tau, \overline{\sigma}_\tau : [a,b] \to [a,b] \quad \text{defined by} \quad \begin{cases} \overline{\sigma}_\tau(s) := \sigma_i & \text{if } s \in (\sigma_{i-1}, \sigma_i], \\ \underline{\sigma}_\tau(s) := \sigma_{i-1} & \text{if } s \in [\sigma_{i-1}, \sigma_i), \end{cases}$$

with $\underline{\sigma}_\tau(b) := b$ and $\overline{\sigma}_\tau(a) := a$. Taking into account the definition (2.23) of the metric derivative $|v'|$, it is a standard matter to check that, on the one hand,

$$\lim_{\tau\downarrow 0} \frac{1}{(\overline{\sigma}_\tau(s) - \underline{\sigma}_\tau(s))} \mathsf{d}(v(\underline{\sigma}_\tau(s)), v(\overline{\sigma}_\tau(s))) \to |v'|(s) \qquad \text{for a.a. } s \in (a,b)\,. \tag{6.99}$$

On the other hand, exploiting the lower semicontinuity of $g$, we observe that for every $s \in [a,b]$ there exists $\sigma_{\min,\tau}(s) \in [\underline{\sigma}_\tau(s), \overline{\sigma}_\tau(s)]$ such that

$$\inf_{\sigma \in [\underline{\sigma}_\tau(s), \overline{\sigma}_\tau(s)]} g(v(\sigma)) = g(v(\sigma_{\min,\tau}(s))).$$

Since $\sigma_{\min,\tau}(s) \to s$ as $\tau \downarrow 0$, by the continuity of $v$ and the lower semicontinuity of $g$ we then have

$$\liminf_{\tau\downarrow 0} g(v(\sigma_{\min,\tau}(s))) \geq g(v(s)) \qquad \text{for all } s \in [a,b].$$

Therefore, by the Fatou Lemma we have

$$\begin{aligned} \mathsf{S} &\geq \liminf_{\tau\downarrow 0} \sum_{i=1}^N \mathsf{d}(v(\sigma_{i-1}), v(\sigma_i)) \inf_{\sigma \in [\sigma_{i-1}, \sigma_i]} g(v(\sigma)) \\ &= \liminf_{\tau\downarrow 0} \int_a^b \frac{1}{(\overline{\sigma}_\tau(s) - \underline{\sigma}_\tau(s))} \mathsf{d}(v(\underline{\sigma}_\tau(s), v(\overline{\sigma}_\tau(s)))\, g(v(\sigma_{\min,\tau}(s)))\,\mathrm{d}s \\ &\geq \int_a^b |v'|(s) g(v(s))\,\mathrm{d}s\,. \end{aligned}$$

and we then conclude (6.98).

We conclude this Appendix by extending the *duality formula* from [1, Lemma 3.1.5] for the (squared) metric slope $|\mathrm{D}\mathscr{E}|^2(t,\cdot)$, $t \in [0,T]$ fixed, namely

$$\begin{aligned} &\frac{1}{2}|\mathrm{D}\mathscr{E}|^2(t,u) = \limsup_{\tau\downarrow 0} \frac{\mathscr{E}(t,u) - \mathscr{E}_\tau(t,u)}{\tau} \quad \text{with} \\ &\mathscr{E}_\tau(t,u) := \inf_{v \in X} \left\{ \frac{1}{2\tau} \mathsf{d}^2(u,v) + \mathscr{E}(t,v) \right\} \\ &\text{the Moreau-Yosida approximation of } \mathscr{E}(t,\cdot) \end{aligned} \tag{6.100}$$

(with slight abuse of notation). We consider the case in which the dissipation potential underlying the definition of Moreau-Yosida approximation is no longer the quadratic $\psi(r) := \frac{1}{2}r^2$, but a general function

$$\psi : [0,\infty) \to [0,\infty) \text{ convex, l.s.c., with } \psi(0) = 0 \text{ and } \lim_{r\uparrow\infty} \frac{\psi(r)}{r} = \infty. \qquad (6.101)$$

With $\psi$ we may associate the *generalized Moreau-Yosida* approximation of the functional $\mathscr{E}(t,\cdot) : X \to \mathbb{R}$, via the formula (again, with slight abuse of notation, we write $\mathscr{Y}_\tau^\psi(\mathscr{E})(t,u)$ in place of $\mathscr{Y}_\tau^\psi(\mathscr{E}(t,\cdot))(u)$)

$$\mathscr{Y}_\tau^\psi(\mathscr{E})(t,u) := \inf_{v\in X} \left( \tau\psi\left(\frac{\mathsf{d}(u,v)}{\tau}\right) + \mathscr{E}(t,v) \right) \qquad (6.102)$$

for $(t,u) \in [0,T] \times X,\ \tau > 0$. Combining the coercivity condition ($\mathrm{E}_1$) with the superlinear growth of $\psi$, it is straightforward to check that for all $(t,u) \in [0,T] \times X$ and for all $\tau > 0$

$$M_\tau^\psi(\mathscr{E})(t,u) := \mathrm{Argmin}_{v\in X} \left( \tau\psi\left(\frac{\mathsf{d}(u,v)}{\tau}\right) + \mathscr{E}(t,v) \right) \neq \emptyset.$$

We have the following counterpart to [1, Lemma 3.1.5].

**Lemma 2.** *There holds for all* $(t,u) \in [0,T] \times X$

$$\psi^*\left(|\mathrm{D}\mathscr{E}|(t,u)\right) = \limsup_{\tau\to 0} \frac{\mathscr{E}(t,u) - \mathscr{Y}_\tau^\psi(\mathscr{E})(t,u)}{\tau}. \qquad (6.103)$$

The *proof* follows by trivially adapting the argument for [1, Lemma 3.1.5]. We conclude this Appendix with the following lower semicontinuity result, which is crucially used in the proof of Proposition 3.

**Lemma 3.** *Assume* ($\mathrm{E}_1$), ($\mathrm{E}_3'$)*, and* (6.101)*. Let* $(\tau_k)_k \subset (0,\infty)$*,* $(t_k)_k \subset [0,T]$*, and* $(u_k)_k \subset X$ *fulfill* $\tau_k \downarrow 0$*,* $t_k \to t$*, and* $u_k \to u$ *for some* $(t,u) \in [0,T] \times X$*, with* $\sup_{k\in\mathbb{N}} \mathscr{E}(t_k,u_k) \leq C$*. Then,*

$$\liminf_{k\to\infty} \frac{\mathscr{E}(t_k,u_k) - \mathscr{Y}_{\tau_k}^\psi(\mathscr{E})(t_k,u_k)}{\tau_k} \geq \psi^*\left(|\mathrm{D}\mathscr{E}|(t,u)\right). \qquad (6.104)$$

*Proof.* For every $k \in \mathbb{N}$, let $u_{\tau_k}^k \in M_{\tau_k}^\psi(\mathscr{E})(t_k,u_k)$. We have that

$$\begin{aligned} \frac{\mathscr{E}(t,u_k) - \mathscr{Y}_{\tau_k}^\psi(\mathscr{E})(t_k,u_k)}{\tau_k} &= \frac{\mathscr{E}(t_k,u_k) - \mathscr{E}(t_k,u_{\tau_k}^k) - \tau_k\psi\left(\frac{\mathsf{d}(u_k,u_{\tau_k}^k)}{\tau_k}\right)}{\tau_k} \\ &\geq \frac{1}{\tau_k}\int_0^{\tau_k} \psi^*\left(|\mathrm{D}\mathscr{E}|(t_k,u_r^k)\right) \mathrm{d}r, \end{aligned}$$

where the latter estimate follows from [20, Lemma 4.5], with $u_r^k$ is a (measurable) selection in $M_r^{\psi}(\mathscr{E})(t_k,u_k)$ for $r \in (0,\tau_k)$. Observe that $\liminf_{k\to\infty} \psi^*\left(|\mathrm{D}\mathscr{E}|(t_k,u_r^k)\right) \geq \psi^*\left(|\mathrm{D}\mathscr{E}|(t,u)\right)$ taking into account that $u_r^k \to u$ as $k \to \infty$ for every $r \in (0,\tau_k)$, cf. the proof of [20, Lemma 4.5], and using the lower semicontinuity of $|\mathrm{D}\mathscr{E}|$ granted by ($\mathrm{E}_3'$). Then, by Fatou's lemma we have

$$\liminf_{k\to\infty} \frac{1}{\tau_k} \int_0^{\tau_k} \psi^*\left(|\mathrm{D}\mathscr{E}|(t_k,u_r^k)\right) \mathrm{d}r \geq \psi^*\left(|\mathrm{D}\mathscr{E}|(t,u)\right),$$

which concludes the proof of (6.104).

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