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ON TWO CLASSES OF GENERALIZED VISCOUS CAHN-HILLIARD EQUATIONS

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Abstract. This paper investigates two classes of *generalized viscous Cahn-Hilliard equations*, featuring two different laws for the mobility, which is assumed to depend on the chemical potential. Both equations can be obtained with the new derivation of equations of Cahn-Hilliard type proposed by M.E. Gurtin [14]. Approximation and compactness tools allow to prove well-posedness and, in one case, regularity results for the equations supplemented with initial and suitable boundary conditions.

1. **Introduction.** This paper is concerned with the analysis of the following fourth-order parabolic equation:

$$\chi_t - \Delta(\alpha(\delta\chi_t - \Delta\chi + \mathcal{W}'(\chi))) = f \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

where $\delta > 0$ is a positive coefficient and Ω is a bounded, connnected domain in $\mathbb{R}^N$, $N = 1, 2, 3$, with smooth boundary Γ, occupied by a two-phase material (for instance, a binary alloy), subject to a phase separation process in the time interval $(0, T)$. The evolution of this phenomenon is described in terms of the *order parameter* χ, representing the local concentration of one of the two components. Furthermore, $\alpha : D(\alpha) \subset \mathbb{R} \to \mathbb{R}$ is a (strictly) increasing function, f may stand for a source term, while $\mathcal{W}'$ is the derivative of a non-convex free energy potential $\mathcal{W}$. Actually, throughout this paper we will assume $\mathcal{W}$ to have the double-well potential form

$$\mathcal{W}(x) = \frac{(x^2-1)^2}{4}, \quad x \in \mathbb{R}. \tag{1.2}$$

We will refer to (1.1) as the *generalized viscous Cahn-Hilliard equation*, which includes as a special case (with the obvious choice $\alpha(r) := \kappa r$ for every $r \in \mathbb{R}$) both the standard *viscous Cahn-Hilliard equation*

$$\chi_t - \kappa\Delta(\chi_t - \Delta\chi + \mathcal{W}'(\chi)) = f \quad \text{in } \Omega \times (0,T), \tag{1.3}$$

(where we have set $\delta = 1$ for simplicity), and the *Cahn-Hilliard equation*

$$\chi_t - \kappa\Delta(-\Delta\chi + \mathcal{W}'(\chi)) = f \quad \text{in } \Omega \times (0,T). \tag{1.4}$$

The latter equation was originally proposed for modelling phase separation phenomena in a binary mixture, quickly quenched from a uniform mixed state into a

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miscibility gap, which gives rise either to a partial nucleation (i.e., the appearance of nucleides in the material), or to a total nucleation (usually referred to as *spinodal decomposition*, i.e., the formation of a spatially separated two-phase structure). Without entering into further detail, let us just mention that (1.4) (with for instance $f \equiv 0$), can be derived by coupling the *mass balance*

$$\frac{\partial \chi}{\partial t} + \operatorname{div} h = 0, \tag{1.5}$$

with the following *constitutive law* for the mass flux h:

$$h = -\kappa \mathbf{I} \, \nabla w,$$

where $\mathbf{I}$ is the $N \times N$ identity matrix, $\kappa \mathbf{I}$ is the *mobility tensor*, and w is the *chemical potential*, defined as the variational derivative of the total free energy functional $\mathcal{F}(\chi) := \int_\Omega \left(\frac{1}{2}|\nabla \chi|^2 + \mathcal{W}(\chi)\right) dx$, i.e.,

$$w = -\Delta \chi + \mathcal{W}'(\chi).$$

Let us refer the interested reader to [20], and the references therein, for a rich survey of the mathematical analysis so far developed for (1.4) and for its *viscous* variant (1.3), which was first proposed in [19] (in the case $f \equiv 0$), to account for viscosity effects in the phase separation of polymer-polymer systems.

We will focus instead on (1.1), which pertains to the class of the *generalized Cahn-Hilliard equations* derived by M.E. Gurtin. As a matter of fact, in the seminal paper [14] a different approach to (1.3) and (1.4) is proposed, based on the consideration that the work of the internal microforces associated with the changes of χ should be taken into account in the model. Then, a new derivation of (1.3) and (1.4) is developed, in which the *macroscopic* mass balance (1.5) is coupled with a microforce balance, and both laws are combined with constitutive equations rigorously deduced from the second law of thermodynamics. In this connection, besides quoting the original paper [14], we also refer the reader to [17] and [18] for a detailed account of the derivation the *generalized (viscous) Cahn-Hilliard equation*

$$\chi_t - \operatorname{div}(\mathbf{M}(Z)\nabla(\delta\chi_t - \Delta\chi + \mathcal{W}'(\chi))) = 0 \quad \text{in } \Omega \times (0,T), \tag{1.6}$$

where $\delta \geq 0$ ($\delta > 0$ in the viscous case), and Z denotes the set of the independent constitutive variables $(\chi, \nabla\chi, \chi_t, w, \nabla w)$ the mobility tensor $\mathbf{M}$ (in general, a positive definite $N \times N$ matrix), is allowed to depend on. Then, our equation (1.1) turns out to be a particular case of (1.6), of course with the choice $\mathbf{M} = \mathbf{M}(w) := \alpha'(w)\mathbf{I}$, α' being the (almost everywhere defined) derivative of the function α in (1.1). Namely, we assume $\mathbf{M}$ to depend on the chemical potential w only. Let us note that a *concentration-dependent* mobility tensor $\mathbf{M}$ has already been considered for the classical Cahn-Hilliard equation (see [13, 6]), possibly supplemented with a source term nonlinearly depending on χ [16]. We may also mention [2], in which a viscous Cahn-Hilliard equation modelling phase separation in a tin-lead solder subject to internal and external mechanical stresses is investigated. Instead, the case of a mobility depending on the chemical potential, which naturally arises in the framework of Gurtin's derivation, has never been tackled so far to the author's knowledge. For the sake of convenience, hereafter we will rephrase (1.1) (for $\delta = 1$), by introducing the new variable $u := \alpha(w)$, so that (1.1) may be split into the system

$$\chi_t - \Delta u = f \quad \text{in } \Omega \times (0,T),$$
$$\chi_t - \Delta \chi + \chi^3 - \chi = \rho(u) \quad \text{in } \Omega \times (0,T),$$

where $\rho := \alpha^{-1}$. In the sequel, we will examine two different choices of α in (1.1), and we will analyse the initial-boundary value problem obtained by supplementing the system above with suitable initial and boundary conditions for χ and u, in accordance with the different properties of α.

Firstly, we will tackle the case α is a *bi-Lipschitz*, strictly increasing function $\alpha_1 : \mathbb{R} \to \mathbb{R}$, leading to the system

$$\chi_t - \Delta u = f \quad \text{in } \Omega \times (0,T), \tag{1.7}$$
$$\chi_t - \Delta \chi + \chi^3 - \chi = \rho_1(u) \quad \text{in } \Omega \times (0,T), \tag{1.8}$$

with of course $\rho_1 = \alpha_1^{-1}$. We will supplement (1.7)-(1.8) with the initial condition

$$\chi(\,\cdot\,, 0) = \chi_0, \tag{1.9}$$

and the *Neumann* boundary conditions

$$\partial_n \chi = 0 \quad \text{and} \quad \partial_n u = g \quad \text{in } \partial\Omega \times (0,T), \tag{1.10}$$

and we will refer to the initial-boundary value problem given by (1.7)-(1.10) as Problem $\mathbf{P_1}$. Let us point out that, as soon as $f \equiv 0$ in (1.7) and $g \equiv 0$ in (1.10), then the *no mass flux* boundary condition (1.10) for u easily yields

$$\frac{1}{|\Omega|}\int_\Omega \chi(x,t)dx = \frac{1}{|\Omega|}\int_\Omega \chi_0(x)dx, \quad \forall t \in (0,T).$$

Namely, χ is a *conserved* parameter (i.e., its (spatial) mean value is constant on $(0,T)$), in compliance with the principle of matter conservation (recall that χ stands in fact for a concentration). Then, if non-zero data f and g are to be included in Problem $\mathbf{P_1}$, we have to require them to fulfil this compatibility condition (cf. Remark 2.2 later on)

$$\int_\Omega f(x,t)dx + \int_\Gamma g(s,t)ds = 0, \quad t \in (0,T), \tag{1.11}$$

in order to retain the mass conservation property.

Secondly, we will consider the fourth-order system given by (1.7), coupled with

$$\chi_t - \Delta \chi + \chi^3 - \chi = \rho_2(u) \quad \text{in } \Omega \times (0,T), \tag{1.12}$$

where $\rho_2 = \alpha_2^{-1}$ and α_2 is a stricly increasing function defined on the halfline $(a, +\infty)$, with $\lim_{r\downarrow a} \alpha_2(r) = -\infty$, and such that ρ_2 is a Lipschitz (but *not bi-Lipschitz*) function. To fix ideas, we can think of

$$\alpha_2(r) := r - \frac{1}{r} \quad \forall r > 0. \tag{1.13}$$

Indeed, this is a particularly meaningful choice: it is straightforward to check that, in this case, the system ((1.7),(1.12)) can be obtained by setting $\varepsilon = 0$ in the system

$$\varepsilon\vartheta_t + \chi_t - \Delta(\vartheta - \frac{1}{\vartheta}) = f \quad \text{in } \Omega \times (0,T), \tag{1.14}$$
$$\chi_t - \Delta \chi + \chi^3 - \chi = -\frac{1}{\vartheta} \quad \text{in } \Omega \times (0,T), \tag{1.15}$$

and formally putting $u := \vartheta - \frac{1}{\vartheta} = \alpha_2(-\frac{1}{\vartheta})$, (so that $-\frac{1}{\vartheta} = \rho_2(u)$). Note that (1.14)-(1.15) is indeed a variant of the well-known *phase field model* proposed by O. Penrose and P.C. Fife (see the pioneering paper [21]), for the modelling of the kinetics of phase transitions. In this framework, ϑ is the absolute temperature of a physical system undergoing a phase transition. Moreover, the evolution of the phenomenon is described in terms of the order parameter χ, depending on the specific phase transition considered (see also [4, Chap. IV] for several examples in this connection). Let us point out that the energy balance equation (1.14) features the heat flux law $\ell(\vartheta) := \vartheta - \frac{1}{\vartheta}$ for $\vartheta > 0$, which was first proposed in [8], on the grounds of physical feasibility, for modelling large temperatures.

Although we have so far highlighted just a *formal* link between the two systems ((1.7),(1.12)) and ((1.14),(1.15)), these considerations still have some interest. As a matter of fact, they fit well in the framework of a series of papers, investigating the (standard) viscous Cahn-Hilliard and Cahn-Hilliard equations as limits of the phase field system proposed by G. Caginalp [5] (see [11, 12, 22, 25]) and as limits of (a variant of) the Penrose-Fife system (see [23]). While referring to [23], and the references therein, for a detailed account of the asymptotic analyses of the Caginalp and Penrose-Fife phase field systems, we just recall here that, for instance, in [25] it is shown that the solutions to the Caginalp phase field system converge as two suitable time relaxation parameters vanish to the solution of the Cahn-Hilliard equation; this kind of analysis is extended to the viscous Cahn-Hilliard equation in [22]. Actually, here we have not been able to prove rigorously that solutions to ((1.14),(1.15)) converge to solutions of ((1.7),(1.12)) (with α_2 given by (1.13)), as $\varepsilon \downarrow 0$. Thus, we have restricted our attention to the *limiting* problem ((1.7),(1.12)), which is still interesting from the analytical point of view, as it features a *singular* law α_2.

We will supplement the system ((1.7),(1.12)) with the initial condition (1.9) and the boundary conditions

$$\partial_n \chi = 0 \quad \text{and} \quad -\partial_n u = \gamma u - h \quad \text{in } \partial\Omega \times (0,T) \tag{1.16}$$

where γ is a positive constant and $h : \Gamma \times (0,T) \to \mathbb{R}$, and we will refer to the initial-boundary value problem ((1.7),(1.12), (1.9), (1.16)) as Problem $\mathbf{P_2}$. Note by the way that in the case of the standard viscous Cahn-Hilliard equation (corresponding to $\alpha_2(r) := r$ for every $r \in \mathbb{R}$), (1.16) in fact prescribes Robin (or third type) boundary conditions on the chemical potential, which are not very usual for this kind of problems. Indeed, in this case the mean value of χ is no longer conserved, and the order parameter χ necessarily has a different physical interpretation. For instance, when Dirichlet boundary conditions, which could also be handled in our framework, are imposed on the chemical potential, the (viscous) Cahn-Hilliard equation models the propagation of a solidification front in a medium at rest with respect to the front. In turn, if we take into account our choice (1.13) of α_2, then the conditions (1.16) appear somehow more natural from an analytical point of view, as we will detail in Remark 4.1 later on. Roughly speaking, a third type boundary condition on u allows us to recover some "coercivity" (namely, an $H^1(\Omega)$-a priori bound), for u from the first equation (1.7): such an estimate proves indeed to be crucial for obtaining a well-posedness result for $\mathbf{P_2}$. Note that this analytical difficulty is instead easily bypassed in the framework of Problem $\mathbf{P_1}$. Indeed, in that case, α_1 (and thus ρ_1) is bi-Lipschitz, which, basically, enables us to estimate the mean value of u from the second equation (1.8). Combining now this estimate with the bound on the $L^2(\Omega)$-norm of ∇u which we infer from (1.7), by Poincaré's inequality we still recover a $H^1(\Omega)$-a priori bound on u as needed.

We will approximate both Problem $\mathbf{P_1}$ and Problem $\mathbf{P_2}$ by inserting the time derivative of u in the first equation of each system; focusing e.g. on $\mathbf{P_1}$, the related approximate system will be given by

$$\nu u_t + \chi_t - \Delta u = f \quad \text{in } \Omega \times (0,T), \tag{1.17}$$

ν being a positive constant, coupled with (1.8). Note that ((1.17),(1.8)) has itself the structure of a phase field system in the variables u and χ. Then, we will show that the sequence of the solutions $\{(\chi_\nu, u_\nu)\}_\nu$ to the initial-boundary value problem for ((1.17),(1.8)) converges to the *unique* solution of Problem $\mathbf{P_1}$, thus establishing our first well-posedness result Theorem 2.1. In the same way, we will prove that Problem $\mathbf{P_2}$ admits a unique solution (cf. Theorem 2.2), and we will investigate its further regularity under some additional assumptions on the data of $\mathbf{P_2}$, thus obtaining Theorem 2.3.

Plan of the paper. Our main results, together with some preliminary material, are presented in Section 2. Section 3 is devoted to the proof of our existence and continuous dependence results for $\mathbf{P_1}$, while the well-posedness and regularity for $\mathbf{P_2}$ are tackled in Section 4.

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2. **Preliminaries and Statement of the Main Results.**

2.1. **Notation.** Our functional setting is given by the spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and} \quad W := \left\{ v \in H^2(\Omega) \ : \ \partial_n v = 0 \right\};$$

we will identify H with its dual space H', so that $W \subset V \subset H \subset V' \subset W'$, with dense and compact embeddings. We will also deal with the Sobolev space $H^{1/2}(\Gamma)$, with dual $H^{-1/2}(\Gamma)$; while referring the reader to [15, Chap. 1] for the definition and properties of $H^{1/2}(\Gamma)$, we just recall that for every $v \in V$, the trace $v|_\Gamma$ is in $H^{1/2}(\Gamma)$.

We will denote by $(\,\cdot\,,\,\cdot\,)_H$ the inner product in H and in H^N, by $\langle\,\cdot\,,\,\cdot\,\rangle$ ($\langle\cdot,\cdot\rangle_\Gamma$, resp.), the duality pairing between V' and V (between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, resp.), and by $|\cdot|_H$, $\|\cdot\|_V$, $\|\cdot\|_{V'}$ the norms in H, in V and in V'; furthermore, let $\mathcal{V}$, $\mathcal{H}$, and $\mathcal{V}'$ be the subspaces of the elements v with zero mean value $m_\Omega(v) = \frac{1}{|\Omega|}\langle v, 1\rangle$ in V, H, and V', respectively. We also consider the operator $A : V \to V'$ given by

$$\langle Au, v\rangle := \int_\Omega \nabla u \nabla v \, dx \quad \forall u, v \in V,$$

and note that $Au \in \mathcal{V}'$ for every $u \in V$. Indeed, the restriction of A to $\mathcal{V}$ is an isomorphism, and we can introduce its inverse operator $\mathcal{N} : \mathcal{V}' \to \mathcal{V}$, defined by

$$A(\mathcal{N}v) = v \quad \forall v \in \mathcal{V}'.$$

Throughout the following Section 3, we will make a systematic use of the relations

$$\langle Au, \mathcal{N}v\rangle = \langle v, u\rangle \quad \forall u \in V, \ \forall v \in \mathcal{V}', \tag{2.1}$$

$$\langle u, \mathcal{N}v\rangle = \int_\Omega \nabla(\mathcal{N}u)\nabla(\mathcal{N}v)\, dx = \langle v, \mathcal{N}u\rangle \quad \forall u, v \in \mathcal{V}'. \tag{2.2}$$

Moreover, we will consider on the spaces V and V' the following norms

$$\|u\|_V^2 := \langle Au, u\rangle + (u, m_\Omega(u))_H \quad \forall u \in V,$$
$$\|v\|_{V'}^2 := \langle v, \mathcal{N}(v - m_\Omega(v))\rangle + (v, m_\Omega(v))_H \quad \forall v \in V',$$

which are equivalent to the standard ones on behalf of Poincaré's inequality for the zero mean value functions. It follows from the above formulae that

$$\|v\|_{V'}^2 = \langle v, \mathcal{N}(v)\rangle = \|\mathcal{N}(v)\|_V^2 \quad \forall v \in \mathcal{V}'. \tag{2.3}$$

In addition, we shall make use of the two following elementary inequalities for the functional $\mathcal{W}'$, ($\mathcal{W}$ being the double well potential in (1.2)),

$$\forall \eta \in [0,1) \quad \exists\, C_\eta \ge 0: \quad \mathcal{W}'(r)r \ge \eta r^4 - C_\eta \quad \forall r \in \mathbb{R}, \tag{2.4}$$
$$\forall \rho > 0 \quad \exists\, C_\rho > 0 \quad \mathcal{W}'(r) \le \rho r^4 + C_\rho \quad \forall r \in \mathbb{R}. \tag{2.5}$$

2.2. **Variational formulation and main results for Problem $\mathbf{P_1}$.** Let us detail our assumptions on Problem $\mathbf{P_1}$ ((1.7)-(1.10)): $\alpha_1 : \mathbb{R} \to \mathbb{R}$ is strictly increasing and bi-Lipschitz, namely

$$\exists m_1, M_1 > 0 \quad \text{s.t.} \quad m_1 \le {\alpha_1}'(r) \le M_1 \quad \text{for a.e. } r \in \mathbb{R}. \tag{2.6}$$

As for the other data of Problem $\mathbf{P_1}$, we will suppose that

$$\chi_0 \in V, \tag{2.7}$$
$$f \in L^2(0,T;V'), \tag{2.8}$$
$$g \in L^2(0,T;H^{-1/2}(\Gamma)). \tag{2.9}$$

We can then define the function $G \in L^2(0,T;V')$ by

$$\langle G(t), v\rangle := \langle f(t), v\rangle + \langle g(t), v\rangle_\Gamma \quad \forall v \in V, \text{ for a.e. } t \in (0,T). \tag{2.10}$$

Remark 2.1. Let $\rho_1 : \mathbb{R} \to \mathbb{R}$ be the inverse function of α_1: it follows from (2.6) that ρ_1 is strictly increasing and fulfils

$$0 < d_1 \le {\rho_1}'(r) \le D_1 \quad \text{for a.e. } r \in \mathbb{R}, \tag{2.11}$$

with of course $d_1 = 1/M_1$, $D_1 = 1/m_1$. Finally, let $\rho_1^\circ := \rho_1(0)$, and let us introduce the primitive

$$\widehat{\rho_1}(u) := \int_0^u (\rho_1(r) - \rho_1^\circ)dr \tag{2.12}$$

Clearly, $\widehat{\rho_1}$ is strictly convex and

$$\widehat{\rho_1}(r) \ge \widehat{\rho_1}(0) = 0 \quad \text{for every } r \in \mathbb{R}. \tag{2.13}$$

Furthermore, it is straightforward to prove that

$$|\widehat{\rho_1}(r)| \le \frac{D_1}{2} r^2 \quad \forall r \in \mathbb{R}. \tag{2.14}$$

We can now give a rigorous variational formulation for the initial boundary value problem $\mathbf{P_1}$.

Problem $\mathbf{P_1}$. *Find* $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ *and* $u \in L^2(0,T;V)$ *such that*

$$\partial_t \chi + Au = G \quad \textit{in } V' \textit{ for a.e. } t \in (0,T), \tag{2.15}$$
$$\partial_t \chi + A\chi + \chi^3 - \chi = \rho_1(u) \quad \textit{in } H \textit{ for a.e. } t \in (0,T), \tag{2.16}$$

and the initial condition (1.9) *holds for* χ.

REMARK 2.2. i). Note that, since the space $H^1(0,T;H) \cap L^2(0,T;W)$ turns out to be continuously embedded in $C^0([0,T];V)$, the solution component χ has the additional regularity $\chi \in C^0([0,T];V)$. Of course, this is in accordance with the requirement $\chi_0 \in V$. Let us also point out that the boundary conditions (1.10) for χ and u are contained in the above variational formulation, on account of (2.10) as well.
ii). Let us further require the function G (2.10) to fulfil

$$\langle G(t), 1\rangle = 0 \quad \text{for a.e. } t \in (0,T),$$

(i.e., (1.11)). Then, *any* solution χ to Problem $\mathbf{P_1}$ enjoys the *conservation property*

$$m_\Omega(\chi(t)) = m_\Omega(\chi_0) \quad \text{for a.e. } t \in (0,T).$$

To see this, it suffices to test (2.15) by 1, so that $\frac{d}{dt}(\langle \chi, 1\rangle) = \langle \partial_t \chi, 1\rangle = 0$.

PROPOSITION 2.1. *Let* (χ_0^1, f_1, g_1) *and* (χ_0^2, f_2, g_2) *be two pairs of data for Problem* $\mathbf{P_1}$ *fulfilling* (2.7)-(2.9)*; let* χ_i*,* $i = 1, 2$*, be the corresponding solutions. Assume that the functions* G_i*, related to* f_i *and* g_i *by* (2.10)*,* $i = 1, 2$*, satisfy*

$$\langle G_1(t) - G_2(t), 1\rangle = 0 \quad \textit{for a.e. } t \in (0,T). \tag{2.17}$$

Set

$$M_* := \max_{i=1,2}\{\|\chi_0^i\|_V + \|f_i\|_{L^2(0,T;V')} + \|g_i\|_{L^2(0,T;H^{-1/2}(\Gamma))}\}.$$

Then there exists a positive constant S_**, depending on* M_**,* T*,* $|\Omega|$*,* m_1*, and* M_1 *only, such that*

$$\|\chi_1 - \chi_2\|_{C^0([0,T];H)\cap L^2(0,T;V)} \le S_*\left(|\chi_0^1 - \chi_0^2|_H + \|f_1 - f_2\|_{L^2(0,T;V')} + \|g_1 - g_2\|_{L^2(0,T;H^{-1/2}(\Gamma))}\right). \tag{2.18}$$

THEOREM 2.1. *Under the assumptions* (2.6)-(2.9)*, Problem* $\mathbf{P_1}$ *admits a unique solution* (χ, u)*.*
The uniqueness statement of the above Theorem is of course a straightforward consequence of Proposition 2.1.

2.3. Variational formulation and main results for Problem $\mathbf{P_2}$.

Functional setting. Let us consider the operator $J : V \to V'$ given by

$$\langle Ju, v\rangle := \int_\Omega \nabla u \cdot \nabla v + \gamma \langle u, v\rangle_\Gamma \quad \forall u, v \in V. \tag{2.19}$$

Of course, J is linear and bounded on V; moreover, a standard version of Poincaré's inequality ensures that there exists a constant K^*, depending on γ and on the geometry of the domain Ω only, such that $K^*|u|_H^2 \le \langle Ju, u\rangle$ for every $u \in V$, so that the operator J is also coercive on V, with bounded inverse $J^{-1} : V' \to V$. Note that

$$\langle Jv, w\rangle = \langle Jw, v\rangle \quad \forall v, w \in V, \quad \text{and} \quad \langle v, J^{-1}w\rangle = \langle w, J^{-1}v\rangle \quad \forall v, w \in V'. \tag{2.20}$$

Throughout this subsection and the following Section 4, we will consider, on the space V (V', respectively), the inner product $(\!(v_1, v_2)\!) := \langle Jv_1, v_2\rangle$ for every v_1, $v_2 \in V$ ($(\!(w_1, w_2)\!)_* := \langle w_1, J^{-1}(w_2)\rangle$ for every w_1, $w_2 \in V'$, resp.). Accordingly, we will endow V and V' with the norms

$$\|v\|_V^2 := \langle Jv, v\rangle \quad \forall v \in V, \quad \|w\|_{V'}^2 := \langle w, J^{-1}(w)\rangle \quad \forall w \in V', \tag{2.21}$$

which are equivalent to the usual norms on V and V'. We will also renorm H so that the aforementioned constant K^* fulfils $K^* = 1$. In this setting, (V, H, V') is still of course a Hilbert triplet, and $J : V \to V'$ turns out to be the duality mapping: therefore, we have

$$\langle Jv, J^{-1}w\rangle = \langle Jw, J^{-1}v\rangle = (v,w)_H \quad \forall v,w \in V,$$
$$\text{whence} \quad \langle Jv, J^{-1}v\rangle = |v|_H^2 \quad \forall v \in V. \tag{2.22}$$

Formulation of Problem $\mathbf{P_2}$. We can now state our main assumptions on the data of $\mathbf{P_2}$: besides (2.7) and (2.8), we will suppose that

$$\alpha_2 : (a,+\infty) \to \mathbb{R},\ a \in \mathbb{R}, \text{ is a strictly increasing function with}$$
$$m_2 := \inf_{r>a} \alpha_2'(r) > 0, \quad \text{and } \lim_{r\downarrow a^+} \alpha_2(r) = -\infty; \tag{2.23}$$
$$h \in L^2(0,T;H^{-1/2}(\Gamma)). \tag{2.24}$$

As in (2.10), we also introduce the function $F \in L^2(0,T;V')$ given by

$$\langle F(t), v\rangle := \langle f(t), v\rangle + \langle h(t), v\rangle_\Gamma, \quad \forall v \in V \text{ for a.e. } t \in (0,T). \tag{2.25}$$

Remark 2.3. It follows from (2.23) that ρ_2, the inverse function of α_2, is defined on $\mathbb{R}$, strictly increasing, and Lipschitz continuous, with Lipschitz constant

$$\|{\rho_2}'\|_\infty = D_2 := \frac{1}{m_2}. \tag{2.26}$$

Furthermore, let us consider the function $\widehat{\rho_2}(u) := \int_0^u (\rho_2(r) - \rho_2^o)dr$, with $\rho_2^o := \rho_2(0)$. Then, $\widehat{\rho_2}$ is strictly convex and $\widehat{\rho_2}(u) \geq \widehat{\rho_2}(0) = 0$; besides, $\widehat{\rho_2}$ fulfils the analogue of (2.14) (replacing of course D_1 by D_2).

In this setting, the variational formulation for Problem $\mathbf{P_2}$ reads as follows.
Problem $\mathbf{P_2}$. *Find $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ ($\subset C^0([0,T];V)$), and $u \in L^2(0,T;V)$ such that*

$$\partial_t \chi + Ju = F \quad \text{in } V' \text{ for a.e. } t \in (0,T), \tag{2.27}$$
$$\partial_t \chi + A\chi + \chi^3 - \chi = \rho_2(u) \quad \text{in } H \text{ for a.e. } t \in (0,T), \tag{2.28}$$

and the initial condition (1.9) *holds for χ.*
In view of (2.19) and (2.25), the boundary conditions (1.16) for χ and u are again contained in the variational formulation (2.27)-(2.28). The following continuous dependence (with respect to the data χ_0, f, and h) result yields in particular that the solution to Problem $\mathbf{P_2}$ is necessarily unique.

Proposition 2.2. *Let (χ_0^1, f_1, h_1) and (χ_0^2, f_2, h_2) be two pairs of data for Problem $\mathbf{P_2}$ fulfilling* (2.7), (2.8), *and* (2.24)*; let F_i, $i = 1,2$, be the associated functions (cf.* (2.25)*), and let χ_i, $i = 1,2$, be the corresponding solutions of Problem $\mathbf{P_2}$. Set*

$$M^* := \max_{i=1,2}\{\|\chi_0^i\|_V + \|f_i\|_{L^2(0,T;V')} + \|h_i\|_{L^2(0,T;H^{-1/2}(\Gamma))}\}. \tag{2.29}$$

Then there exists a positive constant S^*, depending on M^*, T, $|\Omega|$, and m_2, such that

$$\|\chi_1 - \chi_2\|_{C^0([0,T];H)\cap L^2(0,T;V)}$$
$$\leq S^* \left(|\chi_0^1 - \chi_0^2|_H + \|f_1 - f_2\|_{L^2(0,T;V')} + \|h_1 - h_2\|_{L^2(0,T;H^{-1/2}(\Gamma))}\right) \tag{2.30}$$

Our well-posedness result for Problem $\mathbf{P_2}$ reads as follows.

THEOREM 2.2. *Assume that* (2.7)-(2.8) *and* (2.23)-(2.24) *hold. Then, Problem* $\mathbf{P_2}$ *admits a unique solution* (χ, u).
Lastly, we will give a regularity result for Problem $\mathbf{P_2}$ (cf. Remark 4.2 later on), under the following additional regularity assumptions:

$$f \in H^1(0,T;V'), \tag{2.31}$$

$$h \in H^1(0,T;H^{-1/2}(\Gamma)), \tag{2.32}$$

$$\chi_0 \in H^3(\Omega) \cap W. \tag{2.33}$$

Clearly, (2.31) and (2.32) entail that the function F defined in (2.25) fulfils $F \in H^1(0,T;V')$.

THEOREM 2.3. *In the setting of Problem* $\mathbf{P_2}$, *assume that* (2.31)-(2.33) *hold. Furthermore, let the constant* m_2 *in* (2.23) *fulfil* $m_2 > 1$. *Then, the unique solution* (χ, u) *to Problem* $\mathbf{P_2}$ *has the further regularity*

$$\chi \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W) \cap L^\infty(0,T;H^3(\Omega)), \tag{2.34}$$

and $u \in H^1(0,T;V)$.

REMARK 2.4. It follows from (2.34) that $\chi_t \in C^0([0,T];V)$ and that, on behalf of [26, Lemma III.1.4], $\chi \in C^0_{\mathrm{w}}([0,T];H^3(\Omega))$ (the space of $H^3(\Omega)$-valued weakly continuous functions on $[0,T]$). This is in agreement with our additional regularity requirement (2.33) on the initial datum χ_0, as we will see later on in Section 4.3, cf. (4.30).

In the sequel, we will denote by the same symbol C several constants depending only on the quantities specified by the statement of each theorem, and possibly on the initial data; we will point out the few occurring exceptions.

3. Well-posedness for Problem $\mathbf{P_1}$.

3.1. An approximate problem.

Let us first state the rigorous variational formulation for the approximate system ((1.17),(1.8)), which we will supplement with the boundary conditions (1.10), and with initial conditions for χ *and* u featuring for every $\nu > 0$ the *approximate* initial data $\chi_{0\nu}$ and $u_{0\nu}$, with

$$\chi_{0\nu} \in V \quad \text{and} \quad u_{0\nu} \in H \quad \forall \nu > 0. \tag{3.1}$$

Problem $\mathbf{P_{1\nu}}$. *Given the data* $\chi_{0\nu}$, $u_{0\nu}$ *and* G *specified by* (3.1) *and* (2.10), *respectively, find* $u_\nu \in L^2(0,T;V) \cap H^1(0,T;V') \subset C^0([0,T];H)$ *and* $\chi_\nu \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;H) \subset C^0_w([0,T];V)$ *satisfying*

$$\nu\partial_t u_\nu + \partial_t\chi_\nu + Au_\nu = G \quad \text{in } V' \text{ for a.e. } t \in (0,T), \tag{3.2}$$

$$\partial_t\chi_\nu + A\chi_\nu + \chi_\nu^3 - \chi_\nu = \rho_1(u_\nu) \quad \text{in } H \text{ for a.e. } t \in (0,T), \tag{3.3}$$

$$u_\nu(\cdot,0) = u_{0\nu}, \quad \chi_\nu(\cdot,0) = \chi_{0\nu}. \tag{3.4}$$

PROPOSITION 3.1. *Under the above assumptions, Problem* $\mathbf{P_{1\nu}}$ *admits a unique solution* (χ_ν, u_ν) *for every* $\nu > 0$.

Proof. Our argument follows the same outline of the proof of [8, Prop. 3.6], to which we refer the reader for further details. Thus, we first examine equations (3.2) and (3.3) separately.

As for the former, it follows from the well-known result [15, Thm.4.1, p.238] that for any $j \in L^2(0,T;H)$ there exists a unique function $u =: \mathcal{R}_\nu(j) \in L^2(0,T;V) \cap H^1(0,T;V')$ fulfilling

$$\nu\partial_t u + Au = G - j \quad \text{in } V' \text{ for a.e. } t \in (0,T), \tag{3.5}$$

and the first initial condition in (3.4). Moreover, there exists a constant C, depending on the data of the problem only, and not on ν, such that for any pair (j_1, j_2) there holds

$$\nu^2 \|u_1 - u_2\|^2_{L^2(0,t;H)} \le C \|J_1 - J_2\|^2_{L^2(0,t;H)} \quad \text{for a.e. } t \in (0,T), \tag{3.6}$$

where we have set $u_i := \mathcal{R}_\nu(j_i)$ and $J_i(x,t) := \int_0^t j_i(x,s)ds$ for a.e. $(x,t) \in \Omega \times (0,T)$, $i = 1,2$. To check this, it suffices to subtract (3.5) with the source $j = j_2$ from (3.5) with $j = j_1$, integrate the resulting equation on $[0,s]$, $s \in (0,T)$, then test it by $u_1(s) - u_2(s)$ and integrate once again over $[0,t]$, $t \in (0,T)$.

$$\begin{aligned}
&\nu \int_0^t |u_1(s) - u_2(s)|^2_H ds + \int_0^t \Big(\nabla(\int_0^s (u_1(r) - u_2(r))dr), \nabla(u_1(s) - u_2(s))\Big)_H ds \\
&\quad \le \frac{\nu}{2} \int_0^t |u_1(s) - u_2(s)|^2_H ds + \frac{1}{2\nu} \int_0^t \left| \int_0^s (j_1(r) - j_2(r))dr \right|^2_H ds.
\end{aligned}$$

Then, (3.6) follows, noting that the second summand on the left-hand side of the above inequality equals

$$\frac{1}{2} \int_0^t \frac{d}{dt} \Big(\left| \nabla \left(\int_0^s (u_1(r) - u_2(r))dr \right) \right|^2_H \Big) ds = \frac{1}{2} \left| \nabla \left(\int_0^t (u_1(s) - u_2(s))ds \right) \right|^2_H,$$

and is thus positive.

Let us now focus on the second equation (3.3): it is shown in [8, Lemma 3.3] that for any $k \in L^2(0,T;H)$ there exists a unique $\chi =: \mathcal{S}_\nu(k) \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;H)$ fulfilling for a.e. $t \in (0,T)$ (3.3) (with k instead of $\rho_1(u_\nu)$ on the right-hand side), and the second initial condition in (3.4). Moreover, the continuous dependence estimate

$$|\chi_1(t) - \chi_2(t)|^2_H \le C \|k_1 - k_2\|^2_{L^2(0,t;H)}, \tag{3.7}$$

holds for *any* pair k_1, $k_2 \in L^2(0,T;H)$, with $\chi_i = \mathcal{S}_\nu(k_i)$, $i = 1,2$. While referring to [8] for further details, let us only say that, as in the previous case, it suffices to take the difference of (3.3) with data k_1 and k_2, test the resulting equation by the difference $(\chi_1 - \chi_2)$ of the corresponding solutions and exploit the monotonicity of the map $\chi \mapsto \chi^3$, so that a straightforward application of Gronwall's Lemma leads to the estimate (3.7).

We can now define the operator $T_\nu : (L^2(0,T;H))^2 \to (L^2(0,T;H))^2$ given by

$$T_\nu(u,\chi) := (\mathcal{R}_\nu(\partial_t \mathcal{S}_\nu(\rho_1(u))), \mathcal{S}_\nu(\rho_1(u))). \tag{3.8}$$

Let us now consider on the space $(L^2(0,T;H))^2$ the norm

$$\|(v,w)\|^2 := \|v\|^2_{L^2(0,T;H)} + \|w\|^2_{L^2(0,T;H)} \quad \forall (v,w) \in (L^2(0,T;H))^2.$$

We can readily check that for any given pair (u_1, χ_1), $(u_2, \chi_2) \in (L^2(0,T;H))^2$, there holds

$$\|T_\nu(u_1,\chi_1) - T_\nu(u_2,\chi_2)\|^2 = \|\mathcal{R}_\nu(\partial_t \mathcal{S}_\nu(\rho_1(u_1))) - \mathcal{R}_\nu(\partial_t \mathcal{S}_\nu(\rho_1(u_2)))\|^2_{L^2(0,T;H)}$$

$$+\|\mathcal{S}_\nu(\rho_1(u_1)) - \mathcal{S}_\nu(\rho_1(u_2))\|^2_{L^2(0,T;H)} \leq C\nu^{-2} \int_0^T |\mathcal{S}_\nu(\rho_1(u(s))) - \mathcal{S}_\nu(\rho_2(u(s)))|^2_H \, ds$$

$$+CT\|\rho_1'\|^2_{L^\infty(\mathbb{R})} \int_0^T |u_1(s) - u_2(s)|^2_H ds \leq C^2\nu^{-2} T \|\rho_1'\|^2_{L^\infty(\mathbb{R})} \int_0^T |u_1(s) - u_2(s)|^2_H ds$$

$$+CT\|\rho_1'\|^2_{L^\infty(\mathbb{R})} \int_0^T |u_1(s) - u_2(s)|^2_H \, ds \leq K(\nu) T \|(u_1,\chi_1) - (u_2,\chi_2)\|^2. \tag{3.9}$$

Note indeed that, in order to conclude the second inequality, we have used the estimates (3.6) (with of course $j_i = \partial_t \chi_i = \partial_t \mathcal{S}_\nu(\rho_1(u_i))$, $i = 1, 2$), and (3.7) (with $k_i = \rho_1(u_i)$, $i = 1, 2$), also taking into account the Lipschitz continuity of ρ_1. Note that the constant $K(\nu) \nearrow +\infty$ as $\nu \searrow 0$. The same computations as in [3, p.8] yield that the $m-$th iterate of T_ν fulfils

$$\|T_\nu^m(u_1,\chi_1) - T_\nu^m(u_2,\chi_2)\|^2 \leq \frac{(K(\nu)T)^m}{m!} \|((u_1,\chi_1) - (u_2,\chi_2)\|^2$$

for every $(u_1, \chi_1), (u_2, \chi_2) \in (L^2(0,T;H))^2$. Therefore, there exists $m_\nu \in \mathbb{N}$ such that $T_\nu^{m_\nu}$ is a contraction on the space $(L^2(0,T;H))^2$, and has therefore a unique fixed point (u_ν, χ_ν) by the contraction mapping principle. It is straightforward to verify that (u_ν, χ_ν) is also the *unique* fixed point of the mapping T_ν. Recalling the definition of T_ν, we conclude that the pair $(u_\nu, \chi_\nu) \in (L^2(0,T;H))^2$ solves the system (3.2)-(3.4), with the regularity stated in Proposition 3.1. □

3.2. Existence for Problem $\mathbf{P_1}$.

PROPOSITION 3.2. *Let $\{(\chi_\nu, u_\nu)\}_\nu$ be the sequence of solutions to Problem $\mathbf{P_1}_\nu$, corresponding to data $\{(\chi_{0\nu}, u_{0\nu})\}$ and F fulfilling* (3.1) *and* (2.10)*, respectively. Suppose that*

$$\chi_{0\nu} \rightharpoonup \chi_0 \quad \text{in } V \quad \text{and} \quad \nu^{1/2}|u_{0\nu}|_H \to 0 \quad \text{as } \nu \downarrow 0. \tag{3.10}$$

Then, there exist $u \in L^2(0,T;V)$ and $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ such that the following convergences hold as $\nu \downarrow 0$

$$\chi_\nu \rightharpoonup^* \chi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.11}$$

$$\chi_\nu \to \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{3.12}$$

$$u_\nu \rightharpoonup u \quad \text{in } L^2(0,T;V), \tag{3.13}$$

$$\nu u_\nu \to 0 \quad \text{in } L^\infty(0,T;H), \quad \nu u_\nu \rightharpoonup 0 \quad \text{in } H^1(0,T;V'), \tag{3.14}$$

and the pair (χ, u) solves Problem $\mathbf{P_1}$.

The existence statement of Theorem 2.1 is of course a direct consequence of Proposition 3.2.

Proof. As a first step towards the proof of this result, we shall provide some *a priori* estimates on the approximate solutions $\{(\chi_\nu, u_\nu)\}_\nu$, which will enable us to pass to the limit in (3.2)-(3.4).

First estimate. We test (3.2) by $\rho_1(u_\nu) - \rho_1^\circ$ (cf. Remark 2.1), (3.2) by $\partial_t \chi_\nu$, we add the resulting equations and integrate on $(0,t)$, for $t \in (0,T)$ (note that a regularization procedure is needed on the test function $\partial_t \chi_\nu$, since it is not in

V: for further details, we refer the interested reader to the proof of [22, Thm. 3], where such a regularization -originally proposed in [7, Lemma 2.9]- is applied to an analogous estimate). Two terms cancel out, and, taking into account (2.12), we easily obtain

$$\begin{aligned}
&\nu\int_\Omega \widehat{\rho_1}(u_\nu(x,t))dx + \int_0^t\left(\int_\Omega \nabla(u_\nu(x,s))\nabla(\rho_1(u_\nu(x,s)))dx\right)ds \\
&\quad + \int_0^t |\partial_t\chi_\nu(s)|_H^2 ds + \frac{1}{2}|\nabla(\chi_\nu(t))|_H^2 + \frac{1}{4}\int_\Omega \chi_\nu^4(x,t)dx - \frac{1}{2}\int_\Omega \chi_\nu^2(x,t)dx \\
&= \nu\int_\Omega \widehat{\rho_1}(u_{0\nu}(x))dx + \frac{1}{2}|\nabla(\chi_{0\nu})|_H^2 + \frac{1}{4}\int_\Omega \chi_{0\nu}^4(x)dx - \frac{1}{2}\int_\Omega \chi_{0\nu}^2(x)dx \\
&\quad + \int_0^t \langle G(s), \rho_1(u_\nu(s)) - \rho_1^\circ\rangle ds + \rho_1^\circ\int_0^t\int_\Omega \partial_t\chi_\nu(x,s)dxds. \qquad (3.15)
\end{aligned}$$

By (2.6),

$$\int_0^t\left(\int_\Omega \nabla(u_\nu(x,s))\nabla(\rho_1(u_\nu(x,s)))dx\right)ds \geq m_1\int_0^t \nabla(\rho_1(u_\nu(s)))|_H^2 ds, \qquad (3.16)$$

whereas we can deal with the last two terms on the left-hand side of (3.15) recalling (2.4), which yields

$$\frac{1}{4}\int_\Omega \chi_\nu^4(x,t)dx - \frac{1}{2}\int_\Omega \chi_\nu^2(x,t)dx \geq \frac{1}{8}\int_\Omega \chi_\nu^4(x,t)dx - C|\Omega|. \qquad (3.17)$$

Turning to the right-hand side of (3.15), we can estimate the first term therein: indeed,

$$\nu\int_\Omega \widehat{\rho_1}(u_{0\nu}(x))dx \leq \frac{\nu}{2}D_1|u_{0\nu}|_H^2 \leq C, \qquad (3.18)$$

on behalf of (2.14) and (3.10). The latter hypothesis further enables us to control all the terms depending on the initial data $\chi_{0\nu}$. Finally, as for the last two summands in (3.15), we note that

$$\left|\rho_1^\circ\int_0^t\int_\Omega \partial_t\chi_\nu(x,s)dxds\right| \leq C + \frac{1}{4}\|\partial_t\chi_\nu\|_{L^2(0,t;H)}^2, \qquad (3.19)$$

$$\begin{aligned}
\Big|\int_0^t \langle G(s), \rho_1(u_\nu(s)) - \rho_1^\circ\rangle\, ds\Big| &\leq C + \int_0^t |\langle G(s), \rho_1(u_\nu(s)) - m_\Omega(\rho_1(u_\nu(s)))\rangle| ds \\
&\quad + \int_0^t \|G(s)\|_{V'}\|m_\Omega(\rho_1(u_\nu(s)))\|_V\, ds, \qquad (3.20)
\end{aligned}$$

where the constant C in both inequalities depends on ρ_1°, $|\Omega|$, and T, only. The first term on the right-hand side of (3.20) is estimated by

$$\frac{1}{2m_1}\int_0^t \|G(s)\|_{V'}^2\, ds + \frac{m_1}{2}\int_0^t |\nabla(\rho_1(u_\nu(s)))|_H^2\, ds \qquad (3.21)$$

by virtue of Poincaré's inequality for the zero mean value functions, whereas we infer by comparison in (3.3) that

$$\begin{aligned}
&\int_0^t \|G(s)\|_{V'}\|m_\Omega(\rho_1(u_\nu(s)))\|_V\, ds \\
&\leq \Omega|^{1/2}\int_0^t \|G(s)\|_{V'}\Big(|m_\Omega(\partial_t(\chi_\nu(s)))| + |m_\Omega(\chi_\nu^3(s) - \chi_\nu(s))|\Big)ds. \qquad (3.22)
\end{aligned}$$

Let us denote by S_i, $i = 1, 2$, the two summands on the right-hand side of (3.22): using (2.5) as well, we see that

$$S_1 \le c_1 \|G\|^2_{L^2(0,T;V')} + \frac{1}{4}\|\partial_t \chi_\nu\|^2_{L^2(0,t;H)}, \tag{3.23}$$

$$S_2 \le c_2 \int_0^t \|G(s)\|_{V'} \Big(\int_\Omega (1 + \chi_\nu^4(x,s))dx \Big) ds, \tag{3.24}$$

where the constants c_1 and c_2 only depend on $|\Omega|$, T, and on the initial datum χ_0. Collecting (3.16)-(3.24), we infer from (3.15)

$$\begin{aligned}
\nu \int_\Omega \widehat{\rho_1}(u_\nu(x,t))dx + \frac{m_1}{2}\int_0^t |\nabla(\rho_1(u_\nu(s)))|^2_H ds + \int_0^t |\partial_t \chi_\nu(s)|^2_H ds \\
+\frac{1}{2}|\nabla(\chi_\nu(t))|^2_H + \frac{1}{8}\int_\Omega \chi_\nu^4(x,t)dx \le C_0 \left(1 + \|G\|^2_{L^2(0,T;V')}\right) \\
+C_0 \int_0^t \left[\|G(s)\|_{V'} \left(\int_\Omega (1+\chi_\nu^4(x,s))dx \right) \right] ds + \frac{1}{2}\|\partial_t \chi_\nu\|^2_{L^2(0,T;H)},
\end{aligned} \tag{3.25}$$

where the constant C_0 only depends on the initial data $u_{0\nu}$, $\chi_{0\nu}$, cf. (3.10), and on ρ_1°, $|\Omega|$, T. Note that the first term in the left-hand side of the above inequality is positive thanks to (2.13). Then, an easy application of Gronwall's Lemma (see, e.g., [3, Lemma A.4]) to $\|\chi_\nu^4(t)\|_{L^1(\Omega)}$ allows us to conclude that

$$\begin{aligned}
\nu \|\widehat{\rho_1}(u_\nu)\|_{L^\infty(0,T;L^1(\Omega))} + \|\rho_1(u_\nu) - m_\Omega(\rho_1(u_\nu))\|_{L^2(0,T;V)} \\
+\|\chi_\nu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C,
\end{aligned} \tag{3.26}$$

for a constant $C \ge 0$ *independent* of ν.
Arguing by comparison in (3.3), we note that $\|m_\Omega(\rho_1(u_\nu))\|_{L^2(0,T)} \le C$, so that we also infer that

$$\|\rho_1(u_\nu)\|_{L^2(0,T;V)} \le C. \tag{3.27}$$

Second estimate. Let us preliminarily note that

$$|m_\Omega(u_\nu(t))| \le \frac{1}{|\Omega|}\int_\Omega |\alpha_1(\rho_1(u_\nu(x,t))) - \alpha_1(\rho_1^\circ)| dx \le M_1 m_\Omega(|\rho_1(u_\nu(t))|) + M_1|\rho_1^\circ|, \tag{3.28}$$

where we have used the crucial Lipschitz continuity assumption (2.6). Therefore, estimate (3.27) yields that

$$\|m_\Omega(u_\nu)\|_{L^2(0,T)} \le C. \tag{3.29}$$

Thus, we can test (3.2) by u_ν and integrate over $(0,t)$, getting

$$\begin{aligned}
&\frac{\nu}{2}|u_\nu(t)|^2_H + \int_0^t |\nabla(u_\nu(s))|^2_H ds \\
&= \frac{\nu}{2}|u_{0\nu}|^2_H - \int_0^t (\partial_t\chi_\nu(s), u_\nu(s))_H ds + \int_0^t \langle G(s), u_\nu(s)\rangle ds
\end{aligned} \tag{3.30}$$

Moreover,

$$\begin{aligned}
\left|\int_0^t (\partial_t \chi_\nu(s), u_\nu(s))_H ds\right| \le C\Big(\int_0^t |\partial_t \chi_\nu(s)|^2_H ds + \|m_\Omega(u_\nu)\|^2_{L^2(0,T)}\Big) \\
+\frac{1}{4}\int_0^t \|u_\nu(s) - m_\Omega(u_\nu(s))\|^2_V ds.
\end{aligned}$$

We can control the latter summand with the second term on the left-hand side of (3.30), once again by Poincaré's inequality, while the other two terms are bounded in view of the estimate (3.26) for χ_ν, and of (3.29) for $m_\Omega(u_\nu)$. We argue in the same way for the last summand on the right-hand side of (3.30), so that, using (3.10) again, we finally infer

$$\nu^{1/2}\|u_\nu\|_{L^\infty(0,T;H)} + \|u_\nu\|_{L^2(0,T;V)} + \nu\|\partial_t u_\nu\|_{L^2(0,T;V')} \leq C. \tag{3.31}$$

Note that the last estimate in (3.31) follows from a simple comparison argument in (3.2).

In the end, taking into account the previous estimates (3.26) and (3.27), and arguing by comparison in (3.3), we deduce that $\{A\chi_\nu\}$ is bounded in $L^2(0,T;H)$, whence by standard elliptic regularity results

$$\|\chi_\nu\|_{L^2(0,T;W)} \leq C, \tag{3.32}$$

for a constant C independent of ν.

Passage to the limit. The a priori estimates (3.26) and (3.32) for $\{\chi_\nu\}_\nu$ enable us to apply the well-known Lions-Aubin's theorem (see in particular [24, Thm.4, Cor.5]), ensuring that $\{\chi_\nu\}$ is relatively compact for the strong topology of $L^2(0,T;V)$ and $C^0([0,T];H)$. Furthermore, using (3.31) as well, we conclude on behalf of standard weak (and weak-star) compactness results that there exist $u \in L^2(0,T;V)$ and $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ such that the convergences (3.11)-(3.14) hold along some subsequence, which we will not specify for the moment (in particular, (3.14) is an obvious consequence of (3.31)). Note also that the a priori bound for $\|\chi_\nu\|_{L^\infty(0,T;V)}$ implies that there exists $\gamma \in L^\infty(0,T;H)$ such that $\chi_\nu^3 \rightharpoonup^* \gamma$ in $L^\infty(0,T;H)$ along some subsequence. On the other hand, thanks to the strong convergence (3.12), we have that $\chi_\nu^3 \to \chi^3$ a.e. in Q (possibly extracting a further subsequence). Thus, a simple application of the dominated convergence theorem yields

$$\chi_\nu^3 \to \chi^3 \quad \text{in } L^2(0,T;H) \quad \text{and} \quad \chi_\nu^3 \rightharpoonup^* \chi^3 \quad \text{in } L^\infty(0,T;H) \quad \text{as } \nu \downarrow 0, \tag{3.33}$$

always along some subsequence. Finally, it follows from (3.27) that there exists $w \in L^2(0,T;V)$ such that

$$\rho_1(u_\nu) \rightharpoonup w \quad \text{in } L^2(0,T;V). \tag{3.34}$$

Collecting (3.11)-(3.14) and (3.33)-(3.34), we are able to pass to the limit in (3.2)-(3.3), thus obtaining that the triplet (χ, u, w) fulfils (2.15)-(2.16), of course replacing $\rho_1(u)$ with w. We also point out that the initial condition (1.9) holds for χ as well, in view of (3.10), (yielding $\chi_{0\nu} \to \chi_0$ strongly in H), and (3.12).
To conclude the proof, it remains to show that

$$w(x,t) = \rho_1(u(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{3.35}$$

We observe that ρ_1 induces a maximal monotone graph on $L^2(\Omega\times(0,T))$: therefore, by virtue of [1, Prop.1.1, p.42], (3.35) follows if we prove that

$$\limsup_{\nu\downarrow 0} \int_0^t \Big(\int_\Omega \rho_1(u_\nu(x,s))u_\nu(x,s)dx\Big)ds \leq \int_0^t \Big(\int_\Omega w(x,s)u(x,s)dx\Big)ds. \tag{3.36}$$

To this aim, we test (3.3) by u_ν, and we find that

$$\limsup_{\nu\downarrow 0} \int_0^t \Big(\int_\Omega \rho_1(u_\nu(x,s))u_\nu(x,s)dx\Big)ds \leq \limsup_{\nu\downarrow 0} \big(-\nu|u_\nu(t)|_H^2 + \nu|u_{0\nu}|_H^2\big)$$

$$-\liminf_{\nu\downarrow 0}\int_0^t |\nabla(u_\nu(s))|_H^2 ds + \lim_{\nu\downarrow 0}\Big(\int_0^t \langle G(s), u_\nu(s)\rangle ds + \int_0^t (\nabla(\chi_\nu(s)), \nabla(u_\nu(s)))_H ds\Big)$$

$$+\lim_{\nu\downarrow 0}\int_0^t (\chi_\nu^3(s)-\chi_\nu(s),u_\nu(s))_H ds \le -\int_0^t |\nabla(u(s))|_H^2 ds + \int_0^t \langle G(s),u(s)\rangle ds$$

$$+\int_0^t (\nabla(\chi(s)),\nabla(u(s)))_H ds + \int_0^t (\chi^3(s)-\chi(s),u(s))_H ds$$

$$= \int_0^t \langle \partial_t\chi(s)+A\chi(s)+\chi^3(s)-\chi(s),u(s)\rangle ds = \int_0^t (w(s),u(s))_H ds.$$

Note that in the above chain of inequalities, we have substituted (3.2) for $\partial_t\chi_\nu$; then we have exploited (3.10) and combined the strong convergences (3.12) for χ_ν and the convergences (3.13)-(3.14) for u_ν in order to pass to the limit, deducing in the end the last equality.

We have so far obtained that the pair (χ,u) solves Problem $\mathbf{P_1}$, which on the other hand has a unique solution by Proposition 2.1. Then the limit pair (χ,u) does *not* depend on the subsequence that we have extracted, and we conclude that the convergences (3.11)-(3.14) hold indeed for the whole families $\{\chi_\nu\}$ and $\{u_\nu\}$ as $\nu\downarrow 0$. □

3.3. **Continuous dependence on the data for Problem $\mathbf{P_1}$.** The forthcoming proof of Proposition 2.1 relies on the following Lemma.

LEMMA 3.1. *In the framework of* (2.6)-(2.9), *there exists a positive constant* m_*, *depending on* T, Ω, ρ_1^o, m_1, *and* M_1 *only, such that for any solution* (χ,u) *to Problem* $\mathbf{P_1}$, *there holds*

$$\begin{aligned}&\|\chi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\rho_1(u)\|_{L^2(0,T;V)}\\ &\le m_*\left(1+\|\chi_0\|_V + \|f\|_{L^2(0,T;V')} + \|g\|_{L^2(0,T;H^{-1/2}(\Gamma))}\right).\end{aligned} \tag{3.37}$$

Outline of the proof. To check (3.37), we test (2.15) by $\rho_1(u)-\rho_1^o$, (2.16) by $\partial_t\chi$, add the resulting equations and integrate over $(0,t)$, $t\in(0,T)$. Then, we perform exactly the same computations as in the proof of Proposition 3.2, see (3.15)-(3.25). In particular, as therein we have to use the second equation (2.16) in order to estimate $m_\Omega(\rho_1(u))$. In the end, just as in (3.25), a straightforward application of Gronwall's Lemma enables us to conclude. □

Proof of Proposition 2.1. Referring to the notation of the statement of Proposition 2.1, let us set

$$\underline{\chi} := \chi_1-\chi_2, \quad \underline{u} := u_1-u_2, \quad \underline{\chi}_0 := \chi_0^1-\chi_0^2, \quad \underline{G} := G_1-G_2.$$

The pair $(\underline{\chi},\underline{u})$ obviously satisfies

$$\partial_t\underline{\chi} + A\underline{u} = \underline{G}, \quad \text{in } V' \text{ for a.e.}\, t\in(0,T), \tag{3.38}$$

$$\partial_t\underline{\chi} + A\underline{\chi} + \chi_1^3-\chi_2^3-\underline{\chi} = \rho_1(u_1)-\rho_1(u_2), \text{ in } H \text{ for a.e. } t\in(0,T) \tag{3.39}$$

It follows from (3.38) and (2.17) that

$$m_\Omega(\partial_t\underline{\chi}(t)) = m_\Omega(\underline{G}(t)) = 0 \quad \text{for a.e. } t\in(0,T). \tag{3.40}$$

We can thus test (3.38) by $\mathcal{N}\left(\rho_1(u_1(t))-\rho_1(u_2(t))-m_\Omega(\rho_1(u_1(t))-\rho_1(u_2(t)))\right)$, (3.39) by $\mathcal{N}(\partial_t\underline{\chi}(t))$ for a.e. $t\in(0,T)$, add the resulting equations and integrate over $(0,t)$, $t \in (0,T)$. Note that two terms cancel out, since, in view of (2.2) and (3.40), we have

$$\begin{aligned}&\Big\langle \partial_t \underline{\chi}(t), \mathcal{N}\Big(\rho_1(u_1(t)) - \rho_1(u_2(t)) - m_\Omega(\rho_1(u_1(t)) - \rho_1(u_2(t)))\Big)\Big\rangle \\ &= \big\langle \rho_1(u_1(t)) - \rho_1(u_2(t)) - m_\Omega(\rho_1(u_1(t)) - \rho_1(u_2(t)), \mathcal{N}(\partial_t \underline{\chi}(t))\big\rangle \\ &= \big\langle \rho_1(u_1(t)) - \rho_1(u_2(t)), \mathcal{N}(\partial_t \underline{\chi}(t))\big\rangle \quad \text{for a.e. } t \in (0,T),\end{aligned}$$

and, in the same way,

$$\begin{aligned}&\langle \underline{G}(t), \mathcal{N}\left(\rho_1(u_1(t)) - \rho_1(u_2(t)) - m_\Omega(\rho_1(u_1(t)) - \rho_1(u_2(t)))\right)\rangle \\ &= \langle \rho_1(u_1(t)) - \rho_1(u_2(t)), \mathcal{N}(\underline{G}(t))\rangle\,.\end{aligned}$$

On the other hand, (2.1) and (2.6) yield that

$$\begin{aligned}&\langle A(\underline{u}(t)), \mathcal{N}\left(\rho_1(u_1(t)) - \rho_1(u_2(t)) - m_\Omega(\rho_1(u_1(t)) - \rho_1(u_2(t)))\right)\rangle \\ &= (\underline{u}(t), \rho_1(u_1(t)) - \rho_1(u_2(t)))_H - (\underline{u}(t), m_\Omega(\rho_1(u_1(t)) - \rho_1(u_2(t))))_H \\ &\geq m_1|\rho_1(u_1(t)) - \rho_1(u_2(t))|_H^2 - (\underline{u}(t), m_\Omega(\rho_1(u_1(t)) - \rho_1(u_2(t))))_H. \qquad (3.41)\end{aligned}$$

Finally, in view of (2.1), (2.2), and (2.3), we have for a.e. $t \in (0,T)$

$$\langle \partial_t \underline{\chi}(t), \mathcal{N}(\partial_t \underline{\chi}(t))\rangle + \langle A(\underline{\chi}), \mathcal{N}(\partial_t \underline{\chi}(t))\rangle = \|\mathcal{N}(\partial_t \underline{\chi}(t))\|_V^2 + \frac{1}{2}\frac{d}{dt}|\underline{\chi}(t)|_H^2. \qquad (3.42)$$

Collecting (3.41)-(3.42), we may conclude

$$\begin{aligned}&m_1\int_0^t |\rho_1(u_1(s)) - \rho_1(u_2(s))|_H^2 ds + \int_0^t \|\mathcal{N}(\partial_t \underline{\chi}(s))\|_V^2 ds + \frac{1}{2}|\underline{\chi}(t)|_H^2 \\ &= \frac{1}{2}|\underline{\chi}_0|_H^2 + \int_0^t (\underline{u}(s), m_\Omega(\rho_1(u_1(s)) - \rho_1(u_2(s))))_H ds \\ &\quad + \int_0^t \langle \rho_1(u_1(s)) - \rho_1(u_2(s)), \mathcal{N}(\underline{G}(s))\rangle ds - \int_0^t \langle \chi_1^3(s) - \chi_2^3(s), \mathcal{N}(\partial_t \underline{\chi}(s))\rangle ds \\ &\quad - \int_0^t (\underline{\chi}(s), \mathcal{N}(\partial_t \underline{\chi}(s)))_H ds. \qquad (3.43)\end{aligned}$$

Let us first note that, by (3.39),

$$m_\Omega(\rho_1(u_1(t)) - \rho_1(u_2(t))) = m_\Omega(\chi_1^3(t) - \chi_2^3(t)) - m_\Omega(\underline{\chi}(t)) \quad \text{for a.e. } t \in (0,T).$$

Then, the second term on the right-hand side of (3.43) can be obviously estimated by

$$\frac{1}{|\Omega|}\int_0^t\int_\Omega |\underline{u}(x,s)|dx\int_\Omega |\chi_1^3(x,s) - \chi_2^3(x,s)|dx + \frac{1}{|\Omega|}\int_0^t\int_\Omega |\underline{u}(x,s)|dx\int_\Omega |\underline{\chi}(x,s)|dx \qquad (3.44)$$

Now, the first summand in (3.44) can be dealt with by using the elementary inequality

$$|r_1^3 - r_2^3| \leq \frac{3}{2}|r_1 - r_2||r_1^2 + r_2^2| \quad \text{for every } r_1, r_2 \in \mathbb{R}, \qquad (3.45)$$

yielding

$$\begin{aligned}
&\frac{1}{|\Omega|}\int_0^t\int_\Omega|\underline{u}(x,s)|dx\int_\Omega|\chi_1^3(x,s)-\chi_2^3(x,s)|dx\\
&\le C\int_0^t|\underline{u}(s)|_H|\underline{\chi}(s)|_H|\chi_1^2(s)+\chi_2^2(s)|_H ds\le\sigma_1\int_0^t|\underline{u}(s)|_H^2ds\\
&+C_{\sigma_1}\left(\|\chi_1\|^2_{L^\infty(0,T;L^4(\Omega))}+\|\chi_2\|^2_{L^\infty(0,T;L^4(\Omega))}\right)\int_0^t|\underline{\chi}(s)|_H^2ds\\
&\le\sigma_1M_1^2\int_0^t|\rho_1(u_1(s))-\rho_1(u_2)(s)|_H^2ds+C\int_0^t|\underline{\chi}(s)|_H^2ds, \qquad (3.46)
\end{aligned}$$

where we have used Young's inequality for some $\sigma_1>0$ to be specified later. Note also that the last inequality above follows from the Lipschitz continuity (2.6) of α_1 and from (3.37), which yields an *a priori* bound on $\|\chi_i\|_{L^\infty(0,T;L^4(\Omega))}$ in terms of the data χ_0^i, f_i, g_i, $i=1,2$. Concerning the second summand in (3.44), once again by Young's inequality and (2.6), we have

$$\begin{aligned}
&\frac{1}{|\Omega|}\int_0^t\int_\Omega|\underline{u}(x,s)|dx\int_\Omega|\underline{\chi}(x,s)|dx\\
&\le\sigma_2M_1^2\int_0^t|\rho_1(u_1(s))-\rho_1(u_2(s))|_H^2ds+C_{\sigma_2}\int_0^t|\underline{\chi}(s)|_H^2ds. \qquad (3.47)
\end{aligned}$$

The fourth summand on the right-hand side of (3.43) can be coped with arguing in the same way as in (3.46): indeed, (3.45) and Hölder's inequality yield

$$\begin{aligned}
&\left|\int_0^t\langle\chi_1^3(s)-\chi_2^3(s),\mathcal{N}(\partial_t\underline{\chi}(s))\rangle ds\right|\le C\int_0^t|\underline{\chi}(s)|_H\|\chi_1^2(s)+\chi_2^2(s)\|_{L^3(\Omega)}\|\mathcal{N}(\partial_t\underline{\chi}(s))\|_{L^6(\Omega)}ds\\
&\le\frac12\int_0^t\|\mathcal{N}(\partial_t\underline{\chi}(s))\|_V^2ds+C\left(\|\chi_1\|^2_{L^\infty(0,T;L^6(\Omega))}+\|\chi_2\|^2_{L^\infty(0,T;L^6(\Omega))}\right)\int_0^t|\underline{\chi}(s)|_H^2ds. \qquad (3.48)
\end{aligned}$$

Again, the last summand in the line above can be further estimated by means of (3.37). Finally, we note that, also on account of (2.3),

$$\left|\int_0^t(\underline{\chi}(s),\mathcal{N}(\partial_t\underline{\chi}(s)))_H ds\right|\le\frac14\int_0^t|\mathcal{N}(\partial_t\underline{\chi}(s))|_H^2ds+\int_0^t|\underline{\chi}(s)|_H^2ds, \qquad (3.49)$$

$$\begin{aligned}
\left|\int_0^t\langle\rho_1(u_1(s))-\rho_1(u_2(s)),\mathcal{N}(\underline{G}(s))\rangle ds\right|&\le\sigma_3\int_0^t|\rho_1(u_1(s))-\rho_1(u_2(s))|_H^2ds\\
&+C_{\sigma_3}\int_0^t\|\underline{G}(s)\|_{V'}^2ds, \qquad (3.50)
\end{aligned}$$

for some $\sigma_3>0$ to be chosen later.

Hence, collecting (3.43)-(3.50), we obtain

$$\begin{aligned}
&m_1\int_0^t|\rho_1(u_1(s))-\rho_1(u_2(s))|_H^2ds+\frac14\int_0^t\|\mathcal{N}(\partial_t\underline{\chi}(s))\|_V^2ds+\frac12|\underline{\chi}(t)|_H^2\\
&\le\frac12|\underline{\chi}_0|_H^2+C_{\sigma_3}\int_0^t\|\underline{G}(s)\|_{V'}^2ds+(\sigma_1M_1^2+\sigma_2M_1^2+\sigma_3)ds\\
&\int_0^t|\rho_1(u_1(s))-\rho_1(u_2(s))|_H^2+K\int_0^t|\underline{\chi}(s)|_H^2ds, \qquad (3.51)
\end{aligned}$$

where the constant K depends on σ_1, σ_2 and the related constants C_{σ_i}, $i=1,2$, as well as on m_* and the data χ_{0i}, f_i and g_i, $i=1,2$, through the a priori estimate (3.37). Then, we now choose the constants σ_i, $i=1,2,3$, in such a way that $\sigma_1 M_1^2 + \sigma_2 M_1^2 + \sigma_3 < m_1$: a direct application of Gronwall's Lemma allows us to conclude that

$$\|\underline{\chi}\|^2_{L^\infty(0,T;H)} + \|\rho_1(u_1) - \rho_1(u_2)\|^2_{L^2(0,T;H)} \leq s_* \left(|\underline{\chi}_0|^2_H + \|\underline{G}\|^2_{L^2(0,T;V')} \right)$$
$$\leq s_* \left(|\underline{\chi}_0|^2_H + \|f_1 - f_2\|^2_{L^2(0,T;V')} + \|g_1 - g_2\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \right), \tag{3.52}$$

where the constant s_* depends on χ_0^i, f_i and g_i, $i=1,2$ through (3.37), and on the data m_1, M_1, T, and Ω.
Finally, let us test (3.39) by $\underline{\chi}$ and integrate over $(0,t)$, $t \in (0,T)$: we easily obtain

$$\frac{1}{2}|\underline{\chi}(t)|^2_H + \int_0^t \int_\Omega |\nabla(\underline{\chi}(x,s))|^2 dxds + \int_0^t \int_\Omega (\chi_1^3(x,s) - \chi_2^3(x,s))\underline{\chi}(x,s) dxds$$
$$\leq \frac{1}{2}|\underline{\chi}_0|^2_H + \frac{1}{2}\int_0^t |\rho_1(u_1(s)) - \rho_1(u_2(s))|^2_H ds + \frac{3}{2}\int_0^t |\underline{\chi}(s)|^2_H ds.$$

Note that the third summand on the left-hand side of the above inequality is positive thanks to the monotonicity of the map $r \mapsto r^3$: then, taking into account (3.52), we easily deduce the continuous dependence estimate (2.18). □

4. **Well-posedness and Regularity for Problem $\mathbf{P_2}$.** We recall that throughout this section we will refer to the analytical setting specified in Section 2.3.

4.1. **Existence for Problem $\mathbf{P_2}$ via an approximate problem.** As the previous section, we will approximate Problem $\mathbf{P_2}$ with the initial-boundary value problem for the system ((1.12),(1.17)), supplemented with (1.16) and suitable initial conditions.

Problem $\mathbf{P_{2\mu}}$. *Let $\chi_{0\mu}$ and $u_{0\mu}$ fulfil* (3.1) *for every $\mu > 0$, and let F be defined by* (2.10). *Find $u_\mu \in L^2(0,T;V) \cap H^1(0,T;V') (\subset C^0([0,T];H))$ and $\chi_\mu \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;H) (\subset C^0_w([0,T];V))$ satisfying*

$$\mu\partial_t u_\mu + \partial_t \chi_\mu + J u_\mu = F \quad \text{in } V' \text{ for a.e. } t \in (0,T), \tag{4.1}$$
$$\partial_t \chi_\mu + A\chi_\mu + \chi_\mu^3 - \chi_\mu = \rho_2(u_\mu) \quad \text{in } H \text{ for a.e. } t \in (0,T), \tag{4.2}$$
$$u_\mu(\cdot,0) = u_{0\mu}, \quad \chi_\mu(\cdot,0) = \chi_{0\mu}. \tag{4.3}$$

Proposition 4.1. *In this setting, Problem $\mathbf{P_{2\mu}}$ admits a unique solution (χ_μ, u_μ) for every $\mu > 0$.*

Sketch of the proof. Our proof follows the same outline as the argument developed for Proposition 3.1: namely, a fixed point technique is applied, which leads to the analysis of the single equations (4.1) and (4.2). Then, we directly refer the reader to the proof of Proposition 3.1 for the analysis of the latter equation, as well as the definition of the related solution mapping $\mathcal{S}_\mu$. As far as (4.1) is concerned, we can easily show that for any $\ell \in L^2(0,T;H)$ there exists a unique function $u := \mathcal{R}_\mu(\ell) \in L^\infty(0,T;V) \cap H^1(0,T;V')$ such that

$$\mu\partial_t u + Ju = F - \ell \quad \text{in } V' \quad \text{for a.e. } t \in (0,T), \quad u_\mu(\cdot,0) = u_{0\mu}.$$

Indeed, arguing as in [8, Lemma 3.4]), we may tackle the Cauchy problem above by means of the abstract theory of nonlinear semigroups generated by maximal monotone operators, so that its well-posedness follows from the well-known results [1,

Thm. 2.1, p.189] or [3, Thm. 3.6, p.72]. Moreover, the analogues of the continuous dependence estimates (3.6) and (3.7) may be obtained in this case as well, carrying out the same computations as in the proof of Proposition 3.1.
Therefore, we can introduce the *solution operator* $T_\mu : (L^2(0,T;H))^2 \to (L^2(0,T;H))^2$ by obviously adapting the definition (3.8) of the solution operator T_ν for $\mathbf{P_{1\nu}}$. Note that a contraction estimate analogous to (3.9) holds in this setting too, in view of the analogues of (3.6) and (3.7), as well as of the Lipschitz continuity (2.26) of ρ_2. As in the proof of Proposition 3.1, the contraction mapping principle then ensures the well-posedness of Problem $\mathbf{P_{2\mu}}$. □

PROPOSITION 4.2. *Let $\{\chi_{0\mu}, u_{0\mu}\}$ be a sequence of initial data for Problem $\mathbf{P_{2\mu}}$ fulfilling* (3.10)*, and let $\{(\chi_\mu, u_\mu)\}_\mu$ be the corresponding solutions. Then there exist $u \in L^2(0,T;V)$ and $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$ such that the following convergences hold as $\mu \downarrow 0$*

$$\chi_\mu \rightharpoonup^* \chi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{4.4}$$

$$\chi_\mu \to \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{4.5}$$

$$u_\mu \rightharpoonup u \quad \text{in } L^2(0,T;V), \tag{4.6}$$

$$\mu u_\mu \to 0 \quad \text{in } L^\infty(0,T;H), \quad \mu u_\mu \rightharpoonup 0 \quad \text{in } H^1(0,T;V'), \tag{4.7}$$

and the pair (χ, u) solves Problem $\mathbf{P_2}$.

Proof. **A priori estimates.** In order to derive some *a priori* estimates on the sequences of approximate solutions $\{\chi_\mu\}$, $\{u_\mu\}$, we test (4.1) by $u_\mu + \rho_2(u_\mu) - \rho_2^\circ$, (4.2) by $\partial_t \chi_\mu$, add the relations thus obtained and integrate over $(0,t)$, for $t \in (0,T)$. By formal computations (which can be made rigorous by performing the regularization that we mentioned in the proof of Proposition 3.2), we get

$$\begin{aligned}
&\mu \int_\Omega \widehat{\rho_2}(u_\mu(x,t))dx + \frac{\mu}{2}|u_\mu(t)|_H^2 + \int_0^t \|u_\mu(s)\|_V^2 ds \\
&+ \int_0^t (\nabla(u_\mu(s)), \nabla(\rho_2(u_\mu(s))))_H ds + \gamma \int_0^t \int_\Gamma u_\mu(\sigma,s)(\rho_2(u_\mu(\sigma,s)) - \rho_2^\circ) d\sigma ds \\
&+ \int_0^t |\partial_t \chi_\mu(s)|_H^2 ds + \frac{1}{2}|\nabla(\chi_\mu(t))|_H^2 + \frac{1}{4}\int_\Omega \chi_\mu^4(x,t)dx - \frac{1}{2}\int_\Omega \chi_\mu^2(x,t)dx \\
\leq\, &\mu \int_\Omega \widehat{\rho_2}(u_{0\mu}(x))dx + \frac{\mu}{2}|u_{0\mu}|_H^2 + \frac{1}{2}|\nabla(\chi_{0\mu})|_H^2 + \frac{1}{4}\int_\Omega \chi_{0\mu}^4(x)dx \\
&+ \int_0^t \langle F(s), u_\mu(s)\rangle ds + \int_0^t \langle F(s), \rho_2(u_\mu(s)) - \rho_2^\circ\rangle ds - \int_0^t (\partial_t \chi_\mu(s), u_\mu(s))_H ds \\
&+ \rho_2^\circ \int_0^t \int_\Omega \partial_t \chi_\mu(x,s) dx ds,
\end{aligned} \tag{4.8}$$

where we have cancelled out two terms and used (2.21) for the third summand on the left-hand side of (4.8). On the other hand, note that

$$\begin{aligned}
&\int_0^t (\nabla(u_\mu(s)), \nabla(\rho_2(u_\mu(s))))_H ds + \gamma \int_0^t \int_\Gamma u_\mu(\sigma,s)(\rho_2(u_\mu(\sigma,s)) - \rho_2^\circ) d\sigma ds \\
\geq\, &m_2 \left(\int_0^t |\nabla(\rho_2(u_\mu(s)))|_H^2 ds + \gamma \int_0^t \int_\Gamma |\rho_2(u_\mu(\sigma,s)) - \rho_2^\circ|^2 d\sigma ds \right) \\
=\, &m_2 \int_0^t \|\rho_2(u_\mu(s)) - \rho_2^\circ\|_V^2 ds,
\end{aligned} \tag{4.9}$$

in view of the definition of ρ_2° (cf. Remark 2.3), of our assumption (2.23) on α_2, and of (2.21) once again to conclude the last equality. Furthermore, we can deal with the last two terms on the left-hand side of (4.8) as in (3.17). Turning to the right-hand side, all the terms depending on the initial data $\chi_{0\mu}$ and $u_{0\mu}$ are easily controlled thanks to (3.10) and (2.14) (cf. (3.18)). Concerning the remaining summands, we note

$$\left|\int_0^t \langle F(s), u_\mu(s)\rangle ds\right| \leq \int_0^t \|F(s)\|_{V'}^2 ds + \frac{1}{4}\int_0^t \|u_\mu(s)\|_V^2 ds, \tag{4.10}$$

$$\begin{aligned} &\left|\int_0^t \langle F(s), \rho_2(u_\mu(s)) - \rho_2^{\circ}\rangle ds\right| \\ &\leq \frac{1}{2m_2}\int_0^t \|F(s)\|_{V'}^2 ds + \frac{m_2}{2}\int_0^t \|\rho_2(u_\mu(s)) - \rho_2^{\circ}\|_V^2 ds, \end{aligned} \tag{4.11}$$

$$\left|\int_0^t (\partial_t \chi_\mu(s), u_\mu(s))_H ds\right| \leq \frac{1}{2}\int_0^t |\partial_t \chi_\mu(s)|_H^2 ds + \frac{1}{2}\int_0^t \|u_\mu(s)\|_V^2 ds, \tag{4.12}$$

while the last summand in (4.8) can be estimated exactly as in (3.19).
Collecting (4.9)-(4.12), we deduce from (4.8)

$$\begin{aligned} &\mu \int_\Omega \widehat{\rho_2}(u_\mu(x,t))dx + \frac{\mu}{2}|u_\mu(t)|_H^2 + \frac{1}{4}\int_0^t \|u_\mu(s)\|_V^2 ds + \frac{m_2}{2}\int_0^t \|\rho_2(u_\mu(s)) - \rho_2^{\circ}\|_V^2 ds \\ &+\frac{1}{4}\int_0^t |\partial_t \chi_\mu(s)|_H^2 ds + \frac{1}{2}|\nabla(\chi_\mu(t))|_H^2 + \frac{1}{8}\int_\Omega \chi_\mu^4(x,t)dx \leq C^0\Big(1 + \|F\|_{L^2(0,T;V')}^2\Big), \end{aligned}$$

where the constant C^0 only depends on the approximate data $\{u_{0\mu}\}$ and $\{\chi_{0\mu}\}$ through (3.10), as well as on m_2, ρ_2°, $|\Omega|$, and T. Therefore, there exists a positive constant C such that

$$\begin{aligned} &\mu\|\widehat{\rho_2}(u_\mu)\|_{L^\infty(0,T;L^1(\Omega))} + \mu^{1/2}\|u_\mu\|_{L^\infty(0,T;H)} + \|u_\mu\|_{L^2(0,T;V)} + \|\rho_2(u_\mu)\|_{L^2(0,T;V)} \\ &+\mu\|\partial_t u_\mu\|_{L^2(0,T;V')} + \|\chi_\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C \quad \forall \mu > 0, \end{aligned} \tag{4.13}$$

where the estimate for $\rho_2(u_\mu)$ is an obvious consequence of the estimate for $\rho_2(u_\mu) - \rho_2^{\circ}$, while the bound for $\mu\|\partial_t u_\mu\|_{L^2(0,T;V')}$ (for $\|\chi_\mu\|_{L^2(0,T;W)}$, respectively), follows from a comparison in (4.1) (in (4.2), resp., yielding that $\{A\chi_\mu\}$ is bounded in $L^2(0,T;H)$, whence the bound for $\{\chi_\mu\}$ in $L^2(0,T;W)$ by elliptic regularity results).
Passage to the limit. We can now refer the reader to the final part of the proof of Proposition 3.2. Arguing in the same way, we indeed deduce the convergences (4.4)-(4.7) from the a priori estimates (4.13). Then, the passage to the limit in the nonlinear coupling term $\rho_2(u_\mu)$ in (4.2) may be performed by monotonicity exactly as in (3.36). Thus, we can pass to the limit in both (4.1) and (4.2), and conclude that the limit pair (χ, u) is the unique solution to Problem $\mathbf{P_2}$. □

Remark 4.1. A careful comparison of the proofs of Propositions 3.2 and 4.2 somehow justifies our choice (1.16) of the boundary conditions for Problem $\mathbf{P_2}$ from an analytical viewpoint.

Indeed, the main technical difficulty that one encounters in passing to the limit in Problems $\mathbf{P_{1\nu}}$ and $\mathbf{P_{2\mu}}$ is related to the deduction of suitable a priori estimates for the sequences $\{u_\nu\}_\nu$ and $\{u_\mu\}_\mu$, respectively. Focusing on Problem $\mathbf{P_1}$, the (physically preferable) Neumann boundary conditions for u_ν yield poor estimates on the sequence $\{u_\nu\}_\nu$, due to a lack of coercivity in (3.2). Nonetheless, it is possible to recover the H^1-a priori bound that we need for u_ν, by estimating $m_\Omega(u_\nu)$ through

$m_\Omega(\rho_1(u_\nu))$ in the second equation (3.3). Here, the Lipschitz continuity of α_1 plays a key role, cf. (3.28).

In turn, in the case of Problem $\mathbf{P_2}$ we have to compensate for α_2 not being Lipschitz any more. Then, we consider Robin boundary conditions for u (and thus for u_μ), and we are thus able to control $\|u_\mu\|_{L^2(0,T;V)}$, see (4.8).

4.2. Uniqueness via continuous dependence for Problem $\mathbf{P_2}$.

LEMMA 4.1. *In the setting of Problem $\mathbf{P_2}$, there exists a positive constant m^*, depending on T, Ω, ρ_2^o, and m_2 only, such that for any solution (χ, u) to Problem $\mathbf{P_2}$, there holds*

$$\|\chi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\rho_2(u)\|_{L^2(0,T;V)} \\ \le m^* \left(1 + \|\chi_0\|_V + \|f\|_{L^2(0,T;V')} + \|h\|_{L^2(0,T;H^{-1/2}(\Gamma))}\right). \qquad (4.14)$$

The above result can be proved exactly in the same way as Lemma 3.1: namely, (4.14) can be deduced by performing on the system (2.27)-(2.28) the same estimates (cf. (4.8)-(4.12)) developed in the proof of Proposition 4.2, to which we refer the interested reader.

Proof of Proposition 2.2. Let us set

$$\overline{\chi} := \chi_1 - \chi_2, \quad \overline{u} := u_1 - u_2, \quad \overline{\chi}^0 := \chi_0^1 - \chi_0^2, \quad \overline{F} := F_1 - F_2.$$

Clearly, the pair $(\overline{\chi}, \overline{u})$ fulfils

$$\partial_t\overline{\chi} + J\overline{u} = \overline{F} \quad \text{in } V' \text{ for a.e. } t \in (0,T), \qquad (4.15)$$

$$\partial_t\overline{\chi} + A\overline{\chi} + \chi_1^3 - \chi_2^3 - \overline{\chi} = \rho_2(u_1) - \rho_2(u_2) \quad \text{in } H \text{ for a.e. } t \in (0,T) \qquad (4.16)$$

Let us test (4.15) by $J^{-1}(\rho_2(u_1) - \rho_2(u_2))$, (4.16) by $J^{-1}(\partial_t\overline{\chi}) + \overline{\chi}$, add the ensuing relations and integrate over $(0,t)$, $t \in (0,T)$. Note that, by (2.20), for a.e. $t \in (0,T)$

$$\left\langle \partial_t\overline{\chi}(t), J^{-1}(\rho_2(u_1(t)) - \rho_2(u_2(t)))\right\rangle = \left\langle \rho_2(u_1(t)) - \rho_2(u_2(t)), J^{-1}(\partial_t\overline{\chi}(t))\right\rangle$$

so that two terms cancel out and we get

$$\int_0^t (\overline{u}, \rho_2(u_1(s)) - \rho_2(u_2(s)))_H ds + \int_0^t \|J^{-1}(\partial_t\overline{\chi}(s))\|_V^2 ds + \frac{1}{2}|\overline{\chi}(t)|_H^2 \\ + \int_0^t |\nabla(\overline{\chi}(s))|_H^2 ds + \int_0^t \left(\nabla(\overline{\chi}(s)), \nabla(J^{-1}(\partial_t\overline{\chi}(s)))\right)_H ds \\ + \int_0^t \int_\Omega J^{-1}(\partial_t\overline{\chi}(x,s))(\chi_1^3(x,s) - \chi_2^3(x,s)) dx ds \\ + \int_0^t \int_\Omega \overline{\chi}(x,s)(\chi_1^3(x,s) - \chi_2^3(x,s)) dx ds \\ = \frac{1}{2}|\overline{\chi}^0|_H^2 + \int_0^t \int_\Omega \overline{\chi}(x,s) J^{-1}(\partial_t\overline{\chi}(x,s)) dx ds + \int_0^t |\overline{\chi}(s)|_H^2 ds \\ + \int_0^t \langle \rho_2(u_1(s)) - \rho_2(u_2(s)), J^{-1}(\overline{F}(s))\rangle ds + \int_0^t (\rho_2(u_1(s)) - \rho_2(u_2(s)), \overline{\chi}(s))_H ds \qquad (4.17)$$

where we have systematically used (2.20), (2.21), and (2.22). As usual, (2.23) yields that

$$\int_0^t (\overline{u}, \rho_2(u_1(s)) - \rho_2(u_2(s)))_H ds \ge m_2 \int_0^t |\rho_2(u_1(s)) - \rho_2(u_2(s))|_H^2 ds, \qquad (4.18)$$

whereas we can estimate the fourth and the fifth terms on the left-hand side of (4.17) by

$$\left|\int_0^t (\nabla(\overline{\chi}(s)), \nabla(J^{-1}(\partial_t\overline{\chi}(s))))_H ds\right| \leq \frac{1}{2}\int_0^t |\nabla(\overline{\chi}(s))|_H^2 ds + \frac{1}{2}\int_0^t \|J^{-1}(\partial_t\overline{\chi}(s))\|_V^2 ds, \quad (4.19)$$

$$\left|\int_0^t (J^{-1}(\partial_t\overline{\chi}(s)), \chi_1^3(s) - \chi_2^3(s))_H\right| \leq \frac{3}{2}\int_0^t |\overline{\chi}(s)|_H \|\chi_1^2(s) + \chi_2^2(s)\|_{L^3(\Omega)} \|J^{-1}(\partial_t\overline{\chi}(s))\|_{L^6(\Omega)} ds$$

$$\leq \frac{1}{8}\int_0^t \|J^{-1}(\partial_t\overline{\chi}(s))\|_V^2 ds C\left(\|\chi_1\|^2_{L^\infty(0,T;L^6(\Omega))} + \|\chi_2\|^2_{L^\infty(0,T;L^6(\Omega))}\right)\int_0^t |\overline{\chi}(s)|_H^2 ds. \quad (4.20)$$

Note that (4.20) is analogous to (3.48): as in that case, (4.20) has to be integrated using the preliminary estimate (4.14) to control the terms $\|\chi_i\|_{L^\infty(0,T;L^6(\Omega))}$, $i = 1, 2$. Finally, the last summand on the left-hand side of (4.17) is positive by monotonicity. Concerning the remaining terms on the right-hand side, the second and the fourth ones can be estimated exactly as in (3.49)-(3.50), i.e.,

$$\left|\int_0^t (\overline{\chi}(s), J^{-1}(\partial_t\overline{\chi}(s)))_H ds\right| \leq \frac{1}{8}\int_0^t |J^{-1}(\partial_t\overline{\chi}(s))|_H^2 ds + 2\int_0^t |\overline{\chi}(s)|_H^2 ds, \quad (4.21)$$

$$\left|\int_0^t \langle\rho_2(u_1(s)) - \rho_2(u_2(s)), J^{-1}(\overline{F}(s))\rangle ds\right|$$
$$\leq \frac{m_2}{4}\int_0^t |\rho_2(u_1(s)) - \rho_2(u_2(s))|_H^2 ds + C\int_0^t \|\overline{F}(s)\|_{V'}^2 ds. \quad (4.22)$$

In the end, we trivially have

$$\left|\int_0^t (\rho_2(u_1(s)) - \rho_2(u_2(s)), \overline{\chi}(s))_H ds\right|$$
$$\leq \frac{m_2}{2}\int_0^t |\rho_2(u_1(s)) - \rho_2(u_2(s))|_H^2 ds + \frac{1}{2m_2}\int_0^t |\overline{\chi}(s)|_H^2 ds. \quad (4.23)$$

On behalf of (4.18)-(4.23), we infer from (4.17)

$$\frac{m_2}{4}\int_0^t |\rho_2(u_1(s)) - \rho_2(u_2(s))|_H^2 ds + \frac{1}{4}\int_0^t \|J^{-1}(\partial_t\overline{\chi}(s))\|_V^2 ds + \frac{1}{2}|\overline{\chi}(t)|_H^2$$
$$+\frac{1}{2}\int_0^t |\nabla(\overline{\chi}(s))|_H^2 ds \leq C\left(|\overline{\chi}^0|_H^2 + \int_0^t \|\overline{F}(s)\|_{V'}^2 ds + \int_0^t |\overline{\chi}(s)|_H^2 ds\right),$$

where the constant C depends on T, $|\Omega|$, m_2, and M^* (2.29) through (4.14). Then an easy application of Gronwall's lemma to the term $|\overline{\chi}(t)|_H^2$ yields the continuous dependence estimate (2.30). □

4.3. **Regularity for Problem $\mathbf{P_2}$.** The proof of Theorem 2.3 relies on this preliminary regularity result for the approximate Problem $\mathbf{P_{2\mu}}$.

PROPOSITION 4.3. *Assume* (2.31), (2.32), *and suppose that the constant* m_2 *in* (2.23) *fulfils* $m_2 > 1$. *Further, let the initial data* $(\chi_{0\mu}, u_{0\mu})$ *fulfil* (2.33), *as well as*

$$u_{0\mu} \in V \quad \textit{and} \quad F(0) - Ju_{0\mu} \in H \quad \forall\mu > 0. \quad (4.24)$$

Then the unique solution (χ_μ, u_μ) *to Problem* $\mathbf{P_2}_\mu$ *has the further regularity*

$$\chi_\mu \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W) \cap L^\infty(0,T;H^3(\Omega)), \quad (4.25)$$

$$u_\mu \in H^2(0,T;V') \cap W^{1,\infty}(0,T;H) \cap H^1(0,T;V). \quad (4.26)$$

Note that our regularity requirements on $u_{0\mu}$ and $\chi_{0\mu}$ are in agreement with (4.25) (cf. Remark 2.4 and (4.30) below), and with (4.26). Indeed, it follows from the latter that $\partial_t u_\mu \in C^0_w([0,T];H)$, whence we should have

$$\mu\partial_t u_\mu(0) = -\partial_t\chi_\mu(0) + F(0) - Ju_{0\mu} \in H, \quad (4.27)$$

which is in fact ensured by (2.33) and (4.24).

We will not develop the proof of the above result, since it relies on the same estimates we shall perform later on for proving our main regularity result Theorem 2.3. Such estimates could indeed be made rigorous by a suitable approximation scheme for Problem $\mathbf{P_2}_\mu$ (possibly based on an implicit time discretization procedure). Nonetheless, we choose not to detail such a scheme here, in order to avoid too many technical details; besides, these approximation techniques are by now standard in the framework of phase field models.

We will need the following preliminary Lemma.

LEMMA 4.2. *Let* λ *be a positive constant with* $0 < \lambda < 1$. *Then, there exist* ε *and* $\eta > 0$ *fulfilling*

$$\varepsilon + \eta < 1 \quad \text{and} \quad \frac{\lambda}{4\varepsilon} + \frac{1}{4\eta} < 1. \quad (4.28)$$

Proof. It is easy to see that, as $\lambda < 1$, we can choose $0 < \sigma < 3/4$ such that

$$\lambda < 5 - 4(\sigma + (1-\sigma)^{1/2}).$$

Then, the algebraic equation $4x^2 + (\lambda + 4\sigma - 5)x + 1 - \sigma = 0$ has discriminant $D = (\lambda + 4\sigma - 5)^2 - 16(1-\sigma) > 0$, so that for every $y_1 := \frac{5-4\sigma-\lambda-\sqrt{D}}{8} < y < y_2 := \frac{5-4\sigma-\lambda+\sqrt{D}}{8}$ we have

$$4y^2 + (\lambda + 4\sigma - 5)y + 1 - \sigma < 0.$$

Furthermore, we can verify that $y_1 < 1-\sigma$ (as $\sigma < 3/4$), and that $y_2 > 0$: therefore, it is possible to choose $\eta > 0$ such that $0 < \eta < 1 - \sigma$ and

$$4\eta^2 + (\lambda + 4\sigma - 5)\eta + 1 - \sigma < 0. \quad (4.29)$$

Let us now set $\varepsilon := 1-\sigma-\eta$: by construction, $\varepsilon > 0$ and the first inequality in (4.28) is trivially true. Thus, we substitute $1 - \varepsilon - \eta$ for σ in the inequality in (4.29) and obtain $\varepsilon + \lambda\eta < 4\varepsilon\eta$, which is clearly equivalent to the second inequality in (4.28). □

Proof of Theorem 2.3. In order to prove the additional regularity for the solution of Problem $\mathbf{P_2}$, we consider the sequence of solutions $\{(\chi_\mu, u_\mu)\}$ to the approximate Problem $\mathbf{P_2}_\mu$, under the additional regularity hypotheses (2.31)-(2.33) and (4.24) on the data f, h and $\{(\chi_{0\mu}, u_{0\mu})\}_\mu$. In this framework, it is understood that $\partial_t u_\mu(0)$ is given by (4.27), and $\partial_t\chi_\mu(0)$ by

$$\partial_t\chi_\mu(0) = -A\chi_{0\mu} - \chi_{0\mu}^3 + \chi_{0\mu} + \rho_2(u_{0\mu}) \in V. \quad (4.30)$$

Note indeed that the right-hand side of (4.30) is in V thanks to (2.33) and (4.24). Moreover, we will assume that

$$\chi_{0\mu} \rightharpoonup \chi_0 \quad \text{in } H^3(\Omega), \quad \mu^{1/2}|F(0) - Ju_{0\mu}|_H \to 0 \quad \text{as } \mu \downarrow 0, \quad \|u_{0\mu}\|_V \leq C, \quad (4.31)$$

for a constant C independent of μ. Then, we shall derive further uniform bounds on the sequence $\{(\chi_\mu, u_\mu)\}$, pass to the limit and deduce (2.34).

To this aim, we take the time derivative of both (4.1) and (4.2), test them by $\partial_t u_\mu$ and $\partial_t^2 \chi_\mu$, respectively, add the resulting equations and integrate over $(0,t)$, $t \in (0,T)$ (note that both $\partial_t u_\mu$ and $\partial_t^2 \chi_\mu$ are admissible test functions for (4.1)-(4.2) thanks to their further regularity (4.25)-(4.26)). We thus obtain

$$\begin{aligned}&\frac{\mu}{2}|\partial_t u_\mu(t)|_H^2 + \int_0^t (\partial_t^2 \chi_\mu(s), \partial_t u_\mu(s))_H ds + \int_0^t \langle J(\partial_t u_\mu(s)), \partial_t u_\mu(s)\rangle ds \\ &+ \int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds + \frac{1}{2}|\nabla(\partial_t \chi_\mu(t))|_H^2 = \frac{\mu}{2}|\partial_t u_\mu(0)|_H^2 + \frac{1}{2}|\nabla(\partial_t \chi_\mu(0))|_H^2 + \\ &\int_0^t \langle \partial_t F(s), \partial_t u_\mu(s)\rangle ds + \int_0^t (\partial_t^2 \chi_\mu(s), \partial_t \chi_\mu(s))_H ds \\ &- 3\int_0^t \int_\Omega \chi_\mu^2(x,s)\partial_t \chi_\mu(x,s)\partial_t^2 \chi_\mu(x,s) dx ds + \int_0^t (\rho_2'(u_\mu(s))\partial_t u_\mu(s), \partial_t^2 \chi_\mu(s))_H ds.\end{aligned} \tag{4.32}$$

Note that the third term on the left-hand side of (4.32) equals $\|\partial_t u_\mu\|^2_{L^2(0,t;V)}$ in view of (2.21), and that, by (4.27), (4.30), and (4.31), there exists a positive constant C such that

$$\|\partial_t \chi_\mu(0)\|_V + \mu^{1/2}|\partial_t u_\mu(0)|_H \leq C \quad \forall \mu > 0. \tag{4.33}$$

Let us now choose two positive constants ε and η fulfilling (4.28), with the choice $\lambda = D_2^2$, D_2 being the Lipschitz constant of ρ_2, cf. (2.26): in fact, such a choice is admissible, as $D_2^2 < 1$ by assumption. Then, the second summand on the left-hand side and the last term on the right-hand side of (4.32) can be respectively estimated by

$$\left|\int_0^t (\partial_t^2 \chi_\mu(s), \partial_t u_\mu(s))_H ds\right| \leq \eta \int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds + \frac{1}{4\eta}\int_0^t |\partial_t u_\mu(s)|_H^2 ds, \tag{4.34}$$

$$\left|\int_0^t (\rho_2'(u_\mu(s))\partial_t u_\mu(s), \partial_t^2 \chi_\mu(s))_H ds\right| \leq \varepsilon \int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds + \frac{D_2^2}{4\varepsilon}\int_0^t |\partial_t u_\mu(s)|_H^2 ds. \tag{4.35}$$

Concerning the remaining terms on the right-hand side of (4.32), we fix $\zeta_1 < 1 - D_2^2/4\varepsilon - 1/4\eta$ and $\zeta_2 < (1-\varepsilon-\eta)/2$: we can now estimate the third, fourth and fifth term by, respectively,

$$\left|\int_0^t \langle \partial_t F(s), \partial_t u_\mu(s)\rangle ds\right| \leq \zeta_1 \int_0^t \|\partial_t u_\mu(s)\|_V^2 ds + \frac{1}{4\zeta_1}\int_0^t \|\partial_t F(s)\|_{V'}^2 ds, \tag{4.36}$$

$$\left|\int_0^t (\partial_t^2 \chi_\mu(s), \partial_t \chi_\mu(s))_H ds\right| \leq \zeta_2 \int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds + \frac{1}{4\zeta_2}\int_0^t |\partial_t \chi_\mu(s)|_H^2 ds, \tag{4.37}$$

$$\begin{aligned}&\left|\int_0^t (3\chi_\mu^2(s)\partial_t \chi_\mu(s), \partial_t^2 \chi_\mu(s))_H ds\right| \leq 3\int_0^t |\partial_t^2 \chi_\mu(s)|_H \|\partial_t \chi_\mu(s)\|_{L^6(\Omega)} \|\chi_\mu^2\|_{L^3(\Omega)} \\ &\leq \zeta_2 \int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds + \frac{1}{4\zeta_2}\|\chi_\mu\|^2_{L^\infty(0,T;V)} \int_0^t \|\partial_t \chi_\mu(s)\|^2_{L^6(\Omega)} ds \\ &\leq \zeta_2 \int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds + C\int_0^t |\nabla(\partial_t \chi_\mu(s))|_H^2 ds + C\int_0^t |\partial_t \chi_\mu(s)|_H^2 ds.\end{aligned} \tag{4.38}$$

Note that, in order to deduce (4.38), we have used Hölder's and Young's inequalities, as well as the estimate (3.26) for $\|\chi_\mu\|_{L^\infty(0,T;V)}$. Collecting (4.34)-(4.38), taking into account (2.21) and using the elementary estimate

$$\int_0^t |\partial_t \chi_\mu(s)|_H^2 ds \le 2T \left(\int_0^t \|\partial_t^2 \chi_\mu\|_{L^2(0,s;H)}^2 ds + |\partial_t \chi_\mu(0)|_H^2 \right)$$

to control the last summand in (4.37) and (4.38), we finally get

$$\begin{aligned} &\frac{\mu}{2}|\partial_t u_\mu(t)|_H^2 + (1 - \frac{D_2^2}{4\varepsilon} - \frac{1}{4\eta} - \zeta_1) \int_0^t \|\partial_t u_\mu(s)\|_V^2 ds \\ &+(1-\varepsilon-\eta-2\zeta_2)\int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds + \frac{1}{2}|\nabla(\partial_t \chi_\mu(t))|_H^2 \\ &\le C\left(1 + \int_0^t \|\partial_t F(s)\|_{V'}^2 ds + \int_0^t |\nabla(\partial_t \chi_\mu(s))|_H^2 ds + \int_0^t \|\partial_t^2 \chi_\mu\|_{L^2(0,s;H)}^2 ds\right), \end{aligned}$$

where the constant C depends on the initial data $\chi_{0\mu}$ and $u_{0\mu}$ through (4.33). Recalling (2.31)-(2.32) (whence $\partial_t F \in L^2(0,T;V')$), a straightforward application of Gronwall's Lemma to $|\nabla(\partial_t \chi_\mu(t))|_H^2 + \int_0^t |\partial_t^2 \chi_\mu(s)|_H^2 ds$ allows us to conclude that

$$\mu^{1/2}\|\partial_t u_\mu\|_{L^\infty(0,T;H)} + \|\partial_t u_\mu\|_{L^2(0,T;V)} + \|\partial_t^2 \chi_\mu\|_{L^2(0,T;H)} + \|\partial_t \chi_\mu\|_{L^\infty(0,T;V)} \le C,$$

for a constant C independent of μ, whence we deduce

$$\|\chi_\mu\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)} + \|u_\mu\|_{H^1(0,T;V)} \le C \quad \forall \mu > 0. \tag{4.39}$$

In view of (4.39) as well, we infer by comparison in the equation obtained taking the time derivative of (4.2) that $A(\partial_t \chi_\mu)$ is bounded in $L^2(0,T;H)$, whence standard elliptic regularity results ensure that $\partial_t \chi_\mu$ is bounded in $L^2(0,T;W)$, so that

$$\|\chi_\mu\|_{H^1(0,T;W)} \le C \quad \forall \mu > 0. \tag{4.40}$$

Another comparison argument in (4.2), combined with (4.39) once again, entails that $A\chi_\mu$ is bounded in $L^\infty(0,T;V)$, so that the same elliptic regularity results previously invoked give that

$$\|\chi_\mu\|_{L^\infty(0,T;H^3(\Omega))} \le C \quad \forall \mu > 0. \tag{4.41}$$

On account of the *further* a priori bounds (4.39)-(4.41) and of the standard weak (weak-star) compactness results [24, Thm.4, Cor.5]), we can reinforce the convergences (4.4)-(4.7) obtained in Proposition 4.2: of course, the limit pair (χ, u) coincides with the unique solution to Problem $\mathbf{P_2}$. Therefore, we conclude the additional regularity stated in Theorem 2.3. □

Remark 4.2. Let us remark that a regularity result analogous to Theorem 2.3 cannot be obtained for Problem $\mathbf{P_1}$ by means of the above techniques. Indeed, in that case, too, the only possible estimate on the time derivatives of (2.15) and (2.16) is to test them, respectively, by $\partial_t u_\nu$ and $\partial_t^2 \chi_\nu$. Then, it turns out that, by lack of coercivity due to the different boundary conditions, $((2.15))_t$ (i.e., the time derivative of (2.15)), does not provide an estimate on $\|\partial_t u_\nu\|_{L^2(0,T;V)}$ any more, so that we are not able to deal with the coupling term in $((2.16))_t$ (cf. instead with (4.35)).

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