Global attractors for gradient flows in metric spaces ☆

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Abstract

We develop the long-time analysis for gradient flow equations in metric spaces. In particular, we consider two notions of solutions for *metric gradient flows*, namely *energy* and *generalized solutions*. While the former concept coincides with the notion of *curves of maximal slope* of (AGS05), we introduce the latter to include limits of time-incremental approximations constructed via the *Minimizing Movements* approach (dGio93; Amb95).

For both notions of solutions we prove the existence of the global attractor. Since the evolutionary problems we consider may lack uniqueness, we rely on the theory of *generalized semiflows* introduced in (Bal97).

The notions of generalized and energy solutions are quite flexible and can be used to address gradient flows in a variety of contexts, ranging from Banach spaces, to Wasserstein spaces of probability measures. We present applications of our abstract results, by proving the existence of the global attractor for the *energy solutions*, both of abstract doubly nonlinear evolution equations in reflexive Banach spaces, and of a class of evolution equations in Wasserstein spaces, as well as for the *generalized solutions* of some phase-change evolutions driven by mean curvature.

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1. Introduction

Gradient flows in metric spaces have recently been the object of intensive studies. We especially refer to the monograph by AMBROSIO, GIGLI & SAVARÉ (AGS05) for a systematic analysis of *metric gradient flows* from the viewpoint of existence and approximation of solutions, as well as their uniqueness. Moreover, some decay to equilibrium result has also been developed.

The main aim of this paper is to progress further in the analysis of the long-time dynamics for metric gradient flows. In particular, we focus on the existence of the global attractor for the *generalized semiflow* (Bal97) associated with suitable solution notions. Moreover, we present new convergence-to-equilibrium results, and apply the abstract theory to doubly nonlinear equations in Banach spaces, to evolutions in Wasserstein spaces, and to some phase-change problems driven by mean curvature.

Energy and generalized solutions. The possibility of discussing gradient flow dynamics in metric spaces relies on a suitable *scalar* formulation of the evolution problem. Let us develop some heuristics by starting from the classical setting of the Euclidean space $\mathbb{R}^d$ and from a proper, smooth functional ϕ on $\mathbb{R}^d$, driving the gradient flow equation

$$u'(t) + \nabla\phi(u(t)) = 0 \qquad \text{in } \mathbb{R}^d \quad \text{for a.a. } t \in (0,T). \tag{1.1}$$

Now, testing (1.1) by $u'(t)$ and taking into account the *chain rule*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi\circ u)(t) = \langle \nabla\phi(u(t)), u'(t)\rangle \qquad \text{for a.a. } t \in (0,T) \tag{1.2}$$

it is straightforward to check that (1.1) may be equivalently reformulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi\circ u)(t) = -\frac{1}{2}|u'(t)|^2 - \frac{1}{2}|\nabla\phi(u(t))|^2 \qquad \text{for a.a. } t \in (0,T). \tag{1.3}$$

Equation (1.1) needs to be appropriately formulated in a metric space, where a linear structure is missing. Indeed, relation (1.3) is *scalar* and can serve as a possible notion of *metric gradient flow* evolution, as soon as one provides some possible metric surrogate for the *norm* of the time-derivative and the *norm* of a gradient (and not for the *full* time-derivative or the *full* gradient). These are provided by the scalar *metric derivative* $t \mapsto |u'|(t)$

and *slope* $t \mapsto |\partial\phi|(u(t))$, which are suitably defined for the metric-space-valued trajectory $t \mapsto u(t) \in (U, d)$ (see Section 2 for their definitions and properties). In particular, the *metric* (and doubly nonlinear) reinterpretation of (1.3) reads (hereafter, $1 < p < \infty,\ 1/p + 1/p' = 1$)

$$\begin{aligned}&\text{the map } t \in (0,T) \mapsto \phi(u(t)) \text{ is absolutely continuous and}\\ &\frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ u)(t) = -\frac{1}{p}|u'|^p(t) - \frac{1}{p'}|\partial\phi|^{p'}(u(t)) \quad \text{for a.a.} \quad t \in (0,T).\end{aligned} \tag{1.4}$$

Our stronger notion of solvability for metric gradient flows is that of *energy solutions*: namely, curves $t \mapsto u(t)$ fulfilling (1.4), where $|\partial\phi|$ is replaced by a suitable relaxation. A weaker notion of solution we aim to consider is that of *generalized solutions*. The latter are intimately tailored to the concept of *Minimizing Movements* (dGio93; Amb95), namely a natural limiting object arising in connection with time-discretization procedures.

We know from (Amb95; AGS05) that, if

$$\phi \text{ has compact sublevels in } U, \tag{COMP}$$

(and hence ϕ is lower semicontinuous), if some conditional continuity holds (see (CONT) in Section 2 below) and (some relaxation of) $|\partial\phi|$ fulfills a suitable *metric chain-rule inequality* (see (2.7) below), then the time-incremental approximations constructed via the Minimizing Movements scheme converge to an energy solution.

Moreover, the *sole compactness* (COMP) is sufficient to guarantee that approximate solutions converge to some curve $t \mapsto u(t)$ which satisfies a specific energy inequality (see (2.22)). The latter keeps track of the limit $t \mapsto \varphi(t)$ of the energy ϕ along the approximate solutions, which does not, in general, coincide with $\phi \circ u$. We shall call the pair (u, φ) a *generalized solution* of the metric gradient flow. Note that energy solutions $t \mapsto u(t)$ give rise to generalized solutions $t \mapsto (u(t), \phi(u(t)))$.

Main results. In (AGS05, Chap. 4) the large-time behavior of *energy solutions* was analyzed in case $p = 2$, by requiring that the functional

$$\phi \text{ is } \lambda\text{-geodesically convex for some } \lambda > 0. \tag{1.5}$$

Such condition extends the usual notion of λ-convexity to the metric framework (see (2.12)). Under (1.5), it was proved that every trajectory $t \mapsto u(t)$

exponentially converges as $t \to +\infty$ to the unique minimum point $\bar{u}$ of ϕ. Moreover, the exponential decay of the energy $\phi(u(t))$ to the equilibrium energy $\phi(\bar{u})$ was obtained (see also (CmCV03; CmCV06)). In addition, it was shown that, under some structural, convexity-type condition on the ambient metric space, the Cauchy problem for (1.4), in the case $p = 2$, admits a unique solution, generating a λ-contracting semigroup.

In this paper, we shall complement the long-time behavior results of (AGS05) in two directions, namely the construction of the global attractor and the investigation of the convergence of single trajectories to stationary states.

Firstly, we shall prove that, under very general hypotheses on ϕ, no structural assumption on the ambient space, no geodesic convexity on ϕ, and for general $1 < p < \infty$, both the set of energy and the set of generalized solutions admit a *global attractor*, namely a maximal, compact, invariant set attracting all bounded sets in the phase space. Our choice of the phase space is dictated by the energy ϕ. Indeed, the functional ϕ decreases along trajectories and is thus a *Lyapunov function* for the system. In view of this, as already pointed out in (RoSc04; Seg06), as well as in (RSS08) (where the long-time behavior of gradient flows of nonconvex functionals in Hilbert spaces was investigated), it is significant to set our long-time analysis in the metric phase space

$$(\mathcal{U}, d_{\mathcal{U}}), \quad \mathcal{U} := D(\phi), \quad d_{\mathcal{U}}(u,v) := d(u,v) + |\phi(u) - \phi(v)| \tag{1.6}$$

for energy solutions u and in an analogously defined, *augmented*, phase space for generalized solutions (u, φ).

Due to the possible nonconvexity of the functional ϕ and to the *doubly nonlinear character* of (1.4) for $p \neq 2$, uniqueness of energy (and, a fortiori, of generalized) solutions may genuinely fail. In recent years, several approaches have been developed to extend the well-established theory of attractors for semigroups (see, e.g., (Tem88)), to differential problems without uniqueness. In this connection, without claiming completeness we may recall the results in (Sel73; Sel96; ChVi95; MeVa98; MeVa00; CMR03). Here, we shall specifically move within the frame advanced by J. M. Ball (Bal97; Bal04), namely with the theory of *generalized semiflows* (Appendix Appendix A).

The two main abstract theorems of this paper run as follows:

- compactness (COMP) and the boundedness of equilibria entail the existence of the global attractor for the generalized semiflow of *generalized* *solutions* (Thm. 4.6).

- compactness (COMP), conditional continuity (CONT), the metric chain-rule inequality (2.7), and the boundedness of equilibria in (1.6) yield the existence of the global attractor for the generalized semiflow of *energy solutions* (Thm. 4.7).

Note that, apart from the boundedness of equilibria, the existence of a global attractor is obtained *under the very same conditions* ensuring existence of the corresponding solution notions.

Moreover, we shall extend the convergence to equilibrium results of (AGS05) to the case of energy solutions (1.4) with $p \neq 2$. As already pointed out in (AGS05, Rmk. 2.4.7), the key condition is a suitable generalization of (1.5), in which the modulus of convexity depends on the p-power of the distance, i.e.

$$\phi \text{ is } (\lambda, p)\text{-geodesically convex for some } \lambda > 0, \tag{CONV}$$

(see Section 2.2). Hence, in Theorem 4.8 we shall prove that, if ϕ complies with (COMP) and (CONV), then every trajectory $t \mapsto u(t)$ of (1.4) converges as $t \to +\infty$ to the unique minimum point $\bar{u}$ of ϕ exponentially fast, again with energy $\phi(u(t))$ exponentially decaying to $\phi(\bar{u})$. This in particular entails that the global attractor for energy solutions reduces to the singleton $\{\bar{u}\}$. Notice however that uniqueness of solutions to (1.4), even under (CONV) and the convexity structural condition on (U, d) imposed in (AGS05, Chap. 4), still seems to be an open problem.

Application to doubly nonlinear evolutions in Banach spaces. Our first example of energy solutions (see Section 3.1) is provided by abstract doubly nonlinear evolution equations in a reflexive Banach space $\mathcal{B}$, of the type

$$\mathrm{J}_p(u'(t)) + \partial\phi(u(t)) \ni 0 \quad \text{in } \mathcal{B}' \quad \text{for a.a. } t \in (0, T)\,, \tag{1.7}$$

where $\mathrm{J}_p : \mathcal{B} \rightrightarrows \mathcal{B}'$ is the p-duality map (see (3.29) later on), $\phi : \mathcal{B} \to (-\infty, +\infty]$ a proper, lower semicontinuous and convex functional and $\partial\phi$ its subdifferential in the sense of convex analysis.

The long-time behavior of doubly nonlinear equations driven by *nonconvex* functionals has been recently investigated in (Seg06; Aka08), where the existence of the global attractor for (1.7) has been obtained in the case ϕ is

λ-convex and, respectively, when $\partial\phi$ is perturbed by a non-monotone multi-valued operator. We also refer to (SSS07; ScSe08), where convergence to equilibrium for large times of the trajectories of Allen-Cahn type equations has been proved.

In Section 6, following the outline of (RMS08) we shall apply our metric approach to the long-time analysis of

$$\mathrm{J}_p(u'(t)) + \partial_\ell\phi(u(t)) \ni 0 \quad \text{in } \mathscr{B}' \quad \text{for a.a. } t \in (0,T)\,, \tag{1.8}$$

the *limiting subdifferential* $\partial_\ell\phi$ of ϕ being a suitably generalized gradient notion, tailored for nonconvex functionals, cf. (Mor84; Mor06; RoSa06). We shall prove that energy solutions in the sense of (1.4) yield solutions to (1.8) and, in fact, characterize the solutions to (1.8) arising from the metric approach. Then, from our abstract results we shall deduce that the semiflow of the energy solutions to (1.8) admits a global attractor, thus extending to the doubly nonlinear framework the results obtained in (RSS08) in a Hilbert setting, for the gradient flow case $p = 2$. Following (RMS08), we shall further exploit the flexibility of the metric approach to tackle *quasivariational* doubly nonlinear evolutions of the type

$$\partial\Psi_{u'}(u(t), u'(t)) + \partial_\ell\phi(u(t)) \ni 0 \qquad \text{in } \mathscr{B}' \quad \text{for a.a. } t \in (0,T)\,,$$

in which the dissipation functional Ψ depends on the unknown function u.

Application to p-gradient flows in Wasserstein spaces. In a series of pioneering papers (Ott96; JKO98; Ott98; Ott01), F. OTTO proposed a novel variational interpretation for a wide class of diffusion equations of the form

$$\begin{aligned} &\partial_t\rho - \mathrm{div}\left(\rho\nabla\left(\frac{\delta\mathscr{L}}{\partial\rho}\right)\right) = 0 \qquad \text{in } \mathbb{R}^d \times (0,+\infty), \\ &\text{with } \ \rho(x,t) \geq 0 \ \text{ for a.a. } (x,t) \in \mathbb{R}^d \times (0,+\infty), \\ \text{and } \ &\int_{\mathbb{R}^d} \rho(x,t)\,\mathrm{d}x = 1, \int_{\mathbb{R}^d} |x|^2\rho(x,t)\,\mathrm{d}x < +\infty \ \text{ for all } t \in (0,+\infty), \end{aligned} \tag{1.9}$$

where $\delta\mathscr{L}/\partial\rho$ is the first variation of an integral functional

$$\mathscr{L}(\rho) = \int_{\mathbb{R}^d} L(x, \rho(x), \nabla\rho(x))\,\mathrm{d}x$$

associated with a smooth Lagrangian $L : \mathbb{R}^d \times [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$. Indeed, it was shown that (1.9) can be interpreted as a gradient flow in the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$ of the probability measures on $\mathbb{R}^d$ with finite second moment, endowed with the 2-Wasserstein distance (see (Vil03; AGS05; Vil09)). In fact, Otto's formalism paved the way to crucial developments in the study of equations of the type (1.9): in particular, mass-transportation techniques have turned out to be key tools for the study of the asymptotic behavior of solutions. While referring to (AGS05) for a thorough survey of results in this direction, here we mention (CmCV03; CmCV06), investigating the decay rates to equilibrium of the solutions of the following drift-diffusion, nonlocal equation

$$\partial_t \rho - \operatorname{div}\left(\rho \nabla \left(V + I'(\rho) + W * \rho\right)\right) = 0 \qquad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{1.10}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a confinement potential, $I : [0, +\infty) \to \mathbb{R}$ a density of internal energy, $W : \mathbb{R}^d \to \mathbb{R}$ an interaction potential and $*$ denotes the convolution product. On the other hand, the convergence to equilibrium for trajectories of the *doubly nonlinear variant* of (1.10) without the nonlocal term $W * \rho$, i.e.

$$\partial_t \rho - \operatorname{div}\left(\rho j_{p'}\left(\nabla \left(V + I'(\rho)\right)\right)\right) = 0 \qquad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{1.11}$$

($j_{p'}(r) := |r|^{p'-2} r$), was studied in (Agu03). Finally, the metric approach to the analysis of equation (1.10) was systematically developed in (AGS05).

In this direction, in Section 7 we shall tackle the long-time behavior of the *nonlocal doubly nonlinear* drift-diffusion equation

$$\partial_t \rho - \operatorname{div}\left(\rho j_{p'}\left(\nabla \left(V + I'(\rho) + W * \rho\right)\right)\right) = 0 \qquad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{1.12}$$

complementing the results of (CmCV03; CmCV06; Agu03). Indeed, reviewing the discussion in (AGS05), we are going to show that solutions to (1.12) can be obtained from energy solutions (1.4) in the Wasserstein space $(\mathscr{P}_p(\mathbb{R}^d), W_p)$ of the probability measures with finite p-moment. Hence, from our abstract results we shall deduce the existence of the global attractor for the *metric solutions* to (1.12).

Application to phase-change evolutions driven by mean curvature. In Sections 3.2 and 8.1 we shall consider the Stefan-Gibbs-Thomson problem, modeling solid-liquid phase transitions obeying the Gibbs-Thomson law for the temperature evolution at the phase interface. The gradient flow structure of this problem was first revealed in the paper (Luc90) (see also (Vis96)). Therein, the existence of solutions was proved by passing to the limit in the

Minimizing Movements scheme. The related energy functional does not fulfill the chain rule. This calls for the notion of generalized solutions. By relying on our abstract theory, in Theorem 8.5 we shall prove that the semiflow of the generalized solutions of the Stefan-Gibbs-Thomson problem possesses a global attractor.

Finally, in Section 8.2 we shall obtain some partial results for the generalized solutions the Mullins-Sekerka flow (LuSt95; Rog06). However, we point out that, at the moment, the existence of the related global attractor is still open.

Plan of the paper. In Section 2.1, we specify the problem setup and recall the basic notions on evolution in metric spaces which shall be used throughout the paper. Hence, in Sections 2.2–2.3 we define the concepts of *energy* and *generalized* solutions, and discuss their relation. In Section 3.1 we illustrate energy solutions by addressing abstract doubly nonlinear equations in Banach spaces, whereas in Section 3.2 we exemplify the concept of generalized solution via the Stefan-Gibbs-Thomson and the Mullins-Sekerka flows. We state our main results Theorem 4.6, 4.7, and 4.8 on the long-time behavior of generalized and energy solutions in Section 4, and detail all proofs in Section 5.

As for the application of our abstract results, in Section 6 we study the connections between energy solutions and doubly nonlinear equations driven by possibly nonconvex functionals. In Section 7 we prove the existence of the global attractor for the energy solutions of a class of gradient flows in the Wasserstein space $(\mathscr{P}_p(\mathbb{R}^d), W_p)$, while in Section 8 generalized solutions are used to investigate the long-time behavior of the Stefan-Gibbs-Thomson flow.

The Appendix is devoted to a concise presentation of the theory of generalized semiflows by J. M. Ball, to the alternative proof of a result in Section 6.1, and to some recaps on maximal functions.

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2. Solution notions

In this section, we introduce the two notions of gradient flows in metric spaces which we shall consider in the paper, see Definitions 2.2 and 2.10 below.

Preliminarily, for the reader's convenience we briefly recall the tools from analysis in metric spaces on which such definitions rely, referring to (AGS05) for a systematic treatment of these issues.

2.1. Problem setup

The ambient space. Throughout the paper

$$(U, d) \text{ shall denote a metric space, } \sigma \text{ a Hausdorff topology on } U \text{, and } \phi : U \to (-\infty, +\infty] \text{ a proper functional.} \tag{2.1}$$

The symbol $\stackrel{\sigma}{\to}$ shall stand for the convergence in the topology σ, the metric d-convergence being denoted by $\to$ instead. We shall precisely state the links between the d- and the σ-topology in Section 4.1. However, to fix ideas one may think that, in a Banach space setting, d is the distance induced by the norm and σ is the weak/weak* topology.

Absolutely continuous curves and metric derivative. We say that a curve $u : [0, T] \to U$ belongs to $\mathrm{AC}^p([0, T]; U)$, $p \in [1, \infty]$, if there exists $m \in L^p(0, T)$ such that

$$d(u(s), u(t)) \leq \int_s^t m(r) \, \mathrm{d}r \quad \text{for all } 0 \leq s \leq t \leq T. \tag{2.2}$$

For $p = 1$, we simply write $\mathrm{AC}([0, T]; U)$ and refer to *absolutely continuous curves*. A remarkable fact is that, for all $u \in \mathrm{AC}^p([0, T]; U)$, the limit

$$|u'|(t) = \lim_{s \to t} \frac{d(u(s), u(t))}{|t - s|}$$

exists for a.a. $t \in (0, T)$. We will refer to it as the *metric derivative* of u at t. We have that the map $t \mapsto |u'|(t)$ belongs to $L^p(0, T)$ and it is minimal within the class of functions $m \in L^p(0, T)$ fulfilling (2.2), see (AGS05, Sec. 1.1).

The local and the strong relaxed slope. Let $D(\phi) := \{u \in U \; : \; \phi(u) < +\infty\}$ denote the effective domain of ϕ. We define the *local slope* (see (AGS05; Che99; dGMT80)) of ϕ at $u \in D(\phi)$ as

$$|\partial \phi|(u) := \limsup_{v \to u} \frac{(\phi(u) - \phi(v))^+}{d(u, v)}.$$

Remark 2.1. The local slope is a surrogate of the norm of $\nabla\phi$, for it can be shown that, if U is a Banach space $\mathcal{B}$ with norm $\|\cdot\|$ and $\phi : \mathcal{B} \to (-\infty, +\infty]$ is (Fréchet) differentiable at $u \in D(\phi)$, then $|\partial\phi|(u) = \|-\nabla\phi(u)\|_*$.

The function $u \mapsto |\partial\phi|(u)$ cannot be expected to be lower semicontinuous. On the other hand, lower semicontinuity is desirable within limiting procedures. For the purposes of the present long-time analysis, we shall deal with the following relaxation notion

$$|\partial^+\phi|(u) := \inf\big\{ \liminf_{n\to+\infty} |\partial\phi|(u_n) \ : \ u_n \stackrel{\sigma}{\to} u, \ \phi(u_n) \to \phi(u)\big\}, \\ \text{for } u \in D(\phi). \tag{2.3}$$

We refer to $|\partial^+\phi|$ as the *strong relaxed slope* and remark that this relaxation is slightly stronger than the corresponding notion considered in (AGS05, Def. 2.3.1) (where the approximating points u_n are additionally required to belong to a d-ball, but only have bounded energy ϕ, possibly non converging.).

A weaker relaxation for the local slope. In the following we shall make use of a second and weaker relaxation of the local slope. This second notion brings into play the limit of the energy ϕ along the approximating sequences $\{u_n\}$, and to this aim an auxiliary variable is introduced. In particular, let the set

$$\mathcal{X} := \{(u, \varphi) \in D(\phi) \times \mathbb{R} : \ \varphi \geq \phi(u)\}$$

be given and define

$$|\partial^-\phi|(u, \varphi) := \inf\big\{ \liminf_{n\to+\infty} |\partial\phi|(u_n) \ : \ u_n \stackrel{\sigma}{\to} u, \ \phi(u_n) \to \varphi \geq \phi(u)\big\} \tag{2.4}$$

for $(u, \varphi) \in \mathcal{X}$.

As soon as some compactness is assumed (see (A4) below), $|\partial^-\phi|$ turns out to be lower semicontinuous with respect to both its arguments, i.e. for any sequence $\{(u_n, \varphi_n)\} \subset \mathcal{X}$,

$$(u_n \stackrel{\sigma}{\to} u, \quad \varphi_n \to \varphi) \ \Rightarrow \ \liminf_{n\to+\infty} |\partial^-\phi|(u_n, \varphi_n) \geq |\partial^-\phi|(u, \varphi). \tag{2.5}$$

Moreover, we clearly have that $|\partial^+\phi|(u) \geq |\partial^-\phi|(u, \varphi)$ for all $(u, \varphi) \in \mathcal{X}$, and

$$|\partial^+\phi|(u) = |\partial^-\phi|(u, \phi(u)) \quad \forall u \in D(\phi). \tag{2.6}$$

Strong upper gradient. By slightly strengthening (AGS05, Def. 1.2.1), we say that a function $g : U \to [0,+\infty]$ is a *strong upper gradient* for the functional ϕ if, for every curve $u \in \mathrm{AC}_{\mathrm{loc}}([0,+\infty);U)$, the function $g \circ u$ is Borel and

$$|\phi(u(t)) - \phi(u(s))| \le \int_s^t g(u(r))|u'|(r)\,\mathrm{d}r \quad \text{for all } 0 \le s \le t \tag{2.7}$$

(the original (AGS05, Def. 1.2.1) requires (2.7) for $s > 0$ only). Let us explicitly observe that, whenever $(g \circ u)|u'| \in L^1_{\mathrm{loc}}([0,+\infty))$, then $\phi \circ u \in W^{1,1}_{\mathrm{loc}}([0,+\infty))$ and

$$|(\phi \circ u)'(t)| \le g(u(t))|u'|(t) \quad \text{for a.a. } t \in (0,+\infty).$$

Curves of Maximal Slope. Let $g : U \to [0,+\infty]$ be a *strong upper gradient* and $p \in (1,\infty)$. We recall (see (AGS05, Def. 1.3.2, p.32), following (dGMT80; Amb95)) that a curve $u \in \mathrm{AC}^p_{\mathrm{loc}}([0,+\infty);U)$ is said to be a *p-curve of maximal slope* for the functional ϕ with respect to the strong upper gradient g if

$$-(\phi \circ u)'(t) = |u'|^p(t) = g^{p'}(u(t)) \qquad \text{for a.a. } t \in (0,+\infty), \tag{2.8}$$

p' denoting the conjugate exponent of p. In particular, $\phi \circ u$ is locally absolutely continuous in $[0,+\infty)$, $g \circ u \in L^{p'}_{\mathrm{loc}}([0,+\infty))$, and the energy identity

$$\frac{1}{p}\int_s^t |u'|^p(r)\,\mathrm{d}r + \frac{1}{p'}\int_s^t g^{p'}(u(r))\,\mathrm{d}r + \phi(u(t)) = \phi(u(s)) \qquad \text{for all } 0 \le s \le t, \tag{2.9}$$

directly follows.

We are now in the position of defining the two solution notions we shall be dealing with in the sequel. Let us start from the strongest one.

2.2. Energy solutions

Definition 2.2 (Energy solution)**.** *Assume*

$$|\partial^+\phi| \textit{ is a strong upper gradient.}$$

We call energy solution of the metric p-gradient flow of the functional ϕ *(or, simply,* energy solution*), a p-curve of maximal slope for the functional ϕ, with respect to the strong upper gradient $|\partial^+\phi|$.*

In particular, an energy solution u is such that $u \in \mathrm{AC}^p_{\mathrm{loc}}([0,+\infty);U)$, $\phi \circ u$ is locally absolutely continuous on $[0,+\infty)$, the map $t \mapsto |\partial^+\phi|(u(t))$ is in $L^{p'}_{\mathrm{loc}}([0,+\infty))$, and

$$(\phi \circ u)'(t) + \frac{1}{p}|u'|^p(t) + \frac{1}{p'}|\partial^+\phi|^{p'}(u(t)) = 0 \qquad \text{for a.a. } t \in (0,+\infty). \tag{2.10}$$

Remark 2.3. We point out that, being $|\partial^+\phi|$ a strong upper gradient, (2.10) is in fact equivalent to the (integrated) *energy inequality*

$$\frac{1}{p}\int_s^t |u'|^p(r)\,\mathrm{d}r + \frac{1}{p'}\int_s^t |\partial^+\phi|^{p'}(u(r))\,\mathrm{d}r + \phi(u(t)) \le \phi(u(s)) \qquad \text{for all } 0 \le s \le t. \tag{2.11}$$

Geodesically convex functionals. A remarkable case in which the local slope is a strong upper gradient occurs when (cf. (AGS05, Thm. 2.4.9)) the functional ϕ is λ-geodesically convex for some $\lambda \in \mathbb{R}$, i.e.

for all $v_0,\ v_1 \in D(\phi)$ there exists a constant-speed geodesic $\gamma : [0,1] \to U$

(i.e. satisfying $d(\gamma_s,\gamma_t) = (t-s)d(\gamma_0,\gamma_1)$ for all $0 \le s \le t \le 1$), such that

$$\gamma_0 = v_0, \quad \gamma_1 = v_1, \quad \text{and } \phi \text{ is } \lambda\text{-convex on } \gamma, \text{ i.e.}$$

$$\phi(\gamma_t) \le (1-t)\phi(\gamma_0) + t\phi(\gamma_1) - \frac{\lambda}{2}t(1-t)d^2(\gamma_0,\gamma_1) \qquad \text{for all } 0 \le t \le 1. \tag{2.12}$$

Remark 2.4 (λ-geodesic convexity in Banach spaces)**.** When U is a Banach space $\mathcal{B}$ with norm $\|\cdot\|$, λ-geodesic convexity reduces to the usual notion of λ-convexity, i.e.

$$\exists \lambda \in \mathbb{R} \quad \forall\, u_0,\ u_1 \in \mathcal{B} \quad \forall\, \theta \in [0,1]\ :$$
$$\phi((1-\theta)u_0 + \theta u_1) \le (1-\theta)\phi(u_0) + \theta\phi(u_1) - \frac{1}{2}\lambda\theta(1-\theta)\|u_0 - u_1\|^2\,, \tag{2.13}$$

In particular, in the Hilbertian case ϕ is λ-convex if and only if the map $v \mapsto \phi(v) - \frac{\lambda}{2}|v|^2$ is convex.

As pointed out in (AGS05, Rmk. 2.4.7) (see also (Agu03)), in the case of p-curves of maximal slope one should consider a generalized notion of geodesic convexity, in which the modulus of convexity depends on the p-power of the distance d. Also in view of the applications to p-gradient flows in Wasserstein spaces, we thus give the following

Definition 2.5. *Given $\lambda \in \mathbb{R}$ and $p \in (1,\infty)$, we say that a functional $\phi : U \to (-\infty,+\infty]$ is (λ,p)-geodesically convex if*

$$\text{for all } v_0,\, v_1 \in D(\phi) \text{ there exists a constant speed geodesic } \gamma : [0,1] \to U \text{ such that}$$
$$\gamma_0 = v_0, \quad \gamma_1 = v_1, \quad \text{and } \phi \text{ is } \lambda\text{-convex on } \gamma, \text{ i.e.}$$
$$\phi(\gamma_t) \le (1-t)\phi(\gamma_0) + t\phi(\gamma_1) - \frac{\lambda}{p} t(1-t) d^p(\gamma_0,\gamma_1)$$
$$\text{for all } 0 \le t \le 1. \tag{2.14}$$

Remark 2.6 (λ- versus (λ,p)-convexity)**.** Clearly, the notion of (λ,p)-geodesic convexity reduces to (λ,p)-*convexity* in a Banach framework, i.e.

$$\exists \lambda \in \mathbb{R} \quad \forall\, u_0,\, u_1 \in \mathcal{B} \quad \forall\, \theta \in [0,1] \;:$$
$$\phi((1-\theta)u_0 + \theta u_1) \le (1-\theta)\phi(u_0) + \theta\phi(u_1) - \frac{1}{p}\lambda\theta(1-\theta)\|u_0 - u_1\|^p\,, \tag{2.15}$$

Hereafter, in a metric framework we shall always speak of (λ,p)-geodesic convexity, and refer to condition (2.12) as $(\lambda,2)$-geodesic convexity. Instead, in a Banach context we shall simply call condition (2.13) λ-convexity, and speak of (λ,p)-convexity only for $p \neq 2$.

The following result extends (AGS05, Cor. 2.4.10, Lemma 2.4.13, Thm. 2.4.9) to the case of (λ,p)-geodesically convex functionals.

Proposition 2.7. *Let $\phi : U \to (-\infty,+\infty]$ be d-lower semicontinuous and (λ,p)-geodesically convex for some $\lambda \in \mathbb{R}$ and $p \in (1,\infty)$.*

1. *Then, the local slope $|\partial\phi|$ is d-lower semicontinuous and admits the representation*

$$|\partial\phi|(u) = \sup_{v \neq u} \left(\frac{\phi(u) - \phi(v)}{d(u,v)} + \frac{\lambda}{p} d^{p-1}(u,v) \right)^+$$
$$\text{for all } u \in D(\phi). \tag{2.16}$$

Furthermore, $|\partial\phi|$ is a strong upper gradient.

2. *Suppose further that*

the (λ,p)-geodesic convexity condition (2.14) holds with $\lambda > 0$.

Then, the following estimate holds

$$\phi(u) - \inf_U \phi \le \frac{1}{\lambda p'} |\partial^+ \phi|^{p'}(u) \le \frac{1}{\lambda p'} |\partial \phi|^{p'}(u) \quad \textit{for all } u \in D(\phi).$$

Moreover, if $\bar{u} \in D(\phi)$ is the unique minimizer of ϕ, then

$$\frac{\lambda}{p} d^p(u, \bar{u}) \le \phi(u) - \phi(\bar{u}) \le \frac{1}{\lambda p'} |\partial^+ \phi|^{p'}(u) \le \frac{1}{\lambda p'} |\partial \phi|^{p'}(u) \qquad \textit{for all } u \in D(\phi). \tag{2.17}$$

The *proof*, which we choose not to detail, follows from carefully adapting the arguments for (AGS05, Cor. 2.4.10, Lemma 2.4.13, Thm. 2.4.9) to the general (λ, p)-geodesically convex case. Notice that, since the relaxed slope $|\partial^+ \phi|$ is defined in terms of the σ-topology, the d-lower semicontinuity of $|\partial \phi|$ is not sufficient to ensure that the local and strong relaxed slopes coincide, cf. also with Remark 2.12.

2.3. Generalized solutions

We shall introduce *generalized solutions* by highlighting their connections to the notion of Minimizing Movements due E. De Giorgi (dGMT80; dGio93), cf. Definition 2.10 later on. In particular, the discussion developed in the next lines will show that every Generalized Minimizing Movement gives raise to a generalized solution.

2.3.1. **Heuristics for generalized solutions: the Minimizing Movements approach**

The natural way of proving the existence of solutions to the Cauchy problem for (2.10) is to approximate it by time-discretization. Hence, let us consider a partition of $[0, +\infty)$, which we identify with the corresponding vector $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3, \dots)$ of strictly positive time-steps. Note that we indicate with superscripts the elements of a generic vector. In particular τ^j represents the j-th component of the vector $\boldsymbol{\tau}$ (and not the j-th power of the scalar τ). Let $|\boldsymbol{\tau}| = \sup \tau^i$ be the diameter of the partition, which we ask to be finite, $t^0_{\boldsymbol{\tau}} = 0$, and define recursively

$$t^i_{\boldsymbol{\tau}} := t^{i-1}_{\boldsymbol{\tau}} + \tau^i, \quad I^i_{\boldsymbol{\tau}} := [t^{i-1}_{\boldsymbol{\tau}}, t^i_{\boldsymbol{\tau}}) \quad \text{for} \;\; i \ge 1.$$

We shall now consider the following time-incremental minimization problem
(see (AGS05, Chap. II)).

Problem 2.8 (Variational approximation scheme)**.** *Given* $U_{\boldsymbol{\tau}}^0 := u_0$*, find* $U_{\boldsymbol{\tau}}^n$ *such that*

$$U_{\boldsymbol{\tau}}^n \in \operatorname*{Argmin}_{v\in U} \left\{ \frac{d^p(v, U_{\boldsymbol{\tau}}^{n-1})}{p(\tau^i)^{p-1}} + \phi(v) \right\} \quad \textit{for all } n \geq 1. \tag{2.18}$$

Under suitable lower semicontinuity and coercivity conditions on ϕ (cf. with (COMP) in the framework of the σ-topology, see assumptions (A3)–(A4) later on), one can verify that, for all partitions $\boldsymbol{\tau}$ and $u_0 \in D(\phi)$, Problem 2.8 admits at least one solution $\{U_{\boldsymbol{\tau}}^n\}_{n\in\mathbb{N}}$. We then construct approximate solutions by considering the left-continuous piecewise constant interpolant $\overline{U}_{\boldsymbol{\tau}}$ of the values $\{U_{\boldsymbol{\tau}}^n\}_{n\in\mathbb{N}}$, i.e.

$$\overline{U}_{\boldsymbol{\tau}}(t) := U_{\boldsymbol{\tau}}^i \quad \text{for all } \ t \in I_{\boldsymbol{\tau}}^i \quad i \geq 1. \tag{2.19}$$

We shall also deal with the *right-continuous* piecewise constant interpolant $\underline{U}_{\boldsymbol{\tau}}$, defined by $\underline{U}_{\boldsymbol{\tau}}(t) := U_{\boldsymbol{\tau}}^{i-1}$ for all $t \in I_{\boldsymbol{\tau}}^i$ and $i \geq 1$.

Definition 2.9 ((dGio93))**.** *We say that a curve* $u : [0, +\infty) \to U$ *is a* Generalized Minimizing Movement *for* ϕ*, starting from* u_0*, if there exists a sequence* $\{\boldsymbol{\tau}_k\}$*, with* $|\boldsymbol{\tau}_k| \to 0$*, such that*

$$\overline{U}_{\boldsymbol{\tau}_k}(t) \overset{\sigma}{\to} u(t) \qquad \textit{as } k \to \infty \quad \textit{for all } \ t \geq 0. \tag{2.20}$$

We denote by $\mathrm{GMM}(\phi; u_0)$ *the set of Generalized Minimizing Movements with initial datum* u_0*.*

In order to show that $\mathrm{GMM}(\phi; u_0)$ is non-empty, one needs to prove some a priori estimates on the sequence $\{\overline{U}_{\boldsymbol{\tau}_k}\}$. The crucial step in this direction is to observe that approximate solutions satisfy at all nodes $t_{\boldsymbol{\tau}}^i$ the following *discrete energy identity*

$$\begin{aligned} \frac{1}{p}\int_{t_{\boldsymbol{\tau}}^{i-1}}^{t_{\boldsymbol{\tau}}^i} \left(\frac{d\left(\overline{U}_{\boldsymbol{\tau}}(t), \underline{U}_{\boldsymbol{\tau}}(t)\right)}{\boldsymbol{\tau}^i} \right)^p \mathrm{d}t + \frac{1}{p'}\int_{t_{\boldsymbol{\tau}}^{i-1}}^{t_{\boldsymbol{\tau}}^i} |\partial\phi|^{p'} \left(\tilde{U}_{\boldsymbol{\tau}}(t)\right) \mathrm{d}t \\ + \phi\left(\overline{U}_{\boldsymbol{\tau}}(t)\right) = \phi\left(\underline{U}_{\boldsymbol{\tau}}(t)\right), \end{aligned} \tag{2.21}$$

where $\underline{U}_{\boldsymbol{\tau}}$ and $\tilde{U}_{\boldsymbol{\tau}}$ respectively denote the right-continuous piecewise constant and the *De Giorgi variational* interpolants of the values $\{U_{\boldsymbol{\tau}}^n\}_{n\in\mathbb{N}}$, see (AGS05, Def. 3.2.1) for the latter notion. It is immediate to realize that (2.21) is the discrete counterpart to (2.10). Exploiting the discrete energy inequality (2.21) and the coercivity of ϕ, in (AGS05, Secs. 3.3.2 and 3.3.3) suitable a priori estimates are obtained for the approximate sequences $\{\overline{U}_{\boldsymbol{\tau}}\}$ and $\{\tilde{U}_{\boldsymbol{\tau}}\}$, and convergence is shown along a suitable subsequence $\{\boldsymbol{\tau}_k\}$ to some limit curve $u : [0,+\infty) \to U$. Furthermore, (2.21) yields that the functions $\varphi_{\boldsymbol{\tau}_k}(t) := \phi(\overline{U}_{\boldsymbol{\tau}_k}(t))$ form a non-increasing sequence. A suitable generalization of Helly's theorem (cf. (AGS05, Lemma 3.3.3)) gives that, up to the extraction of a further subsequence,

$$\exists\, \varphi(t) := \lim_{k\to\infty} \phi(\overline{U}_{\boldsymbol{\tau}_k}(t)) \geq \phi(u(t)) \quad \text{for all } \ t \geq 0,$$

the latter inequality by the lower semicontinuity of ϕ. Altogether, passing to the limit by lower semicontinuity in (2.21) it is proved in (AGS05) that $\mathrm{GMM}(\phi; u_0) \neq \emptyset$ and that for every $u \in \mathrm{GMM}(\phi; u_0)$ there exists a non-increasing function $\varphi : [0,+\infty) \to \mathbb{R}$ such that

$$\frac{1}{2}\int_s^t |u'|^2(r)dr + \frac{1}{2}\int_s^t |\partial^-\phi|^2(u(r),\varphi(r))dr + \varphi(t) \leq \varphi(s) \ \text{ for all } 0 \leq s \leq t,$$
$$\phi(u(t)) \leq \varphi(t) \quad \text{for all } t \geq 0,$$

where the weak relaxed slope $|\partial^-\phi|$ naturally arises by taking the $\liminf$ of the second integral term on the left-hand side of (2.21).

2.3.2. Definition of generalized solution

Motivated by the above discussion, we give the definition of generalized solution, tailored to include all limits of the time-incremental approximations constructed in (2.18).

Definition 2.10 (Generalized solution)**.** *A pair* (u,φ)*, with* $u \in \mathrm{AC}^p_{\mathrm{loc}}([0,+\infty);U)$ *and* $\varphi : [0,+\infty) \to \mathbb{R}$*, is a* generalized solution of the metric p-gradient flow of the functional ϕ *(or, simply, a* generalized solution*), if*

$$\frac{1}{p}\int_s^t |u'|^p(r)\,\mathrm{d}r + \frac{1}{p'}\int_s^t |\partial^-\phi|^{p'}(u(r),\varphi(r))\,\mathrm{d}r + \varphi(t) \leq \varphi(s) \quad \textit{for all } 0 \leq s \leq t, \tag{2.22}$$

$$\phi(u(t)) \leq \varphi(t) \quad \textit{for all } t \geq 0. \tag{2.23}$$

Remark 2.11. Notice that, if (u, φ) is a generalized solution, then φ is non-increasing and thus of finite pointwise variation. Moreover, letting $\varphi_-(t) := \varphi(t-)$ and $\varphi_+(t) := \varphi(t+)$ denote the pointwise left- and right-limits, respectively, we have that the pairs (u, φ_-) and (u, φ_+) are generalized solutions as well.

From generalized to energy solutions via conditional continuity. The main step to conclude that the Minimizing Movements approach of Section 2.3.1 yields energy solutions is to identify $\phi(u(t))$ as the limit of the sequence $\{\phi(\overline{U}_{\boldsymbol{\tau}_k}(t))\}$, i.e. to prove that

$$\varphi(t) = \phi(u(t)) \qquad \text{for all } t \geq 0. \tag{2.24}$$

Once (2.24) is obtained, in view of equality (2.6) one can replace $|\partial^-\phi|^{p'}(u, \varphi)$, in the second integral term in (2.22), with the strong relaxed slope $|\partial^+\phi|^{p'}(u)$ (see (2.3), (2.6)). In this way, one obtains the energy inequality (2.11), yielding that u is an energy solution, see Remark 2.3.

Indeed, a sufficient condition for (2.24) to hold is the following *conditional continuity* requirement

$$u_n \overset{\sigma}{\to} u, \quad \sup_n \{\phi(u_n), |\partial\phi|(u_n)\} < +\infty \quad \Rightarrow \quad \phi(u_n) \to \phi(u). \tag{CONT}$$

Remark 2.12 (Links between (λ, p)-geodesic convexity and conditional continuity.)**.** In the case σ is the topology induced by d, let $\phi : U \to (-\infty, +\infty]$ be a lower semicontinuous functional complying with the (λ, p)-geodesic convexity condition (2.14) for some constant $\lambda \in \mathbb{R}$. Then, the conditional continuity property (CONT) holds. Indeed, let $\{u_n\}$ be a sequence as in (CONT). It follows from (2.16) that

$$\phi(u_n) - \phi(u) + \frac{\lambda}{p} d^p(u_n, u) \leq |\partial\phi|(u_n) d(u_n, u) \qquad \text{for all } n \in \mathbb{N}, \tag{2.25}$$

whence

$$\begin{aligned} \phi(u) &\leq \liminf_{n\to\infty} \phi(u_n) \leq \limsup_{n\to\infty} \phi(u_n) \\ &\leq \phi(u) + \limsup_{n\to\infty} \left(|\partial\phi|(u_n) d(u_n, u) - \frac{\lambda}{p} d^p(u_n, u) \right) \leq \phi(u)\,, \end{aligned}$$

the first inequality by lower semicontinuity, the third one by (2.25), and the last one by the properties of the sequence $\{u_n\}$, cf. with (CONT).

It is immediate to see that the previous argument also works when the σ- and d-topology do not coincide, provided that ϕ has the following property: along sequences with bounded energy, σ-convergence implies d-convergence.

2.3.3. Comparison between generalized and energy solutions

Under the conditional continuity (CONT), generalized and energy solutions may be compared. The first result in this direction is the following

Lemma 2.13. *Assume that the conditional continuity property* (CONT) *holds. Then,*

$$
\begin{gathered}
|\partial^- \phi|(u, \varphi) < +\infty \;\Rightarrow\; \varphi = \phi(u) \text{ and} \\
|\partial^- \phi|(u, \varphi) = |\partial^- \phi|(u, \phi(u)) = |\partial^+ \phi|(u) \quad \forall (u, \varphi) \in \mathcal{X}.
\end{gathered}
\tag{2.26}
$$

Proof. It follows from the definition (2.4) of $|\partial^- \phi|(u, \varphi)$ that for all $\varepsilon > 0$ there exists a sequence $\{u_n\}$ with $u_n \overset{\sigma}{\to} u$, $\phi(u_n) \to \varphi$, and $|\partial \phi|(u_n) \leq |\partial^- \phi|(u, \varphi) + \varepsilon$. Hence, (CONT) yields that $\varphi = \phi(u)$, and (2.26) follows. □

An immediate consequence of Lemma 2.13 is the following

Proposition 2.14 ($\varphi = \phi \circ u$)**.** *Let* (CONT) *hold. Then, for any generalized solution* (u, φ) *we have* $\varphi = \phi \circ u$ *almost everywhere in* $(0, +\infty)$ *and the following energy inequality holds for all* $t \geq 0$*, for a.a.* $s \in (0, t)$

$$
\frac{1}{p} \int_s^t |u'|^p(r) \, \mathrm{d}r + \frac{1}{p'} \int_s^t |\partial^+ \phi|^{p'}(u(r)) \, \mathrm{d}r + \phi(u(t)) \leq \phi(u(s)) \,. \tag{2.27}
$$

We thus conclude

Proposition 2.15 (Comparison between generalized and energy solutions)**.** *Given any energy solution* u*, the pair* $(u, \phi \circ u)$ *is a generalized solution.*

Conversely, assume (CONT)*, let* $|\partial^+ \phi|$ *be a strong upper gradient for the functional* ϕ*, and* (u, φ) *be a generalized solution. Then,* u *is an energy solution and* $\varphi = \phi \circ u$*.*

Proof. The first part of the proposition is immediate. As for the second one, use Proposition 2.14 in order to get that $\varphi = \phi \circ u$ almost everywhere in $(0, +\infty)$, and replace the energy inequality (2.22) with

$$
\frac{1}{p} \int_s^t |u'|^p(r) \, \mathrm{d}r + \frac{1}{p'} \int_s^t |\partial^+ \phi|^{p'}(u(r)) \, \mathrm{d}r + \varphi(t) \leq \varphi(s) \quad \text{for all } 0 \leq s \leq t.
$$

Hence, as $r \mapsto |\partial^+\phi|(u(r))$ is in $L^{p'}_{\rm loc}([0,+\infty))$, we have that

$$|\partial^+\phi|(u)|u'| \in L^1_{\rm loc}([0,+\infty))$$

and $\phi \circ u$ is locally absolutely continuous since $|(\phi\circ u)'| \leq |\partial^+\phi|(u)|u'|$ almost everywhere. In particular, $\varphi(s) = \phi(u(s))$ for all $s \geq 0$. Finally, inequality (2.27) holds for all $0 \leq s \leq t$, and the converse inequality follows again from $|(\phi\circ u)'| \leq |\partial^+\phi|(u)|u'|$. □

3. Examples of energy and generalized solutions

3.1. Doubly nonlinear equations in Banach spaces: the convex case

Let $(\mathcal{B}, \|\cdot\|)$ be a (separable) reflexive Banach space and suppose that

$$\phi : \mathcal{B} \to (-\infty, +\infty] \ \text{ is a proper, l.s.c. and convex functional.}$$

We now show that the notion of energy solution (of the metric p-gradient flow of ϕ), in the ambient metric space $U = \mathcal{B}$ (with $d(u,v) = \|v-u\|$ and σ the strong topology), relates to the doubly nonlinear equation

$$\mathrm{J}_p(u'(t)) + \partial\phi(u(t)) \ni 0 \qquad \text{in } \mathcal{B}' \quad \text{for a.a. } t \in (0,T)\,, \tag{3.28}$$

where $\mathrm{J}_p : \mathcal{B} \rightrightarrows \mathcal{B}'$ is the p-duality map, defined by

$$\xi \in \mathrm{J}_p(v) \ \Leftrightarrow \ \langle \xi, v\rangle = \|v\|^p = \|\xi\|_*^{p'} = \|v\|\|\xi\|_*\,, \tag{3.29}$$

and $\partial\phi$ is the Fréchet subdifferential of ϕ, defined at a point $u \in D(\phi)$ by

$$\xi \in \partial\phi(u) \quad \Leftrightarrow \quad \phi(v) - \phi(u) \geq \langle \xi, v-u\rangle + o(\|v-u\|) \ \text{ as } v \to u. \tag{3.30}$$

(which, in the present convex case, in fact reduces to the standard subdifferential of ϕ in the sense of convex analysis).

The link between the formulation of Definition 2.2 and equation (3.28) is based on the following key fact: denoting

$$\begin{cases} \partial^\circ\phi(v) := \mathrm{Argmin}\,\{\|\xi\|_* \ : \ \xi \in \partial\phi(v)\} \\ \|\partial^\circ\phi(v)\|_* := \min\,\{\|\xi\|_* \ : \ \xi \in \partial\phi(v)\} \end{cases} \quad \text{for } v \in D(\phi), \tag{3.31}$$

it was proved in (AGS05, Prop. 1.4.4) that

$$|\partial\phi|(v) = \|\partial^\circ\phi(v)\|_* \qquad \text{for all } v \in D(\phi)\,. \tag{3.32}$$

It now follows from (3.32) and from the strong-weak closedness of the graph of $\partial\phi$ that the map $v \mapsto |\partial\phi|(v)$ is lower semicontinuous, so that $|\partial\phi| \equiv |\partial^+\phi|$. Finally, the chain rule for convex functionals

$$\begin{aligned} &\text{if} \quad u \in H^1(0,T;\mathcal{B}),\ \xi \in L^2(0,T;\mathcal{B}'),\ \xi(t) \in \partial\phi(u(t)) \text{ for a.a. } \ t \in (0,T) \\ &\text{then} \quad \phi \circ u \in \mathrm{AC}([0,T]), \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) = \langle \xi(t), u'(t)\rangle \text{ for a.a. } \ t \in (0,T). \end{aligned}$$

ensures that $|\partial\phi| \equiv |\partial^+\phi|$ is a strong upper gradient. Exploiting (3.31), in (AGS05, Prop. 1.4.1) it was shown that a curve $u \in \mathrm{AC}^p([0,T];\mathcal{B})$ is an energy solution if and only if it fulfills

$$\mathrm{J}_p(u'(t)) + \partial^\circ\phi(u(t)) \ni 0 \qquad \text{in } \mathcal{B}' \quad \text{for a.a. } \ t \in (0,T)\,,$$

i.e. $\mathrm{J}_p(u'(t))$ complies with the *minimal section* principle.

3.2. Generalized solutions of the Stefan-Gibbs-Thomson and the Mullins-Sekerka flows

The Stefan-Gibbs-Thomson problem. The Stefan problem coupled with the Gibbs-Thomson law describes the melting and solidification of a solid-liquid system, and also takes into account surface tension effects by imposing that the temperature is equal to the mean curvature at the phase interface. We denote by ϑ the relative temperature of the system, occupying a bounded domain $\Omega \subset \mathbb{R}^d$, and by $\chi \in \{-1, 1\}$ the phase parameter, so that the phases at time $t \in (0,T)$ are

$$E^+(t) = \{x \in \Omega \,:\, \chi(x,t) = 1\}\,, \qquad E^-(t) = \{x \in \Omega \,:\, \chi(x,t) = -1\}\,,$$

and the phase interface $S(t)$ is their common essential boundary. Neglecting external heat sources, the related PDE system is given by the energy balance

$$\partial_t(\vartheta + \chi) + A\vartheta = 0 \qquad \text{in } H^{-1}(\Omega) \quad \text{for a.a. } \ t \in (0,T)\,, \tag{3.33}$$

(A denoting the realization of the Laplace operator with homogeneous Dirichlet boundary conditions), and by the Gibbs-Thomson condition at the phase interface, which formally reads

$$\mathbf{H}(\cdot,t) = \vartheta(\cdot,t)\nu(t) \qquad \text{on } S(t), \tag{3.34}$$

$\mathbf{H}(\cdot, t)$ being the mean curvature vector of $S(t)$ and $\nu(t) : S(t) \to \mathrm{S}^{d-1}$ the inner measure theoretic normal to $E^+(t)$.

In the pioneering paper (Luc90), LUCKHAUS has shown that there exist functions

$$\vartheta \in L^2(0,T; H^1_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \tag{3.35}$$

and

$$\chi : \Omega \times (0,T) \to \{-1,1\}, \quad \text{with} \quad \begin{cases} \chi \in L^\infty(0,T; \mathrm{BV}(\Omega)), \\ u := \vartheta + \chi \in H^1(0,T; H^{-1}(\Omega)), \end{cases} \tag{3.36}$$

fulfilling equation (3.33) for a.a. $t \in (0,T)$, and complying with the weak formulation of the Gibbs-Thomson law (3.34), i.e. for a.a. $t \in (0,T)$

$$\begin{gathered} \int_\Omega \big(\operatorname{div} \zeta - \nu^T(t)\, D\zeta\, \nu(t) \big)\, d|D\chi(\cdot,t)| = \int_\Omega \operatorname{div}(\vartheta(\cdot,t)\zeta)\chi(\cdot,t)\, \mathrm{d}x \\ \text{for all } \ \zeta \in C^2(\overline{\Omega}; \mathbb{R}^d), \quad \zeta \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega\,, \end{gathered} \tag{3.37}$$

$|D\chi(\cdot,t)|$ being the total variation measure associated with the distributional gradient $D\chi(\cdot,t)$. LUCKHAUS's existence proof (see also (Vis96, Chap. VIII)), is based on a time-discretization technique which fits into the general Minimizing Movements scheme introduced in Section 2.3.1. In fact, (the initial-boundary value problem for) system (3.33, 3.34) has a natural gradient flow structure with respect to the functional $\phi_{\mathrm{Ste}} : H^{-1}(\Omega) \to (-\infty, +\infty]$, given by

$$\phi_{\mathrm{Ste}}(u) := \begin{cases} \inf\limits_{\chi \in \mathrm{BV}(\Omega)} \left\{ \int_\Omega \left(\frac{1}{2}|u-\chi|^2 + I_{\{-1,1\}}(\chi) \right) \mathrm{d}x + \int_\Omega |D\chi| \right\} \\ \qquad\qquad \text{if } u \in L^2(\Omega), \\ +\infty \text{otherwise}, \end{cases} \tag{3.38}$$

$I_{\{-1,1\}}$ denoting the indicator function of the set $\{-1,1\}$.

This gradient structure was later exploited in (RoSa06), where it was shown that the Minimizing Movements scheme in $H^{-1}(\Omega)$, driven by ϕ_{Ste} and starting from an initial datum $u_0 \in D(\phi) = L^2(\Omega)$

$$U^0_\tau := u_0, \qquad U^n_\tau \in \operatorname*{Argmin}_{v \in H^{-1}(\Omega)} \left\{ \frac{1}{2\tau} \|v - U^{n-1}_\tau\|^2_{H^{-1}(\Omega)} + \phi_{\mathrm{Ste}}(v) \right\}, \tag{3.39}$$

(i.e. the variational approximation scheme (2.18), for $p = 2$ and constant time-step $\tau > 0$), admits a solution $\{U_\tau^n\}_{n\in\mathbb{N}}$, and in fact coincides with the approximation algorithm constructed in (Luc90) (see (RoSa06, Rmk. 2.7)). Furthermore, it was proved (see (RoSa06, Thm. 2.5) and Section 8.1 later on) that, for every initial datum $u_0 \in L^2(\Omega)$, every $u \in \mathrm{GMM}(\phi_{\mathrm{Ste}}, u_0)$ (cf. with Definition 2.9) gives raise to a solution (ϑ, χ) of (3.33) and (3.37), complying with (3.35) and (3.36), and fulfilling the *Lyapunov inequality* for all $t \in [0, T]$ and for almost all $s \in (0, t)$

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}|u(x,t) - \chi(x,t)|^2 + I_{\{-1,1\}}(\chi(x,t))\Big)\,\mathrm{d}x \\
&+ \int_\Omega |D\chi(t)| + \int_s^t \int_\Omega |\nabla\vartheta(x,r)|^2\,\mathrm{d}x\,\mathrm{d}r \\
&\leq \int_\Omega \Big(\frac{1}{2}|u(x,s) - \chi(x,s)|^2 + I_{\{-1,1\}}(\chi(x,s))\Big)\,\mathrm{d}x + \int_\Omega |D\chi(s)|\,.
\end{aligned} \tag{3.40}$$

In Section 8.1, we shall recover the gradient flow approach of (RoSa06). First of all, we shall make precise the setting in which the (Generalized) Minimizing Movement associated with the functional ϕ_{Ste} yields solutions to the weak formulation (3.33, 3.37) of the Stefan-Gibbs-Thomson problem, see Corollary 8.3. On the other hand, on account of Theorem 4.2 later on, the *Minimizing Movement solutions* constructed in (RoSa06) (i.e. the solutions arising from the time-incremental scheme (3.39)), are in fact generalized solutions (in the sense of Definition 2.10), driven by the functional ϕ_{Ste} (3.38). Using this fact, in Theorem 8.5 we shall obtain the existence of the global attractor for the Minimizing Movements solutions of the Stefan-Gibbs-Thomson problem.

The Mullins-Sekerka flow. The Mullins-Sekerka flow is a variant of the Stefan problem with the Gibbs-Thomson condition, modeling solid-liquid phase transitions in thermal systems with a negligible specific heat. In this setting, instead of (3.33) the internal energy balance reads

$$\partial_t \chi + A\vartheta = 0 \qquad \text{in } H^{-1}(\Omega) \quad \text{for a.a. } t \in (0,T)\,, \tag{3.41}$$

coupled with the Gibbs-Thomson condition in the weak form (3.37).

The global existence of solutions (ϑ, χ) to the Cauchy problem for the weak formulation of the Mullins-Sekerka system, with $\vartheta \in L^\infty(0,T; H_0^1(\Omega)$

and $\chi \in L^\infty(0,T;\mathrm{BV}(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, was first obtained in (LuSt95). The proof was carried out by passing to the limit in the following variational approximation scheme: starting from an initial datum $\chi_0 \in \mathrm{BV}(\Omega;\{-1,1\})$, for a fixed time-step $\tau > 0$ the discrete solutions $\{\chi_\tau^n\}_{n\in\mathbb{N}}$ are constructed via $\chi_\tau^0 := \chi_0$ and, for $n \geq 1$,

$$\begin{aligned}\chi_\tau^n &\in \operatorname*{Argmin}_{\chi \in H^{-1}(\Omega)} \left\{ \frac{1}{2\tau} \int_\Omega \left(A^{-1}\left(\chi - \chi_\tau^{n-1}\right)\left(\chi - \chi_\tau^{n-1}\right) + I_{\{-1,1\}}(\chi)\right) \mathrm{d}x + \int_\Omega |D\chi| \right\} \\ &= \operatorname*{Argmin}_{\chi \in H^{-1}(\Omega)} \left\{ \frac{1}{2\tau} \|\chi - \chi_\tau^{n-1}\|^2_{H^{-1}(\Omega)} + \int_\Omega I_{\{-1,1\}}(\chi)\,\mathrm{d}x + \int_\Omega |D\chi| \right\}.\end{aligned}$$

Indeed, the above scheme (cf. with (2.18)) reveals that problem (3.41, 3.37) has a gradient flow structure w.r.t. the functional $\phi_{\mathrm{MS}} : H^{-1}(\Omega) \to [0,+\infty)$

$$\phi_{\mathrm{MS}}(\chi) := \begin{cases} \displaystyle\int_\Omega I_{\{-1,1\}}(\chi)\,\mathrm{d}x + \int_\Omega |D\chi| & \text{if } \chi \in \mathrm{BV}(\Omega;\{-1,1\}), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.42}$$

In fact, the existence result in (LuSt95) was conditioned to the validity of the convergence of the energy ϕ_{MS} along the approximate solutions. Such condition excludes a loss of surface area for the phase interfaces in the passage to the limit in the time-discrete problem. However, this convergence requirement, akin to our continuity assumption (CONT), is not, in general, fulfilled, as observed in (Rog06). Therein, by use of refined techniques from the theory of integral varifolds, the author could dispense with the additional condition of (LuSt95), however at the price of obtaining in the limit the weak formulation of the Gibbs-Thomson law (3.37) in a generalized, varifold form.

Since ϕ_{MS} does neither fulfill the chain rule nor the conditional continuity, the Minimizing Movements solutions of the Mullins-Sekerka problem only give raise to generalized solutions of the gradient flow driven by ϕ_{MS}. In Section 8.2 we shall initiate some analysis in this direction. However, proving the existence of the global attractor for the (Minimizing Movements solutions of the) Mullins-Sekerka problem remains an open problem.

4. Main results

4.1. Statement of the main assumptions

Topological assumptions.

$$(U,d) \quad \text{is a complete metric space.} \tag{A1}$$

$$\sigma \quad \text{is a Hausdorff topology on } U \text{ compatible with } d \tag{A2a}$$

The latter compatibility means that σ is weaker than the topology induced by d and d is sequentially σ-lower semicontinuous, namely

$$(u_n, v_n) \overset{\sigma\times\sigma}{\to} (u, v) \quad \Rightarrow \quad \liminf_{n\to+\infty} d(u_n, v_n) \geq d(u, v).$$

An example of a choice for σ complying with (A2a) is of course that of σ being the topology induced by d. We shall however keep these two topologies distinct for the sake of later applications.

We also require that

$$\begin{gathered}\text{there exists a distance } d_\sigma \text{ on } U \text{ such that for all } \{u_n\},\ u \in U \\ u_n \overset{\sigma}{\to} u \ \Rightarrow \ d_\sigma(u, u_n) \to 0 \ \text{ as } n \to \infty\end{gathered} \tag{A2b}$$

Namely, the topology induced by d_σ is globally weaker than σ.

Assumptions on the functional $\boldsymbol{\phi}$. Let $p \in (1, \infty)$. We shall ask for the following

(Lower semicontinuity)

$$\phi \ \text{ is sequentially } \sigma\text{-lower semicontinuous.} \tag{A3}$$

(Compactness)

$$\text{the sublevels of } \phi \text{ are relatively } \sigma\text{-sequentially compact.} \tag{A4}$$

Remark 4.1. It can be easily checked that (A3) and (A4) entail that

$$\phi \ \text{ is bounded from below on } U. \tag{4.1}$$

Existence and approximation of generalized solutions. Under the above assumptions, we have the following crucial statement, which subsumes (AGS05, Cor. 3.3.4) and (AGS05, Thm. 2.3.1).

Proposition 4.2 (Generalized Minimizing Movements are generalized solutions). *Assume* (A1)–(A4) *and let a family* Λ *of partitions of* $[0, +\infty)$ *be given with* $\inf_{\boldsymbol{\tau}\in\Lambda} |\boldsymbol{\tau}| = 0$.

Then, GMM$(\phi; u_0) \neq \emptyset$. *Further, every* $u \in$ GMM$(\phi; u_0)$ *is in* $\mathrm{AC}^p_{\mathrm{loc}}([0,+\infty); U)$ *and there exists a non-increasing function* $\varphi : [0,+\infty) \to \mathbb{R}$ *such that,* $\{\boldsymbol{\tau}_k\}$ *being a sequence with* $|\boldsymbol{\tau}_k| \to 0$ *fulfilling* (2.20)*, there holds*

$$\varphi(t) = \lim_{k\to\infty} \phi(\overline{U}_{\boldsymbol{\tau}_k}(t)) \geq \phi(u(t)) \quad \textit{for all} \;\; t \geq 0, \quad \varphi(0) = \phi(u(0)) = \phi(u_0),$$

and the pair (u, φ) *is a generalized solution in the sense of Definition* 2.10.

Remark 4.3. Note that no restriction on the diameter of the partitions is needed for the above convergence statement. Moreover, in (AGS05) a slightly more general class of functionals was considered, and the compactness condition (A4) was in fact required only on d-bounded subsets of sublevels of ϕ. Indeed, the stronger (A4) is needed for the purposes of the present long-time analysis.

4.2. Statement of the main results

4.2.1. **Global attractor for generalized solutions**

We refer the reader to Section Appendix A for the main definitions and results of the theory of global attractors for generalized semiflows, closely following J.M. Ball (Bal97).

We shall apply the theory of generalized semiflows in the framework of the metric phase space

$$\begin{aligned}&\mathcal{X} = \{(u,\varphi) \in D(\phi) \times \mathbb{R} : \; \varphi \geq \phi(u)\}, \quad \text{endowed with the distance}\\ &d_{\mathcal{X}}((u,\varphi),(u',\varphi')) = d_\sigma(u,u') + |\varphi - \varphi'| \quad \forall (u,\varphi),(u',\varphi') \in \mathcal{X}.\end{aligned} \tag{4.2}$$

Our candidate generalized semiflow is the set

$$\mathcal{S} := \Big\{(u,\varphi) : [0,+\infty] \to U \times \mathbb{R} \; : \; (u,\varphi) \;\; \text{is a generalized solution}\Big\}.$$

All of the following results shall be proved in Section 5.

Theorem 4.4 (Generalized solutions form a generalized semiflow)**.** *Assume* (A1)–(A4)*. Then,* $\mathcal{S}$ *is a semiflow on* $(\mathcal{X}, d_{\mathcal{X}})$ *and complies with the* continuity property (C0)*.*

The following Proposition sheds light on the properties of the semiflow $\mathcal{S}$.

Proposition 4.5. *Assume* (A1)–(A4). *Then,*

1. *$\mathcal{S}$ is asymptotically compact;*
2. *$\mathcal{S}$ admits a Lyapunov function;*
3. *the set $Z(\mathcal{S})$ of the rest points for $\mathcal{S}$ is given by*

$$Z(\mathcal{S}) = \left\{(\bar{u}, \bar{\varphi}) \in \mathcal{X} \ : \ |\partial^- \phi|(\bar{u}, \bar{\varphi}) = 0\right\}. \tag{4.3}$$

Theorem 4.6 (Global attractor for generalized solutions)**.** *Under assumptions* (A1)–(A4)*, suppose further that*

$$\textit{the set } Z(\mathcal{S}) \textit{ of the rest points of } \mathcal{S} \textit{ is bounded in } (\mathcal{X}, d_{\mathcal{X}}). \tag{A5}$$

Then, the semiflow $\mathcal{S}$ admits a global attractor A. Moreover, $\omega(u, \varphi) \subset Z(\mathcal{S})$ for every trajectory $(u, \varphi) \in \mathcal{S}$.

4.2.2. **Global attractor for energy solutions**

Throughout this section, we further assume that

$$\phi \text{ complies with the conditional continuity property (CONT)}, \tag{A6}$$

$$|\partial^+ \phi| \text{ is a strong upper gradient.} \tag{A7}$$

Proposition 2.14 and (A6) yield that

$$\mathcal{S} = \Big\{(u, \varphi) \ \text{ generalized solution, with } \varphi(t) = \phi(u(t)) \text{ a.e. on } (0, +\infty)\Big\}.$$

Hence, it is not difficult to check that the set of rest points of $\mathcal{S}$ (4.3) reduces to

$$Z(\mathcal{S}) = \left\{(\bar{u}, \bar{\varphi}) \in \mathcal{X} : \ \ \bar{\varphi} = \phi(\bar{u}), \ \ |\partial^+ \phi|(\bar{u}) = 0\right\}. \tag{4.4}$$

Hereafter, we shall use the following notation

$$\mathcal{E} := \{u \in \mathrm{AC}^p_{\mathrm{loc}}([0, +\infty); U) \,:\, u \text{ is an energy solution}\}. \tag{4.5}$$

Thanks to Proposition 2.14, assumption (A7) gives that $\pi_1(\mathcal{S}) = \mathcal{E}$ (π_1 denoting the projection on the first component).

We now aim to study the long-time behavior of energy solutions in the phase space

$$(D(\phi), d_\phi), \quad \text{with} \quad d_\phi(u,u') := d_\sigma(u,u') + |\phi(u) - \phi(u')| \quad \text{for all} \;\; u, u' \in D(\phi). \tag{4.6}$$

We shall denote by e_ϕ the Hausdorff semidistance associated with the metric d_ϕ, namely, for all non-empty sets A, $B \subset D(\phi)$, we have $e_\phi(A,B) = \sup_{a\in A} \inf_{b\in B} d_\phi(a,b)$. Similarly, we denote by $e_{\mathcal{X}}$ the Hausdorff semidistance associated with the metric $d_{\mathcal{X}}$. Let us introduce the *lifting operator*

$$\mathcal{U} : D(\phi) \to \mathcal{X} \qquad \mathcal{U}(u) := (u, \phi(u)) \quad \text{for all} \;\; u \in D(\phi)$$

and remark that $\mathcal{U}(\pi_1(E)) \subset E$ for all $E \subset \mathcal{X}$ and

$$E \subset \mathcal{U}(D(\phi)) \;\Rightarrow\; E = \mathcal{U}(\pi_1(E)). \tag{4.7}$$

Furthermore, the metric $d_{\mathcal{X}}$ restricted to the set $\mathcal{U}(D(\phi))$ coincides with d_ϕ, namely

$$d_{\mathcal{X}}(\mathcal{U}(u_1), \mathcal{U}(u_2)) = d_\phi(u_1, u_2) \qquad \text{for all} \;\; u_1, u_2 \in D(\phi). \tag{4.8}$$

The following result (which is in fact a corollary of Theorems 4.4 and 4.6) states that all information on the long-time behavior of energy solutions is encoded in the set $\pi_1(A)$, A being the global attractor for generalized solutions.

Theorem 4.7 (Global attractor for energy solutions)**.** *Assume* (A1)–(A4) *and* (A5)–(A7)*. Then,*

1) *the set $\mathcal{E}$ from* (4.5) *is a generalized semiflow in the phase space $(D(\phi), d_\phi)$ defined by* (4.6)*, and fulfills the* continuity properties (C0)–(C3) *(cf. Definition Appendix A.1),*

2) *the set*

$$\pi_1(A) \textit{ is the global attractor for } \mathcal{E}, \tag{4.9}$$

while $\pi_1(Z(\mathcal{S}))$ is the set of its rest points.

4.2.3. Convergence to equilibrium for energy solutions in the (λ, p)-geodesically convex case

The following enhanced result on the convergence to equilibrium for energy solutions is the *doubly nonlinear* counterpart to (AGS05, Thm. 2.4.14), which was proved for gradient flows in the $(\lambda, 2)$-geodesically convex case.

Theorem 4.8 (Exponential decay to equilibrium)**.** *Assume* (A1)–(A4)*, and suppose further that*

$$\phi \text{ fulfills the } (\lambda, p)\text{-geodesic convexity condition (2.14) with } \lambda > 0. \tag{A8}$$

Let $\bar{u} \in D(\phi)$ be the unique minimizer of ϕ. Then, every energy solution fulfills for all $t_0 > 0$ the exponential decay to equilibrium estimate

$$\frac{\lambda}{p} d^p(u(t), \bar{u}) \leq \phi(u(t)) - \phi(\bar{u}) \leq (\phi(u(t_0)) - \phi(\bar{u})) \exp(-\lambda p'(t - t_0)) \quad \text{for all } t \geq t_0. \tag{4.10}$$

Hence, for all $u \in \mathcal{E}$

$$d_\phi(u(t), \bar{u}) \to 0 \quad \text{as } t \to +\infty, \tag{4.11}$$

and the global attractor of the generalized semiflow $\mathcal{E}$ is the singleton $Z(\mathcal{E}) = \{\bar{u}\}$.

Notice that the (λ, p)-geodesic assumption (A8) has replaced (A5)–(A7). Although the *proof* is an adaptation of the argument for (AGS05, Thm. 2.4.14), for the reader's convenience we shall develop it at the end of Section 5.2.

5. Proofs

5.1. Proof of Theorem 4.6

Proof of Theorem 4.4. In order to check (H1), we fix $(u_0, \varphi_0) \in (\mathcal{X}, d_{\mathcal{X}})$. It follows from Theorem 4.2 that there exists a *generalized solution* (u, φ) fulfilling $u(0) = u_0$ and $\varphi(0) = \phi(u_0)$. We let

$$\tilde{\varphi}(t) := \begin{cases} \varphi(t) & \text{for } t > 0, \\ \varphi_0 & \text{for } t = 0. \end{cases}$$

Clearly, the pair $(u, \tilde{\varphi})$ still complies with (2.22)–(2.23) and starts from (u_0, φ_0) as desired. It can be easily checked that $\mathcal{S}$ fulfills the translation and concatenation properties. What we are left with is the proof of the upper semicontinuity (H4). To this end, we fix a sequence $\{(u_0^n, \varphi_0^n)\} \subset \mathcal{X}$ with $d_\sigma(u_0^n, u_0) + |\varphi_0^n - \varphi_0| \to 0$, and we consider a sequence $\{(u_n, \varphi_n)\} \subset \mathcal{S}$ such that $u_n(0) = u_0^n$ and $\varphi_n(0) = \varphi_0^n$. Inequality (2.22) reads for all $n \in \mathbb{N}$

$$\frac{1}{p}\int_s^t |u_n'|^p(r)dr + \frac{1}{p'}\int_s^t |\partial^-\phi|^{p'}(u_n(r), \varphi_n(r))dr + \varphi_n(t) \le \varphi_n(s) \quad \text{for all } \ 0 \le s \le t. \tag{5.1}$$

Since

$$\varphi_n(t) \ge \phi(u_n(t)) \quad \forall t \ge 0 \quad \forall n \in \mathbb{N}, \tag{5.2}$$

using inequality (5.1) for $s = 0$ and (4.1), we deduce that

$$\int_0^t |u_n'|^p(r)dr, \quad \int_0^t |\partial^-\phi|^{p'}(u_n(r), \varphi_n(r))dr, \quad \varphi_n(t), \quad \phi(u_n(t)) \quad \text{are bounded uniformly with respect to } n \text{ and } t \ge 0. \tag{5.3}$$

In view of (A4), we conclude that there exists a σ-sequentially compact set $\mathcal{K} \subset U$ such that $u_n(t) \in \mathcal{K}$ for all $n \in \mathbb{N}$ and $t \ge 0$. Then, by exploiting a suitably refined version of Ascoli's theorem (see (AGS05, Prop. 3.3.1, p. 69)), we find a subsequence $\{u_n\}$ (which we do not relabel), a curve $u \in \mathrm{AC}^p_{\mathrm{loc}}([0, +\infty); U)$, and a function $A \in L^p_{\mathrm{loc}}([0, +\infty))$ such that, also recalling (A3) and (5.3), one has

$$u_n(t) \overset{\sigma}{\to} u(t) \ \text{ for all } \ t \ge 0, \quad u(0) = u_0, \tag{5.4}$$

$$\liminf_{n \to +\infty} \phi(u_n(t)) \ge \phi(u(t)) \ \text{ for all } \ t \ge 0, \quad \phi(u(0)) = \phi(u_0), \tag{5.5}$$

$$|u_n'| \rightharpoonup A \ \text{ in } L^p_{\mathrm{loc}}([0, +\infty)), \quad A \ge |u'| \ \text{ a.e. in } (0, +\infty). \tag{5.6}$$

In particular, (A2b) yields that for all $t \ge 0$

$$d_\sigma(u(t), u_n(t)) \to 0 \qquad \text{as } n \to \infty,$$

On the other hand, since, for all $n \in \mathbb{N}$, the function φ_n is non-increasing, thanks to Helly's compactness theorem (see, e.g., (AGS05, Lemma 3.3.3, p.

70)) and to the a priori estimate (5.3), there exists a (non-relabeled) subsequence of $\{\varphi_n\}$ and a non-increasing map $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$\varphi(t) = \lim_{n\to\infty} \varphi_n(t) \geq \liminf_{n\to\infty} \phi(u_n(t)) \geq \phi(u(t)) \qquad \text{for all } \ t \geq 0,$$

where the second inequality is due to (5.2), whereas the latter one to (A3). Hence, $\varphi(0) = \varphi_0$. We are now in the position of passing to the limit as $n \to \infty$ in the energy inequality (5.1). Relying on (5.4)–(5.5) and (2.5), and exploiting Fatou's Lemma, we conclude that the limit pair (u, φ) fulfills (2.22) for all $0 \leq s \leq t$, and (H4) follows. □

Proof of Proposition 4.5. We fix a sequence $\{(u_j, \varphi_j)\} \subset \mathcal{S}$ with $\{(u_j(0), \varphi_j(0))\}$ $d_{\mathcal{X}}$-bounded and a sequence $t_j \to +\infty$. Inequality (2.22) for $s = 0$ and $t = t_j$ yields that there exists a constant $C > 0$ such that

$$\phi(u_j(t_j)) \leq \varphi_j(t_j) \leq \varphi_j(0) \leq C \quad \forall j \in \mathbb{N}. \tag{5.7}$$

Thanks to (A4) we conclude that there exists a σ-sequentially compact set $\mathcal{K}' \subset U$ such that $u_j(t_j) \in \mathcal{K}'$ for all $j \in \mathbb{N}$. On the other hand, using (5.7) and recalling that ϕ is bounded from below (cf. with (4.1)), we find that $\sup_j |\varphi_j(t_j)| < +\infty$. Hence, the sequence $\{(u_j(t_j), \varphi_j(t_j))\}$ admits a $d_{\mathcal{X}}$-converging subsequence.

The projection on the second component $\pi_2 : \mathcal{X} \to \mathbb{R}$ is a *Lyapunov function* for the semiflow $\mathcal{S}$. Indeed, π_2 is clearly continuous and decreases along the elements of $\mathcal{S}$, since for all $(u, \varphi) \in \mathcal{S}$ the map φ is non-increasing. Moreover, let (v, ψ) be a complete orbit, such that there exists $\bar{\psi} \in \mathbb{R}$ with $\pi_2(v(t), \psi(t)) = \psi(t) \equiv \bar{\psi}$ for all $t \in \mathbb{R}$. Then, (2.22) yields

$$\frac{1}{p}\int_s^t |v'|^p(r)dr + \frac{1}{p'}\int_s^t |\partial^- \phi|(v(r), \bar{\psi})dr \leq 0 \qquad \forall s \leq t.$$

Hence, by the properties of the metric derivative we easily conclude that there exists $\bar{v} \in D(\phi)$ such that $v(t) = \bar{v}$ for all $t \in \mathbb{R}$, so that (v, ψ) is a stationary orbit.

Finally, the check that the set $Z(\mathcal{S})$ in (4.3) is the set of rest points is immediate. □

The *proof* of Theorem 4.6 follows from Theorem Appendix A.6 and Proposition 4.5.

5.2. Proof of Theorem 4.7

[**Ad** 1).] Since $\mathcal{S}$ complies with (H1), so does $\mathcal{E} = \pi_1(\mathcal{S})$ thanks to Proposition 2.14. Properties (H2) and (H3) can be trivially checked too. We also note that, in view of Definition 2.2, any energy solution u is continuous on $[0, +\infty)$ with values in the space $(D(\phi), d_\phi)$. In particular (C0), (C1), and (C3) hold.

In order to prove (H4), we fix a sequence $\{u_n\} \subset \pi_1(\mathcal{S})$ such that $d_\phi(u_n(0), u_0) \to 0$ for some $u_0 \in D(\phi)$. This entails that the lifted sequence $\{(u_n(0), \phi(u_n(0)))\}$ fulfills $d_{\mathcal{X}}(\mathcal{U}(u_n(0)), \mathcal{U}(u_0)) \to 0$ as $n \to +\infty$. Thanks to Theorem 4.4, there exists $(u, \varphi) \in \mathcal{S}$ with $u(0) = u_0$ and $\varphi(0) = \phi(u_0)$, and a subsequence $\{n_k\}$ such that for all $t \geq 0$ $d_{\mathcal{X}}(\mathcal{U}(u_{n_k}(t)), (u(t), \varphi(t))) \to 0$. In view of Propositions 2.14 and 2.15, the curve u is an energy solution and $\varphi(t) = \phi(u(t))$ for all $t > 0$, so that

$$d_\phi(u_{n_k}(t), u(t)) = d_\sigma(u_{n_k}(t), u(t)) + |\phi(u_{n_k}(t)) - \phi(u(t))| \to 0 \\ \text{as } k \to \infty \text{ for all } \ t > 0,$$

and the proof of (H4) is completed. The check of (C2) follows easily from the same arguments as in the proof of Theorem 4.4. In particular, as long as one restricts to energy solutions, it is easy to establish estimate (5.3) and Ascoli's Theorem (AGS05, Prop. 3.3.1, p. 69) in metric spaces entails the desired convergence.

[**Ad** 2).] We shall prove that $\pi_1(A)$ is compact in the phase space $(D(\phi), d_\phi)$ and that it is invariant and attracting for $\mathcal{E}$. To this aim, for every $t \geq 0$, we denote by $\mathcal{T}(t)$ ($T_1(t)$, resp.) the operator associated with the semiflow $\mathcal{S}$ (with $\mathcal{E}$, resp.) by formula (A.1). It follows from Proposition 2.14 that, for all $t > 0$, $\mathcal{T}(t)\mathcal{X} \subset \mathcal{U}(D(\phi))$. Hence $\mathcal{U}(D(\phi))$ is positively invariant for the semiflow $\mathcal{S}$, i.e.

$$\mathcal{T}(t)(\mathcal{U}(D(\phi))) \subset \mathcal{U}(D(\phi)) \qquad \text{for all } \ t \geq 0. \tag{5.8}$$

As a consequence, the global attractor A of $\mathcal{S}$ fulfills $A \subset \mathcal{U}(D(\phi))$. Hence, by (4.7) we have

$$A = \mathcal{U}(\pi_1(A)). \tag{5.9}$$

Moreover, as the projection operator is continuous from $(\mathcal{U}(D(\phi)), d_{\mathcal{X}})$ to $(D(\phi), d_\phi)$, we conclude that the set $\pi_1(A)$ is compact as well. Again using Proposition 2.14, it is not difficult to check that the operators $\mathcal{T}(t)$ and

$T_1(t)$ are related in the following way:

$$T_1(t)(B) = \pi_1(\mathcal{T}(t)(\mathcal{U}(B))) \qquad \text{for all} \quad B \subset D(\phi) \quad \text{for all} \quad t \geq 0. \tag{5.10}$$

Hence, thanks to (5.9) and using that A is invariant for the semiflow $\mathcal{S}$ we have

$$T_1(t)(\pi_1(A)) = \pi_1(\mathcal{T}(t)(\mathcal{U}(\pi_1(A)))) = \pi_1(\mathcal{T}(t)(A)) = \pi_1(A) \quad \forall t \geq 0,$$

so that $\pi_1(A)$ is itself invariant for $\mathcal{E}$. Finally, we fix a bounded set $B \subset (D(\phi), d_\phi)$. Recalling (4.8), we deduce that the lifted set

$$\mathcal{U}(B) \text{ is bounded in } (\mathcal{X}, d_\mathcal{X}). \tag{5.11}$$

Hence,

$$\begin{aligned} \lim_{t\to+\infty} e_\phi(T_1(t)(B), \pi_1(A)) &= \lim_{t\to+\infty} e_\phi(\pi_1(\mathcal{T}(t)(\mathcal{U}(B))), \pi_1(A)) \\ &= \lim_{t\to+\infty} e_\mathcal{X}(\mathcal{U}(\pi_1(\mathcal{T}(t)(\mathcal{U}(B)))), \mathcal{U}(\pi_1(A))) \\ &= \lim_{t\to+\infty} e_\mathcal{X}(\mathcal{T}(t)(\mathcal{U}(B)), A) = 0 \end{aligned} \tag{5.12}$$

where the first identity follows from (5.10), the second one from (4.8), the third one from (4.7), (5.8), and the last one from the fact that A attracts the bounded sets of $(\mathcal{X}, d_\mathcal{X})$ and from (5.11). By (5.12), $\pi_1(A)$ has the same attracting property in $(D(\phi), d_\phi)$, and (4.9) follows. □

Proof of Theorem 4.8. Notice that (A3) and (A4) guarantee that ϕ has at least a minimizer $\bar{u} \in U$, which is unique by the (λ, p)-geodesic convexity (A8). Furthermore, the set of the rest points of the semiflow $\mathcal{E}$ is given by

$$Z(\mathcal{E}) = \{\bar{u}\}. \tag{5.13}$$

Indeed, there holds

$$0 \leq \phi(w) - \phi(\bar{u}) \leq 0 \qquad \text{for all } w \in Z(\mathcal{E}),$$

where the first inequality ensues from the fact that $\bar{u}$ is the minimizer of ϕ, while the second one follows from (2.17) and $|\partial^+\phi|(w) = 0$, being w a rest point. Then, $\phi(w) = \phi(\bar{u})$, whence $w = \bar{u}$ by (A8).

To check (4.10) one argues along the very same lines as in the proof of (AGS05, Thm. 2.4.14). Namely, for every $u \in \mathcal{E}$ and all $t > 0$ one sets $\Delta(t) := \phi(u(t)) - \phi(\bar{u})$, noticing that

$$\Delta'(t) = -|\partial^+\phi|^{p'}(u(t)) \qquad \text{for a.a. } t \in (0, +\infty),$$

cf. with (2.8). Combining this with (2.17), one obtains the differential inequality

$$\Delta'(t) \leq -\lambda p \Delta(t) \qquad \text{for a.a. } t \in (0, +\infty),$$

whence the second inequality in (4.10). The first one ensues from the first of (2.17). Then, (4.11) is a trivial consequence of (4.10) via (A2a). Clearly, (5.13) and (4.11) yield that the attractor for $\mathcal{E}$ is given by $\{\bar{u}\}$. □

6. Applications to doubly nonlinear equations in Banach spaces

Hereafter, we shall denote by

$$\mathcal{B} \text{ a (separable) reflexive Banach space, with norm } \|\cdot\|.$$

6.1. Doubly nonlinear evolutions driven by nonconvex energies

Let $\phi : \mathcal{B} \to (-\infty, +\infty]$ be a proper functional complying with (A3)–(A4) in the ambient space

$$U = \mathcal{B}, \text{ with } d(u, v) = \|v - u\| \text{ and } \sigma \text{ the strong topology} \tag{6.14}$$

(see the following Sec. 6.2 for a different choice). In this framework, we shall extend the discussion of Section 3.1 to the doubly nonlinear differential inclusion

$$\mathrm{J}_p(u'(t)) + \partial_\ell \phi(u(t)) \ni 0 \qquad \text{in } \mathcal{B}' \quad \text{for a.a. } t \in (0, T)\,, \tag{6.15}$$

which features the *limiting subdifferential* $\partial_\ell \phi$ of ϕ (cf. with (6.16)), a generalized gradient notion related to the strong-weak closure of the Fréchet subdifferential (3.30) of ϕ.

Literature on gradient flows with limiting subdifferentials. Being ϕ nonconvex, the Fréchet subdifferential is not, in general, strongly-weakly closed in $\mathcal{B} \times \mathcal{B}'$ in the sense of graphs. It is hence meaningful to consider its strong-weak closure (along sequences with bounded energy) $\partial_\ell \phi$, defined at some $u \in D(\phi)$ by

$$\xi \in \partial_\ell \phi(u) \Leftrightarrow \begin{cases} \exists\, u_n \in \mathcal{B}, \xi_n \in \mathcal{B}' \text{ with } \xi_n \in \partial\phi(u_n) \text{ for all } n \in \mathbb{N}, \\ u_n \to v, \ \xi_n \rightharpoonup \xi \text{ as } n \to +\infty, \quad \sup_n \phi(u_n) < +\infty\,. \end{cases} \tag{6.16}$$

In analogy with (3.31), for $u \in D(\phi)$ we shall use the notation

$$\begin{cases} \|\partial_\ell^\circ \phi(u)\|_* = \inf\{\|\xi\|_* \, : \, \xi \in \partial_\ell^\circ \phi(u)\} \\ \partial_\ell^\circ \phi(u) = \operatorname{Argmin}\{\|\xi\|_* \, : \, \xi \in \partial_\ell^\circ \phi(u)\}\,. \end{cases} \tag{6.17}$$

Notice that, in general, the latter set may be empty. The limiting subdifferential, first introduced in (KrMo80; Mor84), was proposed as a replacement of the Fréchet subdifferential for gradient flow equations in Hilbert spaces, driven by *nonconvex* energy functionals, in the paper (RoSa06). Therein, existence and approximation results for the gradient flow equation

$$u'(t) + \partial_\ell \phi(u(t)) \ni 0 \qquad \text{in } \mathcal{H} \quad \text{for a.a. } t \in (0,T)\,, \tag{6.18}$$

(corresponding to $p = 2$ and to $\mathcal{B} = \mathcal{H}$, $\mathcal{H}$ a separable Hilbert space, in (6.15)) were obtained, and applications to various PDEs were developed, under the standing assumptions (A3)–(A4), the hypothesis that ϕ satisfies the *continuity property*

$$u_n \to u, \quad \sup_n \big(\|\partial_\ell^\circ \phi(u_n)\|_*, \phi(u_n)\big) < +\infty \ \ \Rightarrow \phi(u_n) \to \phi(u), \tag{6.19}$$

and that the chain rule w.r.t. $\partial_\ell \phi$ holds, which we already state in the general (p,p')-case

$$\begin{aligned} &\text{if } \ u \in \mathrm{AC}^p([0,T];\mathcal{B}), \ \xi \in L^{p'}(0,T;\mathcal{B}'), \ \xi(t) \in \partial_\ell \phi(u(t)) \quad \text{for a.a. } t \in (0,T) \\ &\qquad \text{then } \ \phi \circ u \in \mathrm{AC}([0,T]), \quad \tfrac{d}{dt}\phi(u(t)) = \langle \xi(t), u'(t) \rangle \\ &\qquad\qquad \text{for a.a. } t \in (0,T). \end{aligned} \tag{6.20}$$

The subsequent paper (RSS08) addressed the long-time behavior of a slightly less general version of equation (6.18) (see (6.22) below), in which $\partial_\ell \phi$ was replaced by a strengthened variant, the *strong limiting subdifferential* $\partial_s\phi$, defined at some $u \in D(\phi)$ by

$$\xi \in \partial_s\phi(u) \quad \Leftrightarrow \quad \begin{cases} \exists\, u_n \in \mathcal{B}, \xi_n \in \mathcal{B}' \text{ with } \xi_n \in \partial\phi(u_n) \text{ for all } n \in \mathbb{N}, \\ u_n \to v, \ \xi_n \to \xi \text{ as } n \to \infty, \quad \phi(v_n) \to \phi(v)\,. \end{cases}$$

In fact, it was shown in (RSS08, Lemma 1) that $\partial_\ell\phi$ is the (sequential) strong-weak closure of $\partial_s\phi$ along sequences with bounded energy, namely for all $u \in D(\phi)$

$$\xi \in \partial_\ell\phi(u) \iff \exists\, u_k \in \mathcal{B}, \ \ \xi_k \in \mathcal{B}' : \begin{cases} u_k \to u, \, \xi_k \rightharpoonup \xi, \, \sup_k \phi(u_k) < +\infty, \\ \xi_k \in \partial_s\phi(u_k) \text{ for all } k \in \mathbb{N}. \end{cases} \tag{6.21}$$

Assuming (A3)–(A4) and (6.19)–(6.20), the existence of the global attractor for the semiflow associated with

$$u'(t) + \partial_s\phi(u(t)) \ni 0 \qquad \text{in } \mathcal{H} \quad \text{for a.a. } t \in (0,T)\,, \tag{6.22}$$

was proved in (RSS08) under the further condition that the set of the rest points

$$\{u \in D(\phi) \,:\, 0 \in \partial_s\phi(u)\} \text{ is bounded in the phase space (4.6).}$$

Existence of the global attractor for (6.15) **via energy solutions.** Following the outline of Section 3.1, we shall analyze (6.15) from a metric point of view. Namely, in Proposition 6.2 below we shall prove that the energy solutions of the metric p-gradient flow of ϕ, in the ambient space (6.14), yield solutions to (6.15). Further, we shall show that, if ϕ is in addition (λ, q)-convex, namely if (2.15) holds for some $\lambda \in \mathbb{R}$ and $q \in (1,\infty)$, then energy solutions in fact exhaust the set of solutions to (6.15). Hence, we shall deduce from Theorem 4.7 a result on the long-time behavior of the solutions of (6.15), see Theorem 6.4 later on.

Our starting point is the following key proposition.

Proposition 6.1. *In the setting of* (6.14)*, suppose that* $\phi : \mathcal{B} \to (-\infty, +\infty]$ *complies with* (A3)*–*(A4)*.*

1. *Then,*

$$\|\partial_\ell^\circ\phi(u)\|_* \le |\partial^+\phi|(u) \qquad \textit{for all } u \in D(|\partial^+\phi|). \tag{6.23}$$

2. *If, in addition, ϕ is (λ, q)-convex for some $\lambda \in \mathbb{R}$ and $q \in (1, \infty)$, then for all $u \in D(\phi)$*

$$\partial\phi(u) = \partial_s\phi(u) = \partial_\ell\phi(u), \qquad |\partial^+\phi|(u) = |\partial\phi|(u)\,, \tag{6.24a}$$

$$\|\partial^\circ\phi(u)\|_* = \|\partial^\circ_\ell\phi(u)\|_* = |\partial^+\phi|(u) = |\partial\phi|(u)\,. \tag{6.24b}$$

Furthermore, the local slope $|\partial\phi|$ is a strong upper gradient.

Proof. Preliminarily, we recall that, being reflexive, $\mathcal{B}$ has a renorm which is Fréchet differentiable off the origin. Therefore, up to switching to an equivalent norm, we may suppose that the map

$$v \in \mathcal{B} \mapsto \frac{1}{2}\|v\|^2 \quad \text{is Fréchet differentiable on } \mathcal{B} \tag{6.25}$$

(however, in the Appendix we are going to present a proof which does not use (6.25)).
Proof of (6.23)**.** It is not restrictive to suppose that $|\partial^+\phi|(u) < +\infty$: hence, (up to further extractions) we can select a sequence $\{u_k\} \subset \mathcal{B}$ such that

$$u_k \to u, \qquad \phi(u_k) \to \phi(u), \qquad |\partial\phi|(u_k) \to |\partial^+\phi|(u)\,. \tag{6.26}$$

Now, we invoke (AGS05, Lemma 3.1.5), which provides a duality formula for the local slope $|\partial\phi|$: for every $k \in \mathbb{N}$, there exists a sequence $\{r_j^k\}_j$, with $r_j^k \downarrow 0$ as $j \to \infty$, and a selection

$$z_j^k \in \operatorname*{Argmin}_{v\in\mathcal{B}} \left\{ \frac{\|v - u_k\|^2}{2r_j^k} + \phi(v) \right\}, \tag{6.27}$$

such that

$$|\partial\phi|^2(u_k) = \lim_{j\to\infty} \frac{\|z_j^k - u_k\|^2}{(r_j^k)^2}, \quad \text{and} \quad z_j^k \to u_k \text{ as } j \to \infty. \tag{6.28}$$

Notice that (6.27) yields $\phi(z_j^k) \le \phi(u_k)$ for all $j \in \mathbb{N}$, whereas, by the convergence of $\{z_j^k\}_j$ in (6.28) and the lower semicontinuity of ϕ, we gather $\liminf_j \phi(z_j^k) \ge \phi(u_k)$, so that

$$\phi(z_j^k) \to \phi(u_k) \qquad \text{as } j \to \infty. \tag{6.29}$$

Furthermore, (6.25) ensures that there holds the following sum rule for the Fréchet subdifferential

$$\partial \left(\frac{\| \cdot - u_k\|^2}{2r_j^k} + \phi \right) = \mathrm{J}_2 \left(\frac{\cdot - u_k}{r_j^k} \right) + \partial \phi \,.$$

Hence, we may conclude that for all $j \in \mathbb{N}$ z_j^k fulfills the Euler equation

$$\mathrm{J}_2 \left(\frac{z_j^k - u_k}{r_j^k} \right) + \partial \phi(z_j^k) \ni 0 \,,$$

namely

$$\exists\, w_j^k \in \mathrm{J}_2 \left(\frac{z_j^k - u_k}{r_j^k} \right) \cap \left(-\partial \phi(z_j^k) \right) \,. \tag{6.30}$$

Thus, in view of (6.28) and the definition of J_2,

$$\left(|\partial \phi|(u_k) \right)^2 = \lim_{j \to \infty} \|w_j^k\|_*^2 \,. \tag{6.31}$$

By a diagonalization procedure, collecting (6.28), (6.29), and (6.31), we may extract subsequences $\{w_{j_k}^k\}$, $\{z_{j_k}^k\}$ ($\{w_k\}$, $\{z_k\}$ for short) such that for all $k \in \mathbb{N}$

$$\Big(|\|w_k\|_* - |\partial\phi|(u_k)| + \|z_k - u_k\| + |\phi(z_k) - \phi(u_k)| \Big) \leq \frac{1}{k}. \tag{6.32}$$

Thus, in view of (6.26) we find that $z_k \to u$, $\phi(z_k) \to \phi(u)$ and that there exists $w \in \mathcal{B}^*$ such that, up the extraction of a further non-relabeled subsequence, $w_k \rightharpoonup w$ in $\mathcal{B}^*$. We readily conclude from (6.30) and from the definition of $\partial_\ell \phi$ that

$$-w \in \partial_\ell \phi(u). \tag{6.33}$$

On the other hand, with (6.32) and (6.26) we find

$$\|w\|_{\mathcal{B}^*} \leq \liminf_{k \to \infty} \|w_k\|_* = \liminf_{k \to \infty} |\partial \phi|(u_k) = |\partial^+ \phi|(u) \tag{6.34}$$

Combining (6.33) with (6.34) we arrive at (6.23).

Proof of (6.24)**.** Relations (6.24) have been proved in (RMS08, Prop. 5.6) for λ-convex functionals. Mimicking the proof of (RMS08, Prop. 5.6) and taking into account Proposition 2.7, it is easy to extend (6.24) to the (λ, q)-convex case. In the same way, the last statement follows from the very same arguments as in the proof of (RMS08, Prop. 5.11). □

Proposition 6.2. *In the setting of* (6.14)*, suppose that* $\phi : \mathcal{B} \to (-\infty, +\infty]$ *complies with* (A3)–(A4)*.*

1. *If* ϕ *also fulfills the chain rule* (6.20) *with respect to* $\partial_\ell \phi$ *and*

$$\partial_\ell^\circ \phi(u) \neq \emptyset \qquad \text{for all } u \in D(\partial_\ell \phi), \tag{6.35}$$

then

$$|\partial^+ \phi| \text{ is a strong upper gradient.} \tag{6.36}$$

Further, any energy solution $u \in \mathrm{AC}^p([0,T];\mathcal{B})$ *of the metric* p*-gradient flow of* ϕ *is also a solution of the doubly nonlinear equation* (6.15) *and fulfills the minimal section principle*

$$-\partial_\ell^\circ \phi(u(t)) \subset \mathrm{J}_p(u'(t)) \qquad \text{for a.a. } t \in (0,T)\,. \tag{6.37}$$

2. *In addition, if* ϕ *fulfills* (2.15) *for some* $\lambda \in \mathbb{R}$ *and* $q \in (1,\infty)$*, a curve* $u \in \mathrm{AC}^p([0,T];\mathcal{B})$ *is an energy solution* if and only if *the map* $t \mapsto \phi(u(t))$ *is absolutely continuous on* $(0,T)$ *and* u *fulfills* (6.37)*.*

Proof. Property (6.36) is a straightforward consequence of the chain rule (6.20) and of inequality (6.23). The proof of the subsequent statement relies on the same argument as (AGS05, Prop. 1.4.1): we shall however sketch it for the reader's convenience. It follows from (2.10) and (6.23) that any energy solution $u \in \mathrm{AC}^p([0,T];\mathcal{B})$ fulfills (note that u is almost everywhere differentiable, hence $|u'|(\cdot) \equiv \|u'(\cdot)\|$ a.e. in $(0,T)$)

$$\begin{cases} t \mapsto \|\partial_\ell^\circ \phi(u(t))\|_* \in L^{p'}(0,T), \\ (\phi \circ u)'(t) \leq -\dfrac{1}{p}\|u'(t)\| - \dfrac{1}{p'}\|\partial_\ell^\circ \phi(u(t))\|_*^{p'} \quad \text{for a.a. } t \in (0,T) \end{cases} \tag{6.38}$$

On the other hand, arguing in the same way as in the proof of (RoSa06, Lemma 3.4) and also exploiting (6.35) and the first of (6.38), we find that there exists a selection $\xi_{\min} \in L^{p'}(0,T;\mathcal{B}^*)$ in the multi-valued map $t \mapsto \partial_\ell^\circ \phi(u(t))$. The chain rule (6.20) yields $(\phi \circ u)' = \langle \xi_{\min}, u' \rangle$ a.e. in $(0,T)$. Combining this with the inequality in (6.38), we conclude (6.37).

The converse implication may be proved in the (λ, q)-convex case by a completely analogous argument, relying on identity (6.24b). $\square$

Remark 6.3. On behalf of the above result, we are entitled to refer to the energy solutions of the metric p-gradient flow of ϕ as the *metric solutions* of (6.15). It follows from the proof of Proposition 6.2 that a curve $u \in \mathrm{AC}^p([0,T];\mathcal{B})$ is a *metric solution* of (6.15) if and only if

$$-(\phi \circ u)'(t) = \|u'(t)\| |\partial^+\phi|(u(t)) = \|u'(t)\| \|\partial_\ell^\circ \phi(u(t))\|_* \quad \text{for a.a. } t \in (0,T)\,.$$

The following result is a direct consequence of Theorem 4.7 and the previous Proposition 6.2.

Theorem 6.4. *In the setting of* (6.14)*, suppose that* $\phi : \mathcal{B} \to (-\infty, +\infty]$ *complies with* (A3)–(A4)*, with the conditional continuity* (CONT)*, and with* (A5)*. Then,*

1. *if* ϕ *also complies with* (6.35) *and*

$$\text{the set of the rest points } \{u \in D(\phi) \,:\, 0 \in \partial_\ell \phi(u)\} \text{ is bounded in the phase space (4.6),} \tag{6.39}$$

then the semiflow associated with the metric solutions *of equation* (6.15) *(cf. with Remark 6.3) admits a global attractor;*

2. *if* ϕ *is* (λ, q)*-convex for some* $\lambda \in \mathbb{R}$ *and* $q \in (1,\infty)$ *and complies with* (6.39)*, the semiflow generated by the* whole *set of solutions to* (6.15) *admits a global attractor.*

6.2. Outlook to quasivariational doubly nonlinear equations

Finally, we show how our results can be applied to the study of the long-time behavior of a class of doubly nonlinear equations of the form

$$\partial\Psi(u(t), u'(t)) + \partial_\ell \phi(u(t)) \ni 0 \qquad \text{in } \mathcal{B}' \quad \text{for a.a. } t \in (0,T)\,, \tag{6.40}$$

where the dissipation functional $\Psi : \mathcal{B} \times \mathcal{B} \to [0, +\infty)$ also depends on the state variable u, and which are often referred to as *quasivariational*, cf. (RMS08) and the references therein.

In particular, following (RMS08), where the metric approach was applied to prove the existence of solutions to the Cauchy problem for (6.40), we shall focus on the case the functional Ψ is given by

$$\Psi(u,v) = \frac{\eta_u(v)^p}{p} \qquad \text{for all } u,\, v \in \mathcal{B}, \text{ with } 1 < p < \infty, \text{ and} \tag{6.41}$$

$$\begin{aligned} &\{\eta_u\}_{u\in\mathcal{B}} \quad \text{is a family of norms on } \mathcal{B}, \text{ such that} \\ &\exists\, \mathsf{K}>0 \ \ \forall\, u,\, v\in\mathcal{B}: \quad \mathsf{K}^{-1}\|v\| \le \eta_u(v) \le \mathsf{K}\|v\| \end{aligned} \tag{6.42}$$

and the dependence $u \mapsto \eta_u$ is continuous in the sense of MOSCO-convergence (see, e.g., (Att86, Sec. 3.3, p. 295)), namely

$$u_n \to u, \quad v_n \rightharpoonup v \quad \text{in } \mathcal{B} \quad \Rightarrow \quad \liminf_{n\to\infty} \eta_{u_n}(v_n) \ge \eta_u(v), \tag{6.43a}$$

$$u_n \to u, \quad v\in\mathcal{B} \quad \Rightarrow \quad \exists\, v_n \to v: \quad \lim_{n\to\infty} \eta_{u_n}(v_n) = \eta_u(v). \tag{6.43b}$$

For all $u \in \mathcal{B}$, we shall denote by η_{u*} the related dual norm on $\mathcal{B}'$.

Within this framework, the metric approach to (6.40) may be developed by endowing the ambient space

$$\begin{aligned} &U=\mathcal{B} \ \text{ with the Finsler distance induced by } \{\eta_u\}_{u\in\mathcal{B}}, \text{ i.e.} \\ &d_\eta(v,w) := \inf\left\{ \int_0^1 \eta_{u(t)}(u'(t))\,\mathrm{d}t \ : \ u\in \mathrm{AC}([0,1];\mathcal{B}), \right. \\ &\qquad\qquad\qquad\qquad \left. u(0)=v,\, u(1)=w \right\} \end{aligned} \tag{6.44}$$

for all v, $w \in \mathcal{B}$. Then, we have the analogue of Proposition 6.2 (see (RMS08, Prop. 8.2) for the proof).

Proposition 6.5. *In the setting of* (6.42)–(6.44)*, suppose that* $\phi : \mathcal{B} \to (-\infty,+\infty]$ *complies with* (A3)–(A4)*, that* ϕ *fulfills the chain rule* (6.20) *with respect to* $\partial_\ell\phi$*, and that* (6.35) *holds.*

Then, any energy solution $u \in \mathrm{AC}^p([0,T];\mathcal{B})$ *of the metric p-gradient flow of* ϕ *is also a solution of the doubly nonlinear equation* (6.40) *with* Ψ *given by* (6.41)*, and fulfills the minimal section principle*

$$\partial\Psi(u(t),u'(t)) \supset \mathrm{Argmin}\left\{\eta_{u(t)*}(-\xi) : \xi \in \partial_\ell(u(t))\right\} \qquad \textit{for a.a. } t\in(0,T)\,. \tag{6.45}$$

Conversely, if ϕ *fulfills* (2.15) *for some* $\lambda \in \mathbb{R}$ *and* $q \in (1,\infty)$*, a curve* $u \in \mathrm{AC}^p([0,T];\mathcal{B})$ *is an energy solution* if and only if *the map* $t \mapsto \phi(u(t))$ *is absolutely continuous on* $(0,T)$*, and* u *fulfills* (6.45)*.*

Hence, we derive from our general Theorem 4.7 the quasivariational counterpart to Theorem 6.4. To avoid overburdening this paper, we prefer to omit the statement that, under the assumptions of Proposition 6.5 and the boundedness of the set of the rest points (6.39), the generalized semiflow associated with the metric solutions to (6.40) admits a global attractor.

Example 6.6. Our results apply to the following generalized Allen-Cahn equation

$$\rho(u)\,|u_t|^{p-2}u_t - \mathrm{div}(\beta(\nabla u)) + \mathrm{W}'(u) = 0 \quad \text{in } \Omega \times (0,T), \tag{6.46}$$

with $\Omega \subset \mathbb{R}^d$ a bounded domain with sufficiently smooth boundary, $\rho : \mathbb{R} \to (0,+\infty)$ a continuous function, bounded from below and from above by positive constants, $\beta : \mathbb{R}^d \to \mathbb{R}^d$ the gradient of some smooth function j on $\mathbb{R}^d$, and $\mathrm{W} : \mathbb{R} \to \mathbb{R}$ a differentiable function. For simplicity, we supplement (6.46) with homogeneous Dirichlet boundary conditions for u.

The boundary value problem for (6.46) can be recast in the general form (6.40) in the Banach space

$$\mathcal{B} = L^p(\Omega),$$

with the choices

$$\eta_u(v) := \left(\int_\Omega \rho(u(x))|v(x)|^p \,\mathrm{d}x \right)^{1/p} \quad \text{for all } u,\, v \in L^p(\Omega), \tag{6.47}$$

and

$$\phi(u) := \begin{cases} \displaystyle\int_\Omega (j(\nabla u(x)) + \mathrm{W}(u(x)))\,\mathrm{d}x \\ \qquad\qquad \text{if } u \in H^1_0(\Omega),\ j(\nabla u) + \mathrm{W}(u) \in L^1(\Omega), \\ +\infty \quad \text{otherwise.} \end{cases} \tag{6.48}$$

We refer to (RMS08, Sec. 8.2) for the proof of the fact that the family of norms $\{\eta_u\}$ (6.47) fulfills (6.42)–(6.43b), and for the precise statement of the assumptions on the nonlinearities j and W, under which the functional ϕ (6.48) complies with the assumptions of Proposition 6.5. Let us just mention that, in particular, the classical double-well potential $\mathrm{W}(r) = \frac{(r^2-1)^2}{4}$ fits into the frame of such assumptions.

7. Application to Curves of Maximal Slope in Wasserstein spaces

In this section, we aim to apply our general Theorem 4.7 to the long-time analysis of the class of diffusion equations in $\mathbb{R}^d$, $d \geq 1$, mentioned in the Introduction (cf. with (1.12)). We shall systematically follow the *metric approach* to evolutions in Wasserstein spaces developed in (AGS05). In order to make the paper as self-contained as possible, we recall some preliminary definitions and results on gradient flows in Wasserstein spaces in Sec. 7.1 below, referring to (AGS05) for more details and all the proofs.

7.1. Setup in Wasserstein spaces

Hereafter, we shall denote by $\mathscr{L}^d$ the d-dimensional Lebesgue measure and by $\pi^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2$, the projection on the i-component.

In the following, we shall work in the space of probability measures in $\mathbb{R}^d$ with finite p-moment, i.e. we shall take U to be

$$U = \mathscr{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathscr{P}(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} |x|^p \, \mathrm{d}\mu(x) < +\infty \right\}, \tag{7.49}$$

endowed with the p-Wasserstein distance

$$W_p(\mu_1, \mu_2) = \min \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \mathrm{d}\gamma(x, y) \right)^{1/p} \,:\, \gamma \in \Gamma(\mu_1, \mu_2) \right\} \tag{7.50}$$

for all μ_1, $\mu_2 \in \mathscr{P}_p(\mathbb{R}^d)$, where $\Gamma(\mu_1, \mu_2)$ is the set of probability measures $\gamma \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)$ fulfilling the push-forward relations $\pi^1_{\#}\gamma = \mu_1$ and $\pi^2_{\#}\gamma = \mu_2$ (namely, for all Borel subset $B \subset \mathbb{R}^d$ and $i = 1, 2$ $\mu_i(B) = \gamma((\pi^i)^{-1}(B))$). We shall denote by $\Gamma_{\mathrm{o}}(\mu_1, \mu_2)$ the class of the measures $\gamma \in \Gamma(\mu_1, \mu_2)$ attaining the minimum in the definition of W_p. It follows from (AGS05, Prop. 7.1.5) that the metric space $(\mathscr{P}_p(\mathbb{R}^d), W_p)$ is complete, so that (A1) is fulfilled. In this setting,

$$\sigma \text{ is the topology of narrow convergence.} \tag{7.51}$$

We recall that a sequence $(\mu_n) \subset \mathscr{P}(\mathbb{R}^d)$ narrowly converges to some $\mu \in \mathscr{P}(\mathbb{R}^d)$ if for all $f \in \mathrm{C}^0_{\mathrm{b}}(\mathbb{R}^d)$ (the space of continuous and bounded functions on $\mathbb{R}^d$) there holds

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) \, \mathrm{d}\mu_n(x) = \int_{\mathbb{R}^d} f(x) \, \mathrm{d}\mu(x) \, .$$

The narrow topology complies with (A2a) thanks to (AGS05, Lemma 7.1.4), while property (A2b) ensues from (AGS05, Rem. 5.1.1).

A remarkable property of absolutely continuous curves with values in $\mathscr{P}_p(\mathbb{R}^d)$ is that they can be characterized as solutions of the *continuity equation.* More precisely (see (AGS05, Thm. 8.3.1)), with any $\mu \in \mathrm{AC}^p([0,T];\mathscr{P}_p(\mathbb{R}^d))$ we can associate a Borel vector field $v : (x,t) \in \mathbb{R}^d \times (0,T) \mapsto v_t(x) \in \mathbb{R}^d$ such that for a.a. $t \in (0,T)$

$$\begin{aligned} v_t \in L^p(\mu_t;\mathbb{R}^d), \quad \text{and} \quad |\mu'|(t) &= \|v_t\|_{L^p(\mu_t;\mathbb{R}^d)} \\ &= \left(\int_{\mathbb{R}^d} |v_t(x)|^p \,\mathrm{d}\mu_t(x)\right)^{1/p}, \end{aligned} \tag{7.52}$$

and μ_t, v_t fulfill the continuity equation

$$\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \tag{7.53}$$

in the sense of distributions. In fact, the velocity field v_t turns out to be unique in some suitable sense, see (AGS05, Chap. 8).

Finally, we recall that it is possible to introduce a subdifferential notion intrinsic to the Wasserstein framework. In fact, for the sake of simplicity we shall not recall (AGS05, Def. 10.3.1) in its general form, but in a particular case.

Definition 7.1. *Let $\phi : \mathscr{P}_p(\mathbb{R}^d) \to (-\infty,+\infty]$ be a proper and lower semicontinuous functional and let $\mu \in D(\phi)$. Given a Borel vector field $\xi : \mathbb{R}^d \to \mathbb{R}^d$, we say that ξ belongs to the* (extended) Fréchet subdifferential *of ϕ at μ, and write $\xi \in \boldsymbol{\partial}\phi(\mu)$, if*

$$\begin{gathered} \xi \in L^{p'}(\mu;\mathbb{R}^d), \qquad \textit{and, as } \nu \to \mu \textit{ in } \mathscr{P}_p(\mathbb{R}^d), \\ \phi(\nu) - \phi(\mu) \geq \inf_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \xi(x), y-x \rangle \,\mathrm{d}\gamma(x,y) + \mathrm{o}\left(W_p(\mu,\nu)\right). \end{gathered}$$

In agreement with notation (6.17), we define

$$\begin{cases} \|\boldsymbol{\partial}^\circ \phi(\mu)\|_{L^{p'}(\mu;\mathbb{R}^d)} := \inf\{\|\xi\|_{L^{p'}(\mu;\mathbb{R}^d)} \ : \ \xi \in \boldsymbol{\partial}\phi(\mu)\}, \\ \boldsymbol{\partial}^\circ \phi(\mu) := \mathrm{Argmin}\{\|\xi\|_{L^{p'}(\mu;\mathbb{R}^d)} \ : \ \xi \in \boldsymbol{\partial}\phi(\mu)\}. \end{cases}$$

Gradient flows in Wasserstein spaces. Hereafter, we shall focus on proper and lower semicontinuous functionals $\phi : \mathscr{P}_p(\mathbb{R}^d) \to (-\infty, +\infty]$, fulfilling some coercivity condition which we do not specify, for it is implied by our general lower semicontinuity/compactness conditions (A3)–(A4), and such that ϕ is *regular*, i.e. for all $(\mu_n) \subset \mathscr{P}_p(\mathbb{R}^d)$, with $\varphi_n = \phi(\mu_n)$ and $\xi_n \in \boldsymbol{\partial}\phi(\mu_n)$, there holds

$$\begin{cases} \mu_n \to \mu \text{ in } \mathscr{P}_p(\mathbb{R}^d), \\ \varphi_n \to \varphi, \\ \sup_n \|\xi_n\|_{L^{p'}(\mu_n;\mathbb{R}^d)} < +\infty, \\ (i \times \xi_n)_{\#}\mu_n \to (i \times \xi)_{\#}\mu \\ \qquad \text{narrowly in } \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d), \end{cases} \Longrightarrow \begin{cases} \xi \in \boldsymbol{\partial}\phi(\mu), \\ \varphi = \phi(\mu)\,, \end{cases} \tag{7.54}$$

($i : \mathbb{R}^d \to \mathbb{R}^d$ denoting the identity map). Notice that (7.54) is in the same spirit as the conditional continuity (CONT).

Using this (extended) Fréchet subdifferential notion, in (AGS05, Chap. 11) a formulation of gradient flows intrinsic to the Wasserstein framework has been proposed: according to (AGS05, Def. 11.1.1), we say that a curve $\mu \in \mathrm{AC}^p([0,T]; \mathscr{P}_p(\mathbb{R}^d))$ is a solution of the p-gradient flow equation driven by ϕ if its velocity field v_t (7.52)–(7.53) satisfies the inclusion

$$\mathrm{J}_p(v_t) \in -\boldsymbol{\partial}\phi(\mu_t) \qquad \text{for a.a. } t \in (0,T)\,, \tag{7.55}$$

(where $\mathrm{J}_p : L^p(\mu_t;\mathbb{R}^d) \to L^{p'}(\mu_t;\mathbb{R}^d)$ denotes the duality map) and complies with the *minimal section principle*

$$\mathrm{J}_p(v_t) \in -\boldsymbol{\partial}\phi^\circ(\mu_t) \qquad \text{for a.a. } t \in (0,T)\,. \tag{7.56}$$

Under the assumptions that ϕ is lower semicontinuous, coercive, and regular in the sense of (7.54), (AGS05, Thm. 11.1.3) provides the crucial link between curves of maximal slope and gradient flows in Wasserstein spaces: it states that

$\mu \in \mathrm{AC}^p([0,T]; \mathscr{P}_p(\mathbb{R}^d))$ is a p-curve of maximal slope w.r.t.
the local slope $|\partial\phi|$ if and only if
μ_t is a solution of the gradient flow equation (7.56)
and the map $t \mapsto \phi(\mu_t)$ is a.e. equal to a function of bounded variation.

Relying on this result and on our Theorem 4.7, we shall prove the existence of the global attractor for the solutions of the gradient flow equation (7.55), driven by a functional ϕ which is the sum of the internal, potential, and interaction energy, cf. with (7.60) below.

7.2. The sum of the internal, potential, and interaction energy

As in (AGS05, Sec. 10.4.7) (cf. also (CmCV03; CmCV06)), we consider the following functions

$$\begin{gathered} V:\mathbb{R}^d \to (-\infty,+\infty] \ \text{ proper, lower semicontinuous, such that} \\ \text{its proper domain } D(V) \text{ has a non-empty interior } O\subset\mathbb{R}^d, \end{gathered} \tag{V1}$$

$$\begin{gathered} F:[0,+\infty)\to\mathbb{R} \ \text{ convex, differentiable, with } F(0)=0, \\ \text{and satisfying the } \textit{doubling condition} \\ \exists\, C_F>0 \ \ \forall\, z,w\in[0,+\infty): \quad F(z+w)\leq C_F\left(1+F(z)+F(w)\right), \end{gathered} \tag{F1}$$

$$\begin{gathered} W:\mathbb{R}^d\to[0,+\infty) \ \text{ convex, Gâteaux differentiable, even,} \\ \text{and satisfying the } \textit{doubling condition} \\ \exists\, C_W>0 \ \ \forall\, x,y\in\mathbb{R}^d: \quad W(x+y)\leq C_W\left(1+W(x)+W(y)\right), \end{gathered} \tag{W1}$$

which induce the following functionals on $\mathscr{P}_p(\mathbb{R}^d)$:

$$\mathcal{V}(\mu):=\int_{\mathbb{R}^d} V(x)\,\mathrm{d}\mu(x) \qquad \text{(potential energy)}, \tag{7.57}$$

$$\mathcal{F}(\mu):=\begin{cases}\displaystyle\int_{\mathbb{R}^d} F(\rho(x))\,\mathrm{d}x & \text{if } \mu=\rho\mathscr{L}^d,\\ +\infty & \text{otherwise,}\end{cases} \qquad \text{(internal energy)}, \tag{7.58}$$

$$\mathcal{W}(\mu):=\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} W(x,y)\,\mathrm{d}(\mu\otimes\mu)(x,y) \qquad \text{(interaction energy)}. \tag{7.59}$$

Hence, for given constants $c_1\in(0,+\infty)$ and c_2, $c_3\in[0,+\infty)$ we define $\phi:\mathscr{P}_p(\mathbb{R}^d)\to(-\infty,+\infty]$ by

$$\phi(\mu):=c_1\mathcal{V}(\mu)+c_2\mathcal{F}(\mu)+c_3\mathcal{W}(\mu)\,. \tag{7.60}$$

We further assume that the function V is p-coercive, i.e.

$$\limsup_{|x|\to+\infty}\frac{V(x)}{|x|^p}=+\infty\,, \tag{V2}$$

and that (cf. with condition (Agu03, (2)))

$$V \text{ is } (\lambda, p)\text{-convex on } \mathbb{R}^d \text{ for some } \lambda \in \mathbb{R}. \tag{V3}$$

Notice that (V3) yields that V is Gâteaux-differentiable in O. As for F, we require that

$$\begin{gathered} F \text{ has a superlinear growth at infinity, i.e. } \limsup_{s \to +\infty} \frac{F(s)}{s} = +\infty, \text{ and} \\ \liminf_{s \downarrow 0} \frac{F(s)}{s^\alpha} > -\infty \text{ for some } \alpha > \frac{d}{d+p}, \end{gathered} \tag{F2}$$

and finally that

$$\text{the map } s \mapsto s^d F(s^{-d}) \text{ is convex and non-increasing in } (0, +\infty). \tag{F3}$$

We associate with F the function $L_F : [0, +\infty) \to [0, +\infty)$ given by

$$L_F(r) = rF'(r) - F(r) \qquad \text{for all } r \geq 0.$$

Remark 7.2. We point out that (F2) yields

$$\exists\, C_1, C_2 \geq 0 \; \forall\, r \geq 0 : \quad F(r) \geq -C_1 - C_2 r^\alpha, \tag{7.61}$$

(F^- being the negative part of F), and refer to (AGS05, Chap. 9) for further details on the above conditions. We shall just mention (see (AGS05, Rem. 9.3.10)) that examples of functionals complying with (F1, F2, F3) are

$$\begin{gathered} \text{the entropy functional } \; F(s) = s \log(s), \\ \text{the power functional } \; F(s) = \frac{1}{m-1} s^m, \quad \text{for } m > 1. \end{gathered} \tag{7.62}$$

(in fact, the case $1 - 1/d \leq m < 1$ can also be treated, see (AGS05, Rem. 9.3.10)).

Remark 7.3 **((λ, p)-geodesic convexity of ϕ).** It follows from (AGS05, Props. 9.3.5, 9.3.9) that, thanks to (W1) and (F3), the functionals $\mathcal{W}$ and $\mathcal{F}$ are convex along geodesics in $\mathscr{P}_p(\mathbb{R}^d)$. On the other hand, arguing as in the proof of (AGS05, Prop. 9.3.2), one verifies that the potential energy $\mathcal{V}$ is (λ, p)-geodesically convex in $\mathscr{P}_p(\mathbb{R}^d)$. We thus conclude that

$$\phi \text{ from (7.60) is } (\lambda, p)\text{-geodesically convex in } \mathscr{P}_p(\mathbb{R}^d).$$

The following proposition collects the main properties of ϕ. We shall just sketch its proof for the reader's convenience.

Proposition 7.4. *In the setting of* (7.49)–(7.51), *assume* (V1)–(F3). *Then, the functional* $\phi : \mathscr{P}_p(\mathbb{R}^d) \to (-\infty, +\infty]$ *defined by* (7.60) *fulfills* (A3)–(A4) *and* (A6)–(A7); ϕ *is regular in the sense of* (7.54), *there holds*

$$|\partial\phi|(\mu) = |\partial^+\phi|(\mu) \qquad \text{for all } \mu \in D(\phi), \tag{7.63}$$

and the (extended) Fréchet subdifferential $\boldsymbol{\partial}\phi$ *of* $\phi(\mu)$ *admits the following characterization at every* $\mu \in D(\phi)$*:*

$$\begin{gathered}\xi \in \boldsymbol{\partial}\phi(\mu) \quad \text{if and only if} \quad \xi \in L^{p'}(\mu;\mathbb{R}^d) \quad \text{and for all } \nu \in \mathscr{P}_p(\mathbb{R}^d) \\ \phi(\nu) - \phi(\mu) \geq \inf_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \xi(x), y - x \rangle \, \mathrm{d}\gamma(x,y) + \frac{\lambda}{p} W_p^p(\mu,\nu)\,.\end{gathered} \tag{7.64}$$

Furthermore, if $\mu \in \mathrm{AC}^p_{\mathrm{loc}}([0,+\infty);\mathscr{P}_p(\mathbb{R}^d))$ *is an energy solution (in the sense of Definition* 2.2*), then there exists* $\rho : t \in [0,+\infty) \mapsto \rho_t \in L^1(\mathbb{R}^d)$ *such that*

$$\begin{gathered}\int_{\mathbb{R}^d} \rho_t(x)\,\mathrm{d}x = 1, \quad \int_{\mathbb{R}^d} |x|^p \rho_t(x)\,\mathrm{d}x < +\infty, \quad \mu_t = \rho_t \mathscr{L}^d \ \text{ for all } t \in [0,+\infty), \\ L_F(\rho) \in L^1_{\mathrm{loc}}(0,+\infty; W^{1,1}_{\mathrm{loc}}(O)), \\ \text{the map } t \mapsto \left\| c_1 \nabla V + c_2 \frac{\nabla L_F(\rho_t)}{\rho_t} + c_3 \nabla W * \rho_t \right\|_{L^{p'}(\mu_t;\mathbb{R}^d)} \in L^p_{\mathrm{loc}}(0,+\infty)\,,\end{gathered}$$

(where $*$ *denotes the convolution product), satisfying the drift-diffusion equation with nonlocal term*

$$\partial_t \rho_t - \mathrm{div}\left(\rho_t j_{p'} \left(c_1 \nabla V + c_2 \frac{\nabla L_F(\rho_t)}{\rho_t} + c_3 \nabla W * \rho_t \right) \right) = 0 \tag{7.65}$$

in $\mathscr{D}'(\mathbb{R}^d \times (0,+\infty))$ $(j_{p'}(r) := |r|^{p'-2} r)$*, and the energy identity for all* $0 \leq s \leq t$

$$\int_s^t \left\| c_1 \nabla V + c_2 \frac{\nabla L_F(\rho_r)}{\rho_r} + c_3 \nabla W * \rho_r \right\|^p_{L^{p'}(\mu_r;\mathbb{R}^d)} \mathrm{d}r + \phi(\mu_t) = \phi(\mu_s)\,. \tag{7.66}$$

Proof. It follows from (AGS05, Lemma 5.1.7, Example 9.3.1, Rmk. 9.3.8) that the functional ϕ is sequentially lower semicontinuous w.r.t. the narrow convergence. In order to prove (A4) in the case $c_2 > 0$ (the proof for $c_2 = 0$ being analogous), let us consider a sequence $\{\mu_n = \rho_n \mathscr{L}_d\} \subset \mathscr{P}_p(\mathbb{R}^d)$ fulfilling $\sup_n \phi(\mu_n) < +\infty$. Using that W takes nonnegative values and the coercivity (V2) of V, one finds that

$$\sup_n \left(\int_{\mathbb{R}^d} c_2 F(\rho_n(x))\,\mathrm{d}x + M_1 \int_{\mathbb{R}^d} |x|^p \rho_n(x)\,\mathrm{d}x \right) < +\infty\,, \tag{7.67}$$

for some $M_1 > 0$. On the other hand, being $(\alpha p)/(1-\alpha) > d$ by (F2), one has

$$K := \int_{\mathbb{R}^d} (1+|x|)^{-\frac{\alpha p}{1-\alpha}}\,\mathrm{d}x < +\infty,$$

and Hölder's inequality yields

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_n^\alpha(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \rho_n^\alpha(x)(1+|x|)^{\alpha p}(1+|x|)^{-\alpha p}\,\mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^d} \rho_n(x)(1+|x|)^p\,\mathrm{d}x \right)^\alpha K^{1-\alpha}.\end{aligned}$$

Therefore, from (7.61) we infer that

$$\begin{aligned}\int_{\mathbb{R}^d} F(\rho_n(x))\,\mathrm{d}x &\ge -C_1 - C_2 \int_{\mathbb{R}^d} \rho_n^\alpha(x)\,\mathrm{d}x \\ &\ge -C_1 - C_2 K^{1-\alpha} 2^{\alpha(p-1)} \left(1 + \int_{\mathbb{R}^d} \rho_n(x)|x|^p\,\mathrm{d}x \right)^\alpha \\ &\ge -C_1 - C_{M_1} \left(1 + \int_{\mathbb{R}^d} \rho_n(x)|x|^p\,\mathrm{d}x \right) - C,\end{aligned} \tag{7.68}$$

the last passage following from a trivial application of the Young inequality with a suitable constant $C_{M_1} > 0$ depending on M_1 and to be specified later. Combining (7.67) and (7.68), and choosing C_{M_1} in such a way that $c_2 C_{M_1} \le M_1/2$, we conclude that

$$\sup_n \left(\frac{M_1}{2} \int_{\mathbb{R}^d} |x|^p \rho_n(x)\,\mathrm{d}x \right) < +\infty. \tag{7.69}$$

Thus, the integral condition for tightness is satisfied and, by the Prokhorov theorem, $\{\mu_n\}$ admits a narrowly converging subsequence. Furthermore, it follows from (7.69) that for all $\varepsilon > 0$ $\{\mu_n\}$ has uniformly integrable $(p-\varepsilon)$-moments so that, by (AGS05, Prop. 7.1.5), it is $W_{p-\varepsilon}$-relatively compact.

It follows from Remark 7.3 and from Proposition 2.7 that the local slope $|\partial\phi|$ is a strong upper gradient. Furthermore, arguing as in the proof of (AGS05, Lemma 10.3.8) one easily checks that, by (λ, p)-geodesic convexity, the functional ϕ is also regular, and (7.64) can be shown by repeating the calculations of (AGS05, Thm. 10.3.6). Property (7.63) is proved in (AGS05, Prop. 10.4.14). In view of (AGS05, Thms. 10.3.11, 10.4.13), we have that for a given measure $\mu \in \mathscr{P}_p(\mathbb{R}^d)$

$$\begin{aligned} |\partial\phi|(\mu) = |\partial^+\phi|(\mu) < +\infty \text{ if and only if } L_F(\rho) \in W^{1,1}_{\text{loc}}(O), \\ \text{there exists a unique } w \in L^{p'}(\mu;\mathbb{R}^d) \text{ s.t.} \\ \|w\|_{L^{p'}(\mu;\mathbb{R}^d)} = \|\boldsymbol{\partial}^\circ\phi(\mu)\|_{L^{p'}(\mu;\mathbb{R}^d)} = |\partial\phi|(\mu), \end{aligned} \tag{7.70}$$

and there holds

$$\rho w = c_1\rho\nabla V + c_2\nabla L_F(\rho) + c_3\rho(\nabla W) * \rho \qquad \mu - \text{a.e. in } \mathbb{R}^d. \tag{7.71}$$

Thus, the conditional continuity (CONT) follows from combining (7.70) with the regularity property (7.54).

Finally, let $\mu \in \mathrm{AC}^p([0,T];\mathscr{P}_p(\mathbb{R}^d))$ be a p-curve of maximal slope: collecting (7.56), (7.70), and (7.71), we conclude that its velocity field satisfies for a.a. $t \in (0,T)$

$$-j_p(v_t) = c_1\nabla V + c_2\frac{\nabla L_F(\rho_t)}{\rho_t} + c_3\nabla W * \rho_t \qquad \mu_t - \text{a.e. in } \mathbb{R}^d\,.$$

Joint with the continuity equation (7.53), the latter relation yields (7.65). Then, the energy identity (7.66) ensues from (7.52) and (2.9). □

Remark 7.5 (An alternative convexity assumption)**.** Instead of the (λ, p)-convexity condition (V3) for V, one could impose the standard λ-convexity on V, with λ having a different sign depending on p, namely

$$V \ \lambda\text{-convex on } \mathbb{R}^d, \text{ with } \lambda \geq 0 \text{ if } p \leq 2 \text{ or } \lambda \leq 0 \text{ if } p \geq 2. \tag{V3'}$$

It is proved in (AGS05, Prop. 9.3.2) that, under (V3'), the functional $\mathcal{V}$ is $(\lambda, 2)$-geodesically convex on $\mathscr{P}(\mathbb{R}^d)$, and so is ϕ given by (7.60). It is not difficult to verify that the proof of Proposition 7.4 goes through upon replacing (V3) with (V3').

Remark 7.6 (Reduction to the (λ, p)-geodesically convex case, with $\lambda \leq 0$). In view of Proposition 7.4 and Theorem 4.8, if ϕ from (7.60) is (λ, p)-geodesically convex with $\lambda > 0$ (which is implied by V being (λ, p)-convex with $\lambda > 0$), then the global attractor for the semiflow of the energy solutions (of the gradient flow associated with ϕ), reduces to the unique minimum point of ϕ, to which all trajectories converge as $t \to +\infty$ with an exponential rate.

Therefore, henceforth we shall focus on the case in which

$$V \text{ is } (\lambda, p)\text{-convex on } \mathbb{R}^d, \text{ for some } \lambda \leq 0.$$

The following result provides the last ingredient for the proof of the existence of the global attractor for the drift-diffusion equation (7.65).

Lemma 7.7. *In the setting of* (7.49)–(7.51)*, assume* (V1)–(F3)*, and that* (V3) *holds with* $\lambda \leq 0$*. Then, the set of the rest points*

$$\Sigma(\phi) = \left\{\bar{\mu} \in \mathscr{P}_p(\mathbb{R}^d) \,:\, 0 \in \boldsymbol{\partial}\phi(\bar{\mu})\right\}$$
$$= \left\{ \bar{\mu} \in \mathscr{P}_p(\mathbb{R}^d) \,:\, \begin{array}{l} \bar{\mu} = \bar{\rho}\mathscr{L}_d, \ L_F(\bar{\rho}) \in W^{1,1}_{\mathrm{loc}}(O), \\ c_1 \nabla V + c_2 \dfrac{\nabla L_F(\bar{\rho})}{\bar{\rho}} + c_3 \nabla W * \bar{\rho} = 0 \quad \bar{\mu} - a.e. \ in\ \mathbb{R}^d \end{array} \right\}$$

is bounded in the phase space $(D(\phi), d_\phi)$*,* d_ϕ *being as in* (4.6).

Proof. We suppose that $c_2 > 0$, referring to the following Example 7.9 for some ideas on the case $c_2 = 0$. Due to (7.64), for every $\bar{\mu} = \bar{\rho}\mathscr{L}_d \in \Sigma(\phi)$ there holds in particular

$$\phi(\tilde{\mu}) - \phi(\bar{\mu}) \geq \frac{\lambda}{p} W_p^p(\tilde{\mu}, \bar{\mu}) \qquad (7.72)$$
for all $\tilde{\mu} = \tilde{\rho}\mathscr{L}_d \in \mathscr{P}_p(\mathbb{R}^d)$, with $\tilde{\rho} \in \mathrm{C}^\infty(\mathbb{R}^d)$ compactly supported.

Let $\mathrm{t} : \mathbb{R}^d \to \mathbb{R}^d$ be the unique (by (AGS05, Thm. 6.2.4)) optimal transport map between $\bar{\mu}$ and $\tilde{\mu}$, so that

$$W_p^p(\tilde{\mu}, \bar{\mu}) = \int_{\mathbb{R}^d} |\mathrm{t}(x) - x|^p \,\mathrm{d}\bar{\mu}(x) \leq 2^{p-1} \int_{\mathbb{R}^d} |\mathrm{t}(x)|^p \,\mathrm{d}\bar{\mu}(x) + 2^{p-1} \int_{\mathbb{R}^d} |x|^p \,\mathrm{d}\bar{\mu}(x)\,.$$

Hence, it follows from (7.72) (recalling that $\lambda \leq 0$ and that $\tilde{\mu}$ is compactly supported) that

$$\begin{aligned} &\phi(\bar{\mu}) + \frac{\lambda 2^{p-1}}{p} \int_{\mathbb{R}^d} |x|^p \,\mathrm{d}\bar{\mu}(x) \\ &\leq \phi(\tilde{\mu}) - \frac{\lambda 2^{p-1}}{p} \int_{\mathbb{R}^d} |\mathrm{t}(x)|^p \,\mathrm{d}\bar{\mu}(x) \leq \phi(\tilde{\mu}) + C \,, \end{aligned} \tag{7.73}$$

Exploiting now (V2) and (F2) and repeating the calculations developed throughout (7.67)–(7.69) in the proof of Proposition 7.4, it is not difficult to infer from (7.73) that

$$\exists \overline{C} > 0 \ \ \forall \bar{\mu} \in \Sigma(\phi) \ : \ \begin{cases} \phi(\bar{\mu}) \leq \overline{C} \,, \\ W_p^p(\delta_0, \bar{\mu}) \leq \int_{\mathbb{R}^d} |x|^p \,\mathrm{d}\bar{\mu}(x) \leq \overline{C} \,, \end{cases} \tag{7.74}$$

δ_0 denoting the Dirac mass centered at 0, which concludes the proof. □

On behalf of Theorem 4.7, Proposition 7.4, and Lemma 7.7, we thus conclude

Theorem 7.8. *In the setting of* (7.49)–(7.51), *assume* (V1)–(F3), *and that* (V3) *holds with* $\lambda \leq 0$.

Then, the generalized semiflow generated by the energy solutions of the gradient flow driven by ϕ *from* (7.60) *admits a global attractor.*

Example 7.9. In the case $c_2 = c_3 = 0$, the set of the rest points $\Sigma(\phi)$ consists of the measures $\bar{\mu} \in \mathscr{P}_p(\mathbb{R}^d)$ satisfying

$$\bar{\mu}(\mathbb{R}^d \setminus S) = 0, \quad \text{with} \qquad S = \left\{ x \in \mathbb{R}^d \ : \ \ \nabla V(x) = 0 \right\} . \tag{7.75}$$

The boundedness of $\Sigma(\phi)$ follows from (V1), (V2), and (V3) (with $\lambda \leq 0$). Indeed, the latter condition and (7.75) yield that for every fixed $\bar{y} \in O$ there holds

$$V(\bar{y}) \geq V(x) + \frac{\lambda}{p} |\bar{y} - x|^p \qquad \text{for } \bar{\mu} - \text{a.a. } x \in \mathbb{R}^d \,,$$

whence, also using (V2), we find that for all $M_2 > 0$ there exists $C_{M_2} > 0$ with

$$\begin{aligned} V(\bar{y}) - \frac{\lambda 2^{p-1}}{p} |\bar{y}|^p &\geq V(x) + \frac{\lambda 2^{p-1}}{p} |x|^p \\ &\geq \frac{V(x)}{2} + \frac{1}{2} \left(M_2 + \frac{\lambda 2^p}{p} \right) |x|^p - \frac{C_{M_2}}{2} \end{aligned}$$

for $\bar{\mu}$-a.a. $x \in \mathbb{R}^d$. Choosing $M_2 > -\lambda 2^p/p$ and taking into account that V is bounded from below, one immediately deduces (7.74). An analogous argument can be developed in the case V is λ-convex in the standard sense.

Example 7.10. In the case $c_2 \neq 0$ and $c_3 = 0$, V fulfills (V1), (V2), and (V3) (with $\lambda \leq 0$), and F is either the entropy or the power functional (cf. with (7.62)), then $\Sigma(\phi)$ reduces to a singleton. In fact, when $F(s) = s\log(s)$ the stationary equation defining the set of the rest points of ϕ becomes

$$c_1 \nabla V + c_2 \frac{\nabla \bar{\rho}}{\bar{\rho}} = 0 \qquad \bar{\mu} = \bar{\rho}\mathscr{L}^d\text{-a.e. in } \mathbb{R}^d,$$

leading to

$$\bar{\mu} = \frac{1}{Z} e^{-V} \mathscr{L}^d \,,$$

(where $Z = \int_{\mathbb{R}^d} e^{-V(x)}\,\mathrm{d}x$). Similar calculations may be developed in the case of the power functional.

8. Application to phase transition evolutions driven by mean curvature

Throughout this section, we shall assume that

$$\Omega \text{ is a } \mathrm{C}^1\text{, connected, and open set,} \tag{8.76}$$

and take as ambient space

$$U = H^{-1}(\Omega), \quad \sigma \text{ being the norm topology} \tag{8.77}$$

8.1. The Stefan-Gibbs-Thomson problem

It is straightforward to check that, in the setting of (8.77), the functional $\phi_{\mathrm{Ste}} : H^{-1}(\Omega) \to [0, +\infty]$

$$\phi_{\mathrm{Ste}}(u) := \begin{cases} \inf_{\chi \in \mathrm{BV}(\Omega)} \left\{ \int_\Omega \left(\frac{1}{2}|u-\chi|^2 + I_{\{-1,1\}}(\chi) \right) dx + \int_\Omega |D\chi| \right\} \\ \qquad\qquad \text{if } u \in L^2(\Omega), \\ +\infty \quad \text{otherwise} \end{cases}$$

complies with (A3)–(A4). Given $u \in D(\phi_{\mathrm{Ste}}) = L^2(\Omega)$, we denote by $\mathcal{M}(u)$ the non-empty set of the χ's in $\mathrm{BV}(\Omega; \{-1, 1\})$ attaining the minimum in the definition of $\phi_{\rm Ste}$, and by $\partial\phi_{\rm Ste}$ its Fréchet subdifferential with respect to the topology of $H^{-1}(\Omega)$ (which we identify with its dual). Therefore, the limiting subdifferential $\partial_\ell\phi_{\rm Ste}$ is given at every $u \in L^2(\Omega)$ by

$$\partial_\ell\phi_{\rm Ste}(u) = \left\{ \xi \in H^{-1}(\Omega) \ : \ \begin{array}{l} \exists\, u_n \in H^{-1}(\Omega),\ \xi_n \in H^{-1}(\Omega) \text{ with} \\ \xi_n \in \partial\phi_{\rm Ste}(u_n) \text{ for all } n \in \mathbb{N}, \\ u_n \to u \text{ in } H^{-1}(\Omega),\ \xi_n \rightharpoonup \xi \text{ in } H^{-1}(\Omega), \\ \sup_n \phi_{\rm Ste}(u_n) < +\infty \end{array} \right\}.$$

It was proved (cf. with (RoSa06, Thm. 2.5)) that, for every initial datum $u_0 \in L^2(\Omega)$, every $u \in \mathrm{GMM}(\phi_{\rm Ste}, u_0)$ (which is a non-empty set thanks to Theorem 4.2) fulfills the Cauchy problem for the gradient flow equation

$$u'(t) + \partial_\ell\phi_{\rm Ste}(u(t)) \ni 0 \qquad \text{in } H^{-1}(\Omega) \quad \text{for a.a. } t \in (0,T)\,.$$

The following result sheds some light on the properties of $\partial_\ell\phi_{\rm Ste}$.

Proposition 8.1. *The functional $\phi_{\rm Ste}$ fulfills the continuity property* (CONT) *and*

$$\text{for all } u \in L^2(\Omega) \text{ there exists a unique } \bar\chi_u \in \mathrm{BV}(\Omega) \text{ such that} \\ \bar\chi_u \in \mathcal{M}(u) \text{ and } u - \bar\chi_u \in H^1_0(\Omega)\,. \tag{8.78}$$

Further, let $u \in L^2(\Omega)$ fulfill $|\partial^+\phi_{\rm Ste}|(u) < +\infty$. Then, $\partial_\ell\phi_{\rm Ste}(u)$ is non-empty and, more precisely,

$$\partial_\ell\phi_{\rm Ste}(u) = \{A(u - \bar\chi_u)\}, \tag{8.79}$$

(A being the Laplace operator with homogeneous Dirichlet boundary conditions) with $\vartheta := u - \bar\chi_u$ fulfilling the weak form of the Gibbs-Thomson law, i.e. for all $\zeta \in C^2(\overline{\Omega};\mathbb{R}^d)$ with $\zeta\cdot\mathbf{n} = 0$ on $\partial\Omega$ there holds

$$\int_\Omega \big(\operatorname{div}\zeta - \nu^T D\zeta\,\nu\big)\, d|D\bar\chi_u| = \int_\Omega \operatorname{div}(\vartheta\zeta)\bar\chi_u \,\mathrm{d}x, \tag{8.80}$$

where the Radon-Nikodým derivative $\nu = \dfrac{d(D\bar\chi_u)}{d|D\bar\chi_u|}$ is the measure-theoretic inner normal to the boundary of the phase $\{x \in \Omega : \bar\chi_u(x) = 1\}$. Finally, there holds

$$|\partial^+\phi_{\rm Ste}|(u) \ge \|A(u - \bar\chi_u)\|_{H^{-1}(\Omega)} = \|\nabla(u - \bar\chi_u)\|_{L^2(\Omega)}\,. \tag{8.81}$$

Remark 8.2. In view of the conditional continuity (CONT) and of Proposition 2.14, for any generalized solution (u, φ) of the metric gradient flow of ϕ_{Ste} there holds $\varphi = \phi \circ u$. Hence we shall simply refer to any generalized solution (u, φ) as u.

Proof. Property (8.78) (which is somehow underlying some of the arguments in (Luc90)) is explicitly proved in (RoSa06, Sec. 5.2). It is shown in (RoSa06, Prop. 5.3) that, if $\partial\phi_{\mathrm{Ste}}(u) \neq \emptyset$, then $\partial\phi_{\mathrm{Ste}}(u) = \{A(u - \bar{\chi}_u)\}$. It is straightforward to prove that there also holds

$$|\partial\phi_{\mathrm{Ste}}|(u) \leq \|A(u - \bar{\chi}_u)\|_{H^{-1}(\Omega)}. \tag{8.82}$$

On the other hand, we fix $w \in L^2(\Omega)$ and notice that

$$\begin{aligned}
\Lambda(u, w) &:= \limsup_{h\downarrow 0} \frac{(\phi_{\mathrm{Ste}}(u) - \phi_{\mathrm{Ste}}(u + hw))^+}{h\|w\|_{H^{-1}(\Omega)}} \\
&\geq \limsup_{h\downarrow 0} \frac{\left(\frac{1}{2}\|u-\bar{\chi}_u\|^2_{L^2(\Omega)} + \int_\Omega |\nabla\bar{\chi}_u| - \frac{1}{2}\|u+hw-\bar{\chi}_u\|^2_{L^2(\Omega)} - \int_\Omega |\nabla\bar{\chi}_u|\right)^+}{h\|w\|_{H^{-1}(\Omega)}} \\
&= \limsup_{h\downarrow 0} \frac{\left(-\frac{h^2}{2}\|w\|^2_{L^2(\Omega)} + h\int_\Omega w(u - \bar{\chi}_u)\right)^+}{h\|w\|_{H^{-1}(\Omega)}} = \frac{\int_\Omega w(u - \bar{\chi}_u)}{\|w\|_{H^{-1}(\Omega)}}.
\end{aligned}$$

Being $|\partial\phi_{\mathrm{Ste}}|(u) \geq \sup_{w\in L^2(\Omega)} \Lambda(u, w)$, by a density argument we easily conclude that $|\partial\phi_{\mathrm{Ste}}|(u) \geq \|u - \bar{\chi}_u\|_{H^1_0(\Omega)}$, so that, also in view of (8.82), with a slight abuse of notation we may write

$$|\partial\phi_{\mathrm{Ste}}|(u) = \|\partial\phi_{\mathrm{Ste}}(u)\|_{H^{-1}(\Omega)} = \|u - \bar{\chi}_u\|_{H^1_0(\Omega)}. \tag{8.83}$$

Using (8.83) and arguing as in (RoSa06, Prop. 5.3) it is not difficult to check the conditional continuity (CONT), and that (8.79) holds. The latter yields (8.81) through the general inequality (6.23). Finally, the weak Gibbs-Thomson law (8.80) has been proved in (RoSa06, Lemma 5.10). □

Corollary 8.3. *Under the above assumptions, for every* $u \in \mathrm{GMM}(\phi_{\mathrm{Ste}}, u_0)$ *(which is in particular a generalized solution of the metric gradient flow of* ϕ_{Ste}*), and every* $\bar{\chi}_u \in \mathcal{M}(u)$ *the pair* $(\vartheta = u - \bar{\chi}_u, \bar{\chi}_u)$ *is a solution of the weak formulation (3.33, 3.37) of the Stefan-Gibbs-Thomson Problem, fulfilling the Lyapunov inequality* (3.40).

Remark 8.4 (Lyapunov inequality)**.** Indeed, through inequality (8.81), the energy inequality (2.22) for generalized solutions translates into the Lyapunov inequality (3.40). We may observe that, since in this case $|\partial^+\phi_{\mathrm{Ste}}|$ is not an upper gradient (from another viewpoint, the chain rule (6.20) w.r.t. $\partial_\ell\phi_{\mathrm{Ste}}$ does not hold), inequality (3.40) yields the most exhaustive *energetic information* on solutions of the Stefan-Gibbs-Thomson problem.

Existence of the global attractor. We deduce from Theorem 4.6 for generalized solutions the following result, which in particular yields information on the long-time behavior of the *Minimizing Movements solutions* of the Stefan problem with the (weak) Gibbs-Thomson law.

Theorem 8.5. *Under the above assumptions, the semiflow of the generalized solutions of the metric gradient flow driven by the functional ϕ_{Ste} from* (3.38) *admits a global attractor in the phase space $D(\phi_{\mathrm{Ste}}) = L^2(\Omega)$, endowed with the distance $d_{\phi_{\mathrm{Ste}}}$ from* (4.2).

Proof. Since ϕ_{Ste} complies with (A3)–(A4), in order to apply Theorem 4.6 it remains to check that the set of the rest points is bounded in the space $(L^2(\Omega), d_{\phi_{\mathrm{Ste}}})$. Indeed, it follows from the conditional continuity (CONT) (cf. with (4.4)) and from inequality (8.81) that for every rest point $\bar{u}$ of the semiflow generated by ϕ_{Ste} there holds

$$\bar{u} = \bar{\chi}_{\bar{u}} \in \mathcal{M}(\bar{u}).$$

Thus, $|\bar{u}| = 1$ a.e. in Ω, and

$$\phi_{\mathrm{Ste}}(\bar{u}) = \int_\Omega |\nabla\bar{\chi}_{\bar{u}}| \leq \frac{1}{2}\int_\Omega |\bar{u} - 1|^2 \leq 2|\Omega|\,,$$

where the second inequality ensues from choosing $\chi \equiv 1$ in the minimization which defines ϕ_{Ste}. This concludes the proof. □

8.2. Some partial results for the Mullins-Sekerka problem

The functional $\phi_{\mathrm{MS}} : H^{-1}(\Omega) \to [0, +\infty]$

$$\phi_{\mathrm{MS}}(\chi) := \begin{cases} \displaystyle\int_\Omega I_{\{-1,1\}}(\chi)\,\mathrm{d}x + \int_\Omega |D\chi| & \text{if } \chi \in \mathrm{BV}(\Omega; \{-1,1\}), \\ +\infty & \text{if } \chi \in H^{-1}(\Omega) \setminus \mathrm{BV}(\Omega; \{-1,1\}) \end{cases}$$

complies with (A3)–(A4) (cf. with (LuSt95, Lemma 3.1)). On account of Theorem 4.6, the next step towards the existence of the global attractor for the generalized solutions (χ, φ) driven by ϕ_{MS} would be the proof of the boundedness of the rest points set, in the phase space (4.2). In this direction, we present the following result on properties of the local and weak relaxed slopes of ϕ_{MS}, which we believe to have an independent interest.

Proposition 8.6. *In the present setting, for every*

$$\zeta \in C^2(\overline{\Omega};\mathbb{R}^d) \quad \text{with} \quad \zeta \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \tag{8.84}$$

there exists a constant $C_\zeta \geq 0$, depending on $\|\zeta\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}$, such that the following inequalities hold for all $\chi \in D(\phi_{\mathrm{MS}})$ and $\varphi \in \mathbb{R}$ with $\varphi \geq \phi_{\mathrm{MS}}(\chi)$:

$$|\partial\phi_{\mathrm{MS}}|(\chi) \geq C_\zeta \int_\Omega \big(\operatorname{div}\zeta - \nu^T D\zeta\, \nu\big)\, \mathrm{d}|D\chi|\,; \tag{8.85}$$

$$|\partial^-\phi_{\mathrm{MS}}|(\chi,\varphi) \geq C_\zeta \int_\Omega \big(\operatorname{div}\zeta - \nu^T D\zeta\, \nu\big)\, \mathrm{d}|D\chi|\,. \tag{8.86}$$

Proof. We start by proving (8.85) for a fixed field ζ as in (8.84). Arguing as in the proof of (RoSa06, Lemma 5.10), for $s \in \mathbb{R}$ we introduce the flow $\boldsymbol{X}_s : \Omega \to \Omega$ associated with ζ via the ODE system

$$\begin{cases} \frac{d}{ds}\boldsymbol{X}_s(x) = \zeta(\boldsymbol{X}_s(x))\,, \\ \boldsymbol{X}_0(x) = x \end{cases} \qquad \text{for all } (s,x) \in \mathbb{R}\times\Omega. \tag{8.87}$$

Since Ω is regular (cf. with (8.76)) and $\zeta \cdot \mathbf{n} = 0$ on $\partial\Omega$, $\boldsymbol{X}_s(\Omega) = \Omega$ for all $s \in \mathbb{R}$. Further, the map $x \in \Omega \mapsto \boldsymbol{X}_s(x)$ is a C^2 diffeomorphism, with inverse $\boldsymbol{X}_{-s} : \Omega \to \Omega$. Setting

$$\mathsf{D}_s(x) := D_x\boldsymbol{X}_s(x), \quad J_s(x) := \det \mathsf{D}_s(x) \quad \text{for all } x \in \Omega,$$

it is immediate to check that

$$\begin{cases} \frac{d}{ds}J_s(x) = \operatorname{div}\left(\zeta(\boldsymbol{X}_s(x))\right) J_s(x), \\ J_0(x) = 1 \end{cases} \qquad \text{for all } (s,x) \in \mathbb{R}\times\Omega. \tag{8.88}$$

It follows from (8.87) that (recall that $i : \mathbb{R}^d \to \mathbb{R}^d$ denotes the identity map)

$$\|\boldsymbol{X}_s - i\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq |s|\|\zeta\|_{L^\infty(\Omega;\mathbb{R}^d)}\,. \tag{8.89}$$

In the same way, (8.88) and the Gronwall lemma yield for all $s \in [-1,1]$

$$\begin{aligned} &\|J_s\|_{L^\infty(\Omega)} \le \exp(|s|\|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)}),\\ &\|J_s - 1\|_{L^\infty(\Omega)} \le \|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)} \exp(|s|\|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)})\,. \end{aligned} \tag{8.90}$$

Then, for every fixed $\chi \in D(\phi_{\mathrm{MS}})$, we consider its perturbation $\chi_s : \Omega \to \{-1,1\}$, for $s \in \mathbb{R}$, defined by

$$\chi_s(x) := \chi(\boldsymbol{X}_{-s}(x)) \qquad \forall\, x \in \Omega\,.$$

Now, χ_s still belongs to $BV(\Omega;\{-1,1\})$ and it can be verified that $|D\chi_s|$ coincides with the $(m-1)$-dimensional Hausdorff measure restricted to $S_s = \boldsymbol{X}_s(S)$ (S being the essential boundary separating the phases). Therefore, the first variation formula for the area functional (see, e.g., (AFP00, Thm. 7.31)) yields

$$\frac{d}{ds}\Big(\int_\Omega |D\chi_s|\Big)_{s=0} = \int_\Omega \big(\mathrm{div}\,\zeta - \nu^T D\zeta\,\nu\big)\,\mathrm{d}|D\chi|.$$

Hence, we have the following chain of inequalities

$$\begin{aligned} |\partial\phi_{\mathrm{MS}}|(\chi) &= \limsup_{\|v-\chi\|_{H^{-1}(\Omega)}\to 0} \frac{\big(\int_\Omega |D\chi| - \int_\Omega |Dv|\big)^+}{\|v-\chi\|_{H^{-1}(\Omega)}}\\ &\ge \limsup_{s\to 0}\left(\frac{\big(\int_\Omega |D\chi| - \int_\Omega |D\chi_s|\big)^+}{s}\,\frac{s}{\|\chi-\chi_s\|_{H^{-1}(\Omega)}}\right)\\ &\ge \limsup_{s\to 0}\frac{s}{\|\chi-\chi_s\|_{H^{-1}(\Omega)}}\left(\int_\Omega \big(-\mathrm{div}\,\zeta + \nu^T D\zeta\,\nu\big)\,\mathrm{d}|D\chi|\right)\,. \end{aligned}$$

To conclude for (8.85), we are going to show that

$$\limsup_{s\to 0}\frac{s}{\|\chi-\chi_s\|_{H^{-1}(\Omega)}} \ge C_\zeta\,, \tag{8.91}$$

with $C_\zeta > 0$ if ζ is not identically zero, depending on $\|\zeta\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}$. Indeed, to evaluate $\|\chi-\chi_s\|_{H^{-1}(\Omega)}$ we fix $v \in H^1_0(\Omega)$, and for later convenience extend it to a function $\tilde{v} \in H^1(\mathbb{R}^d)$. There holds

$$\begin{aligned}
{}_{H^{-1}(\Omega)}\langle \chi - \chi_s, v\rangle_{H_0^1(\Omega)} &= \int_\Omega (\chi(x) - \chi(\boldsymbol{X}_{-s}(x)))\, v(x)\,\mathrm{d}x \\
&= \int_\Omega \chi(x)\,(v(x) - v(\boldsymbol{X}_s(x))|J_s(x)|)\,\mathrm{d}x \\
&= I_1 + I_2 := \int_\Omega \chi(x)\,(v(x) - v(\boldsymbol{X}_s(x)))\,\mathrm{d}x \\
&\qquad + \int_\Omega \chi(x) v(\boldsymbol{X}_s(x))(J_0(x) - |J_s(x)|)\,\mathrm{d}x\,,
\end{aligned} \tag{8.92}$$

where the second equality follows from the change of variable formula. Then, supposing $|s| \le 1$ and setting $\lambda(\zeta) := \|\zeta\|_{L^\infty(\Omega)}$, we estimate I_1 by means the maximal function $M_{\lambda(\zeta)}$ of v (see Section Appendix C). In view of Lemma Appendix C.2, there holds

$$\|M_{\lambda(\zeta)}(\nabla v)\|^2_{L^2(\Omega;\mathbb{R}^d)} \le C(d,2) \int_{B_{\bar{r}+\lambda(\zeta)}(0)} |\nabla \tilde{v}(x)|^2\,\mathrm{d}x = C(d,2)\|\nabla v\|^2_{L^2(\Omega)} \tag{8.93}$$

where $\bar{r} > 0$ is such that $\Omega \subset B_{\bar{r}}(0)$. Thus, changing variables we have

$$\begin{aligned}
\int_\Omega \left|M_{\lambda(\zeta)}(\nabla v(\boldsymbol{X}_s(x)))\right|^2\,\mathrm{d}x &= \int_\Omega \left|M_{\lambda(\zeta)}(\nabla v(y))\right|^2 |J_{-s}(y)|\,\mathrm{d}y \\
&\le \exp(|s|\|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)}) C(d,2)\|\nabla v\|^2_{L^2(\Omega)}\,,
\end{aligned} \tag{8.94}$$

the latter inequality from the first of (8.90) and (8.93). Using (8.89), Lemma Appendix C.3 and (8.93)–(8.94), we thus have

$$\begin{aligned}
|I_1| &\le C(d) \int_\Omega |\boldsymbol{X}_s(x) - x| \left(M_{\lambda(\zeta)}(\nabla v(x)) + M_{\lambda(\zeta)}(\nabla v(\boldsymbol{X}_s(x)))\right)\mathrm{d}x \\
&\le C(d) C(d,2)^{1/2} \|\boldsymbol{X}_s - i\|_{L^\infty(\Omega;\mathbb{R}^d)} \|\nabla v\|_{L^2(\Omega)} \left(1 + \exp(\frac{1}{2}|s|\|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)})\right) \\
&\le C_d \|\zeta\|_{L^\infty(\Omega)} \left(1 + \exp(\frac{1}{2}|s|\|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)})\right) |s| \|v\|_{H_0^1(\Omega)}\,,
\end{aligned} \tag{8.95}$$

the constant $C_d > 0$ only depending on the dimension. As for I_2 we have

$$\begin{aligned}
|I_2| &\le \|v\|_{L^2(\Omega)} \|J_s - 1\|_{L^\infty(\Omega)} \\
&\le \|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)} \exp(|s|\|\mathrm{div}(\zeta)\|_{L^\infty(\Omega)}) \|v\|_{L^2(\Omega)}\,.
\end{aligned} \tag{8.96}$$

Collecting (8.92) and (8.95)–(8.96), we immediately infer (8.91).

In order to prove (8.86) for all $\chi \in D(\phi_{\rm MS})$ and $\varphi \geq \phi_{\rm MS}(\chi)$, we argue in the following way: according to the definition of $|\partial^- \phi_{\rm MS}|$ we fix a sequence $\{\chi_k\} \subset D(\phi_{\rm MS})$ converging to χ in $H^{-1}(\Omega)$ and such that

$$\big| |\partial\phi_{\rm MS}|(\chi_k) - |\partial^-\phi_{\rm MS}|(\chi, \varphi) \big| \leq \frac{1}{k}, \qquad \left| \int_\Omega |D\chi_k| - \varphi \right| \leq \frac{1}{k}. \tag{8.97}$$

It follows from the latter inequality and (AFP00, Thm. 1.5.9) that $\{D\chi_k\}$ has a weakly* converging subsequence in the sense of measures. Therefore, $D\chi_k \rightharpoonup^* D\chi$ and, thanks to Reshetnyak Theorem (AFP00, Thm. 2.38), there holds

$$\liminf_{k\to\infty} \left(\int_\Omega \big(\operatorname{div}\zeta - \nu^T D\zeta\, \nu \big) \, \mathrm{d}|D\chi_k| \right) \geq \int_\Omega \big(\operatorname{div}\zeta - \nu^T D\zeta\, \nu \big) \, \mathrm{d}|D\chi|$$

Thus, using the first of (8.97) we can pass to the limit in (8.85) as $k \to +\infty$ and conclude (8.86). □

Remark 8.7. As a consequence of Proposition 8.6, for every rest point $(\bar\chi, \bar\varphi)$ of the semiflow of the generalized solutions driven by the functional $\phi_{\rm MS}$ there holds

$$\int_\Omega \big(\operatorname{div}\zeta - \bar\nu^T D\zeta\, \bar\nu \big) \, \mathrm{d}|D\bar\chi| = 0$$
$$\text{for all } \zeta \in C^2(\overline{\Omega}; \mathbb{R}^d) \ \text{ with } \zeta \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \tag{8.98}$$

($\bar\nu = \dfrac{d(D\bar\chi)}{d|D\bar\chi|}$ being the measure-theoretic inner normal to the boundary of the phase $\{\bar\chi \equiv 1\}$). This suggests that, a way to prove that the generalized semiflow associated with $\phi_{\rm MS}$ complies with condition (A5) could be: to deduce from condition (8.98) some universal bound for every rest point $\bar\chi$. However, this presently remains an open problem.

Appendix A. Generalized semiflows

In the following, we shall denote by $(\mathcal{X}, d_{\mathcal{X}})$ a (not necessarily complete) metric space. We recall that the Hausdorff semidistance or *excess* $e_{\mathcal{X}}(A, B)$ of two non-empty subsets $A, B \subset \mathcal{X}$ is given by $e_{\mathcal{X}}(A, B) := \sup_{a\in A} \inf_{b\in B} d_{\mathcal{X}}(a, b)$.

For all $\varepsilon > 0$, we also denote by $B(0,\varepsilon)$ the ball $B(0,\varepsilon) := \{x \in X : d_{\mathcal{X}}(x,0) < \varepsilon\}$, and by $N_\varepsilon(A) := A + B(0,\varepsilon)$ the ε-neighborhood of a subset A.

Definition Appendix A.1 (Generalized semiflow)**.** *A generalized semiflow G on $\mathcal{X}$ is a family of maps $g : [0,+\infty) \to \mathcal{X}$ (referred to as* solutions*), satisfying:*

(Existence)

For any $g_0 \in \mathcal{X}$ there exists at least one $g \in G$ with $g(0) = g_0$. (H1)

(Translates of solutions are solutions)

For any $g \in G$ and $\tau \geq 0$, the map $g^\tau(t) := g(t+\tau)$, $t \in [0,+\infty)$, belongs to G. (H2)

(Concatenation)

For any g, $h \in G$ and $t \geq 0$ with $h(0) = g(t)$, then $z \in G$, z being the map defined by $z(\tau) := g(\tau)$ if $0 \leq \tau \leq t$, and $h(\tau - t)$ if $t < \tau$. (H3)

(Upper semicontinuity with respect to initial data)

If $\{g_n\} \subset G$ and $g_n(0) \to g_0$, then there exist a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in G$ such that $g(0) = g_0$ and $g_{n_k}(t) \to g(t)$ for all $t \geq 0$. (H4)

Furthermore, a generalized semiflow G may fulfill the following measurability *and* continuity *properties:*

(C0) Each $g \in G$ is strongly measurable on $(0,+\infty)$, i.e. there exists a sequence $\{f_j\}$ of measurable countably valued maps converging to g a.e. on $(0,+\infty)$.

(C1) Each $g \in G$ is continuous on $(0,+\infty)$.

(C2) For any $\{g_n\} \subset G$ with $g_n(0) \to g_0$, there exist a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in G$ such that $g(0) = g_0$ and $g_{n_k} \to g$ uniformly in compact subsets of $(0,+\infty)$.

(C3) Each $g \in G$ is continuous on $[0, +\infty)$.

(C4) For any $\{g_n\} \subset G$ with $g_n(0) \to g_0$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in G$ such that $g(0) = g_0$ and $g_{n_k} \to g$ uniformly in compact subsets of $[0, +\infty)$.

$\boldsymbol{\omega}$-limits and attractors. Given a generalized semiflow G on $\mathcal{X}$, we introduce for every $t \geq 0$ the operator $T(t) : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ defined by

$$T(t)E := \{g(t) \ : \ g \in G \ \text{ with } \ g(0) \in E\}, \quad E \subset \mathcal{X}. \tag{A.1}$$

The family of operators $\{T(t)\}_{t \geq 0}$ defines a semigroup on $2^{\mathcal{X}}$, i.e., it fulfills the following property

$$T(t+s)B = T(t)T(s)B \quad \forall s, t \geq 0 \quad \forall B \subset \mathcal{X}.$$

Given a solution $g \in G$, we introduce its *ω-limit* $\omega(g)$ by

$$\omega(g) := \{x \in \mathcal{X} \ : \ \exists \{t_n\}, \ t_n \to +\infty, \text{ such that } \ g(t_n) \to x\}.$$

We say that $w : \mathbb{R} \to \mathcal{X}$ is a *complete orbit* if, for any $s \in \mathbb{R}$, the translate map $w^s \in G$ (cf. (H2)). Finally, the *ω-limit* of E is defined as

$$\omega(E) := \big\{x \in \mathcal{X} \ : \ \exists \{g_n\} \subset G \text{ such that } \{g_n(0)\} \subset E, \\ \{g_n(0)\} \text{ is bounded, and } \ \exists t_n \to +\infty \text{ with } g_n(t_n) \to x\big\}.$$

Given subsets $F, E \subset \mathcal{X}$, we say that F *attracts* E if

$$e_{\mathcal{X}}(T(t)E, F) \to 0 \ \text{ as } \ t \to +\infty.$$

Moreover, we say that F is *positively invariant* if $T(t)F \subset F$ for every $t \geq 0$, that F is *quasi-invariant* if for any $v \in F$ there exists a complete orbit w with $w(0) = v$ and $w(t) \in F$ for all $t \in \mathbb{R}$, and finally that F is *invariant* if $T(t)F = F$ for every $t \geq 0$ (equivalently, if it is both positively and quasi-invariant).

Definition Appendix A.2 (Global Attractor)**.** *Let G be a generalized semiflow. We say that a non-empty set A is a* global attractor *for G if it is compact, invariant, and attracts all bounded sets of $\mathcal{X}$.*

Definition Appendix A.3 (Point dissipative, asymptotically compact)**.** *Let G be a generalized semiflow. We say that G is* point dissipative *if there exists a bounded set $B_0 \subset \mathcal{X}$ such that for any $g \in G$ there exists $\tau \geq 0$ such that $g(t) \in B_0$ for all $t \geq \tau$. Moreover, G is* asymptotically compact *if for any sequence $\{g_n\} \subset G$ such that $\{g_n(0)\}$ is bounded and for any sequence $t_n \to +\infty$, the sequence $\{g_n(t_n)\}$ admits a convergent subsequence.*

Lyapunov function. We say that a complete orbit $g \in G$ is *stationary* if there exists $x \in \mathcal{X}$ such that $g(t) = x$ for all $t \in \mathbb{R}$. Such x is then called a *rest point*. Note that the set of rest points of G, denoted by $Z(G)$, is closed in view of (H4).

Definition Appendix A.4 (Lyapunov function)**.** *$V : \mathcal{X} \to \mathbb{R}$ is a* Lyapunov function *for G if: V is continuous, $V(g(t)) \leq V(g(s))$ for all $g \in G$ and $0 \leq s \leq t$ (i.e., V decreases along solutions), and, whenever the map $t \mapsto V(g(t))$ is constant for some complete orbit g, then g is a stationary orbit.*

We say that a global attractor A for G is *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \subset \mathcal{X}$ with $e_{\mathcal{X}}(E, A) \leq \delta$, then $e_{\mathcal{X}}(T(t)E, A) \leq \varepsilon$ for all $t \geq 0$.

Finally, we recall the following two results, providing necessary and sufficient conditions for a semiflow to admit a global attractor, cf. (Bal97, Thms. 3.3, 5.1, 6.1).

Theorem Appendix A.5. *A generalized semiflow G has a global attractor if and only if it is point dissipative and asymptotically compact. Moreover, the attractor A is unique, it is the maximal compact invariant subset of $\mathcal{X}$, and it can be characterized as*

$$A = \cup\{\omega(B) \ : \ B \subset \mathcal{X}, \ \textit{bounded}\} = \omega(\mathcal{X}).$$

Besides, if G complies with (C1) *and* (C4)*, then A is Lyapunov stable.*

Theorem Appendix A.6. *Assume that G is asymptotically compact, admits a Lyapunov function V, and that the sets of its rest points $Z(G)$ is bounded. Then, G is also point dissipative, and thus admits a global attractor A. Moreover,*

$$\omega(u) \subset Z(G) \quad \textit{for all trajectories } u \in G.$$

Appendix B. The proof of Proposition 6.1 revisited

Preliminarily, we state and prove the following

Lemma Appendix B.1. *Under the assumptions of Proposition 6.1, let z_j^k fulfill*

$$z_j^k \in \operatorname*{Argmin}_{v\in\mathcal{B}} \left\{ \frac{\|v-u_k\|^2}{2r_j^k} + \phi(v) \right\}. \tag{B.1}$$

Then,

$$\mathrm{J}_2\left(\frac{z_j^k - u_k}{r_j^k}\right) + \partial_s\phi(z_j^k) \ni 0. \tag{B.2}$$

Proof. To check (6.30) we notice that, thanks to (Mor06, Lemma 2.32, p. 214),

$$\forall\,\eta>0\ \ \exists\, z_\eta^1 \in \mathcal{B},\ z_\eta^2 \in D(\phi)\ :\quad \begin{cases} \|z_\eta^1 - z_j^k\| + \|z_\eta^2 - z_j^k\| & \le \eta\,, \\ |\|z_\eta^1 - u_k\|^2 - \|z_j^k - u_k\|^2| & \le 2\eta r_j^k\,, \\ |\phi(z_\eta^2) - \phi(z_j^k)| & \le \eta\,, \end{cases}$$

and

$$\exists\, w_\eta \in \mathrm{J}_2\left(\frac{z_\eta^1 - u_k}{r_j^k}\right),\ \exists\, \xi_\eta \in \partial\phi(z_\eta^2),\ \exists\, \zeta_\eta \in \mathcal{B}^*,\ \|\zeta_\eta\|_* \le \eta,$$
$$\text{such that}\ \ w_\eta + \xi_\eta + \zeta_\eta = 0\,.$$

Choosing $\eta = 1/n$, we find sequences $\{z_n^1\}$, $\{z_n^2\}$, $\{w_n\}$, $\{\xi_n\}$, and $\{\zeta_n\}$ fulfilling

$$\begin{cases} \zeta_n \to 0 & \text{in } \mathcal{B}^*, \\ z_n^1 \to z_j^k & \text{in } \mathcal{B}, \\ z_n^2 \to z_j^k & \text{in } \mathcal{B}, \text{ with } \phi(z_n^2) \to \phi(z_j^k). \end{cases} \tag{B.3}$$

Furthermore, being $w_n \in \mathrm{J}_2((z_n^1 - u_k)/r_j^k)$, thanks to the second of (B.3), we have $\sup_n \|w_n\|_* \le C$, for a positive constant C only depending on $\|u_k\|$ and $\|z_j^k\|$. Thus, there exists $\bar w \in \mathcal{B}^*$ such that, up to a subsequence, $w_n \rightharpoonup^* \bar w$ in $\mathcal{B}^*$ as $n \to \infty$. By the strong-weak* closedness of J_2, we find that $\bar w \in \mathrm{J}_2((z_j^k - u_k)/r_j^k)$. On the other hand, by the definition of J_2 and again by the second of (B.3),

$$\|w_n\|_*^2 = \left\langle w_n, \frac{z_n^1 - u_k}{r_j^k} \right\rangle \to \left\langle \bar w, \frac{z_j^k - u_k}{r_j^k} \right\rangle = \|\bar w\|_*^2\,,$$

so that we ultimately deduce that $w_n \to \bar w$ in $\mathcal{B}^*$ as $n \to \infty$. By the first of (B.3), we conclude that $\xi_n \to -\bar w$ in $\mathcal{B}^*$. Combining the latter information with the third of (B.3), we find from the definition of strong limiting subdifferential that $-\bar{w} \in \partial_s\phi(z_j^k)$, so that (6.30) ensues. □

Proof of Proposition 6.1. To prove (6.23), like in Section 6 we exhibit a sequence $\{u_k\} \subset \mathcal{B}$ for which (6.26) holds and, correspondingly, sequences $\{r_j^k\}_j$ and z_j^k as in (B.1), fulfilling (6.28). The only difference with respect to the argument developed in Section 6 is that, without (6.25), the Euler equation corresponding to (B.1) is (B.2), so that

$$\exists\, w_j^k \in \mathrm{J}_2\left(\frac{z_j^k - u_k}{r_j^k}\right) \cap \left(-\partial_s\phi(z_j^k)\right) \tag{B.4}$$

which also fulfills $|\partial\phi|^2(u_k) = \lim_{j\to\infty} \|w_j^k\|_*^2$. Then, again with a diagonalization procedure we find a subsequence w_k with $w_k \rightharpoonup^* w$ in $\mathcal{B}^*$. Using (B.4) and the closure formula (6.21), we again arrive at

$$-w \in \partial_\ell\phi(u)\,,$$

and conclude the proof of (6.23) along the very same lines as in the proof of Proposition 6.1. □

Appendix C. Maximal functions

We recall the definition of the *local maximal function* of a locally finite measure and two related results, referring e.g. to (Ste70) for all details.

Definition Appendix C.1. *Let μ be a (vector-valued) locally finite measure. For every $\lambda > 0$ the* local maximal function $\mathrm{M}_\lambda\mu$ *of μ is defined by*

$$\mathrm{M}_\lambda\mu(x) := \sup_{0<r<\lambda} \frac{|\mu|(B_r(x))}{\mathscr{L}^d(B_r(x))} \qquad \textit{for all } x \in \mathbb{R}^d.$$

In particular, when $\mu = f\mathscr{L}^d$ for some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, we shall use the notation $\mathrm{M}_\lambda f$.

Lemma Appendix C.2. *For all $1 < p < \infty$ there exists a constant $C(d,p) > 0$ such that for every $f \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ there holds*

$$\int_{B_r(0)} |\mathrm{M}_\lambda f(x)|^p \,\mathrm{d}x \le C(d,p) \int_{B_{r+\lambda}(0)} |f(x)|^d \,\mathrm{d}x \qquad \textit{for all } r > 0.$$

Lemma Appendix C.3. *There exists a constant $C(d) > 0$ such that for every $\lambda > 0$ and $v \in \mathrm{BV}(\mathbb{R}^d)$ there holds*

$$|v(x) - v(y)| \leq C(d)|x - y| \left(M_\lambda \nabla v(x) + M_\lambda \nabla v(y)\right)$$

for all $x, y \in \mathbb{R}^d \setminus N_v$ (with $N_v \subset \mathbb{R}^d$ a negligible set) with $|x - y| \leq \lambda$.

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