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Thermodynamics and analysis of rate-independent adhesive contact at small strains

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ABSTRACT

We address a model for adhesive unilateral frictionless Signorini-type contact between bodies of heat-conductive viscoelastic material, in the linear Kelvin–Voigt rheology, undergoing thermal expansion. The flow rule for debonding the adhesion is considered rate-independent and unidirectional, and a thermodynamically consistent model is derived and analysed as far as the existence of a weak solution is concerned.

1. Introduction

We are interested in the modelling of elastic bodies glued together by an adhesive, which can undergo an inelastic process of the so-called delamination (sometimes also mentioned as debonding). "Microscopically" speaking, some macromolecules in the adhesive may break upon loading and we assume that they can never be glued back, i.e., no "healing" is possible. This makes the process *unidirectional*; sometimes it is also referred to as *irreversible*, although this adjective has an alternative thermodynamical meaning as dissipative in general. On the glued surface, we consider the *delamination process as rate-independent* and, in the bulk, we also consider rate-dependent *inertial, viscous-like*, and *thermal-expansion effects*. Moreover, we confine ourselves to *small strains* and, just for the sake of notational simplicity, we restrict the analysis to the case of *two* bodies Ω_+ and Ω_- glued together along the *contact surface* Γ_C. The material in the bulk is considered as heat conductive, and thus the system is completed by the nonlinear heat equation in a thermodynamically consistent way. The contact surface is considered infinitesimally thin, so that the thermal capacity of the adhesive is naturally neglected. The coupling of the mechanical and thermal effects thus results from thermal expansion, dissipative/adiabatic heat production/consumption, and here also from the possible dependence of the heat transfer through the contact surface Γ_C on the delamination itself, and on the possible slot between the bodies if the contact is debonded.

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We consider an elastic response of the adhesive, and then one speaks about *adhesive contact* (in contrast to *brittle contact*, see Remark 3.5). Within the realm of the literature on (frictionless adhesive) contact, in the isothermal case we refer e.g. to [1] in the framework of rate-independent problems. For *rate-dependent* models, we mention [2–7] (cf. the monograph [8] for further references). The anisothermal rate-dependent case has been recently addressed in [9,10]. The present paper extends the analysis in [1] of rate-independent adhesive contact, to encompass inertial, viscous, and thermal effects.

The elastic response in the adhesive will be considered linear, determined by the scalar elastic modulus $\kappa > 0$; cf. Remark 3.4 for a generalization. At a current time, the "surface fraction" of active molecular links will be "macroscopically" described by the *scalar delamination parameter* $z : \Gamma_C \to [0, 1]$. The state $z(x) = 1$ means that the adhesive is still 100% undestroyed and thus fully effective, while the intermediate state $0 < z(x) < 1$ means that there are some molecular links which have been broken, but the remaining ones are effective. Eventually, $z(x) = 0$ means that the surface is already completely debonded at $x \in \Gamma_C$. As already pointed out in [1], one needs a specific energy to break the macromolecular structure of the adhesive, independently of the rate of this process. Thus, delamination is a *rate-independent* and activated phenomenon, governed by the maximum dissipation principle. Accordingly, we shall consider a rate-independent flow rule for z. Activating the delamination process in the adhesive contact at a given point $x \in \Gamma_C$ again needs the (phenomenologically prescribed) energy $a(x)$.

In the thermodynamical context, the energy $a(x)$ needed for delamination is dissipated by the system in two ways: one part a_1 is spent to the chaotic vibration of the atomic lattice of both sides of the delaminating surface Γ_C, which leads "macroscopically" to heat production (cf. also [11, Remark 4.2]), while another part a_0 is spent to create a new delaminated surface (or, "microscopically" speaking, to break the macromolecules of the adhesive). Thus $a(x) = a_0(x) + a_1(x)$.

The mathematical difficulties, arising both from the proper thermodynamical coupling and from hosting a rate-independent process on Γ_C, have been already revealed for other inelastic processes in the bulk in [12]. The essential ingredient is the satisfaction of the energy balance and, for this, the mentioned concept of energetic solutions to rate-independent systems, recently developed in [13–16], and adapted to systems with inertia and viscosity in [17], appears truly essential.

In Section 2, we set up our model and, in Section 3, discuss its thermodynamics and various modifications. After making a suitable transformation of the problem using an enthalpy variable instead of the temperature, and introducing a suitable weak formulation in Section 4, the main existence results are presented in Section 5, and proved throughout Sections 6–9. For this, in Section 6 we set up procedures of regularization of the Signorini-type unilateral contact. As we shall observe in Section 6, such a regularized problem has its own interest. We further approximate it by convexifying some nonlinear terms, and setting up a time-discretization procedure in Section 7. Hence, we prove fine a priori estimates. Ultimately, a careful passage to the limit is executed in two consecutive steps in Sections 8 and 9.

2. The model

Hereafter, we suppose that the elastic body occupies a reference domain

$$\Omega \subset \mathbb{R}^d,\ d = 2 \text{ or } 3, \text{ bounded and with a Lipschitz boundary } \partial\Omega.$$

We assume that

$$\Omega = \Omega_+ \cup \Gamma_C \cup \Omega_-,$$

with Ω_+ and Ω_- disjoint Lipschitz subdomains and Γ_C their common boundary, which represents a prescribed delamination $(d-1)$-dimensional surface. We denote by ν the outward unit normal to $\partial\Omega$, and by $\nu^\pm$ the unit normal to Γ_C, which we consider oriented from Ω_+ to Ω_-. Moreover, given $v \in W^{1,2}(\Omega \setminus \Gamma_C)$, v^+ (respectively, v^-) shall signify the restriction of v to Ω_+ (to Ω_-, resp.). We further suppose that

$$\partial\Omega = \Gamma_D \cup \Gamma_N,$$

with Γ_D and Γ_N open subsets in the relative topology of $\partial\Omega$, disjoint one from each other and each of them with a smooth boundary.

As *state* variables, inside Ω we have the *displacement* $u : \Omega \setminus \Gamma_C \to \mathbb{R}^d$ and the absolute temperature $\theta : \Omega \setminus \Gamma_C \to (0, +\infty)$, while on the contact boundary we consider a *delamination variable* $z : \Gamma_C \to [0, 1]$, having the meaning of the integrity fraction of the adhesive. Namely, $z = 1$ (respectively $z = 0$) means that the adhesive has full (resp. no) integrity. We denote by

$$[\![u]\!] = u^+|_{\Gamma_C} - u^-|_{\Gamma_C} = \text{the jump of } u \text{ across } \Gamma_C.$$

Furthermore, we shall denote by $T = T(u, v, \theta)$ the traction stress on some $(d-1)$-dimensional surface Γ (later, we shall take either $\Gamma = \Gamma_C$ or $\Gamma = \Gamma_N$), i.e.

$$T(u, v, \theta) := \sigma|_\Gamma \nu, \quad \text{with } \sigma := \mathbb{D}e(v) + \mathbb{C}\big(e(u) - \mathbb{E}\theta\big), \tag{2.1}$$

where of course we take as ν the unit normal $\nu^\pm$ to Γ_C, if $\Gamma = \Gamma_C$. In (2.1), σ is the stress (assuming Kelvin–Voigt's rheology and *thermal expansion*, see (3.8) later on).

To describe various general situations in a unified and simple way, we introduce

a closed, convex cone $K(x) \subset \mathbb{R}^d$, possibly depending on $x \in \Gamma_C$,

and assume the boundary conditions on Γ_C in the complementarity form as

$$\left.\begin{array}{l} [\![u]\!] \succeq 0, \\ T(u,\dot{u},\theta) \overset{*}{\succeq} 0, \\ T(u,\dot{u},\theta)\cdot[\![u]\!] = 0 \end{array}\right\} \quad \text{on } \Gamma_C. \tag{2.2}$$

In (2.2), $\succeq$ is the ordering induced by the multivalued, cone-valued mapping $K : \Gamma_C \rightrightarrows \mathbb{R}^d$, in the sense that, for v_1, $v_2 : \Gamma_C \to \mathbb{R}^d$,

$$v_1 \succeq v_2 \ \text{ if and only if } v_1(x) - v_2(x) \in K(x) \text{ for a.a. } x \in \Gamma_C. \tag{2.3}$$

Likewise, $\overset{*}{\succeq}$ is the dual ordering induced by the negative polar cone to K, in the sense that, for $\zeta_1, \zeta_2 : \Gamma_C \to \mathbb{R}^d$,

$$\zeta_1 \overset{*}{\succeq} \zeta_2 \ \text{ if and only if } \zeta_1(x)\cdot v \geq \zeta_2(x)\cdot v \text{ for all } v \in K(x), \text{ for a.a. } x \in \Gamma_C.$$

Possible choices for the cone-valued mapping $K : \Gamma_C \rightrightarrows \mathbb{R}^d$ are

$$K(x) = K = \mathbb{R}^d \quad \text{for a.a. } x \in \Gamma_C, \quad \text{or} \tag{2.4a}$$

$$K(x) = \{v \in \mathbb{R}^d;\ v\cdot\nu^{\pm}(x) \geq 0\} \quad \text{for a.a. } x \in \Gamma_C, \quad \text{or} \tag{2.4b}$$

$$K(x) = \{v \in \mathbb{R}^d;\ v\cdot\nu^{\pm}(x) = 0\} \quad \text{for a.a. } x \in \Gamma_C. \tag{2.4c}$$

In the first case (2.4a), the second of boundary conditions (2.2) translates into $T(u,\dot{u},\theta) = 0$ on Γ_C, while no constraint on $[\![u]\!]$ is imposed. Thus, (2.4a) allows for no interaction of the bodies Ω_+ and Ω_- after a complete delamination. In fact, this model is very simplified because it does not prevent possible interpenetration and delamination can be thus triggered, rather unphysically, by mere compression. Nevertheless, a model like this may be feasible in some situations. In this connection, let us point out that the interpenetration after developed cracks is neglected in several crack models used in mathematical literature (as e.g. [18–20]), too. The case (2.4b) yields the standard model of unilateral frictionless Signorini contact in the normal displacement at $x \in \Gamma_C$. The last case (2.4c) prescribes the normal jump of the displacement, variable at $x \in \Gamma_C$, to zero. Thus, it only allows for a tangential slip along Γ_C. This may be a relevant model under high pressure, when no cavity of Γ_C can be expected anyhow. Such a situation occurs, e.g., on lithospheric faults deep under the earth surface. Note that, both in (2.4a) and in (2.4c), $K(x)$ is a linear manifold for a.a. $x \in \Gamma_C$. As we shall see later, this feature may allow for some special benefits.

Classical formulation of the adhesive contact problem. Beside the force equilibrium, coupled with the heat equation inside $\Omega \setminus \Gamma_C$ and supplemented with standard boundary conditions, we have two complementarity problems on Γ_C, namely

$$\varrho\ddot{u} - \operatorname{div}\big(\mathbb{D}e(\dot{u}) + \mathbb{C}\big(e(u) - \mathbb{E}\theta\big)\big) = F \quad \text{in } Q \setminus \Sigma_C, \tag{2.5a}$$

$$c_{\rm v}(\theta)\dot{\theta} - \operatorname{div}\big(\mathbb{K}(e(u),\theta)\nabla\theta\big) = \mathbb{D}e(\dot{u}) : e(\dot{u}) - \theta\mathbb{C}\mathbb{E} : e(\dot{u}) + G \quad \text{in } Q \setminus \Sigma_C, \tag{2.5b}$$

$$u = 0 \quad \text{on } \Sigma_D, \tag{2.5c}$$

$$T(u,\dot{u},\theta) = f \quad \text{on } \Sigma_N, \tag{2.5d}$$

$$(\mathbb{K}(e(u),\theta)\nabla\theta)\nu = g \quad \text{on } \Sigma, \tag{2.5e}$$

$$[\![\mathbb{D}e(\dot{u}) + \mathbb{C}(e(u) - \mathbb{E}\theta)]\!]\nu^{\pm} = 0 \quad \text{on } \Sigma_C, \tag{2.5f}$$

$$[\![u]\!] \succeq 0 \quad \text{on } \Sigma_C, \tag{2.5g}$$

$$T(u,\dot{u},\theta) + \kappa z[\![u]\!] \overset{*}{\succeq} 0 \quad \text{on } \Sigma_C, \tag{2.5h}$$

$$\big(T(u,\dot{u},\theta) + \kappa z[\![u]\!]\big)\cdot[\![u]\!] = 0 \quad \text{on } \Sigma_C, \tag{2.5i}$$

$$\dot{z} \leq 0 \quad \text{on } \Sigma_C, \tag{2.5j}$$

$$d \leq a_1 + a_0 \quad \text{on } \Sigma_C, \tag{2.5k}$$

$$\dot{z}\,(d - a_0 - a_1) = 0 \quad \text{on } \Sigma_C, \tag{2.5l}$$

$$d \in N_{[0,1]}(z) + \frac{1}{2}\kappa\big|[\![u]\!]\big|^2 \quad \text{on } \Sigma_C, \tag{2.5m}$$

$$\frac{1}{2}\big(\mathbb{K}(e(u),\theta)\nabla\theta\,|_{\Gamma_C}^{+} + \mathbb{K}(e(u),\theta)\nabla\theta|_{\Gamma_C}^{-}\big)\cdot\nu^{\pm} + \eta([\![u]\!],z)[\![\theta]\!] = 0 \quad \text{on } \Sigma_C, \tag{2.5n}$$

$$[\![\mathbb{K}(e(u),\theta)\nabla\theta]\!]\cdot\nu^{\pm} = -a_1\dot{z} \quad \text{on } \Sigma_C, \tag{2.5o}$$

where we have used the notation

$$Q := (0,T)\times\Omega,\qquad \Sigma := (0,T)\times\partial\Omega,\qquad \Sigma_C := (0,T)\times\Gamma_C,\qquad \Sigma_D := (0,T)\times\Gamma_D,\qquad \Sigma_N := (0,T)\times\Gamma_N,$$

$T > 0$ being a fixed time horizon. In (2.5), $F : Q \to \mathbb{R}^d$ is the applied bulk force, $f : \Sigma_N \to \mathbb{R}^d$ the applied traction, while $G : Q \to \mathbb{R}$ and $g : \Sigma \to \mathbb{R}$ are some external heat sources. In addition,

$$\mathbb{C},\ \mathbb{D} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}} \quad \text{are fourth-order positive definite and symmetric tensors,} \tag{2.6}$$

(i.e. $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$, and the same for $\mathbb{D}$), $\mathbb{K} = \mathbb{K}(e,\theta)$ is the positive definite matrix of the heat conduction coefficients, and $\mathbb{E} \in \mathbb{R}^{d\times d}$ is a matrix of thermal-expansion coefficients. Furthermore, the constant $\kappa > 0$ phenomenologically describes the elastic response of the adhesive. The complementarity problem (2.5g)–(2.5i) describes general, possibly unilateral (depending on the choice of the mapping $K : \Gamma_C \rightrightarrows \mathbb{R}^d$) contact, whereas the adhesive contact results from the complementarity conditions (2.5j)–(2.5m). In (2.5k) the coefficient a_0 (resp. a_1) is the phenomenological specific energy (per area) which is stored (resp. dissipated) by disintegrating the adhesive. The overall activation energy to trigger the debonding process in the adhesive is then $a_0 + a_1$. Note that the term $\kappa\big|[\![u]\!]\big|^2$ in (2.5m) is in fact a penalization of the delamination condition $z[\![u]\!] = 0$, cf. with the brittle delamination model (3.18). Moreover, $N_{[0,1]}$ denotes the normal cone to the interval $[0, 1]$, i.e. the subdifferential in the sense of convex analysis of the indicator function $I_{[0,1]}$ of $[0, 1]$, cf. Notation 2.1. Finally, $\eta = \eta(x, [\![u]\!], z) \geq 0$ is a phenomenological *heat-transfer coefficient*, determining the linear heat convection through Γ_C. We shall suppose that η depends affinely on the delamination variable z, cf. (5.1e) below.

Notation 2.1. In what follows, given a set $M \subset \mathbb{R}^d$, $d = 1, 2, 3$, we shall denote by I_M the indicator set of M, defined by $I_M(x) = 0$ if $x \in M$, and $I_M(x) = +\infty$ otherwise. We recall that its subdifferential ∂I_M at $x \in M$ is the so-called normal cone, i.e.

$$\xi \in \partial I_M(x) \Leftrightarrow \langle \xi, y - x\rangle \leq 0 \quad \text{for all } y \in M.$$

3. Thermodynamics of the model and various remarks

Let us briefly present the *thermodynamics* of the boundary value problem (2.5). The underlying overall Helmholtz *free energy* $\Psi : \mathbb{R}^d \times \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ has a bulk and a surface part, i.e.

$$\Psi(u,z,\theta) = \int_{\Omega\setminus\Gamma_C} \psi^{\mathrm{bulk}}(e(u),\theta)\,\mathrm{d}x + \int_{\Gamma_C} \psi^{\mathrm{surf}}([\![u]\!],z)\,\mathrm{d}S, \tag{3.1a}$$

with ψ^{bulk} and ψ^{surf}, respectively being the bulk and the contact surface contributions to the specific Helmholtz energy. One can identify

$$\begin{aligned}\psi^{\mathrm{bulk}}(e,\theta) &= \frac{1}{2}\mathbb{C}(e-\mathbb{E}\theta):(e-\mathbb{E}\theta) - \frac{\theta^2}{2}\mathbb{B}:\mathbb{E} - \psi_0(\theta)\\ &= \frac{1}{2}\mathbb{C}e:e - \theta\mathbb{B}:e - \psi_0(\theta) \quad \text{with } \mathbb{B} := \mathbb{C}\mathbb{E}.\end{aligned} \tag{3.1b}$$

Here $\frac{1}{2}\mathbb{C}e : e$ is the *mechanical part of the internal energy in the bulk*, while $-\psi_0(\theta)$ is the *thermal part of the free energy*. Hereafter, we shall assume that

$$\psi_0 : (0,+\infty) \to \mathbb{R} \text{ a strictly convex function.} \tag{3.1c}$$

The specific *contact surface* energy $\psi^{\mathrm{surf}}([\![u]\!], z)$ is then

$$\psi^{\mathrm{surf}}([\![u]\!],z) = \begin{cases}\displaystyle\int_{\Gamma_C}\Big(\frac{\kappa}{2}z\big|[\![u]\!]\big|^2 - a_0 z\Big)\,\mathrm{d}S & \text{if } [\![v]\!] \geq 0 \text{ and } 0 \leq z \leq 1 \text{ a.e. on } \Gamma_C,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.1d}$$

The other underlying ingredient of the model is the overall *dissipation rate* ξ, which also has bulk and surface contributions ξ^{bulk} and ξ^{surf}, namely:

$$\Xi(\dot e,\dot z) := \int_{\overline{\Omega}} \big[\xi(\dot e,\dot z)\big](\mathrm{d}x) = \int_{\Omega\setminus\Gamma_C} \xi^{\mathrm{bulk}}(\dot e)\,\mathrm{d}x + \int_{\Gamma_C} \xi^{\mathrm{surf}}(\dot z)\mathrm{d}S \tag{3.2}$$

where the specific dissipation rate $\xi(\dot e,\dot z) = \xi^{\mathrm{bulk}}(\dot e)\mathrm{d}x + \xi^{\mathrm{surf}}(\dot z)\mathrm{d}S$ is a measure in general, with absolutely continuous part determined by the (pseudo)potential of viscous-type dissipative forces in the bulk, and a possibly concentrating part, supported on Γ_C, i.e.

$$\xi^{\mathrm{bulk}}(\dot e) = 2\zeta_2(\dot e),\qquad \zeta_2(\dot e) := \frac{1}{2}\mathbb{D}\dot e:\dot e,\qquad \xi^{\mathrm{surf}}(\dot z) = \zeta_1(\dot z) := \begin{cases} a_1|\dot z| & \text{if } \dot z \leq 0 \text{ a.e. in } \Gamma_C,\\ +\infty & \text{otherwise,}\end{cases}$$

the latter term representing the potential (and also the specific dissipation rate) of the rate-independent delamination process on the contact boundary Γ_C.

Standardly, one then defines the *specific entropy* $s = s(\theta, e)$ by the so-called Gibbs' relation $\langle s, \tilde{\theta}\rangle = -\Psi'_\theta(u, z, \theta; \tilde{\theta})$ where $\Psi'_\theta(u, z, \theta; \tilde{\theta})$ is the directional derivative of Ψ at (u, z, θ) in the direction $\tilde{\theta}$. This yields the entropy in the bulk as

$$s = -\frac{\partial \psi^{\mathrm{bulk}}}{\partial \theta}(e(u), \theta) = \mathbb{B} : e(u) + \psi_0'(\theta). \tag{3.3}$$

Further, we shall use the so-called *entropy equation*

$$\theta \dot{s} = \xi^{\mathrm{bulk}}(e(\dot{u})) - \mathrm{div}(j) + G. \tag{3.4}$$

Substituting $\dot{s} = \mathbb{B} : e(\dot{u}) + \psi_0''(\theta)\dot{\theta}$, cf. (3.3), into the entropy equation (3.4) yields the *heat equation*

$$c_{\mathrm{v}}(\theta)\dot{\theta} + \mathrm{div}(j) = 2\zeta_2(e(\dot{u})) - \theta\mathbb{B} : e(\dot{u}) + G \tag{3.5}$$

with the *heat capacity*

$$c_{\mathrm{v}}(\theta) = \theta\psi_0''(\theta). \tag{3.6}$$

Hence, postulating the constitutive relation for the heat flux

$$j := -\mathbb{K}(e(u), \theta)\nabla\theta, \tag{3.7}$$

i.e. *Fourier's law* in an anisotropic medium, one obtains the heat equation in the form (2.5b).

Similar, but simpler thermodynamics can be seen also on the contact boundary by involving ψ^{surf} and ξ^{surf}. As (3.1d) is independent of temperature, the "boundary entropy" $= -\frac{\partial}{\partial\theta}\psi^{\mathrm{surf}}$ is simply zero, and the corresponding entropy equation reduces to $0 = \xi^{\mathrm{surf}}(\dot{z}) - [\![j]\!] \cdot \nu^{\pm}$ (as an analog of (3.4)), which then results in (2.5o). Incorporating the analog of the phenomenological law (3.7), we arrive at (2.5n).

Momentum equation. As in *Kelvin–Voigt rheology*, the total stress σ is postulated as

$$\sigma = \frac{\partial}{\partial e}\zeta_2\big(e(\dot{u})\big) + \frac{\partial}{\partial e}\psi^{\mathrm{bulk}}\big(e(u), \theta\big), \tag{3.8}$$

which just gives σ from (2.1). From *Hamilton's principle*, generalized for dissipative systems as in [21] and with the specific kinetic energy $\frac{1}{2}\rho|\dot{u}|^2$, one then obtains the equilibrium equation (2.5a). For later use, we introduce the indicator functional I_K associated with $K : \Gamma_C \rightrightarrows \mathbb{R}^d$, defined on $L^2(\Gamma_C; \mathbb{R}^d)$ by

$$I_K(v) = \int_{\Gamma_C} I_{K(x)}(v(x))\,\mathrm{d}S \quad \text{for all } v \in L^2(\Gamma_C; \mathbb{R}^d). \tag{3.9}$$

We point out that the complementarity conditions (2.5g)–(2.5i) may be reformulated as the subdifferential inclusion

$$\partial I_K([\![u]\!]) + T(u, \dot{u}, \theta) + \kappa z[\![u]\!] \ni 0 \quad \text{in } \Sigma_C, \tag{3.10}$$

featuring the (convex analysis) subdifferential $\partial I_K : L^2(\Gamma_C; \mathbb{R}^d) \rightrightarrows L^2(\Gamma_C; \mathbb{R}^d)$ of the indicator functional I_K introduced in (3.9).

Evolution of the delamination parameter (a flow rule). Finally, we consider the following differential inclusion for the inelastic evolution of the parameter z

$$\partial\zeta_1(\dot{z}) + \partial_z\psi^{\mathrm{surf}}\big([\![u]\!], z\big) \ni 0 \quad \text{in } \Sigma_C,$$

which in the adhesive case results in

$$\partial I_{(-\infty,0]}(\dot{z}) + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa\big|[\![u]\!]\big|^2 - a_0 - a_1 \ni 0 \quad \text{in } \Sigma_C. \tag{3.11}$$

It is immediate to check that (3.11) is a reformulation of (2.5j)–(2.5m).

The entropy equation (3.4) is designed to balance the total energy, i.e. the sum of the kinetic energy integrated over $\Omega\setminus\Gamma_C$ with the overall dissipated energy (i.e., Ξ from (3.2) integrated in time), and with the bulk *internal energy*

$$\begin{aligned} e^{\mathrm{bulk}}(e, \theta) := \psi^{\mathrm{bulk}} + \theta s &= \frac{1}{2}\mathbb{C}e(u) : e(u) - \psi_0(\theta) - \theta\mathbb{B} : e(u) + \theta(\mathbb{B} : e(u) + \psi_0'(\theta)) \\ &= h(\theta) + \frac{1}{2}\mathbb{C}e(u) : e(u) \quad \text{with } h(\theta) := \theta\psi_0'(\theta) - \psi_0(\theta); \end{aligned} \tag{3.12}$$

we convene to refer to h as the *enthalpy*, see also [12, Section 2]. One can then derive the *total energy balance*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{\Omega\setminus\Gamma_C}\frac{\varrho}{2}|\dot{u}|^2+\frac{1}{2}\mathbb{C}e(u):e(u)+h(\theta)\,\mathrm{d}x}_{\text{kinetic, elastic, and thermal energies}}+\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{\Gamma_C}\frac{\kappa}{2}z\big|[\![u]\!]\big|^2-a_0z\,\mathrm{d}S}_{\text{mechanical energy in the adhesive}}$$
$$=\underbrace{\int_\Omega G+F\cdot\dot{u}\,\mathrm{d}x}_{\substack{\text{power of bulk heat}\\ \text{and mechanical load}}}+\underbrace{\int_\Gamma g\,\mathrm{d}S+\int_{\Gamma_N}f\cdot\dot{u}\,\mathrm{d}S}_{\substack{\text{power of surface heat}\\ \text{and mechanical load}}}. \tag{3.13}$$

Assuming $\theta_0>0$, $G\geq 0$ a.e. in Ω, and $g\geq 0$ a.e. in $\partial\Omega$, we can rely on the fact that $\theta>0$ a.e. in Ω (proved later in Theorem 5.1) and, using (3.4), we derive the *Clausius–Duhem inequality*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s\,\mathrm{d}x=\int_\Omega\left(\frac{\mathrm{div}\,(\mathbb{K}\nabla\theta)}{\theta}+\frac{G}{\theta}\right)\mathrm{d}x+\int_\Omega\frac{\mathbb{D}e(\dot{u}):e(\dot{u})}{\theta}\,\mathrm{d}x$$
$$=\int_\Omega\left(\frac{\mathbb{K}\nabla\theta\cdot\nabla\theta}{\theta^2}+\frac{G}{\theta}\right)\mathrm{d}x+\int_{\partial\Omega}\frac{g}{\theta}\,\mathrm{d}S+\int_{\Gamma_C}\frac{a_1|\dot{z}|}{\theta}\,\mathrm{d}S\geq 0. \tag{3.14}$$

Remark 3.1 (*Partly Linearized Ansatz*)**.** An important feature is that, as a consequence of the partly linearized ansatz (3.1b), the mechanical and thermal variables are additively separated in (3.3), which makes c_v in (3.5) independent of u, and thus makes mathematical analysis much easier.

Remark 3.2 (*Non-homogeneous Boundary Conditions*)**.** We could supplement (2.5a) with non-homogeneous, Dirichlet boundary conditions on Γ_D, i.e. impose

$$u=\omega_D\quad\text{on }\Sigma_D, \tag{3.15}$$

for some prescribed time-dependent loading $\omega_D:[0,T]\to H^{1/2}(\Gamma_D)$. The analysis we are going to perform in the case of homogeneous Dirichlet conditions can be carried over to the case of (3.15) by arguing as in [11], and thus recurring to the additive split $u(t)=\tilde{u}(t)+u_D(t)$ for almost all $t\in(0,T)$, with $\tilde{u}:[0,T]\to W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)=$ the subspace of $W^{1,2}(\Omega\setminus\Gamma_C;\mathbb{R}^d)$ of functions with zero trace on Γ_D, cf. (4.1), and $u_D:[0,T]\to W^{1,2}(\Omega\setminus\Gamma_C;\mathbb{R}^d)$ an extension of ω_D to Ω. It was observed in [11] that, if $\overline{\Gamma}_C\cap\overline{\Gamma}_D=\emptyset$, one can assume that $[\![u_D(t)]\!]=0$, whence $[\![u(t)]\!]=[\![\tilde{u}(t)]\!]$ for almost all $t\in(0,T)$. This allows for a reformulation of the problem in terms of the unknown $\tilde{u}$, hence reducing the analysis to the case with homogeneous Dirichlet conditions.

Remark 3.3 (*Heat-transfer Contact Conditions*)**.** Note that the transient conditions (2.5n)–(2.5o) on Γ_C for the heat equation can equivalently be written as two Robin-type conditions

$$\mathbb{K}(e(u),\theta)\nabla\theta\,|^+_{\Gamma_C}\cdot\nu^\pm+\eta([\![u]\!],z)\theta\,|^+_{\Gamma_C}=\eta([\![u]\!],z)\theta\,|^-_{\Gamma_C}-\frac{1}{2}a_1\dot{z}\quad\text{on }\Sigma_C, \tag{3.16a}$$
$$\mathbb{K}(e(u),\theta)\nabla\theta\,|^-_{\Gamma_C}\cdot\nu^\mp+\eta([\![u]\!],z)\theta\,|^-_{\Gamma_C}=\eta([\![u]\!],z)\theta\,|^+_{\Gamma_C}-\frac{1}{2}a_1\dot{z}\quad\text{on }\Sigma_C, \tag{3.16b}$$

where we have highlighted the unit normals $\nu^\pm$ from Ω_- to Ω_+ and $\nu^\mp$ from Ω_+ to Ω_-. This reveals that the heat generated by delamination is distributed with proportions $\frac{1}{2}$ and $\frac{1}{2}$ into the two subdomains adjacent to Γ_C. In principle, we could also consider a contribution from the Stefan–Boltzmann radiation, which would then result in the condition

$$\frac{1}{2}\big(\mathbb{K}(e(u),\theta)\nabla\theta\,|^+_{\Gamma_C}+\mathbb{K}(e(u),\theta)\nabla\theta|^-_{\Gamma_C}\big)\cdot\nu^\pm+\eta_0([\![u]\!],z)[\![\theta]\!]+\eta_1([\![u]\!],z)[\![\theta^4]\!]=0\quad\text{on }\Sigma_C, \tag{3.17}$$

with $\eta_0([\![u]\!],\cdot)$, $\eta_1([\![u]\!],\cdot)\geq 0$ affine. In fact, this would lead to a modification of the present analysis which is quite routine. Thus, we shall not scrutinize this generalization here. Let us also remark that, in alternative to the dependence of η on $[\![u]\!]$, a dependence on the normal stress is sometimes considered, cf. [22]. However, it seems difficult to adapt the present multidimensional analysis to that case.

Remark 3.4 (*Elastic Response in Adhesive*)**.** One can easily imagine a positive definite $d\times d$-matrix in place of κ, which would more properly describe the phenomenological elastic response of the adhesive. The related analysis would be just a standard modification of the presented one.

Remark 3.5 (*Griffith Concept*)**.** The classical concept of delamination is based on the *Griffith criterion* [23], phenomenologically prescribing the amount of energy a (in J/m^2, in three-dimensional situations) needed to delaminate the surface, independently of the rate of the process. The classical Griffith-type approach considers the adhesive inelastic and one speaks of a *brittle delamination*. Our adhesive contact problem can be viewed as a regularization of this brittle delamination, and in fact makes mathematical analysis and numerical implementation easier. It has its own interpretation and many applications, and it is thus often considered as the original problem, cf. [2,3,9,10,1,7]. In the quasistatic isothermal case, it has been proved in [11] that the adhesive contact approximates the brittle delamination as the elastic modulus $\kappa \to \infty$ in the framework of the so-called *energetic solution* concept. Furthermore, in [24], any energetic solution to the brittle delamination has been proved to be of "Griffith-type" in the sense that z indeed takes either the value 1 or the value 0.

Remark 3.6 (*Engineering Models*)**.** In the engineering literature, the Griffith-type delamination on a prescribed so-called "weak surface" is a quite accepted concept (for example in the framework of the so-called Finite Fracture Mechanics), although it is often combined with the heuristically devised stress criterion, which in some situations seems to provide a better understanding of the initiation of the delamination process, cf. e.g. [25,26]. The initiation of the delamination process can sometimes be triggered by another crack approaching the weak surface, according to the classical so-called Cook–Gordon mechanism [27], which has been confirmed experimentally. The present form of the activation energy $a(x)$ may typically correspond to crack growth in a pure fracture mode (e.g. Mode I). Nonetheless, it is believed that the present approach can be extended to a generalization of this form, in order to cover more complex phenomenological engineering models, working with the so-called "fracture mode mixity" which reflects the character of the load (the ratio of its shear and normal components) on the crack tip.

Remark 3.7 (*Constant Heat Capacity*)**.** The special case $\psi_0(\theta) = c_0\theta\ln(\theta/\theta_0)$, with $c_0 > 0$ and $\theta_0 > 0$ constant, would give $c_{\mathrm{v}}(\theta) = c_0$ in (3.6). However, this case is not within the scope of our analysis, since $c_{\mathrm{v}}(\cdot)$ does not have a *compatible growth*, cf. (5.1b).

Remark 3.8 (*Brittle Delamination Model*)**.** Let us now briefly comment on the model for brittle delamination with thermal effects which would result from the above derivation. As in the case of adhesive contact, we focus on its classical formulation, which couples the momentum equilibrium equation (2.5a), the heat equation (2.5b), the boundary conditions (2.5c)–(2.5f), and (2.5n)–(2.5o) with the two following complementarity problems on Γ_C:

$$[\![u]\!] \succeq 0 \quad \text{on } \Sigma_C, \tag{3.18a}$$
$$T(u,\dot u,\theta) \overset{*}{\succeq} 0 \quad \text{wherever } z(\cdot) = 0 \quad \text{on } \Sigma_C, \tag{3.18b}$$
$$T(u,\dot u,\theta)\cdot[\![u]\!] = 0 \quad \text{on } \Sigma_C, \tag{3.18c}$$
$$z[\![u]\!] = 0 \quad \text{on } \Sigma_C, \tag{3.18d}$$
$$\dot z \le 0 \quad \text{on } \Sigma_C, \tag{3.18e}$$
$$d \le a_1 + a_0 \quad \text{on } \Sigma_C, \tag{3.18f}$$
$$\dot z(d - a_0 - a_1) = 0 \quad \text{on } \Sigma_C, \tag{3.18g}$$
$$d \in N_{[0,1]}(z) + \partial_z J\big([\![u]\!], z\big) \quad \text{on } \Sigma_C. \tag{3.18h}$$

Indeed, (3.18a)–(3.18c) and (3.18d)–(3.18h) respectively correspond to (possibly unilateral) contact and activated delamination. Note that the penalization terms $\kappa[\![u]\!]$ in (2.5h)–(2.5i) and $\kappa/2|[\![u]\!]|^2$ in (2.5m) are no longer present, and the delamination constraint (3.18d) is enforced by the second subdifferential operator in (3.18h), featuring the indicator function

$$J(v,z) = I_{\{vz=0\}}, \quad \text{i.e. } J(v,z) = \begin{cases} 0 & \text{if } vz = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.19}$$

In fact, as function of the two variables v and z J is nonconvex, but separately convex. Hence, the subdifferentials of the convex functions $J(\cdot, z)$ and $J(v,\cdot)$ are well defined, and in particular $\partial_z J([\![u]\!], z)$ is given by

$$\partial_z J\big([\![u]\!], z\big) = \begin{cases} \emptyset & \text{if } z \neq 0 \text{ and } [\![u]\!] \neq 0, \\ 0 & \text{if } z = 0 \text{ and } [\![u]\!] \neq 0, \\ \mathbb{R} & \text{if } z \neq 0 \text{ and } [\![u]\!] = 0. \end{cases}$$

As we have already mentioned, existence for the (global) energetic formulation of the brittle delamination problem in the isothermal quasistatic case has been proved in [11]. In contrast, the analysis of the corresponding thermomechanical model given by (2.5a)–(2.5f), (2.5n) and (2.5o)–(3.18) is for the moment being an open problem. The main difficulties attached to this problem are related to the presence of two multivalued operators in (3.18h), and in particular to the *essentially nonconvex* character of the nonlinearity (3.19).

However, taking into account (3.18h), we clearly identify a drawback of the differential formulation (3.18d)–(3.18h) of brittle delamination. Indeed, in this framework any driving tendency towards delamination is smeared out if $0 < z < 1$, because then the driving force is $d = 0 < a_0 + a_1$, and necessarily, by (3.18g), we have $\dot{z} = 0$. The adhesive contact problem shows a similar behaviour if $\kappa \to \infty$.

4. Enthalpy transformation and energetic solution

Throughout the paper, we shall adopt the notation

$$\begin{aligned} W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d) &= \big\{ v \in W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) : v = 0 \quad \text{on } \Gamma_D \big\}, \\ W^{1,2}_{\Gamma_C}(\Omega \setminus \Gamma_C; \mathbb{R}^d) &= \big\{ v \in W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) : v = 0 \quad \text{on } \Gamma_C \big\}. \end{aligned} \tag{4.1}$$

Furthermore, in the case $K(x)$ is a linear subspace of $\mathbb{R}^d$ for almost all $x \in \Gamma_C$, we shall use the notation

$$W^{1,2}_{K}(\Omega \setminus \Gamma_C; \mathbb{R}^d) = \big\{ v \in W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) : [\![v(x)]\!] \in K(x) \quad \text{for a.a. } x \in \Gamma_C \big\}. \tag{4.2}$$

We shall also extensively exploit that, for $d \le 3$,

$$\begin{aligned} &W^{1,2}(\Omega) \subset L^p(\Omega) \text{ continuously for } 1 \le p \le 6, \\ &u \mapsto u|_\Gamma : W^{1,2}(\Omega) \to H^{1/2}(\Gamma) \subset L^m(\Gamma) \begin{cases} \text{continuously for } 1 \le m \le 4, \\ \text{compactly for } 1 \le m < 4, \end{cases} \end{aligned} \tag{4.3}$$

with $\Gamma = \partial\Omega$, or $\Gamma = \Gamma_C$, or $\Gamma = \Gamma_N$. The same embeddings hold for the Sobolev space $W^{1,2}(\Omega; \mathbb{R}^d)$ of vector-valued functions. Finally, we shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing between the spaces $W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*$ and $W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d)$.

The analysis of the nonlinear heat equation (2.5b), featuring the quadratic coupling terms with the momentum balance equation (2.5a), calls for rather sophisticated techniques and suitable working assumptions. In particular, one may impose some conditions either on the growth of $\mathbb{K}(e, \cdot)$ (cf., e.g., [28] for the analysis of a similar nonlinear heat equation in some phase transition model), or on the growth of $c_{\rm v}$ (cf., e.g., [29,12] and, more specifically, [30, Section 5.4.2] (see also [48]) for contact problems in thermo-viscoelasticity). Under the latter kind of assumptions, the Galerkin approximation method for proving existence of solutions could serve quite effectively, cf. [29].

On the other hand, system (2.5) hosts the delamination rate-independent process on Γ_C. Hence, the Rothe method (i.e. the implicit discretization in time) seems more natural for the analysis, see e.g. [15,13,31]. In turn, the nonlinearity $c_{\rm v}(\cdot)$ makes it technically difficult to implement such a discretization method. This problem can be circumvented by rewriting the original PDE system (2.5) in terms of the enthalpy, instead of the temperature, as e.g. in [29].

Namely, we introduce the so-called *enthalpy transformation*, setting

$$\vartheta = h_0(\theta) := \int_0^\theta c_{\rm v}(r)\, {\rm d}r. \tag{4.4}$$

Thus, h_0 is a primitive function of $c_{\rm v}$, normalized in such a way that $h_0(0) = 0$. In view of (3.6) and (3.12), we have

$$h'(\theta) = (\theta\psi_0'(\theta) - \psi_0(\theta))' = \theta\psi_0''(\theta) + \psi_0'(\theta) - \psi_0'(\theta) = \theta\psi_0''(\theta) = c_{\rm v}(\theta) = h_0'(\theta), \tag{4.5}$$

hence h_0 differs from h just by a constant, namely $\psi_0(0)$. Furthermore, thanks to (3.1c), $c_{\rm v}$ is strictly positive and hence h_0 is strictly increasing. Thus, we are entitled to define

$$\Theta(\vartheta) := \begin{cases} h_0^{-1}(\vartheta) & \text{if } \vartheta \ge 0, \\ 0 & \text{if } \vartheta < 0, \end{cases} \qquad \mathcal{K}(e, \vartheta) := \frac{\mathbb{K}(e, \Theta(\vartheta))}{c_{\rm v}(\Theta(\vartheta))}, \tag{4.6}$$

where h_0^{-1} here denotes the inverse function to h.

Taking into account (4.6), as well as the *subdifferential reformulations* (3.10) and (3.11) of the complementarity problems (2.5g)–(2.5i), and (2.5j)–(2.5m), respectively, the PDE system (2.5) turns into

$$\left.\begin{aligned} &\varrho\ddot{u} - {\rm div}\big(\mathbb{D}e(\dot{u}) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)\big) = F \\ &\dot{\vartheta} - {\rm div}\big(\mathcal{K}(e(u), \vartheta)\nabla\vartheta\big) = \mathbb{D}e(\dot{u}) : e(\dot{u}) - \Theta(\vartheta)\mathbb{B} : e(\dot{u}) + G \end{aligned}\right\} \quad \text{in } Q \setminus \Sigma_C, \tag{4.7a}$$

$$u = 0 \qquad \text{on } \Sigma_D, \tag{4.7b}$$

$$(\mathcal{K}(e(u), \vartheta)\nabla\vartheta)\nu = g \quad \text{on } \Sigma, \tag{4.7c}$$

$$\mathcal{T}(u, \dot{u}, \vartheta) = f \qquad \text{on } \Sigma_N, \tag{4.7d}$$

$$\left.\begin{aligned} &[\![\mathbb{D}e(\dot{u}) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)]\!]\nu^\pm = 0 \\ &\partial I_K([\![u]\!]) + \mathcal{T}(u, \dot{u}, \vartheta) + \kappa z[\![u]\!] \ni 0 \\ &\partial I_{(-\infty,0]}(\dot{z}) + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa\big|[\![u]\!]\big|^2 - a_0 - a_1 \ni 0 \\ &\frac{1}{2}\Big(\mathcal{K}(e(u), \vartheta)\nabla\vartheta\,\big|^+_{\Gamma_C} + \mathcal{K}(e(u), \vartheta)\nabla\vartheta\,\big|^-_{\Gamma_C}\Big) \cdot \nu^\pm + \eta([\![u]\!], z)[\![\Theta(\vartheta)]\!] = 0 \\ &[\![\mathcal{K}(e(u), \vartheta)\nabla\vartheta]\!] \cdot \nu^\pm = -a_1\dot{z} \end{aligned}\right\} \quad \text{on } \Sigma_C, \tag{4.7e}$$

where

$$\mathcal{T}(u, v, \vartheta) := T(u, v, \Theta(\vartheta)) = \big[\mathbb{D}e(v) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)\big]|_{\Gamma}\nu \tag{4.7f}$$

where again we take as ν the unit normal to $\Gamma_C \nu^{\pm}$, if $\Gamma = \Gamma_C$.

Data qualification. Hereafter, the problem data F, G, f, and g shall be qualified by

$$F \in L^1(0, T; L^2(\Omega; \mathbb{R}^d)); \tag{4.8a}$$

$$f \in \begin{cases} W^{1,1}(0, T; L^{4/3}(\Gamma_N; \mathbb{R}^3)) & \text{if } d = 3, \\ W^{1,1}(0, T; L^{1+\epsilon}(\Gamma_N; \mathbb{R}^2)) & \text{for some } \epsilon > 0, \text{ if } d = 2; \end{cases} \tag{4.8b}$$

$$G \in L^1(Q), \quad G \geq 0 \text{ a.e. in } Q; \tag{4.8c}$$

$$g \in L^1(\Sigma), \quad g \geq 0 \text{ a.e. in } \Sigma. \tag{4.8d}$$

The energetic formulation associated with system (4.7) hinges on the following energy functional Φ which is, in fact, the mechanical part of the internal energy (3.1a), and on the dissipation potential $\mathcal{R}$:

$$\Phi(u, z) := \int_{\Omega\setminus\Gamma_C} \frac{1}{2}\mathbb{C}e(u) : e(u)\,\mathrm{d}x + I_K([\![u]\!]) + \int_{\Gamma_C} \Big(\frac{\kappa}{2} z\big|[\![u]\!]\big|^2 + I_{[0,1]}(z) - a_0 z\Big)\,\mathrm{d}S, \tag{4.9}$$

$$\mathcal{R}\big(\tilde{z} - z\big) := \begin{cases} \displaystyle\int_{\Gamma_C} a_1|\tilde{z} - z|\,\mathrm{d}S & \text{if } \tilde{z} \leq z \text{ a.e. in } \Gamma_C, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.10}$$

For notational convenience, for all $v \in L^2(\Omega)$, we also set for the kinetic energy:

$$T^{\varrho}_{\mathrm{kin}}(v) := \frac{1}{2}\int_{\Omega} \varrho\,|v|^2\,\mathrm{d}x.$$

We are now in the position of introducing the notion of weak solution to system (4.7) which shall be analysed throughout this paper. The reader is referred to [12, Proposition 3.2] for some justification of the energetic solution concept in the framework of general thermomechanical rate-independent processes, in particular for the proof of the fact that energetic solutions are also conventional weak solutions whenever $\dot{z}$ is absolutely continuous.

Prior to Definition 4.1, we specify some further notation. Let X be a (separable) Banach space: We denote by $B_{\mathrm{w}*}([0, T]; X)$ and $\mathrm{BV}([0, T]; X)$, respectively, the Banach spaces of the functions from $[0, T]$ with values in X that are bounded and weakly* measurable (if X has a predual), and, respectively, that have bounded variation on $[0, T]$. Notice that all these functions are defined everywhere on $[0, T]$. We denote $\mathcal{M}(\bar{\Omega}) \cong C(\bar{\Omega})^*$ the space of Borel measures on $\bar{\Omega}$.

Definition 4.1 (*Energetic Solution of the Adhesive Contact Problem*)**.** Given a quadruple of initial data $(u_0, \dot{u}_0, z_0, \theta_0)$ satisfying suitable conditions (cf. (5.4) later on), we call a triple (u, z, ϑ) an *energetic solution* to the Cauchy problem for (the enthalpy reformulation of) system (4.7) if

$$u \in W^{1,2}(0, T; W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d)), \tag{4.11a}$$

$$u \in W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d)) \quad \text{if } \varrho > 0, \tag{4.11b}$$

$$z \in L^\infty(\Sigma_C) \cap \mathrm{BV}([0, T]; L^1(\Gamma_C)), \tag{4.11c}$$

$$\vartheta \in L^r(0, T; W^{1,r}(\Omega \setminus \Gamma_C)) \cap L^\infty(0, T; L^1(\Omega)) \cap B_{\mathrm{w}*}(\bar{I}; \mathcal{M}(\bar{\Omega})) \cap \mathrm{BV}([0, T]; W^{1,r'}(\Omega \setminus \Gamma_C)^*) \tag{4.11d}$$

for every $1 \leq r < \frac{d+2}{d+1}$, with r' denoting the conjugate exponent $\frac{r}{r-1}$ of r, and the triple (u, z, ϑ) complies with:

(i) (weak formulation of the) momentum inclusion, i.e.:

$$[\![u]\!] \succeq 0 \quad \text{on } \Sigma_C, \quad \text{and} \tag{4.12}$$

$$\begin{aligned} &\int_{\Omega} \varrho\dot{u}(T)\cdot\big(v(T) - u(T)\big)\,\mathrm{d}x + \int_Q \big(\mathbb{D}e(\dot{u}) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)\big) : e(v - u) - \varrho\dot{u}\cdot\big(\dot{v} - \dot{u}\big)\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\Sigma_C} \kappa z[\![u]\!]\cdot[\![v - u]\!]\mathrm{d}S\mathrm{d}t \geq \int_{\Omega} \varrho\dot{u}_0\cdot\big(v(0) - u(0)\big)\mathrm{d}x + \int_Q F\cdot(v - u)\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_N} f\cdot(v - u)\,\mathrm{d}S\mathrm{d}t \end{aligned} \tag{4.13}$$

for all v in $L^2(0, T; W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d))$ with $[\![v]\!] \succeq 0$ on Σ_C and, if $\varrho > 0$, also in $W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^d))$,

(ii) total energy balance

$$T^{\varrho}_{\rm kin}\big(\dot u(T)\big)+\Phi\big(u(T),z(T)\big)+\int_{\bar\Omega}\vartheta(T,{\rm d}x)=T^{\varrho}_{\rm kin}\big(\dot u_0\big)+\Phi\big(u_0,z_0\big)+\int_{\Omega}\vartheta_0\,{\rm d}x$$
$$+\int_Q F\cdot\dot u\,{\rm d}x{\rm d}t+\int_{\Sigma_N} f\cdot\dot u\,{\rm d}S{\rm d}t+\int_Q G\,{\rm d}x{\rm d}t+\int_\Sigma g\,{\rm d}S{\rm d}t \tag{4.14}$$

(iii) semistability for a.a. $t\in(0,T)$

$$\forall\tilde z\in L^\infty(\Gamma_C):\quad \Phi\big(u(t),z(t)\big)\le\Phi\big(u(t),\tilde z\big)+\mathcal R\big(\tilde z-z(t)\big) \tag{4.15}$$

(iv) (weak formulation of the) enthalpy equation:

$$\int_{\bar\Omega}w(T)\vartheta(T,{\rm d}x)+\int_Q\mathcal K(e(u),\vartheta)\nabla\vartheta\cdot\nabla w-\vartheta\,\dot w\,{\rm d}x{\rm d}t+\int_{\Sigma_C}\eta([\![u]\!],z)[\![\Theta(\vartheta)]\!][\![w]\!]\,{\rm d}S{\rm d}t$$
$$=\int_Q\big(\mathbb De(\dot u):e(\dot u)-\Theta(\vartheta)\mathbb B:e(\dot u)\big)w\,{\rm d}x{\rm d}t+\int_{\overline{\Sigma}_C}\frac{w\,|^+_{\Gamma_C}+w|^-_{\Gamma_C}}{2}\,\xi^{\rm surf}_{\dot z}({\rm d}S{\rm d}t)$$
$$+\int_Q Gw\,{\rm d}x{\rm d}t+\int_\Sigma gw\,{\rm d}S{\rm d}t+\int_\Omega\vartheta_0 w(0)\,{\rm d}x \tag{4.16}$$

for all $w\in {\rm C}^0([0,T];W^{1,r'}(\Omega\setminus\Gamma_C))\cap W^{1,r'}(0,T;L^{r'}(\Omega))$, where $\vartheta_0:=h_0(\theta_0)$ and $\xi^{\rm surf}_{\dot z}$ is a measure (= heat produced by rate-independent dissipation) defined by prescribing its values for every closed set of the type $A:=[t_1,t_2]\times C\subset[0,T]\times\overline{\Gamma}_C$ as

$$\xi^{\rm surf}_{\dot z}(A):=\begin{cases}\displaystyle\int_C a_1\big|z(t_1,x)-z(t_2,x)\big|\,{\rm d}S & \text{if } z(\cdot,x) \text{ nonincreasing on } [t_1,t_2] \text{ for a.a. } x\in C,\\ +\infty & \text{elsewhere,}\end{cases} \tag{4.17}$$

(v) and the remaining initial conditions (in addition to $\dot u(0)=\dot u_0$, already involved in (4.13)), i.e.

$$u(0)=u_0\quad\text{a.e. in }\Omega,\qquad z(0)=z_0\quad\text{a.e. in }\Gamma_C,\qquad \vartheta(0)=\vartheta_0\quad\text{a.e. in }\Omega. \tag{4.18}$$

Remark 4.2. Notice that (4.14) is the integrated version of the total energy balance (3.13). It is immediate to check that, for every closed set of the type $A:=[t_1,t_2]\times C\subset[0,T]\times\overline{\Gamma}_C$, $\xi^{\rm surf}_{\dot z}(A)$ coincides with ${\rm Var}_{\mathcal R}(z|_C;[t_1,t_2])$, where we set

$${\rm Var}_{\mathcal R}(\tilde z;[t_1,t_2]):=\sup\sum_{i=1}^k\mathcal R\big(\tilde z(s_i)-\tilde z(s_{i-1})\big)\quad\forall\tilde z\in L^1(\Gamma_C),\ [t_1,t_2]\subset[0,T], \tag{4.19}$$

the sup taken over all partitions $t_1=s_0<\cdots<s_k=t_2$ of the interval $[t_1,t_2]$. Note also that, since $\vartheta\in{\rm BV}([0,T];W^{1,r'}(\Omega\setminus\Gamma_C)^*)$, then for all $t\in[0,T]$ one has $\vartheta(t)$ well defined in the sense of $W^{1,r'}(\Omega\setminus\Gamma_C)^*$. Combining this with the fact that $\vartheta\in L^\infty(0,T;L^1(\Omega))$ one sees that even $\vartheta(t)\in\mathcal M(\bar\Omega)$ for all $t\in[0,T]$, which has been exploited in (4.14) and (4.16) at the time $t=T$. It should be emphasized that one cannot expect the map $t\mapsto\vartheta(t)$ to be continuous in any sense, because the measure $\xi^{\rm surf}_{\dot z}$ in (4.17) may concentrate at particular time instants.

Now, relying, e.g., on [32, Proposition 1.3.10, Theorem 1.5.6], one can verify that formula (4.17) indeed defines a non-negative Radon measure on $\overline{\Sigma}_C$. Subtracting (4.16) tested by 1 from (4.14) reveals the mechanical energy equality:

$$T^{\varrho}_{\rm kin}\big(\dot u(T)\big)+\Phi\big(u(T),z(T)\big)+\int_Q\mathbb De(\dot u):e(\dot u)\,{\rm d}x{\rm d}t+{\rm Var}_{\mathcal R}(z;[0,T])$$
$$=T^{\varrho}_{\rm kin}\big(\dot u_0\big)+\Phi\big(u_0,z_0\big)+\int_Q F\cdot\dot u+\Theta(\vartheta)\mathbb B:e(\dot u)\,{\rm d}x{\rm d}t+\int_{\Sigma_N}f\cdot\dot u\,{\rm d}S{\rm d}t. \tag{4.20}$$

In particular, $z(\cdot,x)$ must be nonincreasing on $[0,T]$ for a.a. $x\in\Gamma_C$, otherwise ${\rm Var}_{\mathcal R}(z;[0,T])=\infty$ and (4.20) cannot hold.

5. Main results

Assumptions. Hereafter, we shall denote by the symbols C, C' most of the (positive) constants occurring in calculations and estimates. We suppose that

$$c_{\rm v}:[0,+\infty)\to\mathbb R^+\ \text{continuous,} \tag{5.1a}$$

$$\exists\,\omega_1\ge\omega>\frac{2d}{d+2},\ c_1\ge c_0>0\ \forall\theta\in\mathbb R^+:\quad c_0(1+\theta)^{\omega-1}\le c_{\rm v}(\theta)\le c_1(1+\theta)^{\omega_1-1}, \tag{5.1b}$$

$$\mathbb{K} : \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^{d\times d} \text{ is bounded, continuous, and} \tag{5.1c}$$

$$\inf_{(e,\vartheta,\xi)\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}\times\mathbb{R}^d,\ |\xi|=1} \mathcal{K}(e,\vartheta)\xi : \xi = \mathsf{k} > 0, \tag{5.1d}$$

and that $\eta(x, v, \cdot)$ is a non-negative affine function of the delamination parameter $z \in [0, 1]$, i.e.

$$\eta(x, v, z) = \eta_1(x, v)z + \eta_0(x, v) \text{ for } \eta_1, \eta_0 : \Gamma_C \times \mathbb{R}^d \to \mathbb{R}^+ \text{ Carathéodory s.t.}$$
$$\exists\, C_\eta > 0 \quad \forall\, (x, v) \in \Gamma_C \times \mathbb{R}^d \ : \ |\eta_0(x, v)| + |\eta_1(x, v)| \le C_\eta(|v|^{4/3} + 1); \tag{5.1e}$$

in fact, the above growth condition for the functions $\eta_0(x, \cdot)$ and $\eta_1(x, \cdot)$ is not optimal and could be slightly improved, as one can deduce from the proof of Theorem 6.1 in Section 8 later on. It is immediate to deduce from (5.1b) that

$$\exists\, C^1_\theta,\ C^2_\theta > 0 \quad \forall\, w \in \mathbb{R}^+ : \quad C^1_\theta(w^{1/\omega_1} - 1) \le \Theta(w) \le C^2_\theta(w^{1/\omega} - 1). \tag{5.2}$$

Moreover, it follows from (5.1c) and (5.2), and the definition (4.6) of $\mathcal{K}$ that

$$\exists\, C_{\mathcal{K}} > 0 \quad \forall\, \xi, \zeta \in \mathbb{R}^d : \quad |\mathcal{K}(e, \vartheta)\xi : \zeta| \le C_{\mathcal{K}}|\xi||\zeta|. \tag{5.3}$$

Finally, we impose the following on the initial data

$$u_0 \in W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d), \quad [\![u_0]\!] \succeq 0 \quad \text{on } \Sigma_C, \tag{5.4a}$$

$$\dot u_0 \in L^2(\Omega; \mathbb{R}^d) \quad \text{if } \varrho > 0, \tag{5.4b}$$

$$z_0 \in L^\infty(\Gamma_C), \qquad 0 \le z_0 \le 1 \quad \text{a.e. on } \Gamma_C, \tag{5.4c}$$

$$\theta_0 \in L^\omega(\Omega), \qquad \theta_0 \ge 0 \quad \text{a.e. in } \Omega. \tag{5.4d}$$

Theorem 5.1 (*Existence for the Adhesive Contact Problem*)**.** *Let us assume (4.8), (5.1)–(5.4) and*

(i) *if $\varrho = 0$ (such a case is sometimes referred to as quasistatic), let also*

$$F \in \begin{cases} W^{1,1}(0, T; L^{6/5}(\Omega; \mathbb{R}^3)) & \text{if } d = 3, \\ W^{1,1}(0, T; L^{1+\epsilon}(\Omega; \mathbb{R}^2)) & \text{for some } \epsilon > 0, \text{ if } d = 2; \end{cases} \tag{5.5a}$$

$$\mathscr{H}^{d-1}(\partial\Omega_+ \cap \Gamma_D) > 0, \qquad \mathscr{H}^{d-1}(\partial\Omega_- \cap \Gamma_D) > 0, \tag{5.5b}$$

with $\mathscr{H}^{d-1}$ denoting the $(d-1)$-dimensional Hausdorff measure, or

(ii) *if $\varrho > 0$ (such a case is referred to as dynamic), suppose also that*

$$K(x) \text{ is a linear subspace of } \mathbb{R}^d \text{ for a.a. } x \in \Gamma_C. \tag{5.6}$$

Then, there exists an energetic solution (u, z, ϑ) to the adhesive contact problem with the additional regularity

$$u \in W^{2,2}(0, T; W^{1,2}_K(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*) \quad \text{if } \varrho > 0 \tag{5.7}$$

cf. notation (4.2) for the space $W^{1,2}_K(\Omega \setminus \Gamma_C; \mathbb{R}^d)$. Furthermore, in both cases $\varrho > 0$ and $\varrho = 0$ the positivity of the initial temperature

$$\inf_{x\in\Omega} \theta_0 =: \theta^* > 0 \tag{5.8}$$

implies $\inf_{(t,x)\in Q} \theta = \inf_{(t,x)\in Q} \Theta(\vartheta(t, x)) > 0$; in particular, θ is a.e. positive on Q.

Theorem 5.1 shall be proved in Section 9 by passing to the limit in some regularized problem (where the contact conditions on Γ_C are penalized), which we shall present in Section 6. In turn, existence for the latter problem shall be proved in Section 8 by passing to the limit in a further approximation scheme, constructed in Section 7 by a regularized semi-implicit time discretization.

Remark 5.2 (*Energetics in the Dynamical Case*)**.** When $K(x)$ is a linear subspace of $\mathbb{R}^d$ for almost all $x \in \Gamma_C$, (which, in particular, we have to assume in the dynamical case), replacing $v - u$ with v, one can see that (4.13) is equivalent to

$$\begin{aligned} &\int_\Omega \varrho\dot u(T)\cdot v(T)\,\mathrm{d}x + \int_Q \big(\mathbb{D}e(\dot u) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)\big) : e(v) - \varrho\dot u\cdot\dot v\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_C} \kappa z[\![u]\!]\cdot[\![v]\!]\mathrm{d}S\mathrm{d}t \\ &\quad = \int_\Omega \varrho\dot u_0\cdot v(0)\mathrm{d}x + \int_Q F\cdot v\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_N} f\cdot v\,\mathrm{d}S\mathrm{d}t \end{aligned} \tag{5.9}$$

for all v in $L^2(0, T; W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d))$ with $\varrho v \in W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^d))$ with $[\![v]\!] \succeq 0$ on Σ_C. It is further important that, in this case, $\varrho\ddot u$ is in duality with $\dot u$, which allows to make a by-part-integration in (5.9) with $v := \dot u$. This shall eventually reveal the mechanical energy balance (4.20).

Remark 5.3 (*Energetics in General*)**.** In Theorem 5.1 we have distinguished the cases $\varrho > 0$ and $\varrho = 0$ because in the latter case we are able to prove existence for a far larger class of cones yielding the unilateral constraint on the displacement, in particular the Signorini conditions. This stems from the fact that the analysis of the momentum equilibrium equation in which inertia interacts with Signorini boundary conditions is remarkably complex. In particular, in such a framework the existence of solutions complying with the energy balance (which will play a crucial role in the analysis of own adhesive contact system) is, to our knowledge, an open problem in the case of bounded domains. Indeed, only very recently, in [33,34], existence results have emerged for the dynamical viscoelastic equation with Signorini contact conditions, in the one- and three-dimensional case, on *unbounded* domains. In fact, such results have been proved with very sophisticated Fourier analysis techniques. In the one-dimensional framework of [34], it has also been obtained that the solutions satisfy the energy balance.

Remark 5.4. Under (5.6), the qualification $v \in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^d))$ for the test functions in (4.13) might be relaxed to

$$v \in W^{1,2}(0,T;W^{1,2}_K(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*). \tag{5.10}$$

Indeed, thanks to (4.12) and to the linearity of $K(x)$ for almost all $x \in \Gamma_C$, the function u fulfilling (4.13) is such that $\dot u \in L^2(0,T;W^{1,2}_K(\Omega\setminus\Gamma_C;\mathbb{R}^d))$. Now, the spaces $L^2(0,T;W^{1,2}_K(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*)$ and $L^2(0,T;W^{1,2}_K(\Omega\setminus\Gamma_C;\mathbb{R}^d))$ are in duality. Hence, (5.10) is sufficient to give meaning to the term $\int_\Omega \dot u\cdot\dot v\,\mathrm{d}x$. A similar extension will also apply for the test functions of (6.7) below.

Remark 5.5. In fact, within the proof of Theorem 5.1 we shall also obtain that the mechanical energy equality (4.20) holds on *any* time interval $[0,t]\subset[0,T]$. To wit, a variant of (4.14) on a generic interval $[0,t]$ in place of $[0,T]$, i.e.

$$\begin{aligned} &T^{\varrho}_{\mathrm{kin}}\big(\dot u(t)\big) + \Phi\big(u(t),z(t)\big) + \int_{\bar\Omega}\vartheta(t,\mathrm{d}x) = T^{\varrho}_{\mathrm{kin}}\big(\dot u_0\big) + \Phi\big(u_0,z_0\big) + \int_\Omega \vartheta_0\,\mathrm{d}x \\ &\quad + \int_0^t\left(\int_\Omega F\cdot\dot u + G\,\mathrm{d}x + \int_{\Gamma_N} f\cdot\dot u\,\mathrm{d}S + \int_\Gamma g\,\mathrm{d}S\right)\mathrm{d}t; \end{aligned}\tag{5.11}$$

deserves some discussion. It does not seem to be fulfilled along arbitrary energetic solutions. However, some energetic solutions may satisfy it, and we may refer to them as "special energetic solutions'. For this, a careful selection of ϑ in the limit procedure based on Helly's principle is needed. Actually, assuming $\tau = 2^{-N}T$, $N\in\mathbb{N}$, and arguing as in the same way as one derives (4.14) (cf. Section 9), one can verify (5.11) at any mesh point of the form $t = k2^{-N}T$, $k = 1,\ldots,2^N$. The set of such t's is dense in $[0,T]$. One also uses that the map $t\mapsto\langle\vartheta(t,\cdot),1\rangle = \int_{\bar\Omega}\vartheta(t,\mathrm{d}x)$ is in $\mathrm{BV}([0,T])$ (although $\vartheta(t,\cdot)$ hardly can be expected in $\mathrm{BV}([0,T];\mathcal{M}(\bar\Omega))$). Finally, one exploits the heat energy balance

$$\begin{aligned} &\int_{\bar\Omega}\vartheta(t,\mathrm{d}x) - \int_\Omega\vartheta_0\,\mathrm{d}x - \int_0^t\left(\int_\Omega\big(\mathbb{D}e(\dot u):e(\dot u) - \Theta(\vartheta)\mathbb{B}:e(\dot u) + G\big)\,\mathrm{d}x\right)\mathrm{d}t \\ &\quad + \int_\Gamma g\,\mathrm{d}S - \int_{[0,t]\times\overline{\Gamma}_C}\xi^{\mathrm{surf}}_{\dot z}(\mathrm{d}S\mathrm{d}t) =: r(t) = 0. \end{aligned}\tag{5.12}$$

i.e. (4.16), considered on the time interval $[0,t]$, with test function $w = 1$. As $r\in\mathrm{BV}([0,T])$, we have $r(t) = 0$ not only at the t's in the aforementioned dense subset, but also at each continuity point of r. There are, however, at most countable points of discontinuity of r, at which one can define the limit ϑ suitably in such a way that, eventually, $r = 0$ everywhere. Adding (5.12) and the mechanical energy balance (4.20) on the generic interval $[0,t]$ then yields (5.11). We thus conclude the existence of *special energetic solutions*.

6. Regularization

We shall approximate (the enthalpy reformulation) of the adhesive contact system (4.7) by penalizing the contact condition $[\![u]\!]\succeq 0$. This is a well-established routine in the analysis of contact problems, see e.g. [30]. We should emphasize that the penalized problems themselves have their own practical usage because they allow, first, for combination of inertia and the unilateral-type elastic contact condition and, second, for a more physical interpretation of the coupling through the heat-transfer coefficient, cf. Remark 6.2 below.

Thus, we shall replace the subdifferential operator ∂I_K in the differential inclusion (3.10) (equivalent to the complementarity problem (2.5g)–(2.5i) on Σ_C), with its Yosida regularization (see, e.g., [35–37]). We recall that, for a fixed $\varepsilon > 0$, the Yosida approximation of the indicator functional I_K with respect to L^2-norm is the lower semicontinuous, convex, and Fr'echet differentiable functional given by

$$I^\varepsilon_K : L^2(\Gamma_C;\mathbb{R}^d)\to[0,+\infty) \quad \text{given by } I^\varepsilon_K(v) = \frac{1}{2\varepsilon}\min_{w\succeq 0}\|v-w\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}; \tag{6.1}$$

cf. definition (2.3) for the ordering $\succeq$. We point out that I_K^ε Mosco-converges to I_K in $L^2(\Gamma_C;\mathbb{R}^d)$; see, e.g., [35, Section 3.3] for the definition of Mosco-convergence and [35, Theorem 3.66] for the link with Yosida regularizations. In particular, this entails that

$$v_\varepsilon \rightharpoonup v \quad \text{in } L^2(\Gamma_C;\mathbb{R}^d) \Rightarrow \liminf_{\varepsilon\to 0} I_K^\varepsilon(v_\varepsilon) \geq I_K(v). \tag{6.2}$$

The Yosida regularization of ∂I_K is the Fréchet derivative $(I_K^\varepsilon)' : L^2(\Gamma_C;\mathbb{R}^d) \to L^2(\Gamma_C;\mathbb{R}^d)$ of the functional I_K^ε. It is well known that

$$(I_K^\varepsilon)' = \frac{1}{\varepsilon}\left(\mathrm{Id} - \mathrm{P}_K\right), \tag{6.3}$$

where $\mathrm{Id} : L^2(\Gamma_C;\mathbb{R}^d) \to L^2(\Gamma_C;\mathbb{R}^d)$ is the identity operator and $\mathrm{P}_K : L^2(\Gamma_C;\mathbb{R}^d) \to L^2(\Gamma_C;\mathbb{R}^d)$ is the projection associated with the multivalued mapping $K : \Gamma_C \rightrightarrows \mathbb{R}^d$. For later use, we recall that, being P_K a contraction on $L^2(\Gamma_C;\mathbb{R}^d)$, there holds

$$\|(I_K^\varepsilon)'(v)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leq \frac{2}{\varepsilon}\|v\|_{L^2(\Gamma_C;\mathbb{R}^d)} \quad \text{for all } v \in L^2(\Gamma_C;\mathbb{R}^d). \tag{6.4}$$

Hence, we shall consider the following regularized conditions on Γ_C, where (3.10) is approximated by *Yosida regularization*:

$$\left.\begin{array}{l}
[\![\mathbb{D}e(\dot u) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)]\!]\nu^\pm = 0\\
\kappa z[\![u]\!] + (I_K^\varepsilon)'([\![u]\!]) + \mathcal{T}(u,\dot u,\vartheta) = 0\\
\partial I_{(-\infty,0]}(\dot z) + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa\left|[\![u]\!]\right|^2 - a_0 - a_1 \ni 0\\
\frac{1}{2}\left(\mathcal{K}(e(u),\vartheta)\nabla\vartheta\,|_{\Gamma_C}^+ + \mathcal{K}(e(u),\vartheta)\nabla\vartheta\,|_{\Gamma_C}^-\right)\cdot\nu^\pm + \eta([\![u]\!],z)[\![\Theta(\vartheta)]\!] = 0\\
[\![\mathcal{K}(e(u),\vartheta)\nabla\vartheta]\!]\cdot\nu^\pm = -a_1\dot z
\end{array}\right\} \quad \text{on } \Sigma_C. \tag{6.5}$$

The resulting regularized stored energy is then

$$\Phi_\varepsilon(u,z) := \int_{\Omega\setminus\Gamma_C}\frac{1}{2}\mathbb{C}e(u) : e(u)\,\mathrm{d}x + I_K^\varepsilon([\![u]\!]) + \int_{\Gamma_C}\left(\tfrac{\kappa}{2}z\left|[\![u]\!]\right|^2 + I_{[0,1]}(z) - a_0 z\right)\mathrm{d}S. \tag{6.6a}$$

The main result of this section ensures the existence of energetic solutions to the initial boundary value problem for the adhesive contact model supplemented with the regularized contact conditions (6.5).

Theorem 6.1 (*Existence of Energetic Solutions to the Regularized Problem*)**.** *Under assumptions (4.8) and (5.1)–(5.5), for every $\varepsilon > 0$ there exists a triple $(u_\varepsilon, z_\varepsilon, \vartheta_\varepsilon)$ as in (4.11), and such that, in addition,*

$$u_\varepsilon \in W^{2,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*) \quad \text{if } \varrho > 0,$$

which solves the energetic formulation of the Cauchy problem for system (4.7a)–(4.7d) and (6.5), namely the initial conditions (4.18) hold, as well as

(i) *the (weak formulation of the) momentum equation:*

$$\int_Q\left(\mathbb{D}e(\dot u_\varepsilon) + \mathbb{C}e(u_\varepsilon) - \mathbb{B}\Theta(\vartheta_\varepsilon)\right) : e(v) - \varrho\dot u_\varepsilon\cdot\dot v\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_C}\left(\kappa z_\varepsilon[\![u_\varepsilon]\!] + (I_K^\varepsilon)'([\![u_\varepsilon]\!])\right)\cdot[\![v]\!]\,\mathrm{d}S\mathrm{d}t$$
$$+ \int_\Omega \varrho\dot u_\varepsilon(T)\cdot v(T)\,\mathrm{d}x = \int_\Omega\varrho\dot u_0\cdot v(0)\,\mathrm{d}x + \int_Q F\cdot v\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_N} f\cdot v\,\mathrm{d}S\mathrm{d}t \tag{6.7}$$

for all v in $L^2(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d))$ and, in the case $\varrho > 0$, in $W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^d))$,

(ii) *the total energy identity (4.14), with Φ_ε and $(u_\varepsilon, z_\varepsilon, \vartheta_\varepsilon)$ in place of Φ and (u,z,ϑ),*

(iii) *z_ε complies with the semistability condition (4.15) for a.a. $t \in (0,T)$, again with Φ_ε and $(u_\varepsilon, z_\varepsilon)$, in place of Φ and (u,z)*

(iv) *the weak formulation (4.16) of the enthalpy equation.*

Furthermore, if (5.8) holds, then

$$\inf_{\varepsilon>0,(t,x)\in Q}\vartheta_\varepsilon(t,x) > 0. \tag{6.8}$$

Remark 6.2 (*Signorini contact Case*)**.** Let us point out that, if $K : \Gamma_C \rightrightarrows \mathbb{R}^d$ is of the type (2.4b), i.e. corresponding to unilateral frictionless Signorini contact on Γ_C, the Yosida regularization of ∂I_K is given by $(I_K^\varepsilon)'(u_\varepsilon) = -\frac{1}{\varepsilon}[u_\varepsilon]_\nu^-$ with $[u_\varepsilon]_\nu := [\![u_\varepsilon]\!]\cdot\nu^\pm$ and with $(\cdot)^- = -\min\{0,\cdot\}$, and the second of (6.5) reduces to

$$\kappa z[\![u_\varepsilon]\!] - \frac{1}{\varepsilon}[\![u_\varepsilon]\!]_\nu^-\,\nu^\pm + T(u_\varepsilon,\dot u_\varepsilon,\theta_\varepsilon) = 0 \quad \text{on } \Sigma_C,$$

viz., a *normal compliance* type condition. It follows from the above relation that, for fixed $\varepsilon > 0$, in the case $z(t,x) > 0$ one can express $[\![u]\!]$ as a function of the traction stress $T(u,\dot{u},\theta)$. This also holds for $[u]_\nu$ in the case for $z(t,x) = 0$ and $T_\nu(u,\dot{u},\theta) > 0$, while the tangential stress $T_t(u,\dot{u},\theta) = 0$ because there is no friction. Let us suppose that the heat-transfer coefficient $\eta(\cdot,z)$ vanishes if there is no contact, i.e. on the set $\{[\![u]\!];\ [u]_\nu < 0\}$: then, by continuity, $\eta(\cdot,z)$ vanishes also on $\{[\![u]\!];\ [u]_\nu \le 0\}$, and hence by substitution one can express the heat-transfer coefficient as a function of the normal stress and of z. We point out that the mentioned condition on $\eta(\cdot,z)$ is, to some extent, a natural assumption, also advocated in the engineering literature, cf. e.g. [38]. Such an approach does not seem mathematically amenable for the multidimensional non-penalized Signorini problem; for $d = 1$ we refer to [22].

Scheme of the proof. The proof of Theorem 6.1 shall be developed in the next sections by pursuing the following scenario. First, in Section 7 we shall devise a semi-implicit time discretization (with a further regularization in the momentum equation), and prove existence of solutions to the time-discrete problem. Next, we shall derive refined a priori estimates, enabling us to perform the limit passage as the discretization time-step τ goes to 0 in Section 8. In this way, we shall conclude the proof of Theorem 6.1.

7. Semi-implicit time discretization

We perform a semi-implicit time discretization using an equidistant partition of $[0,T]$, with time-step $\tau > 0$ and nodes $t_\tau^k := k\tau$, $k = 0,\dots,K_\tau$.

Hereafter, given any sequence $\{\phi^j\}_{j\ge1}$, we shall use the following notation for the *backward difference operator* and its iteration by, respectively,

$$\mathrm{D}_t\phi^k := \frac{\phi^k-\phi^{k-1}}{\tau}, \qquad \mathrm{D}_t^2\phi^k := \mathrm{D}_t\big(\mathrm{D}_t\phi^k\big) = \frac{\phi^k-2\phi^{k-1}+\phi^{k-2}}{\tau^2}. \tag{7.1}$$

Secondly, we recall the notion of piecewise constant and piecewise linear interpolants: for a given K_τ-tuple $\{b_\tau^k\}_{k=1}^{K_\tau} \subset \mathcal{B}$, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ being some Banach space, the left-continuous piecewise constant interpolant $\overline{b}_\tau : (0,T) \to \mathcal{B}$, the right-continuous piecewise constant interpolant $\underline{b}_\tau : (0,T) \to \mathcal{B}$, and the piecewise linear interpolant $b_\tau : (0,T) \to \mathcal{B}$ of the elements $\{b_\tau^k\}_{k=1}^{K_\tau}$ are the functions respectively defined by

$$\overline{b}_\tau(t) = b_\tau^k, \qquad \underline{b}_\tau(t) = b_\tau^{k-1}, \qquad b_\tau(t) = \frac{t-t_\tau^{k-1}}{\tau}b_\tau^k + \frac{t_\tau^k - t}{\tau}b_\tau^{k-1} \quad \text{for } t \in (t_\tau^{k-1}, t_\tau^k]. \tag{7.2}$$

Thirdly, we shall denote by $\bar{\mathrm{t}}_\tau$ and by $\underline{\mathrm{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition, i.e. $\bar{\mathrm{t}}_\tau(t) = t_\tau^k$ if $t_\tau^{k-1} < t \le t_\tau^k$ and $\underline{\mathrm{t}}_\tau(t) = t_\tau^{k-1}$ if $t_\tau^{k-1} \le t < t_\tau^k$. For later use, we recall the following elementary inequalities for all $t \in [0,T]$

$$\|b_\tau(t)\|_{\mathcal{B}} \le \|\overline{b}_\tau(t)\|_{\mathcal{B}} + \|\underline{b}_\tau(t)\|_{\mathcal{B}} = \|\overline{b}_\tau(t)\|_{\mathcal{B}} + \|\overline{b}_\tau\left(\underline{\mathrm{t}}_\tau(t)\right)\|_{\mathcal{B}}, \tag{7.3}$$

$$\left\|\overline{b}_\tau(t) - b_\tau(t)\right\|_{\mathcal{B}} \le \int_{\underline{\mathrm{t}}_\tau(t)}^{\bar{\mathrm{t}}_\tau(t)} \|b_\tau'(s)\|_{\mathcal{B}}\,\mathrm{d}s. \tag{7.4}$$

Time-discrete problem. We approximate the data F, f by local means, i.e. setting for all $k = 1,\dots,K_\tau$

$$F_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} F(s)\,\mathrm{d}s, \qquad f_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} f(s)\,\mathrm{d}s,$$

and consider the interpolants $\overline{F}_\tau$, $\overline{f}_\tau$, and f_τ of the K_τ-tuples $\{F_\tau^k\}_{k=1}^{K_\tau}$, $\{f_\tau^k\}_{k=1}^{K_\tau}$. In view of (4.8a)–(4.8b), the following estimates and strong convergences hold as $\tau \to 0$

$$\overline{F}_\tau \to F \begin{cases} \text{in } L^1(0,T;L^2(\Omega;\mathbb{R}^d)) & \text{if } \varrho > 0, \\ \text{in } L^p(0,T;L^{6/5}(\Omega;\mathbb{R}^d)) \text{ for all } 1 \le p < \infty & \text{if } \varrho = 0; \end{cases} \tag{7.5a}$$

$$\begin{aligned} &\exists\, C > 0 \quad \forall\, \tau > 0 : \ \|\overline{f}_\tau\|_{L^\infty(0,T;L^{4/3}(\Gamma_N;\mathbb{R}^d))} \le C\|f\|_{L^\infty(0,T;L^{4/3}(\Gamma_N;\mathbb{R}^d))}, \\ &\overline{f}_\tau \to f \quad \text{in } L^p(0,T;L^{4/3}(\Gamma_N;\mathbb{R}^d)) \text{ for all } 1 \le p < \infty \text{ as } \tau \to 0, \\ &\exists\, C > 0 \quad \forall\, \tau > 0 : \ \|\dot{f}_\tau\|_{L^1(0,T;L^{4/3}(\Gamma_N;\mathbb{R}^d))} \le 2\|\dot{f}\|_{L^1(0,T;L^{4/3}(\Gamma_N;\mathbb{R}^d))}. \end{aligned} \tag{7.5b}$$

Notice that the exponents of the Lebesgue spaces in the second of (7.5a) and in (7.5b) are suited both to the case $d = 2$ and to the three-dimensional case $d = 3$. Furthermore, we shall approximate G and g with suitably constructed discrete data $\{G_\tau^k\}_{k=1}^{K_\tau}$, $\{g_\tau^k\}_{k=1}^{K_\tau}$ with

$$G_\tau^k \in W^{1,2}(\Omega)^*, \qquad g_\tau^k \in H^{1/2}(\partial\Omega)^* \quad \text{for all } k = 1,\dots,K_\tau, \tag{7.6a}$$

and such that

$$\overline{G}_\tau \to G \quad \text{in } L^1(Q), \qquad \overline{g}_\tau \to g \qquad \text{in } L^1(\Sigma) \text{ as } \tau \to 0, \tag{7.6b}$$

and approximate the initial datum u_0 with a sequence $\{u_{0,\tau}\} \subset W^{1,\gamma}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d)$ (with $\gamma > \max\{4, \frac{2\omega}{\omega-1}\}$ as assumed in Problem 7.1) such that

$$\lim_{\tau \downarrow 0} \sqrt[\gamma]{\tau} \|e(u_{0,\tau})\|_{L^\gamma(\Omega;\mathbb{R}^d)} = 0, \qquad u_{0,\tau} \to u_0 \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^d) \text{ as } \tau \to 0. \tag{7.7}$$

We are now in the position of formulating the time-discrete problem, which we again write in the classical formulation for notational simplicity.

Problem 7.1. Let $\gamma > \max\{4, \frac{2\omega}{\omega-1}\}$. Given $\mu \in (4,5)$, $\alpha, \beta \in (0,1)$, and

$$u^0_\tau = u_{0,\tau}, \qquad u^{-1}_\tau = u_{0,\tau} - \tau \dot{u}_0, \qquad z^0_\tau = z_0, \qquad \vartheta^0_\tau = \vartheta_0, \tag{7.8}$$

find $\{(u^k_{\varepsilon\tau}, \vartheta^k_{\varepsilon\tau}, z^k_{\varepsilon\tau})\}_{k=1}^{K_\tau}$ fulfilling for $k = 1, \dots, K_\tau$ the equations in $\Omega \setminus \Gamma_C$

$$\varrho \mathrm{D}^2_t u^k_{\varepsilon\tau} - \mathrm{div}\Big(\mathbb{D}e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) + \mathbb{C}e(u^k_{\varepsilon\tau}) - \mathbb{B}\Theta(\vartheta^k_{\varepsilon\tau}) + \tau \big|e(u^k_{\varepsilon\tau})\big|^{\gamma-2} e(u^k_{\varepsilon\tau})\Big) = F^k_\tau, \tag{7.9a}$$

$$\mathrm{D}_t \vartheta^k_{\varepsilon\tau} - \mathrm{div}\big(\mathcal{K}(e(u^k_{\varepsilon\tau}), \vartheta^k_{\varepsilon\tau})\nabla\vartheta^k_{\varepsilon\tau}\big) = \frac{2-\sqrt{\tau}}{2}\mathbb{D}e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) : e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) - \Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B} : e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) + G^k_\tau, \tag{7.9b}$$

with the boundary conditions

$$u^k_{\varepsilon\tau} = 0 \quad \text{on } \Gamma_D, \tag{7.10a}$$

$$\Big(\mathbb{D}e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) + \mathbb{C}e(u^k_{\varepsilon\tau}) - \Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B} + \tau \big|e(u^k_{\varepsilon\tau})\big|^{\gamma-2} e(u^k_{\varepsilon\tau})\Big)\nu = f^k_\tau \quad \text{on } \Gamma_N, \tag{7.10b}$$

$$\big(\mathcal{K}(e(u^k_{\varepsilon\tau}), \vartheta^k_{\varepsilon\tau})\nabla\vartheta^k_{\varepsilon\tau}\big) \cdot \nu = g^k_\tau \quad \text{on } \partial\Omega, \tag{7.10c}$$

and the conditions on the contact boundary

$$\left.\begin{array}{l} \partial I_{(-\infty,0]}\big(\mathrm{D}_t z^k_{\varepsilon\tau}\big) - a_0 - a_1 + \dfrac{\kappa}{2}\big|[\![u^k_{\varepsilon\tau}]\!]\big|^2 + \tau^\alpha z^k_{\varepsilon\tau} + r(z^k_{\varepsilon\tau}) \ni 0 \\ r(z^k_{\varepsilon\tau}) \in \partial I_{[0,1]}(z^k_{\varepsilon\tau}) \end{array}\right\} \quad \text{on } \Gamma_C, \tag{7.11a}$$

$$[\![\mathbb{D}e(\mathrm{D}_t u^k_{\varepsilon\tau}) + \mathbb{C}e(u^k_{\varepsilon\tau}) - \Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B} + \tau \big|e(u^k_{\varepsilon\tau})\big|^{\gamma-2} e(u^k_{\varepsilon\tau})]\!]\nu^\pm = 0 \quad \text{on } \Gamma_C, \tag{7.11b}$$

$$\begin{aligned} &\kappa z^k_{\varepsilon\tau}[\![u^k_{\varepsilon\tau}]\!] + (I^\varepsilon_K)'([\![u^k_{\varepsilon\tau}]\!]) + \Big(\mathbb{D}e(\mathrm{D}_t u^k_{\varepsilon\tau}) + \mathbb{C}e(u^k_{\varepsilon\tau}) - \Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B} \\ &\qquad + \tau \big|e(u^k_{\varepsilon\tau})\big|^{\gamma-2} e(u^k_{\varepsilon\tau})\Big)\nu^\pm + \tau^\beta\big(1 + \big|[\![u^k_{\varepsilon\tau}]\!]\big|^2\big)^{\frac{\mu}{2}-1}[\![u^k_{\varepsilon\tau}]\!] = 0 \quad \text{on } \Gamma_C, \end{aligned} \tag{7.11c}$$

$$\frac{1}{2}\big(\mathcal{K}(e(u^k_{\varepsilon\tau}), \vartheta^k_{\varepsilon\tau})\nabla\vartheta^k_{\varepsilon\tau}\,|^+_{\Gamma_C} + \mathcal{K}(e(u^k_{\varepsilon\tau}), \vartheta^k_{\varepsilon\tau})\nabla\vartheta^k_{\varepsilon\tau}|^-_{\Gamma_C}\big) \cdot \nu^\pm + \eta([\![u^{k-1}_{\varepsilon\tau}]\!], z^k_{\varepsilon\tau})[\![\Theta(\vartheta^k_{\varepsilon\tau})]\!] = 0 \quad \text{on } \Gamma_C, \tag{7.11d}$$

$$[\![\mathcal{K}(e(u^k_{\varepsilon\tau}), \vartheta^k_{\varepsilon\tau})\nabla\vartheta^k_{\varepsilon\tau}]\!]\nu^\pm = -a_1 \mathrm{D}_t z^k_{\varepsilon\tau} \quad \text{on } \Gamma_C. \tag{7.11e}$$

For later use, we recall the variational formulation of (7.9a), supplemented with conditions (7.10b), (7.10c), (7.11b) and (7.11c), viz.

$$\begin{aligned} &\int_\Omega \varrho \mathrm{D}^2_t u^k_{\varepsilon\tau} \cdot v \,\mathrm{d}x + \int_\Omega \Big(\mathbb{D}e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) + \mathbb{C}e(u^k_{\varepsilon\tau}) - \mathbb{B}\Theta(\vartheta^k_{\varepsilon\tau}) + \tau\big|e(u^k_{\varepsilon\tau})\big|^{\gamma-2}e(u^k_{\varepsilon\tau})\Big) : e(v)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_C}\Big(\kappa z^k_{\varepsilon\tau}[\![u^k_{\varepsilon\tau}]\!] + (I^\varepsilon_K)'([\![u^k_{\varepsilon\tau}]\!]) + \tau^\beta\big(1+\big|[\![u^k_{\varepsilon\tau}]\!]\big|^2\big)^{\frac{\mu}{2}-1}[\![u^k_{\varepsilon\tau}]\!]\Big) \cdot [\![v]\!]\,\mathrm{d}S = \int_\Omega F^k_\tau \cdot v\,\mathrm{d}x + \int_{\Gamma_N} f^k_\tau \cdot v\,\mathrm{d}S \end{aligned} \tag{7.12}$$

for all $v \in W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d)$. Furthermore, the variational formulation of (7.11a) reads, for all $\tilde{z} \in L^\infty(\Gamma_C)$,

$$\int_{\Gamma_C} \zeta_1(\tilde{z}) + \Big(\tau^\alpha z^k_{\varepsilon\tau} + r(z^k_{\varepsilon\tau}) + \frac{\kappa}{2}\big|[\![u^k_{\varepsilon\tau}]\!]\big|^2 - a_0\Big)\left(\tilde{z} - \frac{z^k_{\varepsilon\tau} - z^{k-1}_{\varepsilon\tau}}{\tau}\right)\mathrm{d}S \geq \int_{\Gamma_C} \zeta_1\left(\frac{z^k_{\varepsilon\tau} - z^{k-1}_{\varepsilon\tau}}{\tau}\right)\mathrm{d}S. \tag{7.13}$$

Remark 7.2 (*Semi-implicit Discretization*)**.** The value $u^{k-1}_{\varepsilon\tau}$ at the level $k-1$ in (7.11d) makes the above scheme semi-implicit, not just fully implicit as it would be if $u^k_{\varepsilon\tau}$ were in place of $u^{k-1}_{\varepsilon\tau}$ in (7.11d). This makes the proof of Lemma 7.4 easier.

Remark 7.3 (*Regularization*)**.** Like in [12], a regularizing term $\tau|e(u)|^{\gamma-2}e(u)$ was added to the momentum equation in the bulk and to the corresponding boundary/contact conditions, too. Its role is to compensate the growth of the right-hand side terms in the momentum equation, cf. the proof of Lemma 7.4. Moreover, with the aim of obtaining some suitable semiconvexity property of the approximate stored energy (cf. Lemma 7.5 below), we have also introduced the monotone terms

$$\tau^\alpha z \quad \text{and} \quad \tau^\beta\big(1+\big|[\![u]\!]\big|^2\big)^{\frac{\mu}{2}-1}[\![u]\!], \quad \text{with } 4<\mu<5 \quad \text{and} \quad \alpha,\beta\in(0,1) \tag{7.14}$$

in the differential inclusion for the delamination parameter and in the boundary conditions for u on Γ_C, respectively; see Lemma 7.5 for some further specification of the exponents α and β.

Lemma 7.4 (*Existence of Weak Solutions to* Problem 7.1)**.** *Under the assumptions of Theorem 6.1, for every $k=1,\ldots,K_\tau$ there exists a triple $(u^k_{\varepsilon\tau},z^k_{\varepsilon\tau},\vartheta^k_{\varepsilon\tau})\in W^{1,\gamma}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)\times L^\infty(\Gamma_C)\times W^{1,2}(\Omega\setminus\Gamma_C)$, fulfilling the weak formulation of the boundary value problem (7.9)–(7.11). Moreover, $\vartheta^k_{\varepsilon\tau}\geq 0$ a.e. in Ω. If, in addition, (5.8) holds, then there exists some constant $\chi^*>0$ (cf. (7.26) below) such that, for sufficiently small τ,*

$$\vartheta^k_{\varepsilon\tau}\geq\chi^*>0 \quad \textit{a.e. in } \Omega \quad \textit{for every } k=1,\ldots,K_\tau. \tag{7.15}$$

Proof. The existence of a weak solution to the boundary value problem (7.9)–(7.11) can be proved relying on the standard theory of pseudomonotone set-valued operators (see e.g. [39, Chap. 2]). In particular, we may apply Leray–Lions-type theorems. Indeed, the strict monotonicity of the main part of the operator which comes into play in the weak formulation of problem (7.9)–(7.11) derives from the presence of the term $\tau|e(u)|^{\gamma-2}e(u)$. The latter counteracts the quadratic nonlinearity in $e(u)$ of the dissipative heat source in (7.9b).

We now show that this operator is coercive w.r.t. the norm of $W^{1,\gamma}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)\times L^\infty(\Gamma_C)\times W^{1,2}(\Omega\setminus\Gamma_C)$. To this aim, first of all we test equation (7.9a), with the boundary conditions (7.10b)–(7.10c), and the contact conditions (7.11a) and (7.11c), by $u^k_{\varepsilon\tau}$. Thus, estimating the term

$$\int_\Omega \mathbb{D}\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big):e(u^k_{\varepsilon\tau})\,\mathrm{d}x\geq\frac{1}{2\tau}\int_\Omega\mathbb{D}|e(u^k_{\varepsilon\tau})|^2\,\mathrm{d}x-\frac{1}{2\tau}\int_\Omega\mathbb{D}|e(u^{k-1}_{\varepsilon\tau})|^2\,\mathrm{d}x, \tag{7.16}$$

we find

$$\begin{aligned}
&\frac{\varrho}{2\tau^2}\|u^k_{\varepsilon\tau}\|^2_{L^2(\Omega;\mathbb{R}^d)}+\frac{\mathsf{d}}{2\tau C_{K,2}}\|u^k_{\varepsilon\tau}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}+\frac{\tau}{C_{K,\gamma}}\|u^k_{\varepsilon\tau}\|^\gamma_{W^{1,\gamma}(\Omega;\mathbb{R}^d)}\\
&\quad+\kappa\int_{\Gamma_C}z^k_{\varepsilon\tau}\big|[\![u^k_{\varepsilon\tau}]\!]\big|^2\,\mathrm{d}S+\int_{\Gamma_C}(I^\varepsilon_K)'([\![u^k_{\varepsilon\tau}]\!])\cdot[\![u^k_{\varepsilon\tau}]\!]\,\mathrm{d}S+\tau^\beta\int_{\Gamma_C}\Big(1+\big|[\![u^k_{\varepsilon\tau}]\!]\big|^2\Big)^{\mu/2-1}\big|[\![u^k_{\varepsilon\tau}]\!]\big|^2\,\mathrm{d}S\\
&\quad\leq C\Big(\|u^{k-1}_{\varepsilon\tau}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}+\varrho\|u^{k-2}_{\varepsilon\tau}\|^2_{L^2(\Omega;\mathbb{R}^d)}+\|F^k_\tau\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)^*}+\|f^k_\tau\|^2_{H^{1/2}(\Gamma_N;\mathbb{R}^d)^*}\Big)+4\tau|\mathbb{E}|^2\|\Theta(\vartheta^k_{\varepsilon\tau})\|^2_{L^2(\Omega)},
\end{aligned} \tag{7.17}$$

where $\mathsf{d}:=\inf_{\xi\in\mathbb{R}^{d\times d},\,|\xi|=1}\mathbb{D}\xi:\xi>0$, cf. (2.6), and where we have also used Korn's inequality in the form

$$\exists\, C_{K,\gamma}=C_{K,\gamma}(\Omega,\Gamma_D)\quad\forall\, v\in W^{1,\gamma}_{\Gamma_D}(\Omega;\mathbb{R}^d)\ :\qquad \|v\|_{W^{1,\gamma}(\Omega;\mathbb{R}^d)}\leq C_{K,\gamma}\|e(v)\|_{L^\gamma(\Omega;\mathbb{R}^{d\times d})}. \tag{7.18}$$

Also taking into account the monotonicity of the operator $(I^\varepsilon_K)'$, we have that the fourth, the fifth and the sixth term on the left-hand side of (7.17) are non-negative. Secondly, we test (7.11a) by $z^k_{\varepsilon\tau}$. Noting that $\int_{\Gamma_C}\partial I_{(-\infty,0]}\big(\mathrm{D}_t z^k_{\varepsilon\tau}\big)z^k_{\varepsilon\tau}\,\mathrm{d}S\geq 0$ (where, to avoid too heavy notation, we have formally dealt with $\partial I_{(-\infty,0]}\big(\mathrm{D}_t z^k_{\varepsilon\tau}\big)$ as if it were a singleton), we obtain with trivial calculations

$$\frac{\kappa}{2}\int_{\Gamma_C}z^k_{\varepsilon\tau}\big|[\![u^k_{\varepsilon\tau}]\!]\big|^2\,\mathrm{d}S+\tau^\alpha\|z^k_{\varepsilon\tau}\|^2_{L^2(\Gamma_C)}+\int_{\Gamma_C}r(z^k_{\varepsilon\tau})z^k_{\varepsilon\tau}\,\mathrm{d}S\leq(a_1+a_0)\|z^k_{\varepsilon\tau}\|_{L^1(\Gamma_C)}. \tag{7.19}$$

Note that the third term on the left-hand side of (7.19) is non-negative by monotonicity of the operator $\partial I_{[0,1]}$. Finally, we test (7.9b) by $\vartheta^k_{\varepsilon\tau}$, thus obtaining

$$\begin{aligned}
&\frac{1}{2\tau}\|\vartheta^k_{\varepsilon\tau}\|^2_{L^2(\Omega)}+\mathsf{k}\int_\Omega|\nabla\vartheta^k_{\varepsilon\tau}|^2\,\mathrm{d}x+\int_{\Gamma_C}\eta([\![u^{k-1}_{\varepsilon\tau}]\!],z^k_{\varepsilon\tau})[\![\Theta(\vartheta^k_{\varepsilon\tau})]\!][\![\vartheta^k_{\varepsilon\tau}]\!]\,\mathrm{d}S\\
&\quad\leq\frac{1}{2\tau}\|\vartheta^{k-1}_{\varepsilon\tau}\|^2_{L^2(\Omega)}+I_1+I_2+I_3+I_4
\end{aligned} \tag{7.20}$$

where we have used that $\mathcal{K}$ is positive definite (cf. with (5.1d)). As for the remaining terms I_j, $j = 1, \dots, 4$, we have

$$
\begin{aligned}
I_1 &= \frac{2-\sqrt{\tau}}{2\tau^2} \int_{\Omega\setminus\Gamma_C} \mathbb{D}(e(u^k_{\varepsilon\tau}) - e(u^{k-1}_{\varepsilon\tau})) : (e(u^k_{\varepsilon\tau}) - e(u^{k-1}_{\varepsilon\tau}))\vartheta^k_{\varepsilon\tau}\, \mathrm{d}x \\
&\le \frac{1}{4\tau}\|\vartheta^k_{\varepsilon\tau}\|^2_{L^2(\Omega)} + C\left(\|e(u^k_{\varepsilon\tau})\|^4_{L^4(\Omega;\mathbb{R}^{d\times d})} + \|e(u^{k-1}_{\varepsilon\tau})\|^4_{L^4(\Omega;\mathbb{R}^{d\times d})}\right) \\
&\le \frac{1}{4\tau}\|\vartheta^k_{\varepsilon\tau}\|^2_{L^2(\Omega)} + \frac{\tau}{4C_{K,\gamma}}\|u^k_{\varepsilon\tau}\|^\gamma_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} + C\|u^{k-1}_{\varepsilon\tau}\|^\gamma_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} + C',
\end{aligned}
\tag{7.21}
$$

where we have used Hölder's inequality, the fact that $\gamma > 4$, and Young's inequality. Furthermore, relying on (5.2), and setting $p_\omega = 2\omega/(\omega-1)$, we find

$$
\begin{aligned}
I_2 &= \left|\frac{1}{\tau}\int_{\Omega\setminus\Gamma_C} \Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B} : \big(e(u^k_{\varepsilon\tau}) - e(u^{k-1}_{\varepsilon\tau})\big)\vartheta^k_{\varepsilon\tau}\, \mathrm{d}x\right| \\
&\le \frac{1}{16\tau}\|\vartheta^k_{\varepsilon\tau}\|^2_{L^2(\Omega)} + C\int_\Omega \left|e(u^k_{\varepsilon\tau}) - e(u^{k-1}_{\varepsilon\tau})\right|^2 \left|(\vartheta^k_{\varepsilon\tau})^{2/\omega} + 1\right| \mathrm{d}x \\
&\le \frac{1}{8\tau}\|\vartheta^k_{\varepsilon\tau}\|^2_{L^2(\Omega)} + C\left(\|e(u^k_{\varepsilon\tau})\|^{p_\omega}_{L^{p_\omega}(\Omega;\mathbb{R}^{d\times d})} + \|e(u^{k-1}_{\varepsilon\tau})\|^{p_\omega}_{L^{p_\omega}(\Omega;\mathbb{R}^{d\times d})} + 1\right) \\
&\le \frac{1}{8\tau}\|\vartheta^k_{\varepsilon\tau}\|^2_{L^2(\Omega)} + \frac{\tau}{8C_{K,\gamma}}\|u^k_{\varepsilon\tau}\|^\gamma_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} + C\left(\|u^{k-1}_{\varepsilon\tau}\|^\gamma_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} + 1\right),
\end{aligned}
\tag{7.22}
$$

where we have successively used Hölder's and Young inequalities, and that $\gamma > p_\omega$ due to our assumption that $\gamma > \max\{4, \frac{2\omega}{\omega-1}\}$. Besides, using that $0 \le z^k_{\varepsilon\tau} \le 1$ and $0 \le z^{k-1}_{\varepsilon\tau} \le 1$ a.e. in Γ_C, and the continuous embedding (4.3), we find

$$
I_3 = -a_1\int_{\Gamma_C} \mathrm{D}_t z^k_{\varepsilon\tau}\frac{\vartheta^k_{\varepsilon\tau}|^+_{\Gamma_C} + \vartheta^k_{\varepsilon\tau}|^-_{\Gamma_C}}{2}\,\mathrm{d}S \le C\|\mathrm{D}_t z^k_{\varepsilon\tau}\|_{L^2(\Gamma_C)}\|\vartheta^k_{\varepsilon\tau}\|_{L^2(\Gamma_C)} \le \rho_1\|\vartheta^k_{\varepsilon\tau}\|^2_{W^{1,2}(\Omega)} + C_{\rho_1}
$$

where we choose $\rho_1 > 0$ (C_{ρ_1} being some constant depending on $\rho_1 > 0$) in such a way as to absorb $\|\vartheta^k_{\varepsilon\tau}\|^2_{W^{1,2}(\Omega)}$ into the left-hand side of (7.20). Finally, we have

$$
I_4 = \int_\Omega G^k_\tau\vartheta^k_{\varepsilon\tau}\,\mathrm{d}x + \int_\Gamma g^k_\tau\vartheta^k_{\varepsilon\tau}\,\mathrm{d}S \le \rho_2\|\vartheta^k_{\varepsilon\tau}\|^2_{W^{1,2}(\Omega)} + C_{\rho_2}\left(\|G^k_\tau\|^2_{W^{1,2}(\Omega)^*} + \|g^k_\tau\|^2_{H^{1/2}(\partial\Omega)^*}\right),
\tag{7.23}
$$

in which we again choose a suitably small ρ_2. Collecting (7.17)–(7.23), we readily conclude an estimate for $\|u^k_{\varepsilon\tau}\|_{W^{1,\gamma}(\Omega;\mathbb{R}^d)}$, $\|z^k_{\varepsilon\tau}\|_{L^\infty(\Gamma_C)}$ and $\|\vartheta^k_{\varepsilon\tau}\|_{W^{1,2}(\Omega)}$.

Since $\vartheta^k_{\varepsilon\tau} \in W^{1,2}(\Omega\setminus\Gamma_C)$, we have that $-[\vartheta^k_{\varepsilon\tau}]^- \in W^{1,2}(\Omega\setminus\Gamma_C)$ is a legal test function for (7.9b). Hence, we use recursively that $\vartheta^{k-1}_{\varepsilon\tau} \ge 0$ a.e. in Ω (starting from the initial condition $\vartheta^0_{\varepsilon\tau} = \vartheta_0 \ge 0$ a.e. in Ω, cf. (5.4d)), the fact that $G^k_\tau \ge 0$ a.e. in Ω and $g^k_\tau \ge 0$ a.e. in $\partial\Omega$ (cf. with (4.8c)–(4.8d)), and that $-a_1\mathrm{D}_t z^k_{\varepsilon\tau} \ge 0$ a.e. in Γ_C, and finally the property $[\vartheta^k_{\varepsilon\tau}]^-\Theta(\vartheta^k_{\varepsilon\tau}) = 0$ a.e. in Ω, due to the fact that Θ is non-decreasing (cf. (4.6)). Thus, we conclude that $[\vartheta^k_{\varepsilon\tau}]^- = 0$ a.e. in Ω, whence $\vartheta^k_{\varepsilon\tau} \ge 0$ a.e. in Ω.

Finally, we prove (7.15) by adapting to the time-discrete setting a comparison argument from [28, Section 4.2.1]. Exploiting the fact that $G^k_{\varepsilon\tau} \ge 0$ a.e. in Ω, we deduce from (7.9b) that

$$
\mathrm{D}_t\vartheta^k_{\varepsilon\tau} - \mathrm{div}\big(\mathcal{K}(e(u^k_{\varepsilon\tau}), \vartheta^k_{\varepsilon\tau})\nabla\vartheta^k_{\varepsilon\tau}\big) \ge \frac{\mathsf{d}}{4}\left|e(\mathrm{D}_t u^k_{\varepsilon\tau})\right|^2 - C|\Theta(\vartheta^k_{\varepsilon\tau})|^2 \ge -C'|\vartheta^k_{\varepsilon\tau}|^2 \quad \text{in } \Omega\setminus\Gamma_C
\tag{7.24}
$$

for any $k = 1, \dots, K_\tau$ and some $C' > 0$ independent of τ and ε, where $\mathsf{d} > 0$ is the positive-definiteness constant of $\mathbb{D}$, cf. (2.6), and where the last inequality ensues from (4.6) and (5.1b). We compare (7.24) with the finite difference equation

$$
\mathrm{D}_t\chi_k = -C'|\chi_k|^2 \quad \forall k = 1, \dots, K_\tau,
\tag{7.25}
$$

with C' being the same constant as in (7.24). In fact, this is an implicit discretization of the ordinary differential equation $\dot\chi + C'|\chi|^2 = 0$ which, for $\chi(0) = h_0(\theta^*) > 0$ with θ^* from (5.8), gives a sub-solution of the (continuous) heat equation. This initial-value problem has the solution $\chi(t) = 1/(C't + 1/h_0(\theta^*))$ so that, in particular, $\chi(\cdot) \ge 1/(C'T + 1/h_0(\theta^*)) > 0$ on $[0, T]$. Now we solve (7.25) recursively starting from the initial datum $\chi_0 = h_0(\theta^*) > 0$. In this way we obtain an approximate solution to the mentioned initial-value problem which, for $\tau \to 0$, converges uniformly on the considered finite interval $[0, T]$. In particular, for $\tau > 0$ sufficiently small, we may take for granted that, say,

$$
\chi_k \ge \chi^* := \frac{1}{C'T + 1/h_0(\theta^*) + 1} > 0 \quad \forall k = 1, \dots, K_\tau.
\tag{7.26}
$$

For every $k = 1, \ldots, K_\tau$ we subtract (7.25) from (7.24) (the latter supplemented with the boundary conditions (7.10c), (7.11d) and (7.11e)), and we test the resulting inequality by $-(\vartheta_{\varepsilon\tau}^k - \chi_k)^-$. Thus, for all $k = 1, \ldots, K_\tau$,

$$\frac{1}{2}\mathrm{D}_t\big((\vartheta_{\varepsilon\tau}^k - \chi_k)^-\big)^2 \le -(\vartheta_{\varepsilon\tau}^k - \chi_k)^-\mathrm{D}_t(\vartheta_{\varepsilon\tau}^k - \chi_k) \le C'(|\vartheta_{\varepsilon\tau}^k|^2 - |\chi_k|^2)(\vartheta_{\varepsilon\tau}^k - \chi_k)^- \le 0 \tag{7.27}$$

in $\Omega \setminus \Gamma_C$; the last inequality is also due to the previously proved positivity $\vartheta_{\varepsilon\tau}^k \ge 0$ a.e. in Ω. Summing (7.27) over $k = 1, \ldots, K_\tau$, we easily conclude that $(\vartheta_{\varepsilon\tau}^k(x) - \chi_k)^- = 0$ for almost all $x \in \Omega$ and for every $k = 0, \ldots, K_\tau$, whence $\vartheta_{\varepsilon\tau}^k \ge \chi_k \ge \chi^* > 0$ a.e. in Ω. This concludes the proof of (7.15). □

Approximate solutions. In accordance with notation (7.2), for all $\tau > 0$ we shall denote by

- $\overline{u}_{\varepsilon\tau}, \underline{u}_{\varepsilon\tau}, \overline{\vartheta}_{\varepsilon\tau}, \underline{\vartheta}_{\varepsilon\tau}$, and $\overline{z}_{\varepsilon\tau}$, the piecewise constant interpolants of the elements $\{u_{\varepsilon\tau}^k\}_{k=1}^{K_\tau}$, $\{\vartheta_{\varepsilon\tau}^k\}_{k=1}^{K_\tau}$, and $\{z_{\varepsilon\tau}^k\}_{k=1}^{K_\tau}$;
- by $u_{\varepsilon\tau}, \vartheta_{\varepsilon\tau}$, and $z_{\varepsilon\tau}$, the related piecewise linear interpolants.

We shall now state the weak formulations of (the boundary value problems for) equations (7.9a) and (7.9b), in terms of the interpolants so far introduced by using "discrete test functions". Indeed, one verifies that for every K_τ-tuples $\{v_\tau^k\}_{k=1}^{K_\tau} \subset W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d)$ and $\{w_\tau^k\}_{k=1}^{K_\tau} \subset W^{1,2}(\Omega \setminus \Gamma_C)$, the approximate solutions $(\overline{u}_{\varepsilon\tau}, \overline{\vartheta}_{\varepsilon\tau}, \overline{z}_{\varepsilon\tau}, u_{\varepsilon\tau}, \vartheta_{\varepsilon\tau}, z_{\varepsilon\tau})$ fulfil the following: *the discrete (weak) momentum balance equation*

$$\begin{aligned}
&\int_Q \big(\mathbb{D}e(\dot{u}_{\varepsilon\tau}) + \mathbb{C}e(\overline{u}_{\varepsilon\tau}) - \mathbb{B}\Theta(\overline{\vartheta}_{\varepsilon\tau}) + \tau|e(\overline{u}_{\varepsilon\tau})|^{\gamma-2}e(\overline{u}_{\varepsilon\tau})\big) : e(\overline{v}_\tau)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Sigma_C} \Big(\kappa\overline{z}_{\varepsilon\tau}[\![\overline{u}_{\varepsilon\tau}]\!] + (I_K^\varepsilon)'([\![\overline{u}_{\varepsilon\tau}]\!]) + \tau^\beta\big(1 + \big|[\![\overline{u}_{\varepsilon\tau}]\!]\big|^2\big)^{\mu/2-1}[\![\overline{u}_{\varepsilon\tau}]\!]\Big) \cdot [\![\overline{v}_\tau]\!]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_\tau^T \int_\Omega \varrho\dot{u}_{\varepsilon\tau}(\cdot - \tau) \cdot \dot{v}_\tau\,\mathrm{d}x\mathrm{d}t + \varrho\int_\Omega \dot{u}_{\varepsilon\tau}(T) \cdot v_\tau(T)\,\mathrm{d}x \\
&= \rho\int_\Omega \dot{u}_{0,\tau} \cdot v_\tau(\tau)\,\mathrm{d}x + \int_Q \overline{F}_\tau \cdot \overline{v}_\tau\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_N} \overline{f}_\tau \cdot \overline{v}_\tau\,\mathrm{d}S\mathrm{d}t
\end{aligned} \tag{7.28}$$

(where we have used the notation $\dot{u}_{0,\tau} = \frac{u_{\varepsilon\tau}^0 - u_{\varepsilon\tau}^{-1}}{\tau} = \dot{u}_0$), which can be obtained from (7.9a), (7.10a), (7.10b), (7.11b) and (7.11c) by using a suitable discrete "by-part" summation formula (cf. [12, Formula (4.49)]);
the discrete (weak) enthalpy equation

$$\begin{aligned}
&\int_\Omega \vartheta_{\varepsilon\tau}(T)w_\tau(T)\,\mathrm{d}x + \int_Q \mathcal{K}(e(\overline{u}_{\varepsilon\tau}), \overline{\vartheta}_{\varepsilon\tau})\nabla\overline{\vartheta}_{\varepsilon\tau} \cdot \nabla\overline{w}_\tau\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Sigma_C} \eta([\![\underline{u}_{\varepsilon\tau}]\!], \overline{z}_{\varepsilon\tau})[\![\Theta(\overline{\vartheta}_{\varepsilon\tau})]\!][\![\overline{w}_\tau]\!]\,\mathrm{d}S\mathrm{d}t - \int_\tau^T \underline{\vartheta}_{\varepsilon\tau}\dot{w}_\tau\,\mathrm{d}x\mathrm{d}t \\
&= \int_Q \Big(\frac{2 - \sqrt{\tau}}{2}\mathbb{D}e(\dot{u}_{\varepsilon\tau}) : e(\dot{u}_{\varepsilon\tau}) - \Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B} : e(\dot{u}_{\varepsilon\tau})\Big)\overline{w}_\tau\,\mathrm{d}x\mathrm{d}t + \int_Q \overline{G}_\tau\overline{w}_\tau\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{\Sigma_C} a_1\dot{z}_{\varepsilon\tau}\frac{\overline{w}_\tau|^+_{\Gamma_C} + \overline{w}_\tau|^-_{\Gamma_C}}{2}\,\mathrm{d}S\mathrm{d}t + \int_\Omega \vartheta_0\overline{w}_\tau(\tau)\,\mathrm{d}x + \int_\Sigma \overline{g}_\tau\overline{w}_\tau\,\mathrm{d}S\mathrm{d}t,
\end{aligned} \tag{7.29}$$

again obtained from (7.9b), (7.10c) and (7.11c)–(7.11e) by the use of the summation (cf. [12, Formula (4.51)]);
the discrete flow rule of the delamination parameter (cf. (7.11a))

$$\left.\begin{array}{l}
\partial I_{(-\infty,0]}\big(\dot{z}_{\varepsilon\tau}(t)\big) + \dfrac{\kappa}{2}\big|[\![\overline{u}_{\varepsilon\tau}(t)]\!]\big|^2 + \tau^\alpha\overline{z}_{\varepsilon\tau}(t) + r(\overline{z}_{\varepsilon\tau}(t)) - a_0 - a_1 \ni 0 \\
r(\overline{z}_{\varepsilon\tau}(t)) \in \partial I_{[0,1]}(\overline{z}_{\varepsilon\tau}(t))
\end{array}\right\} \quad \text{on } \Sigma_C. \tag{7.30}$$

A priori estimates. Like in [12, Lemma 4.1], we derive some further energetic information on the approximate solutions (see Lemma 7.6 later on) by recurring to an auxiliary minimization problem. With this aim, we first proceed to the validation of a suitable (strict) semiconvexity property of the stored energy.

We further introduce the short-hand notation for the regularized stored energy

$$\begin{aligned}
\Phi_{\varepsilon\tau}(u, z) := &\int_{\Omega\setminus\Gamma_C} \Big(\frac{1}{2}\mathbb{C}e(u) : e(u) + \frac{\tau}{\gamma}|e(u)|^\gamma\Big)\,\mathrm{d}x + I_K^\varepsilon([\![u]\!]) \\
&+ \int_{\Gamma_C} \Big(\frac{\kappa}{2}z\big|[\![u]\!]\big|^2 + \frac{\tau^\alpha}{2}|z|^2 + \frac{\tau^\beta}{\mu}\big(1 + \big|[\![u]\!]\big|^2\big)^{\mu/2} + I_{[0,1]}(z) - a_0z\Big)\,\mathrm{d}S.
\end{aligned} \tag{7.31a}$$

Lemma 7.5. *Under the assumptions of Theorem 6.1, suppose further that the exponents $\mu \in (4,5)$, $\alpha,\ \beta \in (0,1)$ in (7.14) comply with*

$$\alpha(\mu-2)+2\beta < \frac{\mu-4}{2}. \tag{7.32}$$

Then, for every $\kappa > 0$ there exists $\tau_\kappa > 0$ such that for all $0 < \tau < \tau_\kappa$ the function on $W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)\times L^\infty(\Gamma_C)$ given by

$$(u,z)\mapsto \Phi_{\varepsilon\tau}(u,z)+\int_\Omega \frac{\mathbb{D}e(u):e(u)}{2\sqrt{\tau}}\,\mathrm{d}x \quad \textit{is strictly convex.} \tag{7.33}$$

Proof. Following the calculations in [40], we prove (7.33) by investigating the monotonicity of the multivalued mapping

$$W^{1,2}(\Omega\setminus\Gamma_C;\mathbb{R}^d)\times L^2(\Gamma_C)\rightrightarrows W^{1,2}(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*\times L^2(\Gamma_C):\ (u,z)\mapsto \partial\Phi_{\varepsilon\tau}(u,z).$$

To this goal, we have to estimate from below

$$\begin{aligned}&\langle\partial\Phi_{\varepsilon\tau}(u_1,z_1)-\partial\Phi_{\varepsilon\tau}(u_2,z_2),(u_1-u_2,z_1-z_2)\rangle\\ &\quad=\int_{\Omega\setminus\Gamma_C}\big(\mathbb{C}e(u_1-u_2)+\tau|e(u_1)|^{\gamma-2}e(u_1)-\tau|e(u_2)|^{\gamma-2}e(u_2)\big):e(u_1-u_2)\,\mathrm{d}x\\ &\qquad+\int_{\Gamma_C}\mathcal{L}\big([\![u_1]\!],[\![u_2]\!],z_1,z_2\big)\,\mathrm{d}S.\end{aligned} \tag{7.34}$$

As for the latter term, using the short-hand notation $s_i=[\![u_i]\!]$ for $i=1,2$, and $r(z_i)\in\partial I_{[0,1]}(z_i)$ as in (7.11a), we can estimate the last term in (7.34) as

$$\begin{aligned}\mathcal{L}(s_1,s_2,z_1,z_2)&=\kappa(z_1s_1-z_2s_2)\cdot(s_1-s_2)+\big((I_K^\varepsilon)'(s_1)-(I_K^\varepsilon)'(s_2)\big)\cdot(s_1-s_2)+\frac{\kappa}{2}(z_1-z_2)(|s_1|^2-|s_2|^2)\\ &\quad+(r(z_1)-r(z_2))(z_1-z_2)+\tau^\alpha|z_1-z_2|^2+\tau^\beta\big((1+|s_1|^2)^{\mu/2-1}s_1-(1+|s_2|^2)^{\mu/2-1}s_2\big)\cdot(s_1-s_2) \qquad (7.35)\\ &\ge\kappa z_1|s_1-s_2|^2+\frac{\kappa}{2}(z_1-z_2)(s_1+3s_2)\cdot(s_1-s_2)+\tau^\alpha(z_1-z_2)^2\\ &\quad+\tau^\beta\big((1+|s_1|^2)^{\mu/2-1}s_1-(1+|s_2|^2)^{\mu/2-1}s_2\big)\cdot(s_1-s_2)\\ &\ge\frac{\tau^\alpha}{2}(z_1-z_2)^2-\frac{\kappa^2}{8\tau^\alpha}|s_1-s_2|^2|s_1+3s_2|^2+\tau^\beta\big((1+|s_1|^2)^{\mu/2-1}s_1-(1+|s_2|^2)^{\mu/2-1}s_2\big)\cdot(s_1-s_2)\\ &\ge\frac{\tau^\alpha}{2}\big(z_1-z_2\big)^2-S_{\kappa,\tau}\big|s_1-s_2\big|^2 \qquad (7.36)\end{aligned}$$

for some positive constant $-S_{\kappa,\tau}$. Indeed, the first inequality follows from the positivity (by monotonicity) of the second and fourth term on the right-hand side of (7.35), and from simple algebraic manipulations. So does the second inequality. To conclude the final inequality (7.36) for some constant $S_{\kappa,\tau}>0$ depending on κ and τ, we have used that (cf. [41, Lemma 5.2])

$$\exists\, C_\mu>0:((1+|s_1|^2)^{\mu/2-1}s_1-(1+|s_2|^2)^{\mu/2-1}s_2)\cdot(s_1-s_2)\ge C_\mu(|s_1|^{\mu-2}+|s_2|^{\mu-2})|s_1-s_2|^2$$

for all $s_1,s_2\in\mathbb{R}^d$ and that (since $\mu>4$)

$$\forall\kappa,\ \tau>0\ \exists S_{\kappa,\tau}>0\quad\forall s_1,s_2\in\mathbb{R}^d: C_\mu\tau^\beta(|s_1|^{\mu-2}+|s_2|^{\mu-2})-\frac{\kappa^2}{8\tau^\alpha}|s_1+3s_2|^2\ge -S_{\kappa,\tau}. \tag{7.37}$$

Combining (7.34) with (7.36), the boundedness of the jump operator $u\mapsto[\![u]\!]$ from $W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)$ onto $L^2(\Gamma_C;\mathbb{R}^d)$, as well as using Korn's inequality, we conclude that

$$\langle\partial\Phi_{\varepsilon\tau}(u_1,z_1)-\partial\Phi_{\varepsilon\tau}(u_2,z_2),(u_1-u_2,z_1-z_2)\rangle\ge\frac{\tau^\alpha}{2}\|z_1-z_2\|^2_{L^2(\Gamma_C)}-CS_{\kappa,\tau}\|e(u_1)-e(u_2)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}$$

with the constant C depending on the positive-definiteness constant of $\mathbb{C}$ (cf. (2.6)), on the norm of the trace operator from $W^{1,2}(\Omega\setminus\Gamma_C;\mathbb{R}^d)$ to $L^2(\Gamma_C;\mathbb{R}^d)$, and on the constant in Korn's inequality (7.18). Finally, the key observation is that, for $\kappa>0$ fixed, the constant $S_{\kappa,\tau}$ in (7.37) has the following qualitative behaviour

$$S_{\kappa,\tau}\sim\frac{1}{\tau^{\alpha\frac{\mu-2}{\mu-4}+\frac{2\beta}{\mu-4}}}\quad\text{as }\tau\to0.$$

Thus, using condition (7.32), it can be verified that for all $\kappa>0$ there exists $\tau_\kappa>0$ such that for $0<\tau<\tau_\kappa$ there holds $CS_{\kappa,\tau}\le \mathsf{d}/\sqrt{\tau}$; again $\mathsf{d}>0$ is the positive-definiteness constant of $\mathbb{D}$. This yields (7.33). $\square$

Lemma 7.6 (*First a Priori Information*). *Under the assumptions of Theorem 6.1, for all $\varrho \geq 0$ and for every $\varepsilon > 0$ there is τ_ε such that for all $0 < \tau < \tau_\varepsilon$ the approximate solutions $(\overline{u}_{\varepsilon\tau}, \overline{\vartheta}_{\varepsilon\tau}, \overline{z}_{\varepsilon\tau}, u_{\varepsilon\tau}, \vartheta_{\varepsilon\tau}, z_{\varepsilon\tau})$ fulfil the following "discrete mechanical energy" inequality*

$$\begin{aligned} &T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_{\varepsilon\tau}(t)\big) + \Phi_{\varepsilon\tau}\big(\overline{u}_{\varepsilon\tau}(t), \overline{z}_{\varepsilon\tau}(t)\big) + \int_0^{\bar{\mathrm{t}}_\tau(t)} \left(\int_\Omega \frac{2-\sqrt{\tau}}{2} \mathbb{D}e\big(\dot{u}_{\varepsilon\tau}(s)\big) : e\big(\dot{u}_{\varepsilon\tau}(s)\big)\,\mathrm{d}x + \int_{\Gamma_C} \zeta_1\big(\dot{z}_{\varepsilon\tau}(s)\big)\,\mathrm{d}S \right) \mathrm{d}s \\ &\quad \leq T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_{0,\tau}\big) + \Phi_{\varepsilon\tau}\big(u_{0,\tau}, z_0\big) \\ &\qquad + \int_0^{\bar{\mathrm{t}}_\tau(t)} \left(\int_\Omega \Theta(\overline{\vartheta}_{\varepsilon\tau}(s))\mathbb{B} : e\big(\dot{u}_{\varepsilon\tau}(s)\big)\,\mathrm{d}x + \int_\Omega \overline{F}_\tau(s)\cdot \dot{u}_{\varepsilon\tau}(s)\,\mathrm{d}x + \int_{\Gamma_N} \overline{f}_\tau(s)\cdot\dot{u}_{\varepsilon\tau}(s)\,\mathrm{d}S \right) \mathrm{d}s, \end{aligned} \tag{7.38}$$

as well as the following "discrete total energy" inequality

$$\begin{aligned} &T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_{\varepsilon\tau}(t)\big) + \Phi_{\varepsilon\tau}\big(\overline{u}_{\varepsilon\tau}(t), \overline{z}_{\varepsilon\tau}(t)\big) + \int_\Omega \overline{\vartheta}_{\varepsilon\tau}(t)\,\mathrm{d}x \leq T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_{0,\tau}\big) + \Phi_{\varepsilon\tau}\big(u_{0,\tau}, z_0\big) + \int_\Omega \vartheta_0\,\mathrm{d}x \\ &\quad + \int_0^{\bar{\mathrm{t}}_{\varepsilon\tau}(t)} \left(\int_\Omega \overline{F}_\tau(s)\cdot\dot{u}_{\varepsilon\tau}(s)\,\mathrm{d}x + \int_{\Gamma_N} \overline{f}_\tau(s)\cdot\dot{u}_{\varepsilon\tau}(s)\,\mathrm{d}S + \int_\Omega \overline{G}_\tau(s)\,\mathrm{d}x + \int_{\partial\Omega} \overline{g}_\tau(s)\,\mathrm{d}S \right) \mathrm{d}s \end{aligned} \tag{7.39}$$

and also the "discrete semistability" for a.a. $t \in (0, T)$ (where $\Phi_{\varepsilon\tau}$ is from (7.31))

$$\Phi_{\varepsilon\tau}\big(\overline{u}_{\varepsilon\tau}(t), \overline{z}_{\varepsilon\tau}(t)\big) \leq \Phi_{\varepsilon\tau}\big(\overline{u}_{\varepsilon\tau}(t), \tilde{z}\big) + \mathcal{R}\left(\tilde{z} - \overline{z}_{\varepsilon\tau}(t)\right) \quad \text{for all } \tilde{z} \in L^\infty(\Gamma_C). \tag{7.40}$$

Proof. Let us now fix a solution $(u^k_{\varepsilon\tau}, z^k_{\varepsilon\tau}, \vartheta^k_{\varepsilon\tau})$ of Problem 7.1. Recall that such a triple exists thanks to Lemma 7.4. Let us consider an auxiliary minimization problem, namely

$$\left.\begin{aligned} &\text{minimize } \int_\Omega \varrho \mathrm{D}^2_t u^k_{\varepsilon\tau}\cdot u + \big(1-\sqrt{\tau}\big)\mathbb{D}e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) : e(u) \\ &\quad + \frac{\tau^{3/2}}{2}\mathbb{D}e\left(\frac{u-u^{k-1}_{\varepsilon\tau}}{\tau}\right) : e\left(\frac{u-u^{k-1}_{\varepsilon\tau}}{\tau}\right) - \Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B} : e(u)\,\mathrm{d}x \\ &\quad + \tau\int_{\Gamma_C}\zeta_1\left(\frac{z-z^{k-1}_{\varepsilon\tau}}{\tau}\right)\mathrm{d}S + \Phi_{\varepsilon\tau}(u,z) - \int_\Omega F^k_\tau\cdot u\,\mathrm{d}x - \int_{\Gamma_N} f^k_\tau\cdot u\,\mathrm{d}S \\ &\text{subject to } (u,z) \in W^{1,\gamma}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)\times L^\infty(\Gamma_C). \end{aligned}\right\} \tag{7.41}$$

By convexity of $\Phi_{\varepsilon\tau}$ and coercivity (cf. the calculations developed in the proof of Lemma 7.4), it is immediate to check that the minimization problem (7.41) has a solution which we denote by $(\tilde{u}^k_{\varepsilon\tau}, \tilde{z}^k_{\varepsilon\tau})$; of course, it depends on the pair $(u^k_{\varepsilon\tau}, \vartheta^k_{\varepsilon\tau})$ in general. Writing optimality conditions for $(\tilde{u}^k_{\varepsilon\tau}, \tilde{z}^k_{\varepsilon\tau})$ gives, for all $v \in W^{1,\gamma}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)$, that

$$\begin{aligned} &\int_\Omega \varrho\mathrm{D}^2_t u^k_{\varepsilon\tau}\cdot v + \left(\sqrt{\tau}\mathbb{D}e\left(\frac{\tilde{u}^k_{\varepsilon\tau}-u^{k-1}_{\varepsilon\tau}}{\tau}\right) + \mathbb{C}e(\tilde{u}^k_{\varepsilon\tau}) + \tau\big|e(\tilde{u}^k_{\varepsilon\tau})\big|^{\gamma-2}e(\tilde{u}^k_{\varepsilon\tau})\right) : e(v)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_C}\left(\kappa\tilde{z}^k_{\varepsilon\tau}[\![\tilde{u}^k_{\varepsilon\tau}]\!] + (I^\varepsilon_K)'([\![\tilde{u}^k_{\varepsilon\tau}]\!]) + \tau^\beta\Big(1+\big|[\![\tilde{u}^k_{\varepsilon\tau}]\!]\big|^2\Big)^{\mu/2-1}[\![\tilde{u}^k_{\varepsilon\tau}]\!]\right)\cdot[\![v]\!]\,\mathrm{d}S \\ &= \int_\Omega \big(\sqrt{\tau}-1\big)\mathbb{D}e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big) : e(v) + \Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B} : e(v) + F^k_\tau\cdot v\,\mathrm{d}x + \int_{\Gamma_N} f^k_\tau\cdot v\,\mathrm{d}S \end{aligned} \tag{7.42a}$$

and, for all $\tilde{z} \in L^\infty(\Gamma_C)$, that

$$\int_{\Gamma_C}\zeta_1(\tilde{z}) + \left(\tau^\alpha\tilde{z}^k_{\varepsilon\tau} + r(\tilde{z}^k_{\varepsilon\tau}) + \frac{\kappa}{2}\big|[\![\tilde{u}^k_{\varepsilon\tau}]\!]\big|^2 - a_0\right)\left(\tilde{z} - \frac{\tilde{z}^k_{\varepsilon\tau}-z^{k-1}_{\varepsilon\tau}}{\tau}\right)\mathrm{d}S \geq \int_{\Gamma_C}\zeta_1\left(\frac{\tilde{z}^k_{\varepsilon\tau}-z^{k-1}_{\varepsilon\tau}}{\tau}\right)\mathrm{d}S. \tag{7.42b}$$

Now, we test the difference of (7.12) and (7.42a) by $u^k_{\varepsilon\tau} - \tilde{u}^k_{\varepsilon\tau}$ and the difference of (7.13) and (7.42b) by $z^k_{\varepsilon\tau} - \tilde{z}^k_{\varepsilon\tau}$ and sum up the resulting relations. Using that the underlying potential, namely the functional

$$(u,z) \mapsto \Phi_{\varepsilon\tau}(u,z) + \int_{\Gamma_C}\zeta_1(z - z^{k-1}_{\varepsilon\tau})\,\mathrm{d}S + \int_\Omega\frac{\mathbb{D}e(u) : e(u)}{2\sqrt{\tau}}\,\mathrm{d}x,$$

is strictly convex on $W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)\times L^\infty(\Gamma_C)$ by Lemma 7.5, we conclude that $u^k_{\varepsilon\tau} = \tilde{u}^k_{\varepsilon\tau}$, $z^k_{\varepsilon\tau} = \tilde{z}^k_{\varepsilon\tau}$. Then, the functional in (7.41) must have a bigger or equal value on $(u^{k-1}_{\varepsilon\tau}, z^{k-1}_{\varepsilon\tau})$ than on $(\tilde{u}^k_{\varepsilon\tau}, \tilde{z}^k_{\varepsilon\tau}) = (u^k_{\varepsilon\tau}, z^k_{\varepsilon\tau})$, which gives a discrete mechanical energy inequality (compare with (4.20)), namely

$$T^{\varrho}_{\rm kin}\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big)+\Phi_{\varepsilon\tau}\big(u^k_{\varepsilon\tau},z^k_{\varepsilon\tau}\big)+\tau\int_{\Gamma_C}\zeta_1\big(\mathrm{D}_t z^k_{\varepsilon\tau}\big)\,\mathrm{d}S+\frac{\tau}{2}\int_\Omega\big(2-\sqrt{\tau}\,\big)\mathbb{D}e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big):e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big)\,\mathrm{d}x$$
$$\leq T^{\varrho}_{\rm kin}\big(\mathrm{D}_t u^{k-1}_{\varepsilon\tau}\big)+\Phi_{\varepsilon\tau}\big(u^{k-1}_{\varepsilon\tau},z^{k-1}_{\varepsilon\tau}\big)+\tau\int_\Omega\Theta(\vartheta^k_{\varepsilon\tau})\mathbb{B}:e\big(\mathrm{D}_t u^k_{\varepsilon\tau}\big)+F^k_\tau\cdot\mathrm{D}_t u^k_{\varepsilon\tau}\mathrm{d}x+\tau\int_{\Gamma_N}f^k_\tau\cdot\mathrm{D}_t u^k_{\varepsilon\tau}\mathrm{d}S \tag{7.43}$$

when also employing the algebraic inequality

$$\mathrm{D}^2_t u^k_{\varepsilon\tau}\cdot\mathrm{D}_t u^k_{\varepsilon\tau}\geq\frac{1}{2}|\mathrm{D}_t u^k_{\varepsilon\tau}|^2-\frac{1}{2}|\mathrm{D}_t u^{k-1}_{\varepsilon\tau}|^2.$$

Upon summation over k, we conclude (7.38).

Now, to get (7.39), we add to (7.43) the relation obtained testing (the weak formulation of) the boundary value problem (7.9b), (7.10c), (7.11d) and (7.11e) by τ. Developing all calculations, one sees that, thanks to our carefully designed discretization, the fourth term on the left-hand side of (7.43) and the first dissipative/adiabatic term on the right-hand side of (7.9b) mutually cancel out. So do the third term on the right-hand side of (7.43) and the second right-hand side term in (7.9b). Again upon summation over k, we arrive at (7.39).

Finally, to check (7.40), it just suffices to realize that the functional minimized in (7.41) has a lower value in $(u^k_{\varepsilon\tau},z^k_{\varepsilon\tau})$ than in $(u^k_{\varepsilon\tau},\tilde z)$ for any $\tilde z\in L^\infty(\Gamma_C)$, which gives

$$\Phi_{\varepsilon\tau}\big(u^k_{\varepsilon\tau},z^k_{\varepsilon\tau}\big)+\int_{\Gamma_C}\tau\zeta_1\left(\frac{z^k_{\varepsilon\tau}-z^{k-1}_{\varepsilon\tau}}{\tau}\right)\mathrm{d}S\leq\Phi_{\varepsilon\tau}\big(u^k_{\varepsilon\tau},\tilde z\big)+\int_{\Gamma_C}\tau\zeta_1\left(\frac{\tilde z-z^{k-1}_{\varepsilon\tau}}{\tau}\right)\mathrm{d}S.$$

Then, by using that ζ_1 is homogeneous degree 1 and thus satisfies the triangle inequality $\zeta_1(\tilde z-z^{k-1}_{\varepsilon\tau})\leq\zeta_1(\tilde z-z^k_{\varepsilon\tau})+\zeta_1(z^k_{\varepsilon\tau}-z^{k-1}_{\varepsilon\tau})$, we find

$$\Phi_{\varepsilon\tau}\big(u^k_{\varepsilon\tau},z^k_{\varepsilon\tau}\big)\leq\Phi_{\varepsilon\tau}\big(u^k_{\varepsilon\tau},\tilde z\big)+\int_{\Gamma_C}\zeta_1\big(\tilde z-z^{k-1}_{\varepsilon\tau}\big)-\zeta_1\big(z^k_{\varepsilon\tau}-z^{k-1}_{\varepsilon\tau}\big)\,\mathrm{d}S\leq\Phi_{\varepsilon\tau}\big(u^k_{\varepsilon\tau},\tilde z\big)+\int_{\Gamma_C}\zeta_1\big(\tilde z-z^k_{\varepsilon\tau}\big)\,\mathrm{d}S.$$

Being $k=1,\ldots,K_\tau$ arbitrary, we conclude (7.40). □

Lemma 7.7 (*A Priori Estimates*)**.** *Under the assumptions of Lemma 7.5, there exist constants S_0 and, for every $1\leq r<\frac{d+2}{d+1}$, S_r such that for all $\varrho\geq0$, $\varepsilon>0$ and for all $0<\tau<\tau_\kappa$ (τ_κ being as in Lemma 7.5), for all approximate solutions $(\overline u_{\varepsilon\tau},\overline\vartheta_{\varepsilon\tau},\overline z_{\varepsilon\tau},u_{\varepsilon\tau},\vartheta_{\varepsilon\tau},z_{\varepsilon\tau})$ the following estimates hold*

$$\big\|\overline u_{\varepsilon\tau}\big\|_{L^\infty(0,T;W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d))}\leq S_0, \tag{7.44a}$$

$$\big\|u_{\varepsilon\tau}\big\|_{W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d))}\leq S_0, \tag{7.44b}$$

$$\varrho^{1/2}\big\|u_{\varepsilon\tau}\big\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))}\leq S_0, \tag{7.44c}$$

$$\big\|\overline u_{\varepsilon\tau}\big\|_{L^\infty(0,T;W^{1,\gamma}_{\Gamma_D}(\Omega;\mathbb{R}^d))}\leq\frac{S_0}{\sqrt[\gamma]{\tau}}, \tag{7.44d}$$

$$\big\|\overline z_{\varepsilon\tau}\big\|_{L^\infty(\Sigma_C)}\leq S_0, \tag{7.44e}$$

$$\big\|z_{\varepsilon\tau}\big\|_{\mathrm{BV}([0,T];L^1(\Gamma_C))}\leq S_0, \tag{7.44f}$$

$$\big\|\overline\vartheta_{\varepsilon\tau}\big\|_{L^\infty(0,T;L^1(\Omega))}\leq S_0, \tag{7.44g}$$

$$\big\|\vartheta_{\varepsilon\tau}\big\|_{L^r(0,T;W^{1,r}(\Omega))}\leq S_r\quad\textit{for any }1\leq r<\frac{d+2}{d+1}, \tag{7.44h}$$

$$\Big\|\dot\vartheta_{\varepsilon\tau}\Big\|_{L^1(0,T;W^{1,r'}(\Omega)^*)}\leq S_0, \tag{7.44i}$$

$$\varrho\,\Big\|\dot u_{\varepsilon\tau}\Big\|_{\mathrm{BV}([0,T];W^{1,\gamma}_{\Gamma_D}(\Omega;\mathbb{R}^d)^*)}\leq S_0, \tag{7.44j}$$

(recall that $r'=\frac{r}{r-1}$ is the conjugate exponent of r), where S_0 and S_r neither depend on ε nor on τ. Estimates (7.44e)–(7.44h) respectively hold for $z_{\varepsilon\tau}$, $\overline z_{\varepsilon\tau}$, $\vartheta_{\varepsilon\tau}$ and $\overline\vartheta_{\varepsilon\tau}$, as well.

Proof. Some of the calculations we shall develop hereafter are analogous to the ones in the proof of [12, Prop. 4.2], to which we shall systematically refer.

First of all, we use the "discrete total energy" balance (7.39). Indeed, on the one hand, by definition (7.31) of $\Phi_{\varepsilon\tau}$, the second term on the left-hand side of (7.39) is bounded from below and, thanks to positive-definiteness of $\mathbb{C}$ and Korn's inequality (7.18), it provides a bound for $\|\overline{u}_{\varepsilon\tau}(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}$ and for $\tau\|\overline{u}_{\varepsilon\tau}(t)\|^\gamma_{W^{1,\gamma}(\Omega;\mathbb{R}^d)}$ uniformly w.r.t. $t \in [0, T]$. Further, being $\overline{\vartheta}_{\varepsilon\tau} \geq 0$ a.e. in Ω thanks to (7.15), the third term on the left-hand side of (7.39) estimates $\|\overline{\vartheta}_{\varepsilon\tau}\|_{L^\infty(0,T;L^1(\Omega))}$. To estimate the right-hand side of (7.39), we employ the discrete "by-part" summation [12, Formula (4.51)], to the effect that

$$
\begin{aligned}
&\int_0^{\bar{t}_\tau(t)}\int_{\Gamma_N}\overline{f}_\tau(s)\cdot\dot{u}_{\varepsilon\tau}(s)\,\mathrm{d}S\mathrm{d}s = \int_{\Gamma_N}\overline{f}_\tau(t)\cdot\overline{u}_{\varepsilon\tau}(t)\,\mathrm{d}S - \int_{\Gamma_N}\overline{f}_\tau(\tau)\cdot u_{0,\tau}\,\mathrm{d}S - \int_\tau^{\bar{t}_\tau(t)}\int_{\Gamma_N}\dot{f}_\tau(s)\cdot\underline{u}_{\varepsilon\tau}(s)\,\mathrm{d}S\mathrm{d}s\\
&\quad\leq \rho_3\|\overline{u}_{\varepsilon\tau}(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} + C_{\rho_3}\left(\|u_{0,\tau}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} + \|\overline{f}_\tau\|^2_{L^\infty(0,T;L^{4/3}(\Gamma_N))}\right)\\
&\qquad + C\int_0^{\bar{t}_\tau(t)}\|\dot{f}_\tau(s)\|_{L^{4/3}(\Gamma_N)}\|\overline{u}_{\varepsilon\tau}(s)\|_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s,
\end{aligned}
\tag{7.45}
$$

where inequality (7.45) is also due to the continuous embedding (4.3) and ρ_3 is chosen in such a way as to absorb the first term on the right-hand side into the term $\|\overline{u}_{\varepsilon\tau}(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}$ on the left-hand side of (7.39). Furthermore, in the case $\varrho > 0$ we estimate the fourth term on the right-hand side of (7.39) by

$$
\int_0^{\bar{t}_\tau(t)}\int_\Omega\overline{F}_\tau(s)\cdot\dot{u}_{\varepsilon\tau}(s)\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^{\bar{t}_\tau(t)}\|\overline{F}_\tau(s)\|_{L^2(\Omega;\mathbb{R}^d)}\,\|\dot{u}_{\varepsilon\tau}(s)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s.
\tag{7.46}
$$

We then combine (7.39) and (7.46), and (7.45), and use (7.5)–(7.6b) for $\overline{F}_\tau$, $\overline{f}_\tau$, $\dot{f}_\tau$, $\overline{G}_\tau$, and $\overline{g}_\tau$. Applying the Gronwall Lemma, we conclude estimates (7.44a), (7.44c)–(7.44e) and (7.44g) (the estimates for $z_{\varepsilon\tau}$ and $\vartheta_{\varepsilon\tau}$ following from the bounds for $\overline{z}_{\varepsilon\tau}$ and $\overline{\vartheta}_{\varepsilon\tau}$, and from (7.3)). In the case $\varrho = 0$, the only change in the above calculations is that, under the additional assumption (5.5a), we estimate the fourth term on the right-hand side of (7.39) by use of the aforementioned discrete by-part summation formula. Namely, on account of the Sobolev embedding (4.3)

$$
\begin{aligned}
&\int_0^{\bar{t}_\tau(t)}\int_\Omega\overline{F}_\tau(s)\cdot\dot{u}_{\varepsilon\tau}(s)\,\mathrm{d}x\mathrm{d}s = \int_\Omega\overline{F}_\tau(t)\cdot\overline{u}_{\varepsilon\tau}(t)\,\mathrm{d}x - \int_\Omega\overline{F}_\tau(\tau)\cdot u_{0,\tau}\,\mathrm{d}x - \int_\tau^{\bar{t}_\tau(t)}\int_\Omega\dot{F}_\tau(s)\cdot\underline{u}_{\varepsilon\tau}(s)\,\mathrm{d}x\mathrm{d}s\\
&\quad\leq \rho_4\|\overline{u}_{\varepsilon\tau}(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} + C_{\rho_4}\left(\|u_{0,\tau}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} + \|\overline{F}_\tau\|^2_{L^\infty(0,T;L^{6/5}(\Omega;\mathbb{R}^d))}\right)\\
&\qquad + C\int_0^{\bar{t}_\tau(t)}\|\dot{F}_\tau(s)\|_{L^{6/5}(\Omega;\mathbb{R}^d)}\|\overline{u}_{\varepsilon\tau}(s)\|_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s,
\end{aligned}
\tag{7.47}
$$

where again the positive constant ρ_4 is such that the first term on the right-hand side of (7.47) is controlled by $\|\overline{u}_{\varepsilon\tau}(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}$ on the left-hand side of (7.39).

For later use, we point out that, due to (7.44a) (which yields $\big\|[\![\,\overline{u}_{\varepsilon\tau}]\!]\big\|_{L^\infty(0,T;L^4(\Gamma_C;\mathbb{R}^d))} \leq C$ for some constant independent of $\varepsilon > 0$ and $\tau > 0$), estimate (7.44e), and assumption (5.1e) on η, there holds

$$
\sup_{\varepsilon,\tau}\|\eta(\underline{u}_{\varepsilon\tau}, \overline{z}_{\varepsilon\tau})\|_{L^\infty(0,T;L^3(\Gamma_C))} \leq C.
\tag{7.48}
$$

Secondly, again arguing as for [12, Proposition 4.2], we make use of the technique by Boccardo & Gallouët [42], with the simplification devised in [43]. Hence, we test the heat equation (7.29) by $\Pi(\overline{\vartheta}_{\varepsilon\tau})$, where $\Pi : [0, +\infty) \to [0, 1]$ is the map

$$
w \mapsto \Pi(w) = 1 - \frac{1}{(1+w)^\varsigma}, \quad \varsigma > 0;
$$

note that $\Pi(\overline{\vartheta}_{\varepsilon\tau}) \in W^{1,2}(\Omega \setminus \Gamma_C)$, because Π is Lipschitz continuous. With the same calculations as in [12], taking into account (5.1d) we find

$$
\begin{aligned}
\varsigma\,\mathsf{k}\int_Q\frac{|\nabla\overline{\vartheta}_{\varepsilon\tau}|^2}{(1+\overline{\vartheta}_{\varepsilon\tau})^{1+\varsigma}}\,\mathrm{d}x\mathrm{d}t &\leq \int_Q\mathcal{K}(e(\overline{u}_{\varepsilon\tau}), \overline{\vartheta}_{\varepsilon\tau})\nabla\overline{\vartheta}_{\varepsilon\tau}\cdot\nabla\Pi(\overline{\vartheta}_{\varepsilon\tau})\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \int_{\Sigma_C}\eta([\![\underline{u}_{\varepsilon\tau}]\!], \overline{z}_{\varepsilon\tau})[\![\Theta(\overline{\vartheta}_{\varepsilon\tau})]\!][\![\Pi(\overline{\vartheta}_{\varepsilon\tau})]\!]\,\mathrm{d}S\mathrm{d}t + \int_\Omega\widehat{\Pi}(\overline{\vartheta}_{\varepsilon\tau}(T,\cdot))\,\mathrm{d}x\\
&\leq \int_\Omega\widehat{\Pi}(\vartheta_0)\,\mathrm{d}x + \|\overline{G}_\tau\|_{L^1(Q)} + \|\overline{g}_\tau\|_{L^1(\Sigma)}\\
&\qquad + C\big(\|\mathbb{D}e(\dot{u}_{\varepsilon\tau}) : e(\dot{u}_{\varepsilon\tau})\|_{L^1(Q)} + \|\Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B} : e(\dot{u}_{\varepsilon\tau})\|_{L^1(Q)} + \|\zeta_1(\dot{z}_{\varepsilon\tau})\|_{L^1(\Sigma_C)}\big)
\end{aligned}
\tag{7.49}
$$

where $\widehat{\Pi}$ is the primitive function of Π such that $\widehat{\Pi}(0) = 0$. Note that inequality (7.49) follows from the fact that $\eta(\underline{u}_{\varepsilon\tau}, \overline{z}_{\varepsilon\tau})[\![\Theta(\overline{\vartheta}_{\varepsilon\tau})]\!][\![\Pi(\overline{\vartheta}_{\varepsilon\tau})]\!] \geq 0$ a.e. in Σ_C (by the positivity of η and the monotonicity of Θ and Π), from the "discrete chain rule" [12, Formula (4.30)] for $\widehat{\Pi}$, and from $0 \leq \Pi(\overline{\vartheta}_{\varepsilon\tau}) \leq 1$. Combining (7.49) with the Gagliardo–Nirenberg inequality, we find, for all $1 \leq r < (d+2)/(d+1)$, that

$$\left\|\nabla\overline{\vartheta}_{\varepsilon\tau}\right\|^r_{L^r(Q;\mathbb{R}^d)} \leq C_r\Big(1 + \|\mathbb{D}e(\dot{u}_{\varepsilon\tau}) : e(\dot{u}_{\varepsilon\tau})\|_{L^1(Q)} + \|\Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B} : e(\dot{u}_{\varepsilon\tau})\|_{L^1(Q)} + \|\zeta_1(\dot{z}_{\varepsilon\tau})\|_{L^1(\Sigma_C)}\Big) \tag{7.50}$$

for some positive constant C_r, depending on r and also on the function η, cf. (5.1e).

Then, we multiply (7.50) by a constant $\rho_5 > 0$ and add it to (7.38) (in which we set $t = T$). Now, by positive-definiteness of $\mathbb{C}$, the third term on the left-hand side of (7.38) is bounded from below by $\mathsf{d}/2\|e(\dot{u}_{\varepsilon\tau})\|^2_{L^2(Q;\mathbb{R}^{d\times d})}$, whereas the fourth term controls $\|\zeta_1(\dot{z}_{\varepsilon\tau})\|_{L^1(\Sigma_C)}$. Thus, we choose ρ_5 small enough in such a way to absorb the first and the third term on the right-hand side of (7.50) into the left-hand side of (7.38). Hence, we find

$$\begin{aligned}&\frac{\mathsf{d}}{4}\|e(\dot{u}_{\varepsilon\tau})\|^2_{L^2(Q;\mathbb{R}^{d\times d})} + (1-\rho_5)\|\zeta_1(\dot{z}_{\varepsilon\tau})\|_{L^1(\Sigma_C)} + \rho_5\left\|\nabla\overline{\vartheta}_{\varepsilon\tau}\right\|^r_{L^r(Q;\mathbb{R}^d)} \leq T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_{0,\tau}\big) + \Phi_{\varepsilon\tau}\big(u_{0,\tau}, z_{0,\tau}\big)\\ &\qquad + \int_0^T\int_\Omega \overline{F}_\tau\cdot\dot{u}_{\varepsilon\tau}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Gamma_N}\overline{f}_\tau\cdot\dot{u}_{\varepsilon\tau}\,\mathrm{d}S\mathrm{d}t + (\rho_5 C_r + 1)\|\Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B}e(\dot{u}_{\varepsilon\tau})\|_{L^1(Q)}.\end{aligned} \tag{7.51}$$

The first two terms on the right-hand side of (7.51) are estimated in view of (7.5)–(7.6b) and (7.7), whereas, taking into account the Sobolev embedding (4.3) and Korn's inequality (7.18), we have

$$\begin{aligned}&\int_0^T\int_\Omega \overline{F}_\tau\cdot\dot{u}_{\varepsilon\tau}\,\mathrm{d}x\mathrm{d}t \leq C\int_0^T \|\overline{F}_\tau(s)\|^2_{L^{6/5}(\Omega;\mathbb{R}^d)} + \frac{\mathsf{d}}{16}\int_0^T \|e(\dot{u}_{\varepsilon\tau})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\,\mathrm{d}t,\\ &\int_0^T\int_{\Gamma_N} \overline{f}_\tau\cdot\dot{u}_{\varepsilon\tau}\,\mathrm{d}S\mathrm{d}t \leq C\int_0^T \|\overline{f}_\tau(s)\|^2_{L^{4/3}(\Gamma_N;\mathbb{R}^d)} + \frac{\mathsf{d}}{16}\int_0^T \|e(\dot{u}_{\varepsilon\tau})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\,\mathrm{d}t;\end{aligned}$$

here again $\mathsf{d} > 0$ is the positive-definiteness constant of $\mathbb{D}$. To estimate the last summand, we use the boundedness of $\mathbb{C}$ and (5.2), finding

$$\begin{aligned}(\rho_5 C_r + 1)\|\Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B} : e(\dot{u}_{\varepsilon\tau})\|_{L^1(Q)} &\leq \rho_6\,\|e(\dot{u}_{\varepsilon\tau})\|^2_{L^2(Q;\mathbb{R}^{d\times d})} + C_{\rho_6}\|\Theta(\overline{\vartheta}_{\varepsilon\tau})\|^2_{L^2(Q)}\\ &\leq \rho_6\,\|e(\dot{u}_{\varepsilon\tau})\|^2_{L^2(Q;\mathbb{R}^{d\times d})} + C_{\rho_6}\big(\|\overline{\vartheta}_{\varepsilon\tau}\|^{2/\omega}_{L^{2/\omega}(Q)} + 1\big)\end{aligned} \tag{7.52}$$

in which we choose the positive constant ρ_6 small enough, again to absorb the first term on the right-hand side of (7.52) into the left-hand side of (7.51). In order to estimate $\|\overline{\vartheta}_{\varepsilon\tau}\|_{L^{2/\omega}(Q)}$, we again employ the Gagliardo–Nirenberg inequality. Indeed, with the same calculations as throughout [12, Formulae (4.39)–(4.43)], and relying on the restriction of ω in (5.1b) and on the bound for $\|\overline{\vartheta}_{\varepsilon\tau}\|_{L^\infty(0,T;L^1(\Omega))}$, we conclude

$$\int_0^T \|\overline{\vartheta}_{\varepsilon\tau}\|^{2/\omega}_{L^{2/\omega}(\Omega)} \leq \rho_7\int_0^T \left\|\nabla\overline{\vartheta}_{\varepsilon\tau}\right\|^r_{L^r(\Omega;\mathbb{R}^d)} + C_{\rho_7} \tag{7.53}$$

for a suitably small $\rho_7 > 0$. Then, we plug (7.53) into (7.52), and the latter into (7.51), and choose ρ_7 in such a way as to absorb $\left\|\nabla\overline{\vartheta}_{\varepsilon\tau}\right\|^r_{L^r(Q;\mathbb{R}^d)}$ into the left-hand side of (7.51). Thus, we conclude estimate (7.44b), as well as an estimate for $\|\zeta(\dot{z}_{\varepsilon\tau})\|_{L^1(\Sigma_C)}$ (yielding (7.44f)), and a bound for $\nabla\overline{\vartheta}_{\varepsilon\tau}$ in $L^r(Q;\mathbb{R}^d)$. Combining the latter information with the estimate for $\overline{\vartheta}_{\varepsilon\tau}$ in $L^\infty(0,T;L^1(\Omega))$, we infer (7.44h) (the estimate for $\vartheta_{\varepsilon\tau}$ due to the bound for $\overline{\vartheta}_{\varepsilon\tau}$ and to (7.3)). As a by-product of the above calculations, we find

$$\|\Lambda_{\varepsilon\tau}\|_{L^1(Q)} \leq C, \quad \text{with } \Lambda_{\varepsilon\tau} := \frac{2-\sqrt{\tau}}{2}\mathbb{D}e(\dot{u}_{\varepsilon\tau}) : e(\dot{u}_{\varepsilon\tau}) + \Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B} : e(\dot{u}_{\varepsilon\tau}). \tag{7.54}$$

For later convenience, we also remark that (7.44h) yields

$$\|\Theta(\overline{\vartheta}_{\varepsilon\tau})\|_{L^{\omega r}(0,T;L^{\omega q}(\Gamma_C))} \leq C \quad \text{for all } 1 \leq q \leq \frac{dr}{d-r}, \tag{7.55}$$

where we have also used the continuous embedding $W^{1,r}(\Omega) \subset L^q(\Gamma_C)$ for q ranging in the above-mentioned index interval, as well as the growth restriction (5.2) imposed on Θ.

To prove (7.44i), we argue by comparison in (7.29), to the effect that

$$\|\dot{\vartheta}_{\varepsilon\tau}\|_{L^1(0,T;W^{1,r'}(\Omega)^*)} = \sup_{\|w\|_{L^\infty(0,T;W^{1,r'}(\Omega))}\leq 1} (I_5 + I_6 + I_7 + I_8), \quad \text{where}$$

$$I_5 = \int_Q \Lambda_{\varepsilon\tau}w \leq \|\Lambda_{\varepsilon\tau}\|_{L^1(Q)}\|w\|_{L^\infty(Q)} \leq C\|w\|_{L^\infty(0,T;W^{1,r'}(\Omega))}$$

thanks to (7.54) and the continuous embedding $W^{1,r'}(\Omega) \subset L^\infty(\Omega)$ (since $r' > d+2$), while

$$I_6 = -\int_Q \mathcal{K}(e(\overline{u}_{\varepsilon\tau}), \overline{\vartheta}_{\varepsilon\tau})\nabla\overline{\vartheta}_{\varepsilon\tau} \cdot \nabla w \le C_{\mathcal{K}} \|\nabla\overline{\vartheta}_{\varepsilon\tau}\|_{L^r(Q)} \|\nabla w\|_{L^{r'}(Q)} \le C\|w\|_{L^\infty(0,T;W^{1,r'}(\Omega))}$$

due to (5.3) and (7.44g). Further,

$$\begin{aligned} I_7 &= -\int_{\Sigma_C} \left(\eta([\![\underline{u}_{\varepsilon\tau}]\!], \overline{z}_{\varepsilon\tau})[\![\Theta(\overline{\vartheta}_{\varepsilon\tau})]\!][\![w]\!] + a_1 \dot{z}_{\varepsilon\tau} \frac{w|^+_{\Gamma_C} + w|^-_{\Gamma_C}}{2} \right) \mathrm{d}S\mathrm{d}t \\ &\le \left(\|\eta(\underline{u}_{\varepsilon\tau}, \overline{z}_{\varepsilon\tau})\|_{L^\infty(0,T;L^3(\Gamma_C))} \|\Theta(\overline{\vartheta}_{\varepsilon\tau})\|_{L^{\omega r}(0,T;L^{\omega q}(\Gamma_C))} + \|\zeta_1(\dot{z}_{\varepsilon\tau})\|_{L^1(\Sigma_C)} \right) \|w\|_{L^\infty(\Sigma_C)} \\ &\le C\|w\|_{L^\infty(0,T;W^{1,r'}(\Omega))} \end{aligned}$$

thanks to (7.44e), (7.48) and (7.55), and the continuous embedding $W^{1,r'}(\Omega) \subset L^\infty(\Gamma_C)$, and, finally,

$$I_8 = \int_Q \overline{G}_\tau w \, \mathrm{d}x\mathrm{d}t + \int_\Sigma \overline{g}_\tau w \, \mathrm{d}S\mathrm{d}t \le \left(\|\overline{G}_\tau\|_{L^1(Q)} + \|\overline{g}_\tau\|_{L^1(\Sigma)} \right) \|w\|_{L^\infty(0,T;W^{1,r'}(\Omega))}.$$

Collecting the above calculations, we conclude (7.44i).

Finally, for (7.44j) we use that $\ddot{u}_{\varepsilon\tau}$ is a measure on $[0,T]$, supported at the jumps of $\dot{u}_{\varepsilon\tau}$, and we estimate the norm $\varrho\|\ddot{u}_{\varepsilon\tau}\|_{\mathrm{M}(0,T;W^{1,\gamma}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*)}$, where $\mathrm{M}(0,T;W^{1,\gamma}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)^*)$ denotes the space of Radon measures on $[0,T]$ with values in $W^{1,\gamma}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)^*$, arguing by comparison in (7.28); see the proof of [12, Prop. 4.2], where similar calculations were carried out. □

8. Limit passage with $\tau \to 0$ and proof of Theorem 6.1

Throughout this section, we shall keep $\varepsilon > 0$ fixed, and let $\tau \to 0$. We shall develop a proof of the passage to the limit unifying the cases $\varrho > 0$ and $\varrho = 0$.

Step 0: *selection of convergent subsequences.* First of all, it follows from estimates (7.44b), (7.44c) and (7.44j), from the Banach selection principle, and from the Aubin–Lions theorem (see, e.g., [44, Theorem 5, Corollary 4] and [39, Corollary 7.9] for the generalization to the case of time derivatives as measures), that there exist a (not relabelled) sequence $\tau \to 0$ and a limit function $u_\varepsilon \in W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d))$ such that the following weak, weak*, and strong convergences hold as $\tau \to 0$:

$$u_{\varepsilon\tau} \rightharpoonup u_\varepsilon \quad \text{in } W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)), \tag{8.1a}$$

$$u_{\varepsilon\tau} \to u_\varepsilon \quad \text{in } \mathrm{C}^0([0,T];W^{1-\epsilon,2}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)) \quad \forall \epsilon \in (0,1], \tag{8.1b}$$

$$\text{if } \varrho > 0, \quad u_{\varepsilon\tau} \overset{*}{\rightharpoonup} u_\varepsilon \quad \text{in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{8.1c}$$

In the case $\varrho > 0$ we also have $u_{\varepsilon\tau} \to u_\varepsilon$ in $W^{1,2}(0,T;W^{1-\epsilon,2}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)) \cap W^{1,q}(0,T;L^2(\Omega;\mathbb{R}^d))$ for all $\epsilon \in (0,1]$ and $1 \le q < \infty$. Furthermore, estimate (7.44j) and a generalization of Helly's principle (see [45] as well as [15, Theorem 6.1]) yield that $\dot{u}_\varepsilon \in \mathrm{BV}([0,T];W^{1,\gamma}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)^*)$ and, in addition, $\dot{u}_{\varepsilon\tau}(t) \rightharpoonup \dot{u}_\varepsilon(t)$ in $W^{1,\gamma}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)^*$ for all $t \in [0,T]$. By virtue of estimate (7.44c) and of a trivial compactness argument, this pointwise weak convergence improves to

$$\dot{u}_{\varepsilon\tau}(t) \rightharpoonup \dot{u}_\varepsilon(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \text{ for all } t \in [0,T], \text{ in the case } \varrho > 0. \tag{8.1d}$$

Combining (8.1a) and (8.1b) with the general inequality (7.4), we conclude that, up to the extraction of a further subsequence, for all $\epsilon \in (0,1]$,

$$\begin{aligned} &\overline{u}_{\varepsilon\tau} \overset{*}{\rightharpoonup} u_\varepsilon \quad \text{in } L^\infty(0,T;W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)), \qquad \overline{u}_{\varepsilon\tau} \to u_\varepsilon \quad \text{in } L^\infty(0,T;W^{1-\epsilon,2}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d)), \\ &\overline{u}_{\varepsilon\tau}(t) \to u_\varepsilon(t) \quad \text{in } W^{1-\epsilon,2}_{\Gamma_D}(\Omega \setminus \Gamma_C;\mathbb{R}^d) \text{ for all } t \in [0,T], \end{aligned} \tag{8.1e}$$

the latter pointwise convergence due to (8.1b) and (7.4).

With the aforementioned compactness results, we deduce from estimates (7.44e) and (7.44f) that there exists a function $z_\varepsilon \in L^\infty(\Sigma_C) \cap \mathrm{BV}([0,T];\mathcal{Z})$, ($\mathcal{Z}$ being some reflexive space such that $L^1(\Gamma_C) \subset \mathcal{Z}$ with a continuous embedding, for example $\mathcal{Z} = W^{1,2+\epsilon}(\Gamma_C)^*$ for some $\epsilon > 0$), such that (possibly along a subsequence)

$$\overline{z}_{\varepsilon\tau} \overset{*}{\rightharpoonup} z_\varepsilon \quad \text{in } L^\infty(\Sigma_C), \tag{8.2a}$$

and, again by [15, Theorem 6.1, Proposition 6.2], $\overline{z}_{\varepsilon\tau}(t) \rightharpoonup z_\varepsilon(t)$ in $\mathcal{Z}$ for all $t \in [0,T]$. In view of (7.44e), we indeed have pointwise weak* convergence in $L^\infty(\Gamma_C)$, i.e.

$$\overline{z}_{\varepsilon\tau}(t) \overset{*}{\rightharpoonup} z_\varepsilon(t) \quad \text{in } L^\infty(\Gamma_C) \text{ for all } t \in [0,T]. \tag{8.2b}$$

Finally, arguing as in the proof of [15, Theorem 6.1] with Helly's selection principle and taking into account (8.2b), we conclude that for all $0 \le s \le t \le T$

$$\mathrm{Var}_{\mathcal{R}}(z_\varepsilon; [s,t]) \le \lim_{\tau\to 0} \int_s^t \int_{\Gamma_C} \zeta_1\big(\dot z_{\varepsilon\tau}(r)\big)\,\mathrm{d}S\mathrm{d}r, \tag{8.2c}$$

recall definition (4.19) of $\mathrm{Var}_{\mathcal{R}}$. Ultimately, this yields that there exists $S_0' > 0$ such that

$$z_\varepsilon \in \mathrm{BV}([0,T]; L^1(\Gamma_C)) \quad \text{and} \quad \|z_\varepsilon\|_{\mathrm{BV}([0,T];L^1(\Gamma_C))} \le S_0' \quad \text{for all } \varepsilon > 0. \tag{8.2d}$$

Thirdly, by the same tokens we conclude from estimates (7.44g)–(7.44i) that there exists $\vartheta_\varepsilon \in L^r(0,T; W^{1,r}(\Omega\setminus\Gamma_C)) \cap \mathrm{BV}([0,T]; W^{1,r'}(\Omega\setminus\Gamma_C)^*)$ such that

$$\overline{\vartheta}_{\varepsilon\tau}, \vartheta_{\varepsilon\tau} \rightharpoonup \vartheta_\varepsilon \quad \text{in } L^r(0,T; W^{1,r}(\Omega\setminus\Gamma_C)), \tag{8.3a}$$

$$\overline{\vartheta}_{\varepsilon\tau}, \vartheta_{\varepsilon\tau} \to \vartheta_\varepsilon \quad \text{in } L^r(0,T; W^{1,r-\epsilon}(\Omega\setminus\Gamma_C)) \cap L^q(0,T; L^1(\Omega)), \tag{8.3b}$$

for all $\epsilon \in (0, r-1]$ and $1 \le q < \infty$, as well as, by Helly's selection principle,

$$\vartheta_{\varepsilon\tau}(t) \rightharpoonup \vartheta_\varepsilon(t) \quad \text{in } W^{1,r'}(\Omega\setminus\Gamma_C)^* \text{ for all } t \in [0,T]. \tag{8.3c}$$

Then, by the a priori bound of $\vartheta_{\varepsilon\tau}(t)$ in $L^1(\Omega)$, also

$$\vartheta_{\varepsilon\tau}(t) \stackrel{*}{\rightharpoonup} \vartheta_\varepsilon(t) \quad \text{in } \mathcal{M}(\bar\Omega) \text{ for all } t \in [0,T]. \tag{8.3d}$$

Notice that, under condition (5.8) on θ_0, convergence (8.3b) and (7.15) yield (6.8). It also follows from [15, Theorem 6.1] that $\mathrm{Var}_{\|\cdot\|_{W^{1,r'}(\Omega)^*}}(\vartheta_\varepsilon; [s,t]) \le \lim_{\tau\to 0} \mathrm{Var}_{\|\cdot\|_{W^{1,r'}(\Omega)^*}}(\vartheta_{\varepsilon\tau}; [s,t])$, with $\mathrm{Var}_{\|\cdot\|_{W^{1,r'}(\Omega)^*}}$ denoting the total variation w.r.t. the norm $\|\cdot\|_{W^{1,r'}(\Omega)^*}$. This entails that

$$\|\vartheta_\varepsilon\|_{\mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)} \le S_0' \quad \text{for all } \varepsilon > 0. \tag{8.3e}$$

For later purposes, we also point out that, in view of estimate (7.44g) and of (8.3b), there holds

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le S_0 \quad \text{for all } \varepsilon > 0, \tag{8.3f}$$

S_0 being the same constant as in estimates (7.44).

Besides, (7.44d) yields that

$$\tau \Big\| |e(\overline{u}_{\varepsilon\tau})|^{\gamma-2} e(\overline{u}_{\varepsilon\tau}) \Big\|_{L^{\gamma/(\gamma-1)}(Q;\mathbb{R}^{d\times d})} \le S_0 \tau^{1/\gamma} \to 0 \quad \text{as } \tau \to 0. \tag{8.4a}$$

In view of (4.3) and the second of (8.1e), it is not difficult to verify that, for all $\epsilon \in (0,3]$,

$$\left.\begin{array}{l} [\![\overline{u}_{\varepsilon\tau}]\!] \to [\![u_\varepsilon]\!] \quad \text{in } L^\infty(0,T; L^{4-\epsilon}(\Gamma_C;\mathbb{R}^d)),\\ [\![\overline{u}_{\varepsilon\tau}(t)]\!] \to [\![u_\varepsilon(t)]\!] \quad \text{in } L^{4-\epsilon}(\Gamma_C;\mathbb{R}^d) \text{ for all } t\in[0,T]. \end{array}\right\} \tag{8.4b}$$

Furthermore, using that $(I_K^\varepsilon)'$ is given by (6.3), and recalling (6.4), from (7.44b) we easily infer that

$$\exists S_1 = S_1(\varepsilon) > 0 \quad \forall \tau > 0: \quad \big\|(I_K^\varepsilon)'\big([\![\overline{u}_{\varepsilon\tau}]\!]\big)\big\|_{L^\infty(0,T;L^2(\Gamma_C;\mathbb{R}^d))} \le S_1, \tag{8.4c}$$

with $S_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$; more specifically, due to (6.4) we have $S_1(\varepsilon) = \mathcal{O}(1/\varepsilon)$. Combining (8.4b) with the strong–weak closedness of the graph of the operator $(I_K^\varepsilon)'$, up to the extraction of a further subsequence we find that

$$(I_K^\varepsilon)'\big([\![\overline{u}_{\varepsilon\tau}]\!]\big) \stackrel{*}{\rightharpoonup} (I_K^\varepsilon)'\big([\![u_\varepsilon]\!]\big) \quad \text{in } L^\infty(0,T; L^2(\Gamma_C;\mathbb{R}^d)). \tag{8.4d}$$

Moreover, using that $|(1+|[\![\overline{u}_\varepsilon\tau]\!]|^2)^{\frac{\mu}{2}-1}[\![\overline{u}_\varepsilon\tau]\!]| \le 2^{\frac{\mu}{2}-2}\big(\big|[\![\overline{u}_{\varepsilon\tau}]\!]\big| + \big|[\![\overline{u}_{\varepsilon\tau}]\!]\big|^{\mu-1}\big)$ a.e. in Σ_C, as well as estimate (7.44b), one sees that the sequence $\{(1+|[\![\overline{u}_{\varepsilon\tau}]\!]|^2)^{\frac{\mu}{2}-1}[\![\overline{u}_{\varepsilon\tau}]\!]\}$ is bounded in $L^\infty(0,T; L^{4/(\mu-1)}(\Gamma_C;\mathbb{R}^d))$. Thus,

$$\tau^{\beta/2}\Big(1+\big|[\![\overline{u}_{\varepsilon\tau}]\!]\big|^2\Big)^{\frac{\mu}{2}-1}[\![\overline{u}_{\varepsilon\tau}]\!] \to 0 \quad \text{in } L^\infty(0,T; L^{4/(\mu-1)}(\Gamma_C;\mathbb{R}^d)). \tag{8.4e}$$

Next, let us point out that, in the case the space dimension is $d = 3$, (7.44h) holds for all $1 \le r < 5/4$, so that (8.3b) yields by interpolation

$$\overline{\vartheta}_{\varepsilon\tau} \to \vartheta_\varepsilon \quad \text{in } L^{15/7-\epsilon}(Q) \text{ for all } \epsilon \in \left(0, \frac{8}{7}\right]. \tag{8.5a}$$

In particular, $\Theta(\overline{\vartheta}_{\varepsilon\tau}) \to \Theta(\vartheta_\varepsilon)$ a.e. in Q. Combining this information with (5.2) (note that, by (5.1b), $\omega > \frac{6}{5}$ for $d = 3$), it is immediate to deduce from (8.5a) that, for example,

$$\Theta(\overline{\vartheta}_{\varepsilon\tau}) \to \Theta(\vartheta_\varepsilon) \quad \text{in } L^2(Q). \tag{8.5b}$$

Furthermore, using standard trace theorems we also deduce from (8.3b) that for all $\epsilon \in (0, \frac{3}{7}]$

$$\overline{\vartheta}^{+}_{\varepsilon\tau}|_{\Gamma_C} \to \vartheta^{+}_{\varepsilon}|_{\Gamma_C} \quad \text{and} \quad \overline{\vartheta}^{-}_{\varepsilon\tau}|_{\Gamma_C} \to \vartheta^{-}_{\varepsilon}|_{\Gamma_C} \quad \text{in } L^r(0,T; L^{10/7-\epsilon}(\Gamma_C)),$$

so that, again by (5.2), for $d = 3$, using that $\omega > 6/5$, we conclude that

$$[\![\Theta(\overline{\vartheta}_{\varepsilon\tau})]\!] \to [\![\Theta(\vartheta_{\varepsilon})]\!] \quad \text{in } L^r(0,T; L^{12/7-\epsilon}(\Gamma_C))\ \forall\, \epsilon \in \left(0, \frac{5}{7}\right]. \tag{8.5c}$$

Similar calculations leading to (8.5b) and (8.5c) can be performed in the case $d = 2$.

In the end, we are now going to show that

$$\Phi_\varepsilon(u_\varepsilon(t), z_\varepsilon(t)) \leq \liminf_{\tau\to 0} \Phi_{\varepsilon\tau}(\overline{u}_{\varepsilon\tau}(t), \overline{z}_{\varepsilon\tau}(t)) \quad \text{for all } t \in [0,T]. \tag{8.6}$$

Indeed, taking into account (8.1e) it is not difficult to deduce that for all $t \in [0, T]$

$$\liminf_{\tau\to 0} \int_{\Omega\setminus\Gamma_C} \frac{1}{2}\mathbb{C}e\big(\overline{u}_{\varepsilon\tau}(t)\big) : e\big(\overline{u}_{\varepsilon\tau}(t)\big)\,\mathrm{d}x \geq \int_{\Omega\setminus\Gamma_C} \frac{1}{2}\mathbb{C}e\big(u_\varepsilon(t)\big) : e\big(u_\varepsilon(t)\big)\,\mathrm{d}x.$$

Combining (8.2b) with (8.4b), we have

$$\liminf_{\tau\to 0} \int_{\Gamma_C} \frac{\kappa}{2}\overline{z}_{\varepsilon\tau}(t)\big|[\![\overline{u}_{\varepsilon\tau}(t)]\!]\big|^2\,\mathrm{d}S \geq \int_{\Gamma_C} \frac{\kappa}{2} z_\varepsilon(t)\big|[\![u_\varepsilon(t)]\!]\big|^2\,\mathrm{d}S \quad \text{for all } t \in [0,T].$$

Besides, taking into account that I_K^ε is lower semicontinuous on $L^2(\Gamma_C; \mathbb{R}^d)$ (cf. (6.1)), we immediately conclude

$$\liminf_{\tau\to 0} I_K^\varepsilon([\![\overline{u}_{\varepsilon\tau}(t)]\!]) \geq I_K^\varepsilon([\![u_\varepsilon(t)]\!]) \quad \text{for all } t \in [0,T].$$

Collecting the above inequalities and also relying on (8.2b), we infer (8.6).

Step 1: *passage to the limit in the momentum equation.* As a first step, we shall take the limit as $\tau \to 0$ of the discrete momentum equation (7.28) and of the discrete heat equation (7.29) with more regular test functions, which, for technical reasons, we shall need to approximate carefully. More precisely, for the momentum balance equation (6.7) we shall use test functions

$$v \in L^2(0,T; W^{1,2+\lambda}_{\Gamma_D}(\Omega\setminus\Gamma_C; \mathbb{R}^d)) \cap W^{1,1}(0,T; L^2(\Omega;\mathbb{R}^d)) \quad \text{for some } \lambda > 0, \tag{8.7a}$$

and we shall approximate them with discrete approximations $\{v_\tau^k\}$, such that the related piecewise constant and linear interpolants fulfil as $\tau \to 0$,

$$\left.\begin{array}{ll} v_\tau \to v & \text{in } W^{1,1}(0,T; L^2(\Omega;\mathbb{R}^d)), \\ \overline{v}_\tau \to v & \text{in } L^2(0,T; W^{1,2+\lambda}(\Omega\setminus\Gamma_C;\mathbb{R}^d)) \text{ for some } \lambda > 0, \\ \tau^{1/\gamma} e(\overline{v}_\tau) \to 0 & \text{in } L^\gamma(Q; \mathbb{R}^{d\times d}), \\ \tau^{\beta/2}\|\overline{v}_\tau\|_{L^1(0,T;L^{4/(5-\mu)}(\Gamma_C))} \leq C. & \end{array}\right\} \tag{8.7b}$$

Now, combining (8.1a) and (8.5b) with the second of (8.7b) and (8.4a) with the third of (8.7b), we pass to the limit as $\tau \to 0$ in the first integral term on the left-hand side of (7.28). Secondly, (8.2a) and (8.4b) yield

$$\kappa\overline{z}_{\varepsilon\tau}[\![\overline{u}_{\varepsilon\tau}]\!] \overset{*}{\rightharpoonup} \kappa z_\varepsilon[\![u_\varepsilon]\!] \quad \text{in } L^\infty(0,T; L^{4-\epsilon}(\Gamma_C)) \text{ for all } \epsilon \in [0,3),$$

which we combine with the second of (8.7b). Also taking into account (8.4d) and (8.4e), together with the fourth of (8.7b), we take the limit of the second integral term on the left-hand side of (7.28). In the case $\varrho > 0$, we take the limit of the third and fourth terms on the left-hand side, and of the first term on the right-hand side of (7.28) by means of (8.1c) (combined with the first of (8.7b)), of (8.1d), and of (8.1e). Finally, using (7.5a) and (7.5b) we handle the second and third right-hand side terms in (7.28). We thus conclude that the triple $(u_\varepsilon, z_\varepsilon, \vartheta_\varepsilon)$ fulfils equation (6.7), first with test functions as in (8.7a) and ultimately, by a density argument, with test functions $v \in L^2(0,T; W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)) \cap W^{1,1}(0,T; L^2(\Omega;\mathbb{R}^d))$.

Step 2: *passage to the limit in the semistability condition.* We consider a subset $\mathcal{N} \subset (0,T)$ of full measure such that for all $t \in \mathcal{N}$ the approximate stability condition (7.40) holds independently of $\tau \to 0$. Then we fix $t \in \mathcal{N}$ and $\tilde{z} \in L^\infty(\Gamma_C)$. We may suppose without loss of generality that $\mathcal{R}(\tilde{z} - z_\varepsilon(t)) < +\infty$, hence

$$\tilde{z}(x) \leq z_\varepsilon(t,x) \quad \text{for a.a. } x \in \Gamma_C. \tag{8.8}$$

We then construct the following recovery sequence

$$\tilde{z}_{\varepsilon\tau}(t,x) := \begin{cases} \overline{z}_{\varepsilon\tau}(t,x)\dfrac{\tilde{z}(x)}{z_\varepsilon(t,x)} & \text{where } z_\varepsilon(t,x) > 0, \\ 0 & \text{where } z_\varepsilon(t,x) = 0. \end{cases} \tag{8.9}$$

Now, using (8.8) and (8.2b) one immediately sees that

$$\tilde{z}_{\varepsilon\tau}(\cdot,t) \le \overline{z}_{\varepsilon\tau}(\cdot,t) \quad \text{a.e. in } \Gamma_C, \qquad \tilde{z}_{\varepsilon\tau}(t) \overset{*}{\rightharpoonup} \tilde{z} \quad \text{in } L^\infty(\Gamma_C). \tag{8.10}$$

Plugging $\tilde{z}_{\varepsilon\tau}$ in (7.40), we find

$$\begin{aligned}
0 &\le \limsup_{\tau\to 0}\Big(\Phi_{\varepsilon\tau}(\overline{u}_{\varepsilon\tau}(t),\tilde{z}_{\varepsilon\tau}(t)) + \mathcal{R}(\tilde{z}_{\varepsilon\tau}(t)-\overline{z}_{\varepsilon\tau}(t)) - \Phi_{\varepsilon\tau}(\overline{u}_{\varepsilon\tau}(t),\overline{z}_{\varepsilon\tau}(t))\Big)\\
&= \limsup_{\tau\to 0}\int_{\Gamma_C}\left(\frac{\tau^\alpha}{2}\left(|\tilde{z}_{\varepsilon\tau}(t)|^2 - |\overline{z}_{\varepsilon\tau}(t)|^2\right) + \frac{\kappa}{2}\big|[\![\overline{u}_{\varepsilon\tau}(t)]\!]\big|^2\left(\tilde{z}_{\varepsilon\tau}(t)-\overline{z}_{\varepsilon\tau}(t)\right) - (a_0+a_1)\big(\tilde{z}_{\varepsilon\tau}(t)-\overline{z}_{\varepsilon\tau}(t)\big)\right)\mathrm{d}S\\
&\le \int_{\Gamma_C}\left(\frac{\kappa}{2}\big|[\![u_\varepsilon(t)]\!]\big|^2\left(\tilde{z}(t)-z_\varepsilon(t)\right) - (a_0+a_1)\big(\tilde{z}_\varepsilon(t)-\overline{z}_\varepsilon(t)\big)\right)\mathrm{d}S\\
&= \Phi_\varepsilon(u_\varepsilon(t),\tilde{z}(t)) + \mathcal{R}(\tilde{z}(t)-z_\varepsilon(t)) - \Phi_\varepsilon(u_\varepsilon(t),z_\varepsilon(t)),
\end{aligned} \tag{8.11}$$

where the second inequality ensues from (8.10) and (8.4b).

Step 3: *passage to the limit in the mechanical and total energy inequalities.* Using (8.1a), (8.1d), (8.2a), (8.2c) and (8.6), we pass to the limit in the left-hand side of the discrete mechanical energy inequality (7.38) by weak lower semicontinuity. To take the limit of the right-hand side, we employ (7.7), the weak convergence (8.1a) and the strong convergence (8.5b), which yield

$$\Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B} : e(\dot{u}_{\varepsilon\tau}) \rightharpoonup \Theta(\vartheta_\varepsilon)\mathbb{B} : e(\dot{u}_\varepsilon) \quad \text{weakly in } L^1(Q). \tag{8.12}$$

Also using (7.5a)–(7.5b), we conclude that the triple $(u_\varepsilon, z_\varepsilon, \vartheta_\varepsilon)$ complies for all $t \in [0,T]$ with

$$\begin{aligned}
&T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_\varepsilon(t)\big) + \Phi_\varepsilon\big(u_\varepsilon(t),z_\varepsilon(t)\big) + \int_0^t\int_\Omega \mathbb{D}e\big(\dot{u}_\varepsilon(s)\big) : e\big(\dot{u}_\varepsilon(s)\big)\,\mathrm{d}x\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(z_\varepsilon;[0,t]) \le T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_0\big)\\
&\quad + \Phi_\varepsilon\big(u_0,z_0\big) + \int_0^t\left(\int_\Omega \Theta(\vartheta_\varepsilon(s))\mathbb{B} : e\big(\dot{u}_\varepsilon(s)\big)\,\mathrm{d}x + \int_\Omega F(s)\cdot\dot{u}_\varepsilon(s)\,\mathrm{d}x + \int_{\Gamma_N} f(s)\cdot\dot{u}_\varepsilon(s)\,\mathrm{d}S\right)\mathrm{d}s.
\end{aligned} \tag{8.13}$$

We also pass to the limit in the discrete total energy inequality (7.39). Indeed, one tackles the left-hand side by the above-mentioned lower-semicontinuity arguments (also using (8.3c)), and passes to the limit in the right-hand side by convergences (7.7) and (7.5)–(7.6b). In this way, one concludes that (4.14) holds as an inequality.

Step 4: *mechanical energy equality.* Like in [12], we prove that, in the limit, the mechanical energy inequality (8.13) in fact holds as an equality, obtaining

$$\begin{aligned}
&T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_\varepsilon(t)\big) + \Phi_\varepsilon\big(u_\varepsilon(t),z_\varepsilon(t)\big) + \int_0^t\int_\Omega \mathbb{D}e\big(\dot{u}_\varepsilon\big) : e\big(\dot{u}_\varepsilon\big)\,\mathrm{d}x\mathrm{d}t + \mathrm{Var}_{\mathcal{R}}(z_\varepsilon;[0,t]) = T^{\varrho}_{\mathrm{kin}}\big(\dot{u}_0\big) + \Phi_\varepsilon\big(u_0,z_0\big)\\
&\quad + \int_0^t\left(\int_\Omega \Theta(\vartheta_\varepsilon)\mathbb{B} : e\big(\dot{u}_\varepsilon\big)\,\mathrm{d}x + \int_\Omega F\cdot\dot{u}_\varepsilon\,\mathrm{d}x + \int_{\Gamma_N} f(s)\cdot\dot{u}_\varepsilon(s)\,\mathrm{d}S\right)\mathrm{d}s \quad \text{for all } t\in[0,T].
\end{aligned} \tag{8.14}$$

To this aim, we develop the same calculations as throughout [12, Formulae (4.69)–(4.76)]. The first step of the argument is a sophisticated trick based on the previously proved semistability condition (see also, e.g., [19,31] for the use of such a technique in a rate-independent context), which allows us to prove the following inequality for all $t \in [0,T]$

$$\begin{aligned}
&\Phi_\varepsilon\big(u_\varepsilon(t),z_\varepsilon(t)\big) - \Phi_\varepsilon\big(u_0,z_0\big) + \mathrm{Var}_{\mathcal{R}}(z_\varepsilon;[0,t]) \ge \int_0^t \langle(\Phi_\varepsilon)'_u(u_\varepsilon,z_\varepsilon),\dot{u}_\varepsilon\rangle\,\mathrm{d}s\\
&= \int_0^t\int_{\Omega\setminus\Gamma_C}\mathbb{C}e(u_\varepsilon) : e(\dot{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\Gamma_C}\Big(\kappa z_\varepsilon[\![u_\varepsilon]\!]\cdot[\![\dot{u}_\varepsilon]\!] + (I^\varepsilon_K)'([\![u_\varepsilon]\!])\cdot[\![\dot{u}_\varepsilon]\!]\Big)\,\mathrm{d}S\mathrm{d}s,
\end{aligned} \tag{8.15}$$

where $(\Phi_\varepsilon)'_u$ denotes the partial Gâteaux derivative with respect to u of the functional $\Phi_\varepsilon : W^{1,2}(\Omega\setminus\Gamma_C)\times L^\infty(\Gamma_C)\to\mathbb{R}$, and the equality follows from the definition (6.6a) of Φ_ε. The second step consists of testing of the momentum balance equation (6.7) by $\dot{u}_\varepsilon$. In the case $\varrho = 0$, $\dot{u}_\varepsilon \in L^2(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d))$ is an admissible test function for (6.7). In the case $\varrho > 0$, the test by $\dot{u}_\varepsilon$ may be performed after proving that $\ddot{u}_\varepsilon \in L^2(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*)$, cf. Remark 5.4. In fact, a comparison argument in (6.7) readily yields that $\ddot{u}_\varepsilon \in L^2(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*)$. Choosing $\dot{u}_\varepsilon$ as a test function in (6.7) and integrating on $(0,t)$ for all $t\in[0,T]$ leads, after an integration by parts, to

$$\begin{aligned}
&\frac{\varrho}{2}\int_\Omega|\dot{u}_\varepsilon(t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega\mathbb{D}e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega\mathbb{C}e(u_\varepsilon) : e(\dot{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}s\\
&\quad + \int_0^t\int_{\Gamma_C}\Big(\kappa z_\varepsilon[\![u_\varepsilon]\!] + (I^\varepsilon_K)'([\![u_\varepsilon]\!])\cdot[\![\dot{u}_\varepsilon]\!]\Big)\,\mathrm{d}S\mathrm{d}s\\
&= \frac{\varrho}{2}\int_\Omega|\dot{u}_0|^2\,\mathrm{d}x + \int_0^t\left(\int_\Omega\Theta(\vartheta_\varepsilon)\mathbb{B} : e\big(\dot{u}_\varepsilon\big)\,\mathrm{d}x + \int_\Omega F\cdot\dot{u}_\varepsilon\,\mathrm{d}x + \int_{\Gamma_N} f(s)\cdot\dot{u}_\varepsilon(s)\,\mathrm{d}S\right)\mathrm{d}s.
\end{aligned} \tag{8.16}$$

Combining (8.15) with (8.16), we obtain the reverse inequality in (8.13) and thus conclude (8.14).

Step 5: *passage to the limit in the enthalpy equation.* First of all, we observe that the following chain of inequalities holds for all $t \in [0, T]$:

$$
\begin{aligned}
&\mathrm{Var}_{\mathcal{R}}(z_\varepsilon; [0, t]) + \int_0^t \int_\Omega \mathbb{D}e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t \le \liminf_{\tau\to 0} \int_0^t \int_{\Gamma_C} \zeta_1\left(\dot{z}_{\varepsilon\tau}\right)\,\mathrm{d}S\mathrm{d}t + \int_0^t \int_\Omega \mathbb{D}e(\dot{u}_{\varepsilon\tau}) : e(\dot{u}_{\varepsilon\tau})\,\mathrm{d}x\mathrm{d}t \\
&\quad \le \limsup_{\tau\to 0} T^{\varrho}_{\mathrm{kin}}(\dot{u}_{0,\tau}) + \Phi_{\varepsilon\tau}(u_{0,\tau}, z_0) - T^{\varrho}_{\mathrm{kin}}(\dot{u}_{\varepsilon\tau}(t)) - \Phi_{\varepsilon\tau}(\overline{u}_{\varepsilon\tau}(t), \overline{z}_{\varepsilon\tau}(t)) \\
&\qquad + \int_0^t \left(\int_\Omega \Theta(\overline{\vartheta}_{\varepsilon\tau})\mathbb{B} : e\big(\dot{u}_{\varepsilon\tau}\big) + \int_\Omega \overline{F}_\tau \cdot \dot{u}_{\varepsilon\tau}\,\mathrm{d}x + \int_{\Gamma_N} \overline{f}_\tau \cdot \dot{u}_{\varepsilon\tau}\,\mathrm{d}S \right) \mathrm{d}t \\
&\quad \le T^{\varrho}_{\mathrm{kin}}(\dot{u}_0) + \Phi_\varepsilon(u_0, z_0) - T^{\varrho}_{\mathrm{kin}}(\dot{u}_\varepsilon(t)) - \Phi_\varepsilon(u_\varepsilon(t), z_\varepsilon(t)) + \int_0^t \left(\int_\Omega \Theta(\vartheta_\varepsilon)\mathbb{B} : e\big(\dot{u}_\varepsilon\big) + F \cdot \dot{u}_\varepsilon\,\mathrm{d}x \right. \\
&\qquad \left. + \int_{\Gamma_N} f \cdot \dot{u}_\varepsilon\,\mathrm{d}S \right) \mathrm{d}t = \mathrm{Var}_{\mathcal{R}}(z_\varepsilon; [0, t]) + \int_0^t \int_\Omega \mathbb{D}e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{8.17}
$$

Indeed, the first inequality ensues from (8.1a) and (8.2c), the second one from the discrete mechanical energy inequality (7.38), the third one from (7.7), (8.1d), (8.6) and (8.12), and from (7.5a)–(7.5b), cf. also Step 3. Finally, the last equality ensues from (8.14). Thus, all of the above inequalities turn out to hold as equalities. By a standard lim inf /limsup argument, this entails that

$$
\left.\begin{aligned}
&\lim_{\tau\to 0} T^{\varrho}_{\mathrm{kin}}(\dot{u}_{\varepsilon\tau}(t)) = T^{\varrho}_{\mathrm{kin}}(\dot{u}_\varepsilon(t)), \\
&\lim_{\tau\to 0} \Phi_{\varepsilon\tau}(\overline{u}_{\varepsilon\tau}(t), \overline{z}_{\varepsilon\tau}(t)) = \Phi_\varepsilon(u_\varepsilon(t), z_\varepsilon(t))
\end{aligned}\right\} \quad \text{for all } t \in [0, T], \tag{8.18}
$$

as well as

$$
\mathbb{D}e(\dot{u}_{\varepsilon\tau}) : e(\dot{u}_{\varepsilon\tau}) \to \mathbb{D}e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon) \quad \text{strongly in } L^1(Q). \tag{8.19}
$$

As pointed out in Remark 8.1, it follows from (8.19) that $u_{\varepsilon\tau} \to u_\varepsilon$ in $W^{1,2}(0, T; W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d))$. Therefore, (7.4) yields that

$$
\overline{u}_{\varepsilon\tau} \to u_\varepsilon \quad \text{in } L^\infty(0, T; W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d)). \tag{8.20}
$$

Furthermore, arguing as in [12], from (8.17) holding as an equality we conclude that

$$
\zeta_1\big(\dot{z}_{\varepsilon\tau}\big) \overset{*}{\rightharpoonup} \xi^{\mathrm{surf}}_{\dot{z}_\varepsilon} \quad \text{in measure on } \overline{\Sigma}_C, \tag{8.21}
$$

with $\xi^{\mathrm{surf}}_{\dot{z}_\varepsilon}$ being the measure introduced in (4.17).

We are now in the position of taking the limit of (7.29), where we shall use test functions

$$
w \in \mathrm{C}^0(0, T; W^{1,r'+\varsigma}(\Omega \setminus \Gamma_C)) \cap W^{1,r'}(0, T; L^{r'}(\Omega)) \quad \text{for some } \varsigma > 0, \tag{8.22}
$$

and we shall approximate them with discrete approximations $\{w^k_\tau\}$, such that, as $\tau \to 0$, the related interpolants fulfil as $\overline{w}_\tau \to w$ in $\mathrm{C}^0(0, T; W^{1,r'+\varsigma}(\Omega \setminus \Gamma_C))$ for some $\varsigma > 0$, and $w_\tau \to w$ in $W^{1,r'}(0, T; L^{r'}(\Omega))$. Then, we pass to the limit in the first integral term on the left-hand side by exploiting (8.3c) and the aforementioned convergence for the test functions $\overline{w}_\tau$. To deal with the second term we observe that, due to (8.20), to (8.3b), and to the boundedness of the function $\mathcal{K} : \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^{d\times d}$, there holds $\mathcal{K}(e(\overline{u}_{\varepsilon\tau}), \overline{\vartheta}_{\varepsilon\tau}) \to \mathcal{K}(e(u_\varepsilon), \vartheta_\varepsilon)$ in $L^q(Q)$ for all $1 \le q < \infty$, which we combine with the weak convergence (8.3a) for $\overline{\vartheta}_{\varepsilon\tau}$ and with the convergence for $\overline{w}_\tau$. It follows from (5.1e) and from convergences (8.2a) and (8.4b) that $\eta([\![\underline{u}_{\varepsilon\tau}]\!], \overline{z}_{\varepsilon\tau}) \overset{*}{\rightharpoonup} \eta([\![u_\varepsilon]\!], z_\varepsilon)$ in $L^\infty(0, T; L^{3-\epsilon}(\Gamma_C))$ for all $\epsilon \in (0, 2]$, which we exploit with (8.5c) to take the limit of the third integral term. The passage to the limit in the fourth term results from (8.3a) and the convergence for $\vartheta_{\varepsilon\tau}$. As for the right-hand side of (7.29), to deal with the first integral term we exploit (8.19) and the convergence for $\overline{w}_\tau$, which in particular yields $\overline{w}_\tau \to w$ in $\mathrm{C}^0(\overline{Q})$. Relying on this convergence and on (8.21), we also infer

$$
\lim_{\tau\to 0} \left(- \int_{\Sigma_C} a_1 \dot{z}_{\varepsilon\tau} \frac{\overline{w}_\tau|^+_{\Gamma_C} + \overline{w}_\tau|^-_{\Gamma_C}}{2}\,\mathrm{d}S\mathrm{d}t \right) = \int_{\Sigma_C} \frac{w|^+_{\Gamma_C} + w|^-_{\Gamma_C}}{2}\,\xi^{\mathrm{surf}}_{\dot{z}_\varepsilon}(\mathrm{d}S\mathrm{d}t).
$$

Finally, employing (7.6b), one takes the limit of the last three terms on the right-hand side of (7.29), thus finding that the triple $(u_\varepsilon, z_\varepsilon, \vartheta_\varepsilon)$ fulfils the weak formulation (4.16) of the enthalpy equation for all test functions as in (8.22). Again by a density argument, we conclude that $(u_\varepsilon, z_\varepsilon, \vartheta_\varepsilon)$ fulfil (4.16) with test functions $w \in \mathrm{C}^0([0, T]; W^{1,r'}(\Omega \setminus \Gamma_C)) \cap W^{1,r'}(0, T; L^{r'}(\Omega))$.

Step 6: *total energy identity*. We test the weak formulation (4.16) of the enthalpy equation by 1, and add it to the mechanical energy equality (8.14). Thus, we conclude that the total energy identity (4.14) holds for the ε-approximate problem.

This concludes the proof that $(u_\varepsilon, z_\varepsilon, \vartheta_\varepsilon)$ solves the approximate problem, i.e. Theorem 6.1. □

Remark 8.1 (*Strong Convergence*)**.** Let us observe that, in the case $\varrho > 0$, by a classical argument based on a Korn-type inequality and the uniform convexity of the space $W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d))$, convergence (8.19) joint with (8.1d) allows us to conclude that $u_{\varepsilon\tau}\to u_\varepsilon$ in $W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d))$. Likewise, if $\varrho>0$ (8.18) gives $\dot u_{\varepsilon\tau}(t)\to\dot u_\varepsilon(t)$ in $L^2(\Omega;\mathbb{R}^d)$ for all $t\in[0,T]$.

Remark 8.2 (*Numerics*)**.** We point out that our method of proof may yield some strategy for numerical analysis after making a spatial discretization, although the non-variational structure of (7.9) (preserved if discretized in space) would still require some iterative procedure for numerical solution, cf. [46].

9. Limit passage with $\varepsilon\to 0$ and proof of Theorem 5.1

In passing to the limit in the ε-approximate problem as $\varepsilon\to 0$, we shall follow the steps of the proof of Theorem 6.1. Thus, we shall sketch most of the arguments, referring to the detailed calculations developed in Section 8, and dwell with some detail only on the passages to the limit as $\varepsilon\to 0$ in the momentum equation, and on the proof of the mechanical energy equality.

Step 0: *a priori estimates and compactness.* The sequence $(u_\varepsilon,z_\varepsilon,\vartheta_\varepsilon)_\varepsilon$ inherits the a priori estimates of Lemma 7.7, i.e. now

$$\|u_\varepsilon\|_{W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d))}+\varrho^{1/2}\|u_\varepsilon\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))}+\varrho\|\ddot u_\varepsilon\|_{L^2(0,T;W^{1,2}_{\Gamma_C}(\Omega;\mathbb{R}^d)^*)}\le S, \tag{9.1a}$$

$$\|z_\varepsilon\|_{L^\infty(\Sigma_C)}+\|z_\varepsilon\|_{\mathrm{BV}([0,T];L^1(\Gamma_C))}\le S, \tag{9.1b}$$

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}+\|\vartheta_\varepsilon\|_{L^r(0,T;W^{1,r}(\Omega))}+\|\vartheta_\varepsilon\|_{\mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)}\le S'_r\quad\text{for any }1\le r<\frac{d+2}{d+1}, \tag{9.1c}$$

$$\sup_{t\in[0,T]}\Phi_\varepsilon\left(u_\varepsilon(t),z_\varepsilon(t)\right)\le S \tag{9.1d}$$

for some $S>0$ and $S'_r>0$ depending on $1\le r<\frac{d+2}{d+1}$. Indeed, the first two estimates in (9.1a), the first of (9.1b), and the second of (9.1c) respectively follow from (7.44b), (7.44c), (7.44e) and (7.44h) via lower semicontinuity. The second of (9.1b) and (9.1c) have been proved throughout Section 8, cf. with (8.2d), (8.3e) and (8.3f). The third of estimates (9.1a) follows by testing (6.7) by functions $v\in L^2(0,T;W^{1,2}_{\Gamma_C}(\Omega\setminus\Gamma_C;\mathbb{R}^d))\cap W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^d))$ and taking into account (9.1a), the second of (9.1c). Finally, (9.1d) is a direct consequence of the total energy inequality.

By the Banach and the Helly selection principles, there is a subsequence (for simplicity, denoted by the same indexes) and (u,z,ϑ) such that the following convergences hold:

$$u_\varepsilon\rightharpoonup u\quad\text{in }W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d)), \tag{9.2a}$$

$$u_\varepsilon\to u\quad\text{in }\mathrm{C}^0([0,T];W^{1-\epsilon,2}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d))\quad\forall\epsilon\in(0,1], \tag{9.2b}$$

$$\text{if }\varrho>0,\quad u_\varepsilon\overset{*}{\rightharpoonup}u\text{ in }W^{1,\infty}(0,T;L^2(\Omega\setminus\Gamma_C;\mathbb{R}^d))\cap W^{2,2}(0,T;W^{1,2}_{\Gamma_C}(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*), \tag{9.2c}$$

$$\text{if }\varrho>0,\ u_\varepsilon\to u\quad\text{in }W^{1,2}(0,T;W^{1,2-\epsilon}_{\Gamma_D}(\Omega\setminus\Gamma_C;\mathbb{R}^d))\cap W^{1,q}(0,T;L^2(\Omega;\mathbb{R}^d))\quad\forall\,0<\epsilon\le 1,\ 1\le q<\infty, \tag{9.2d}$$

$$\text{if }\varrho>0,\ \dot u_\varepsilon(t)\rightharpoonup\dot u(t)\quad\text{in }L^2(\Omega;\mathbb{R}^d)\text{ for all }t\in[0,T], \tag{9.2e}$$

$$z_\varepsilon\overset{*}{\rightharpoonup}z\quad\text{in }L^\infty(\Sigma_C), \tag{9.2f}$$

$$z_\varepsilon(t)\overset{*}{\rightharpoonup}z(t)\quad\text{in }L^\infty(\Gamma_C)\text{ for all }t\in[0,T], \tag{9.2g}$$

$$\vartheta_\varepsilon\rightharpoonup\vartheta\quad\text{in }L^r(0,T;W^{1,r}(\Omega\setminus\Gamma_C)), \tag{9.2h}$$

$$\vartheta_\varepsilon\to\vartheta\quad\text{in }L^r(0,T;W^{1,r-\epsilon}(\Omega\setminus\Gamma_C))\cap L^q(0,T;L^1(\Omega))\quad\forall\epsilon\in(0,r-1],\ 1\le q<\infty, \tag{9.2i}$$

$$\vartheta_\varepsilon(t)\rightharpoonup\vartheta(t)\quad\text{in }\mathcal{M}(\bar\Omega)\text{ for all }t\in[0,T], \tag{9.2j}$$

$$\left.\begin{array}{ll}[\![u_\varepsilon]\!]\to[\![u]\!] & \text{in }L^\infty(0,T;L^{4-\epsilon}(\Gamma_C;\mathbb{R}^d)),\\ [\![u_\varepsilon(t)]\!]\to[\![u(t)]\!] & \text{in }L^{4-\epsilon}(\Gamma_C;\mathbb{R}^d)\text{ for any }t\in[0,T],\end{array}\right\}\quad\text{for all }\epsilon\in(0,3] \tag{9.2k}$$

$$\Theta(\vartheta_\varepsilon)\to\Theta(\vartheta)\quad\text{in }L^2(Q), \tag{9.2l}$$

$$[\![\Theta(\vartheta_\varepsilon)]\!]\to[\![\Theta(\vartheta)]\!]\quad\text{in }L^r(0,T;L^{3/2}(\Gamma_C)). \tag{9.2m}$$

Convergences (9.2) can be deduced from estimates (9.1) arguing in the very same way as throughout (8.1a)–(8.5c) in Section 8: in particular, cf. (8.5c) for (9.2m).

Step 1: *passage to the limit in the momentum equation.* First of all, notice that (9.1d), (9.2k) and (6.2), and the constraint $z\in[0,1]$ a.e. on Γ_C, yield that there exists some $C>0$ such that for all $t\in[0,T]C\ge\liminf_{\varepsilon_j\to 0}I^\varepsilon_K\left([\![u_\varepsilon(t)]\!]\right)\ge I_K\left([\![u(t)]\!]\right)$, whence (4.12). We now exploit (9.2a)–(9.2f), (9.2k) and (9.2l) to pass to the limit in (6.7) with $\varrho \geq 0$, tested by $v - u_\varepsilon$, for any v in $L^2(0, T; W^{1,2}_{\Gamma_D}(\Omega \setminus \Gamma_C; \mathbb{R}^d))$ (and in $W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^d))$ if $\varrho > 0$), with $[\![v]\!] \succeq 0$ on Σ_C, i.e. for any test function in (4.13). Using the mentioned convergences, we find

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{\Sigma_C} (I_K^{\varepsilon_j})'\big([\![u_\varepsilon]\!]\big) \cdot [\![u_\varepsilon - v]\!] \,\mathrm{d}S\mathrm{d}t = \limsup_{\varepsilon\to 0} \int_\Omega \varrho \dot{u}_0 \cdot \big(u_\varepsilon(0) - v(0)\big) - \varrho \dot{u}_\varepsilon(T) \cdot (u_\varepsilon(T) - v(T)) \,\mathrm{d}x \\
&\quad + \int_{\Sigma_N} f \cdot (u_\varepsilon - v) \,\mathrm{d}S\mathrm{d}t - \int_{\Sigma_C} \kappa z_\varepsilon [\![u_\varepsilon]\!] \cdot [\![u_\varepsilon - v]\!] \,\mathrm{d}S\mathrm{d}t \\
&\quad + \int_Q F \cdot (u_\varepsilon - v) - \big(\mathbb{D}e(\dot{u}_{\varepsilon_j}) + \mathbb{C}e(u_\varepsilon) - \mathbb{B}\Theta(\vartheta_{\varepsilon_j})\big) : e(u_\varepsilon - v) + \varrho \dot{u}_\varepsilon \cdot (\dot{u}_\varepsilon - v) \,\mathrm{d}x\mathrm{d}t \\
&\leq \int_\Omega \varrho \dot{u}_0 \cdot (u_0 - v(0)) - \varrho \dot{u}(T) \cdot (u(T) - v(T)) \,\mathrm{d}x + \int_{\Sigma_N} f \cdot (u - v) \,\mathrm{d}S\mathrm{d}t - \int_{\Sigma_C} \kappa z [\![u]\!] \cdot [\![u - v]\!] \,\mathrm{d}S\mathrm{d}t \\
&\quad + \int_Q F \cdot (u - v) - \big(\mathbb{D}e(\dot{u}) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)\big) : e(u - v) + \varrho \dot{u} \cdot (\dot{u} - v) \,\mathrm{d}x\mathrm{d}t. \qquad (9.3)
\end{aligned}$$

On the other hand, recalling formula (6.3) for the Yosida regularization $(I_K^\varepsilon)'$, we see that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{\Sigma_C} (I_K^{\varepsilon_j})'\big([\![u_\varepsilon]\!]\big) \cdot \big([\![u_\varepsilon]\!] - [\![v]\!]\big) \mathrm{d}S\mathrm{d}t \geq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\Sigma_C} \big([\![u_\varepsilon]\!] - \mathrm{P}_K\big([\![u_\varepsilon]\!]\big)\big) \cdot \big(\mathrm{P}_K\big([\![u_\varepsilon]\!] - [\![v]\!]\big)\big) \mathrm{d}S\mathrm{d}t \\
&\quad + \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\Sigma_C} \big([\![u_\varepsilon]\!] - \mathrm{P}_K\big([\![u_\varepsilon]\!]\big)\big) \cdot \big([\![u_\varepsilon]\!] - \mathrm{P}_K\big([\![u_\varepsilon]\!]\big)\big) \,\mathrm{d}S\mathrm{d}t \geq 0 \qquad (9.4)
\end{aligned}$$

the latter inequality holding due to the properties of the projection operator and the fact that $v \succeq 0$ on Σ_C. Combining (9.3) and (9.4), and rearranging some terms, we readily conclude the weak formulation (4.13) of the momentum inclusion.

Step 2: *passage to the limit in the semistability condition.* It can be performed by the very same recovery sequence trick devised in Step 2 of the proof of Theorem 6.1.

Step 3: *passage to the limit in the mechanical and total energy inequalities.* It follows from (9.2a), (9.2f), (9.2k) and (6.2) that

$$\Phi(u(t), z(t)) \leq \liminf_{\varepsilon_j\to 0} \Phi_{\varepsilon_j}(u_\varepsilon(t), z_\varepsilon(t)) \quad \text{for all } t \in [0, T]. \qquad (9.5)$$

Combining (9.5) with convergences (9.2) and arguing exactly like in Step 3 of the proof of Theorem 6.1, we pass to the limit by lower semicontinuity in conclude that (u, z, ϑ) complies for all $t \in [0, T]$ with the mechanical energy inequality (8.13), with Φ in place of Φ_ε. Likewise, we conclude the total energy inequality.

Step 4: *mechanical energy equality*. Arguing like in Section 8 (cf. [12, Formulae (4.69)–(4.76)]), we first of all prove that for all $t \in [0, T]$

$$\begin{aligned}
&\Phi\big(u(t), z(t)\big) - \Phi\big(u_0, z_0\big) + \mathrm{Var}_{\mathcal{R}}(z; [0, t]) \geq \int_0^t \langle \lambda, \dot{u} \rangle \,\mathrm{d}s \\
&\quad \text{for any } \lambda \in L^2(0, T; W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*) \text{ with } \lambda(t) \in \partial_u \Phi(u(t), z(t)) \text{ for a.a. } t \in (0, T), \qquad (9.6)
\end{aligned}$$

where $\partial_u \Phi : W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) \rightrightarrows W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*$ denotes the subdifferential w.r.t. u of the functional $\Phi : W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) \times L^\infty(\Gamma_C) \to \mathbb{R}$. It follows from definition (4.9) that the operator $\partial_u \Phi$ is given by

$$\begin{aligned}
&\lambda \in \partial_u \Phi(u, z) \quad \text{if and only if } \exists\, \ell \in \partial \mathcal{I}_K(u); \quad \forall\, v \in W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) : \\
&\quad \langle \lambda, v \rangle = \int_{\Omega\setminus\Gamma_C} \mathbb{C}e(u) : e(v) \,\mathrm{d}x + \int_{\Gamma_C} \kappa z [\![u]\!] \cdot [\![v]\!] \,\mathrm{d}S + \langle \ell, v \rangle, \qquad (9.7)
\end{aligned}$$

where we have introduced for notational convenience the functional $\mathcal{I}_K : W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) \to [0, +\infty]$ defined by $\mathcal{I}_K(u) = I_K([\![u]\!])$ its subdifferential $\partial \mathcal{I}_K : W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d) \rightrightarrows W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*$. Notice that $\partial \mathcal{I}_K = J^* \circ \partial I_K \circ J$, where J denotes the jump operator $J(u) = [\![u]\!]$ and J^* its adjoint. Now, let us observe that

$$\begin{aligned}
&\int_0^t \langle \ell, \dot{u} \rangle \,\mathrm{d}s = \mathcal{I}_K(u(t)) - \mathcal{I}_K(u(0)) = I_K\big([\![u(t)]\!]\big) - I_K\big([\![u(0)]\!]\big) = 0 \\
&\quad \text{for all } \ell \in L^2(0, T; W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*) \text{ such that } \ell(s) \in \partial \mathcal{I}_K\big([\![u(s)]\!]\big) \text{ for a.a. } s \in (0, T), \qquad (9.8)
\end{aligned}$$

by the chain rule for the convex functional $\mathcal{I}_K$ (cf. [47, Proposition XI.4.11]), and by (5.4a) and (4.12). Combining (9.6)–(9.8), we conclude the following inequality for all $t \in [0, T]$

$$\Phi\big(u(t), z(t)\big) - \Phi\big(u_0, z_0\big) + \mathrm{Var}_{\mathcal{R}}(z; [0, t]) \geq \int_0^t \left(\int_\Omega \mathbb{C}e(u) : e(\dot{u}) \,\mathrm{d}x\mathrm{d}s + \int_{\Gamma_C} \kappa z [\![u]\!] \cdot [\![\dot{u}]\!] \,\mathrm{d}S \right) \mathrm{d}s. \qquad (9.9)$$

We now distinguish the two cases $\varrho = 0$ and $\varrho > 0$.

Case $\varrho > 0$: arguing by comparison in (4.13) we may readily check that $\ddot{u} \in L^2(0,T; W^{1,2}_K(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*)$, which is in duality with $\dot{u} \in L^2(0,T; W^{1,2}_K(\Omega \setminus \Gamma_C; \mathbb{R}^d))$. As pointed out in Remark 5.4, this entails that $\dot{u}$ is an admissible test function for the momentum balance inclusion (4.13) (notice that, since $K(x)$ is a linear subspace of $\mathbb{R}^d$ for almost all $x \in \Gamma_C$, we also have $[\![\dot{u}]\!] \succeq 0$ on Σ_C). Further, (4.13) is equivalent to (5.9), cf. Remark 5.2. Then, upon testing (5.9) by $\dot{u}$ and again using (9.8) we conclude for all $t \in [0,T]$ that

$$\begin{aligned} &\frac{\varrho}{2}\int_\Omega |\dot{u}(t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega \mathbb{D}e(\dot{u}) : e(\dot{u})\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega \mathbb{C}e(u) : e(\dot{u})\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\Gamma_C} \kappa z[\![u]\!]\cdot[\![\dot{u}]\!]\,\mathrm{d}S\mathrm{d}s \\ &\quad = \frac{\varrho}{2}\int_\Omega |\dot{u}_0|^2\,\mathrm{d}x + \int_0^t \left(\int_\Omega \Theta(\vartheta)\mathbb{B} : e\big(\dot{u}\big)\,\mathrm{d}x + \int_\Omega F\cdot\dot{u}\,\mathrm{d}x + \int_{\Gamma_N} f\cdot\dot{u}\,\mathrm{d}S\right)\mathrm{d}s. \end{aligned} \tag{9.10}$$

Combining (9.10) with (9.9), we get the reverse of the mechanical energy inequality, which leads to the desired mechanical energy equality.

Case $\varrho = 0$: From the previously proved estimates, one can see that the functional

$$\ell : v \mapsto \int_Q F\cdot v\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_N} f\cdot v\,\mathrm{d}S\mathrm{d}t - \int_Q \big(\mathbb{D}e(\dot{u}) + \mathbb{C}e(u) - \mathbb{B}\Theta(\vartheta)\big) : e(v)\,\mathrm{d}x\mathrm{d}t - \int_{\Sigma_C} \kappa z[\![u]\!]\cdot[\![v]\!]\,\mathrm{d}S\mathrm{d}t \tag{9.11}$$

is in $L^2(0,T; W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*)$, and fulfils (cf. (4.13))

$$\int_{\Sigma_C} I_K\big([\![v]\!]\big)\mathrm{d}S\mathrm{d}t \geq \int_{\Sigma_C} I_K\big([\![u]\!]\big)\,\mathrm{d}S\mathrm{d}t + \int_0^T \langle \ell, v-u\rangle\,\mathrm{d}t. \tag{9.12}$$

Hence, $\ell(t) \in \partial\mathcal{I}_K\big([\![u(t)]\!]\big)$ for almost all $t \in (0,T)$. Thus, (9.8) yields $\int_0^t \langle \ell, \dot{u}\rangle\,\mathrm{d}s = 0$ for all $t \in [0,T]$, which is just relation (9.10) with $\varrho = 0$. Again, we combine the latter with (9.9), and conclude the mechanical energy equality.

Step 5: *passage to the limit in the enthalpy equation.* It can be developed in the very same way as in the proof of Theorem 6.1. We point out that, if (5.8) holds, convergence (9.2i) and (6.8) yield for almost all $(t,x) \in Q$ the strict positivity of $\theta(t,x) = \Theta(\vartheta(t,x))$.

Step 6: *total energy identity.* We test the weak formulation (4.16) of the enthalpy equation, by 1, and add it to the mechanical energy equality. Thus, we conclude the total energy balance (4.14). □

Remark 9.1 (*Convergence of the Reaction Force*)**.** Notice that, if $\varrho = 0$, there exists $S' > 0$ such that, for all $\varepsilon > 0$,

$$\left\| J^* \circ (I_K^\varepsilon)'([\![u_\varepsilon]\!])\right\|_{L^2(0,T;W^{1,2}(\Omega\setminus\Gamma_C;\mathbb{R}^d)^*)} = \sup_{\|v\|_{L^2(0,T;W^{1,2}(\Omega\setminus\Gamma_C;\mathbb{R}^d))}\leq 1} \left| \int_{\Sigma_C} (I_K^\varepsilon)'\big([\![u_\varepsilon]\!]\big)\cdot[\![v]\!]\,\mathrm{d}S\mathrm{d}t\right| \leq S',$$

which ensues from a comparison in (6.7) by using (9.1) (in spite of the blow-up of the reaction force $(I_K^\varepsilon)'([\![u_\varepsilon]\!])$ in $L^\infty(0,T; L^2(\Gamma_C; \mathbb{R}^d))$, cf. (8.4d)). Passing to the limit in (6.7), we can see that, for the subsequence selected in Step 0, $J^* \circ (I_K^\varepsilon)'([\![u_\varepsilon]\!])$ weakly converges in $L^2(0,T; W^{1,2}(\Omega \setminus \Gamma_C; \mathbb{R}^d)^*)$ to the function ℓ defined in (9.11).

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References

[1] M. Kočvara, A. Mielke, T. Roubíček, A rate-independent approach to the delamination problem, Math. Mech. Solids 11 (2006) 423–447.
[2] E. Bonetti, G. Bonfanti, R. Rossi, Global existence for a contact problem with adhesion, Math. Methods Appl. Sci. 31 (2008) 1029–1064.
[3] E. Bonetti, G. Bonfanti, R. Rossi, Well-posedness and long-time behaviour for a model of contact with adhesion, Indiana Univ. Math. J. 56 (2007) 2787–2820.

[4] M. Frémond, Equilibre des structures qui adhèrent à leur support, C. R. Acad. Sci. Paris 295 (1982) 913–916.
[5] M. Frémond, Adhèrence des solides, J. Mech. Theor. Appl. 6 (1987) 383–407.
[6] N. Point, Unilateral contact with adherence, Math. Methods Appl. Sci. 10 (1988) 367–381.
[7] M. Raous, L. Cangémi, M. Cocu, A consistent model coupling adhesion, friction, and unilateral contact, Comput. Methods Appl. Mech. Eng. 177 (1999) 383–399.
[8] M. Sofonea, W. Han, M. Shillor, Analysis and Approximation of Contact Problems with Adhesion or Damage, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[9] E. Bonetti, G. Bonfanti, R. Rossi, Thermal effects in adhesive contact: modelling and analysis, Nonlinearity 22 (2009) 2697–2731.
[10] E. Bonetti, G. Bonfanti, R. Rossi, Long-time behaviour of a thermomechanical model for adhesive contact, Discrete Contin. Dyn. Syst. Ser. S 4 (2011) 273–309.
[11] T. Roubíček, L. Scardia, C. Zanini, Quasistatic delamination problem, Contin. Mech. Thermodyn. 21 (2009) 223–235.
[12] T. Roubíček, Thermodynamics of rate independent processes in viscous solids at small strains, SIAM J. Math. Anal. 40 (2010) 256–297.
[13] A. Mielke, Evolution in rate-independent systems, in: C.M. Dafermos, E. Feireisl (Eds.), Handbook of Differential Equations, Evolutionary Equations, 2, Elsevier, Amsterdam, 2005, pp. 461–559.
[14] A. Mielke, F. Theil, A mathematical model for rate-independent phase transformations with hysteresis, in: H.-D. Alber, R.M. Balean, R. Farwig (Eds.), Proceedings of the Workshop on Models of Continuum Mechanics in Analysis and Engineering, Shaker-Verlag, Aachen, 1999, pp. 117–129.
[15] A. Mielke, F. Theil, On rate-independent hysteresis models, Nonlinear Differential Equations Appl. 11 (2004) 151–189. Accepted July 2001.
[16] A. Mielke, F. Theil, V. Levitas, A variational formulation of rate–independent phase transformations using an extremum principle, Arch. Ration. Mech. Anal. 162 (2002) 137–177.
[17] T. Roubíček, Rate independent processes in viscous solids at small strains, Math. Methods Appl. Sci. 32 (2009) 825–862.
[18] A. Chambolle, A. Giacomini, M. Ponsiglione, Crack initiation in brittle materials, Arch. Ration. Mech. Anal. 188 (2008) 309–349.
[19] G. Dal Maso, G. Francfort, R. Toader, Quasistatic crack growth in nonlinear elasticity, Arch. Ration. Mech. Anal. 176 (2005) 165–225.
[20] A. Giacomini, M. Ponsiglione, A Γ-convergence approach to stability of unilateral minimality properties in fracture mechanics and applications, Arch. Ration. Mech. Anal. 180 (2006) 399–447.
[21] A. Bedford, Hamilton's Principle in Continuum Mechanics, Pitman, Boston, 1985.
[22] K.T. Andrews, K.L. Kuttler, M. Rochdi, M. Shillor, One-dimensional dynamic thermoviscoelastic contact with damage, J. Math. Anal. Appl. 272 (2002) 249–275.
[23] A.A. Griffith, The phenomena of rupture and flow in solids, Philos. Trans. R. Soc. Lond. Ser. A. Math. Phys. Eng. Sci. 221 (1921) 163–198.
[24] A. Mielke, T. Roubíček, M. Thomas, From damage to delamination in nonlinearly elastic materials at small strain. (WIAS Preprint no. 1542, Berlin, 2010) J. Elasticity (submitted for publication).
[25] D. Leguillon, Strength or toughness? A criterion for crack onset at a notch, Eur. J. Mech. A Solids 21 (2002) 61–72.
[26] V. Mantič, Interface crack onset at a circular cylindrical inclusion under a remote transverse tension, Int. J. Solids Struct. 46 (2009) 1287–1304.
[27] J. Cook, J.E. Gordon, C.C. Evans, D.M. Marsh, A mechanism for the control of crack propagation in all-brittle systems, Proc. R. Soc. Lond. Ser. A 282 (1964) 508–520.
[28] E. Feireisl, H. Petzeltová, E. Rocca, Existence of solutions to a phase transition model with microscopic movements, Math. Methods Appl. Sci. 32 (2009) 1345–1369.
[29] T. Roubíček, Thermo-visco-elasticity at small strains with L^1-data, Quar. Appl. Math. 67 (2009) 47–71.
[30] C. Eck, J. Jarušek, M. Krbec, Unilateral Contact Problems, Chapman & Hall/CRC, Boca Raton, 2005.
[31] G. Francfort, A. Mielke, Existence results for a class of rate-independent material models with nonconvex elastic energies, J. Reine Angew. Math. 595 (2006) 55–91.
[32] D.H. Fremlin, Measure Theory. Vol. 3. Measure Algebras, Torres Fremlin, Colchester, 2004, Corrected second printing of the 2002 original.
[33] A. Petrov, M. Schatzman, Mathematical results on existence for viscoelastodynamic problems with unilateral constraints, SIAM J. Math. Anal. 40 (2009) 1882–1904.
[34] A. Petrov, M. Schatzman, A pseudodifferential linear complementarity problem related to a one dimensional viscoelastic model with Signorini conditions. Arch. Ration. Mech. Anal., in print (2010).
[35] H. Attouch, Variational Convergence for Functions and Operators, Pitman, Boston, 1984.
[36] H. Brézis, Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam, 1973.
[37] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden, 1976.
[38] P. Sadowski, S. Stupkiewicz, A model of thermal contact conductance at high real contact area fraction, Wear 268 (2010) 77–85.
[39] T. Roubíček, Nonlinear Partial Differential Equations with Applications, Birkhäuser, Basel, 2005.
[40] A. Mielke, T. Roubíček, Rate-independent systems, Theory and Application (in preparation).
[41] A. Mielke, M. Thomas, Damage of nonlinearly elastic materials at small strain: existence and regularity results, ZAMM 90 (2010) 88–112.
[42] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1989) 149–169.
[43] E. Feireisl, J. Málek, On the Navier–Stokes equations with temperature-dependent transport coefficients, Differ. Equ. Nonlinear Mech. (2006) 14 pp. (electronic), Art. ID 90616.
[44] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.
[45] V. Barbu, T. Precupanu, Convexity and Optimization in Banach Spaces, 2nd edition, D. Reidel Pub. Co, Dordrecht, 1986.
[46] S. Bartels, T. Roubíček, Thermo-visco-elasticity with rate-independent plasticity in isotropic materials undergoing thermal expansion, Math. Modelling Numer. Anal. 45 (2011) 477–504.
[47] A. Visintin, Models of Phase Transitions, Birkhäuser Boston, Inc., Boston, 1996.
[48] C. Eck, Existence of solutions to a thermo-viscoelastic contact problem with Coulomb friction, Math. Models Methods Appl. Sci. 12 (2002) 1491–1511.