

WED functionals for gradient flows in metric spaces: the convex case

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work in progress with

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Outline

- ▶ Gradient flows and WED functionals in Hilbert spaces
- ▶ Recaps on gradient flows in metric spaces
- ▶ The WED approach in metric spaces: difficulties
- ▶ A different viewpoint
- ▶ Convergence of the WED approximation in the convex case

Gradient flows in Hilbert spaces

Setup:

- ▶ $(\mathcal{H}, \|\cdot\|)$ separable Hilbert space
- ▶ $\varphi : \mathcal{H} \rightarrow (-\infty, +\infty]$ (proper), lower semicontinuous, convex functional
- ▶ $u_0 \in \text{dom}(\varphi)$

Let's consider the Cauchy problem

$$\begin{cases} u'(t) + \partial\varphi(u(t)) \ni 0 & \text{in } \mathcal{H}, \quad t \in (0, T), \\ u(0) = u_0 \end{cases} \quad (\text{P})$$

for the gradient flow equation **in the autonomous case**

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with $\partial\varphi$ **convex-analysis subdifferential of φ** :

$$u \in D(\varphi), \xi \in \partial\varphi(u) \Leftrightarrow \varphi(w) - \varphi(u) \geq \langle \xi, w - u \rangle \quad \forall w \in \mathcal{H}$$

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“Classical” results on (P)

Existence, uniqueness, approximation of solutions: ([Kōmura'67, Crandall-Pazy'69, Brézis'73])

The WED functional for the gradient flow

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For fixed $\varepsilon > 0$, $I_\varepsilon : H^1(0, T; \mathcal{H}) \rightarrow (-\infty, \infty]$

$$I_\varepsilon(v) := \int_0^T e^{-t/\varepsilon} \left(\frac{1}{2} \|v'(t)\|^2 + \frac{1}{\varepsilon} \varphi(v(t)) \right) dt$$

is the **WED=Weighted Energy Dissipation** functional for (P)

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Every minimizer $u_\varepsilon \in H^1(0, T; \mathcal{H})$ satisfies

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(this can be easily seen for **smooth** φ , calculating the first variation of I_ε)

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Problem: for $\varepsilon \searrow 0$ (**causal limit**), does (u_ε) converge to u , solution of (P)?

YES, see [\[Mielke-Stefanelli, ESAIM-COCV 2010\]](#).

Sketch of the proof (I)

$$\min_{v \in H^1(0, T; \mathcal{H}), v(0) = u_0} \int_0^T e^{-t/\varepsilon} \left(\frac{1}{2} \|v'(t)\|^2 + \frac{1}{\varepsilon} \varphi(v(t)) \right) dt \quad (\text{MIN})$$

Facts:

- ▶ φ convex & l.s.c. $\Rightarrow I_\varepsilon$ is (uniformly) convex on $H^1(0, T; \mathcal{H})$ & l.s.c.
- ▶ I_ε has a minimizer u_ε
- ▶ u_ε satisfies the Euler-Lagrange equation (can be checked via convex analysis)

$$\begin{cases} -\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0 & \text{in } \mathcal{H}, t \in (0, T) & \text{(Eul.-Lagr. equation),} \\ u_\varepsilon(0) = u_0 & & \text{(initial condition),} \\ u_\varepsilon'(T) = 0 & & \text{(final condition)} \end{cases}$$

- ▶ Crucial estimate on (u_ε) :

$$\exists C > 0 : \varepsilon \|u_\varepsilon''\|_{L^2(0, T; \mathcal{H})} + \varepsilon^{1/2} \|u_\varepsilon'\|_{L^\infty(0, T; \mathcal{H})} + \|u_\varepsilon'\|_{L^2(0, T; \mathcal{H})} + \|\partial\varphi(u_\varepsilon)\|_{L^2(0, T; \mathcal{H})} \leq C$$

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under the further assumption $\varphi \in C^{1,1}(\mathcal{H})$.

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- **Conclusion of the proof:**
 - ▶ $(u_\varepsilon)_\varepsilon$ is a **Cauchy sequence** in $C^0([0, T]; \mathcal{H})$
 - ▶ **passage to the limit** in (EUL) as $\varepsilon \searrow 0$

Outcome and applications of the results

The convergence of the WED minimizers (u_ε) to u (unique) solution of the gradient flow, is interesting

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2. **general theoretical frame**: approach to evolution problems via minimization of functionals on spaces of trajectories: [Brézis-Ekeland principle 1976], [Nayroles 1976]; monography by [Ghoussoub 2008], [Stefanelli 2008–, Visintin 2008]

The WED approach to metric gradient flows

Gradient flows in metric spaces

- ▶ [De Giorgi, Marino, Saccon, Tosques, Degiovanni, Ambrosio '80 ~ '90] \rightsquigarrow theory of **Curves of Maximal Slope** and **Minimizing Movements**
- ▶ [*Gradient flows in metric spaces*, Ambrosio-Gigli-Savaré 2005] \rightsquigarrow systematic treatment and refinement of the theory on existence, approximation and uniqueness, application to gradient flows in Wasserstein spaces

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WED functional on $(0, +\infty)$

We address a **different problem**: convergence as $\varepsilon \searrow 0$ of

$$u_\varepsilon \in \operatorname{Argmin} \left\{ \int_0^{+\infty} e^{-t/\varepsilon} \left(\frac{1}{2} \|v'(t)\|^2 + \frac{1}{\varepsilon} \varphi(v(t)) \right) dt \right\}$$

\rightsquigarrow links with the theory of **optimal control**, dynamic programming principle

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Further motivation

Obtain an **“alternative” proof of existence** of curves of maximal slope: the **WED functional has a “scalar character”**, hence it's suited to analysis in metric spaces, where derivatives are replaced by **“scalar surrogate”** quantities.

Heuristics for the metric formulation of gradient flows

Suppose **for the time being** that $\varphi : \mathcal{H} \rightarrow (-\infty, +\infty]$ is **smooth**

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 &\Leftrightarrow \|u'(t)\|^2 + \|D\varphi(u(t))\|^2 + 2\frac{d}{dt}\varphi(u(t)) = 0
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So we get the **equivalent formulation**:

$$\frac{d}{dt}\varphi(u(t)) = -\frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|D\varphi(u(t))\|^2$$

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involving **norms of derivatives**, rather than derivatives!!

Heuristics for the metric formulation of gradient flows

Suppose for the time being that $\varphi : \mathcal{H} \rightarrow (-\infty, +\infty]$ is **smooth**

$$\begin{aligned} u'(t) + D\varphi(u(t)) = 0 &\Leftrightarrow \|u'(t) + D\varphi(u(t))\|^2 = 0 \\ &\Leftrightarrow \|u'(t)\|^2 + \|D\varphi(u(t))\|^2 + \underbrace{2\langle u'(t), D\varphi(u(t)) \rangle}_{\text{chain rule}} = 0 \\ &\Leftrightarrow \|u'(t)\|^2 + \|D\varphi(u(t))\|^2 + 2\frac{d}{dt}\varphi(u(t)) = 0 \end{aligned}$$

So we get the **equivalent formulation**:

$$\frac{d}{dt}\varphi(u(t)) = -\frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|D\varphi(u(t))\|^2$$

involving **norms of derivatives**, rather than derivatives!!

Hence this formulation can be given in a metric context, using suitable **“surrogate notions” of (norms of) derivatives**.

“Derivatives” in metric spaces (I)

- **Ambient space:** (X, d) complete metric space

Metric derivative & geodesics

Given an **absolutely continuous curve** $u : [0, T] \rightarrow X$ ($u \in AC([0, T]; X)$), its **metric derivative** is

$$|u'| (t) := \lim_{h \rightarrow 0} \frac{d(u(t), u(t+h))}{|h|} \quad \text{for a.a. } t \in (0, T),$$

$$\boxed{\|u'(t)\| \rightsquigarrow |u'| (t)}$$

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A curve $\gamma : [0, 1] \rightarrow X$ is a (constant-speed) **geodesic** if:

$$\frac{d(\gamma(s), \gamma(t))}{|t-s|} = d(\gamma(0), \gamma(1)) \doteq |\gamma'| \quad \forall s, t \in [0, 1].$$

“Derivatives” in metric spaces (II)

- **Ambient space:** (X, d) complete metric space

Local slope & Chain rule

- Given $\varphi : X \rightarrow (-\infty, +\infty]$ (proper), and $u \in D(\varphi)$, the **local slope** of φ at u is

$$|\partial\varphi|(u) := \limsup_{v \rightarrow u} \frac{(\varphi(u) - \varphi(v))^+}{d(u, v)} \quad u \in D(\varphi)$$

$$\| -D\varphi(u) \| \rightsquigarrow |\partial\varphi|(u)$$

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- We say that $\varphi : X \rightarrow (-\infty, +\infty]$ satisfies the **chain rule** with respect to $|\partial\varphi|$ if $\forall v \in AC([0, T]; D(\varphi))$, the map $t \mapsto \varphi(v(t))$ is **absolutely continuous** and

$$-\frac{d}{dt}\varphi(v(t)) \leq |v'(t)| |\partial\varphi|(v(t)) \quad \text{for a.a. } t \in (0, T).$$

Metric formulation of gradient flows

Curves of Maximal Slope

An **absolutely continuous** curve $u : [0, T] \rightarrow X$ is a **Curve of Maximal Slope** for φ (w.r.t. the local slope) if

$$\boxed{\frac{d}{dt} \varphi(u(t)) = -\frac{1}{2} |u'|^2(t) - \frac{1}{2} |\partial\varphi|^2(u(t)) \quad \text{a.e. in } (0, T).} \quad (\text{CMS})$$

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- In view of the chain rule $\boxed{-\frac{d}{dt} \varphi \circ v \leq |v'| |\partial\varphi(v)|}$, (CMS) is equivalent to the **inequality**

$$\frac{d}{dt} \varphi(u(t)) \leq -\frac{1}{2}|u'|^2(t) - \frac{1}{2}|\partial\varphi|^2(u(t)) \quad \text{a.e. in } (0, T).$$

Existence for Curves of Maximal Slope

Theorem [Ambrosio-Gigli-Savaré '05]

IF $\varphi : X \rightarrow (-\infty, +\infty]$

1. is proper, lower semicontinuous
2. “coercive” $\sim \varphi$ has compact sublevels
3. the local slope $u \mapsto |\partial\varphi|(u)$ is l.s.c.
4. φ satisfies the chain rule w.r.t. $|\partial\varphi|$

THEN for every $u_0 \in D(\phi)$ there **exists** a Curves of Maximal Slope u for φ , with $u(0) = u_0$.

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THEN for every $u_0 \in D(\varphi)$ there **exists** a Curves of Maximal Slope u for φ , with $u(0) = u_0$.

Remark

- A sufficient condition for properties (3) & (4) is

φ is (geodesically-)convex

i.e. for all $u_0, u_1 \in D(\varphi)$ there exists a geodesic $\gamma : [0, 1] \rightarrow X$ such that

$$\varphi(\gamma(\theta)) \leq (1 - \theta)\varphi(\gamma(0)) + \theta\varphi(\gamma(1)) \quad \forall \theta \in [0, 1].$$

WED functionals on $(0, +\infty)$

Basic assumptions on $\varphi : X \rightarrow (-\infty, +\infty]$

1. φ is proper, lower semicontinuous, positive (bdd from below)
2. φ has compact sublevels

♣ For fixed $\varepsilon > 0$, we consider $J_\varepsilon : AC(0, +\infty; X) \rightarrow (-\infty, \infty]$

$$J_\varepsilon(v) := \int_0^{+\infty} e^{-t/\varepsilon} \left(\frac{1}{2} |v'|^2(t) + \frac{1}{\varepsilon} \varphi(v(t)) \right) dt$$

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Fact (I)

The WED minimum problem

$$\min \{ \mathcal{J}_\varepsilon(v) : v \in AC(0, +\infty; X), v(0) = u_0 \}.$$

has at least a solution u_ε .

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has at least a solution u_ε .

Indeed, let (v_n) be an infimizing sequence:

- ▶ (integral) estimate for $\varphi(v_n)$ + coercivity of $\varphi \Rightarrow$ “compactness” for (v_n)
- ▶ (integral) estimate for $|v_n'| \Rightarrow$ “equicontinuity” for (v_n)

The Euler-Lagrange equation for the WED minimization

$$u_\varepsilon \in \operatorname{Argmin} \left\{ \int_0^{+\infty} e^{-t/\varepsilon} \left(\frac{1}{2} |v'|^2(t) + \frac{1}{\varepsilon} \varphi(v(t)) \right) dt : v \in AC(0, +\infty; X), v(0) = u_0 \right\}$$

- Euler-Lagrange equation in the Hilbert case:

$$-\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + \partial\varphi(u_\varepsilon(t)) \ni 0 \text{ in } \mathcal{H}, t \in (0, +\infty) \quad (\text{EUL})$$

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u_ε fulfils the Euler-Lagrange equation

$$\frac{d}{dt} \left(\varphi(u(t)) - \frac{\varepsilon}{2} |u_\varepsilon'|^2(t) \right) + |u_\varepsilon'|^2(t) = 0 \quad \text{for a.a. } t \in (0, +\infty)$$

viz. the **metric version** of (EUL).

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$$\underbrace{-\varepsilon u_\varepsilon''(t) \times u_\varepsilon'(t)}_{\frac{d}{dt}(-\varepsilon \|u_\varepsilon'(t)\|^2)} + u_\varepsilon'(t) \times u_\varepsilon'(t) + \underbrace{\partial \varphi(u_\varepsilon(t)) \times u_\varepsilon'(t)}_{\frac{d}{dt} \varphi(u_\varepsilon(t))} = 0 \quad t \in (0, +\infty)$$

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Analytical difficulties (I)

From $\frac{d}{dt} \left(\varphi(u(t)) - \frac{\varepsilon}{2} |u'|^2(t) \right) + |u'|^2(t) = 0$ we get the *energy inequality*

$$\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2} |u'_\varepsilon|^2(t) + \int_0^t |u'_\varepsilon|^2(s) \, ds \leq \varphi(u_0) \quad \text{for all } t > 0 \quad (\text{ENE})$$

which is **NOT sufficient** for taking the limit $\varepsilon \searrow 0!!$

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♠ Lack of

- ▶ **pointwise** estimates for $\varphi(u_\varepsilon) \rightsquigarrow$ “compactness”
- ▶ **integral** estimates for $|u'_\varepsilon| \rightsquigarrow$ “equicontinuity”

& it is not possible to reproduce in the **metric context** the estimates of [Mielke-Stefanelli, ESAIM-COCV 2010]!!

◇ **Problem 1:** deduce $\boxed{\text{further estimates for } u_\varepsilon}$

Analytical difficulties (II)

$$\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u'_\varepsilon|^2(t) + \int_0^t |u'_\varepsilon|^2(s) \, ds \leq \varphi(u_0) \quad \text{per ogni } t > 0 \quad (\text{ENE})$$

♠ (ENE) does not have the “right structure”: to conclude that u_ε converges to a **Curve of Maximal Slope** for φ , we should pass to the limit

$$\begin{aligned} \varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2}|u'_\varepsilon|^2(t) + \int_0^t |u'_\varepsilon|^2(s) \, ds &\leq \varphi(u_0) \\ \downarrow \quad \text{as } \varepsilon \searrow 0 & \end{aligned}$$

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BUT (ENE) does not contain information on $|\partial\varphi|(u)!!$

◇ **Problem 2:** deduce another energy inequality where we can take the limit as $\varepsilon \searrow 0$

A new viewpoint

We study the **Value Functional**

$$\begin{aligned} V_\varepsilon(\bar{u}) &:= \min \{J_\varepsilon(v) : v \in AC(0, +\infty; X), v(0) = \bar{u}\} \\ &= \min \left\{ \int_0^{+\infty} e^{-t/\varepsilon} \left(\frac{1}{2}|v'|^2(t) + \frac{1}{\varepsilon}\varphi(v(t)) \right) dt : v \in AC(0, +\infty; X), v(0) = \bar{u} \right\} \end{aligned}$$

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♣ Remark: $V_\varepsilon(\bar{u}) \rightarrow \varphi(\bar{u})$ as $\varepsilon \searrow 0$

A new argument for taking the limit as $\varepsilon \searrow 0$

- ▶ Use the **Dynamic Programming Principle** for V_ε and deduce that for all $\varepsilon > 0$

u_ε is a Curve of Maximal Slope for V_ε

- ▶ under the assumption that φ is (geodesically)-convex obtain further estimates for $(u_\varepsilon)_\varepsilon$
- ▶ take the limit as $\varepsilon \searrow 0$

in the gradient flow equation of V_ε

instead of the Euler-Lagrange equation for the WED minimization!!

The Dynamic Programming Principle in a metric context

♣ It can be shown that for all $t > 0$

$$V_\varepsilon(\bar{u}) = \min_{v \in AC(0, +\infty; X), v(0) = \bar{u}} \left[\int_0^t e^{-s/\varepsilon} \left(\frac{1}{2} |v'(s)|^2 + \frac{1}{\varepsilon} \varphi(v(s)) \right) ds + V_\varepsilon(v(t)) e^{-t/\varepsilon} \right]$$

viz., to achieve the minimum cost $V_\varepsilon(\bar{u})$ it is necessary & sufficient to:

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3. pay what remains to pay in an **optimal way**
4. **minimize** over all possible trajectories

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hence the identity

$$- \frac{d}{dt} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2} |u'_\varepsilon|^2(s) + \frac{1}{\varepsilon} \varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon} V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0, T).$$

The Hamilton-Jacobi-Bellman equation in a metric context

For all $\varepsilon > 0$ there holds

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) = \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in X} \quad (\text{H-J})$$

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♣ Let's prove $\boxed{\leq}$ with the **Dynamic Programming Principle**: for any curve $u_{\min} \in \text{Argmin}_{v(0)=\bar{u}} \mathcal{J}_\varepsilon(v)$ there holds

$$V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta))e^{-\delta/\varepsilon} = \int_0^\delta e^{-t/\varepsilon} \left(\frac{1}{2}|u'_{\min}|^2(t) + \frac{1}{\varepsilon}\varphi(u_{\min}(t)) \right) dt$$

The Hamilton-Jacobi-Bellman equation in a metric context

For all $\varepsilon > 0$ there holds

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) = \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in X} \quad (\text{H-J})$$

♣ Let's prove $\boxed{\leq}$ with the **Dynamic Programming Principle**: for any curve $u_{\min} \in \text{Argmin}_{v(0)=\bar{u}} \mathcal{J}_\varepsilon(v)$ there holds

$$V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta))e^{-\delta/\varepsilon} = \int_0^\delta e^{-t/\varepsilon} \left(\frac{1}{2}|u'_{\min}|^2(t) + \frac{1}{\varepsilon}\varphi(u_{\min}(t)) \right) dt$$

hence

$$\frac{1}{\delta} \int_0^\delta e^{-t/\varepsilon} \left(\frac{1}{2}|u'_{\min}|^2(t) + \frac{1}{\varepsilon}\varphi(u_{\min}(t)) \right) dt - \frac{1 - e^{-\delta/\varepsilon}}{\delta} V_\varepsilon(u_{\min}(\delta)) = \frac{V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta))}{\delta}.$$

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and then we take the $\boxed{\lim_{\delta \searrow 0}}$ for $\boxed{\text{fixed } \varepsilon > 0}$.

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There holds

$$\underbrace{\liminf_{\delta \searrow 0} \frac{1}{\delta} \int_0^\delta e^{-t/\varepsilon} \left(\frac{1}{2}|u'_{\min}|^2(t) + \frac{1}{\varepsilon}\varphi(u_{\min}(t)) \right) dt}_{\geq \frac{1}{2}|u'_{\min}|^2(0) + \frac{1}{\varepsilon}\varphi(\bar{u})} \leq \underbrace{-\lim_{\delta \searrow 0} \frac{1 - e^{-\delta/\varepsilon}}{\delta} V_\varepsilon(u_{\min}(\delta))}_{= -\frac{1}{\varepsilon}V_\varepsilon(\bar{u})}$$

$$\leq \limsup_{\delta \searrow 0} \frac{V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta))}{\delta}$$

$$\leq \limsup_{\delta \searrow 0} \frac{(V_\varepsilon(\bar{u}) - V_\varepsilon(u_{\min}(\delta)))^+}{d(u_{\min}(\delta), \bar{u})} \frac{d(u_{\min}(\delta), \bar{u})}{\delta} \leq |\partial V_\varepsilon|(\bar{u})|u'_{\min}|(0)$$

Hence

$$\frac{1}{2}|u'_{\min}|^2(0) + \frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) \leq |\partial V_\varepsilon|(\bar{u})|u'_{\min}|(0)$$

whence

$$\boxed{\frac{1}{\varepsilon}\varphi(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) \leq \frac{1}{2}|\partial V_\varepsilon|^2(\bar{u})}$$

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- ♣ The proof of \leq only uses lower semicontinuity and coercivity of φ & the Dynamic Programming Principle
- ♠ The proof of \geq instead relies on

φ (geodesically-)convex on X

The gradient flow of the WED minimizers

♣ Let $u_\varepsilon \in \operatorname{Argmin}_{v(0)=u_0} \mathcal{J}_\varepsilon(v)$: Combining

$$-\frac{d}{dt} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2} |u'_\varepsilon|^2(s) + \frac{1}{\varepsilon} \varphi(u_\varepsilon(t)) - \frac{1}{\varepsilon} V_\varepsilon(u_\varepsilon(t)) \quad \text{for a.a. } t \in (0, T).$$

with the Hamilton-Jacobi-Bellman equation

$$\frac{1}{\varepsilon} \varphi(\bar{u}) - \frac{1}{\varepsilon} V_\varepsilon(\bar{u}) = \frac{1}{2} |\partial V_\varepsilon|^2(\bar{u}) \quad \text{for all } \bar{u} \in X$$

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we conclude

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viz. for all $\varepsilon > 0$, u_ε is a Curve of Maximal Slope for the Value functional V_ε .

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viz. for all $\varepsilon > 0$, u_ε is a Curve of Maximal Slope for the Value functional V_ε .

✓ It's in (EQV) that one has to take the limit as $\varepsilon \searrow 0!!!$

Energy estimates in the convex case

Assumptions on φ

- ▶ $\varphi : X \rightarrow (-\infty, +\infty]$ positive, l.s.c., **coercive**,
- ▶ $\varphi : X \rightarrow (-\infty, +\infty]$ (geodesically-)convex

Then $u_\varepsilon \in \operatorname{Argmin}_{v(0)=u_0} \mathcal{J}_\varepsilon(v)$ has the further properties:

- ▶ $t \mapsto |u'_\varepsilon|(t)$ is decreasing;
- ▶ $t \mapsto \varphi(u_\varepsilon(t))$ is convex;

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- ▶ $t \mapsto |u'_\varepsilon|(t)$ is decreasing;
- ▶ $t \mapsto \varphi(u_\varepsilon(t))$ is convex;
- ▶ $t \mapsto \varphi(u_\varepsilon(t))$ is decreasing: hence we get an **energy estimate**

$$\sup_{\varepsilon > 0, t > 0} \varphi(u_\varepsilon(t)) \leq \varphi(u_0)$$

From the gradient flow of V_ε to the gradient flow of φ

- Passage to the limit as $\varepsilon \searrow 0$ in

$$-\frac{d}{dt} V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2} |u'_\varepsilon|^2(s) + \frac{1}{2} |\partial V_\varepsilon|^2(u_\varepsilon(t))$$

i.e. in

$$\frac{1}{2} \int_0^t |u'_\varepsilon|^2(s) ds + \frac{1}{2} \int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s)) ds + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(u_0)$$

From the gradient flow of V_ε to the gradient flow of φ

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$$\frac{1}{2} \int_0^t |u'_\varepsilon|^2(s) \, ds + \frac{1}{2} \int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s)) \, ds + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(u_0)$$

- ▶ A priori estimates:

$$\sup_{\varepsilon > 0, t > 0} \varphi(u_\varepsilon(t)) \leq \varphi(u_0) \quad \rightsquigarrow \quad \text{compactness}$$

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- ▶ **Ascoli-Arzelà Theorem** (in metric spaces):

$$\exists (u_{\varepsilon_k})_k, \exists u \in \text{AC}(0, +\infty; X) : \quad u_{\varepsilon_k}(t) \rightarrow u(t) \quad \forall t \in (0, +\infty).$$

From the gradient flow of V_ε to the gradient flow of φ

- Passage to the limit as $\varepsilon \searrow 0$ in

$$\frac{1}{2} \int_0^t |u'_\varepsilon|^2(s) \, ds + \frac{1}{2} \int_0^t |\partial V_\varepsilon|^2(u_\varepsilon(s)) \, ds + V_\varepsilon(u_\varepsilon(t)) = V_\varepsilon(u_0)$$

- ♣ It can be checked that

$$\begin{aligned} \frac{1}{2} \int_0^t |u'|^2(s) \, ds &\leq \liminf_{k \rightarrow \infty} \frac{1}{2} \int_0^t |u'_{\varepsilon_k}|^2(s) \, ds \\ \varphi(u(t)) &\leq \liminf_{k \rightarrow \infty} V_{\varepsilon_k}(u_{\varepsilon_k}(t)), \quad V_{\varepsilon_k}(u_0) \rightarrow \varphi(u_0) \end{aligned}$$

From the gradient flow of V_ε to the gradient flow of φ

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From the gradient flow of V_ε to the gradient flow of φ

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- Hence $u \in AC(0, +\infty; X)$ fulfils for all $t \in (0, +\infty)$

$$\frac{1}{2} \int_0^t |u'|^2(s) ds + \frac{1}{2} \int_0^t |\partial \varphi|^2(u(s)) ds + \varphi(u(t)) \leq \varphi(u_0)$$

equivalent (via chain rule) to
$$\frac{1}{2} |u'|^2(t) + \frac{1}{2} |\partial \varphi|^2(u(t)) = -\frac{d}{dt} \varphi(u(t)) \quad \text{a.e. in } (0, +\infty)$$

The main result

Theorem (R., Savaré, Segatti, Stefanelli 2011)

Suppose that $\varphi : X \rightarrow (-\infty, +\infty]$ is

- ▶ positive, l.s.c., **coercive**,
- ▶ (geodesically-) **convex**;

let $u_0 \in D(\varphi)$.

Then,

any sequence of WED minimizers (u_ε) with $u_\varepsilon(0) = u_0$
has a subsequence (u_{ε_k}) which converges as $\varepsilon_k \searrow 0$ to
 $u \in AC(0, +\infty; X)$, Curve of Maximal Slope for φ such that $u(0) = u_0$.

Conclusions & future developments

- ▶ φ (geodesically-)convex $\rightsquigarrow \varphi$ (geodesically-) λ -convex, with $\lambda < 0$
- ▶ u_ε WED minimizer $\rightsquigarrow u_\varepsilon$ approximate minimizer for the WED functional
- ▶ combine WED approximation with time-discretization (Minimizing Movements)
- ▶

Future development: φ **NOT (geodesically-) λ -convex**..... it's an open problem in the Hilbert case as well!