Global attractors for gradient flows in metric spaces

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Recent advances in Free Boundary Problems and related topics

Levico, September 14–16 2006

Evolution PDEs of diffusive type and the Wasserstein metric

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$$L = L(x, \rho, \nabla \rho) : \mathbb{R}^n \times (0,+\infty) \times \mathbb{R}^n \to \mathbb{R} \quad \text{(Lagrangian)}$$

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$$\begin{cases} \partial_t \rho - \operatorname{div}\left(\rho \nabla (\frac{\delta \mathcal{L}}{\delta \rho})\right) = 0 \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ \rho(x,t) \geq 0, \quad \int_{\mathbb{R}^n} \rho(x,t)\,\mathrm{d}x = 1 \ \ \forall\,(x,t) \in \mathbb{R}^n \times (0,+\infty), \\ \int_{\mathbb{R}^n} |x|^2 \rho(x,t)\,\mathrm{d}x < +\infty \quad \forall\, t \geq 0, \end{cases}$$

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For t fixed, identify $\rho(\cdot,t)$ with the probability measure $\mu_t := \rho(\cdot,t)dx$

then $\mathcal{L}$ can be considered defined on $\mathscr{P}_2(\mathbb{R}^n)$ (the space of probability measures on $\mathbb{R}^n$ with finite second moment)

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Otto, Jordan & Kinderlehrer and Otto ['97–'01]
showed that (PDE) can be interpreted as
the **gradient flow** of $\mathcal{L}$ in $\mathscr{P}_2(\mathbb{R}^n)$
w.r.t. the Wasserstein distance W_2 on $\mathscr{P}_2(\mathbb{R}^n)$

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Examples

The potential energy functional $\rightsquigarrow$ **The linear transport equation**

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Ex.3: The internal energy functional

$$\mathcal{L}_3(\rho) := \int_{\mathbb{R}^n} \frac{1}{m-1}\rho^m(x)\,\mathrm{d}x, \quad m \neq 1$$

Examples

The potential energy functional $\rightsquigarrow$ The linear transport equation

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$$\mathcal{L}_3(\rho) := \int_{\mathbb{R}^n} \frac{1}{m-1}\rho^m(x)\,\mathrm{d}x, \quad \begin{cases} L_3(x,\rho,\nabla\rho) = \frac{1}{m-1}\rho^m, \\ \frac{\delta \mathcal{L}_3}{\delta \rho} = \partial_\rho L_3(\rho) = \frac{m}{m-1}\rho^{m-1}, \end{cases}$$

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The internal functional $\rightsquigarrow$ The porous media equation

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$$\partial_t \rho - \Delta\rho^m = 0 \quad \text{Otto '01}$$

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Ex.4: The (Entropy+ Potential) energy functional

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Ex.4: The (Entropy+ Potential) energy functional

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Entropy+Potential $\rightsquigarrow$ The Fokker-Planck equation

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$$\partial_t\rho - \Delta\rho - \mathrm{div}(\rho\nabla V) = 0 \quad \text{Jordan-Kinderlehrer-Otto '97}$$

New insight

- This **gradient flow approach** has brought several developments in:
  - **approximation algorithms**
  - **asymptotic behaviour** of solutions (new contraction and energy estimates)
  - applications to functional inequalities.....

[Agueh, Brenier, Carlen, Carrillo, Dolbeault, Gangbo, Ghoussoub,McCann, Otto, Vazquez, Villani..]

Towards gradient flows in metric spaces

- In **[Gradient flows in metric and in the Wasserstein spaces** AMBROSIO, GIGLI, SAVARÉ **'05]**, these **existence, approximation, long-time behaviour** results were recovered for **general**

  Gradient Flows in Metric Spaces

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- Metric spaces are a **suitable framework** for rigorously interpreting diffusion PDE as gradient flows in the Wasserstein spaces ($\rightsquigarrow$ no linear structure!)

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• The approach of [AGS'05] based on the theory of **Minimizing Movements & Curves of Maximal Slope** [DE GIORGI, MARINO, TOSQUES, DEGIOVANNI, AMBROSIO.. '80∼'90]

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• [AGS'05]: $\Rightarrow$ **refined existence, approximation, long-time behaviour** for Curves of Maximal Slope & applications to gradient flows in Wasserstein spaces

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• [R.-SEGATTI-STEFANELLI'06]: **complement** AGS's results on the **long-time behaviour** of Curves of Maximal Slope

Gradient flows in metric spaces: heuristics

Data:

- A complete metric space (X, d),
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To get some insight, let us go back to the euclidean case...

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So we get the **equivalent** formulation:

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Metric derivatives and slopes

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$\Rightarrow$ definition of the space of **absolutely continuous curves** $(\mathrm{AC}(0, T; X))$;

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The local slope in an **upper gradient** for ϕ, i.e. for any curve $v \in \mathrm{AC}(0, T; X)$ s.t. $\phi \circ u$ is absolutely continuous on $(0, T)$

$$\left| \frac{d}{dt}\phi(u(t)) \right| \leq |u'|(t)\ |\partial\phi|(u(t)) \quad \text{a.e. in } (0, T).$$

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2-curves of maximal slope in $\mathscr{P}_2(\mathbb{R}^n)$ lead (for a suitable ϕ) to the **linear** transport equation

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p-curves of maximal slope in $\mathscr{P}_p(\mathbb{R}^n)$ lead (for a suitable ϕ) to the **"doubly linear"** transport equation

$$\partial_t \rho - \nabla \cdot (\rho j_q(\nabla V)) = 0 \qquad j_q(r) := \begin{cases} |r|^{q-2} r & r \neq 0, \\ 0 & r = 0. \end{cases}$$

An existence result

- Given an **initial datum** $u_0 \in X$, does there exist a $p-$curve of maximal slope u on $(0, T)$ fulfilling $u(0) = u_0$?

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$$\phi(\gamma_t) \leq (1-t)\phi(v_0) + t\phi(v_1) - \frac{\lambda}{2}t(1-t)d^2(v_0, v_1) \quad \forall\, t \in [0,1].$$

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Theorem [Ambrosio-Gigli-Savaré '05]

• $\lambda > 0$:
exponential convergence of the solution as $t \to +\infty$ **to the unique minimum point** $\bar{u}$ of ϕ:

$$d(u(t), \bar{u}) \leq e^{-\lambda t} d(u_0, \bar{u}) \quad \forall\, t \geq 0$$

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Theorem [Ambrosio-Gigli-Savaré '05]

• $\lambda = 0 \quad + \quad \phi$ has compact sublevels:
convergence to (an) equilibrium as $t \to +\infty$

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"Fill in the gaps" in the study of the long-time behaviour of p-curves of maximal slope:

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Study the general case:

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- p general

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Namely, we comprise the cases:

1. $p = 2$, $\lambda < 0$ $\rightsquigarrow$ uniqueness: **YES**
2. $p \neq 2$, $\lambda \in \mathbb{R}$ $\rightsquigarrow$ uniqueness: **NO**

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Not the study of the convergence to equilibrium as $t \to +\infty$ of **a single trajectory**

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In [**R., Segatti, Stefanelli**, *Global attractors for curves of maximal slope*, in preparation]: Ball's theory of **generalized semiflows**

Generalized Semiflows: definition

Phase space: a metric space $(\mathcal{X}, d_{\mathcal{X}})$

Generalized Semiflows: definition

A **generalized semiflow** $\mathcal{S}$ on $\mathscr{X}$ is a family of maps $g : [0, +\infty) \to \mathscr{X}$ ("solutions"), s. t.

(Existence) $\forall\, g_0 \in \mathscr{X}\ \exists$ **at least one** $g \in \mathcal{S}$ with $g(0) = g_0$,

(Translation invariance) $\forall\, g \in \mathcal{S}$ and $\tau \geq 0$, the map $g^\tau(\cdot) := g(\cdot + \tau)$ is in $\mathcal{S}$,

(Concatenation) $\forall\, g,\ h \in \mathcal{S}$ and $t \geq 0$ with $h(0) = g(t)$, then $z \in \mathcal{S}$, where

$$z(\tau) := \begin{cases} g(\tau) & \text{if } 0 \leq \tau \leq t, \\ h(\tau - t) & \text{if } t < \tau, \end{cases}$$

(U.s.c. w.r.t. initial data) If $\{g_n\} \subset \mathcal{S}$ and $g_n(0) \to g_0$, $\exists$ subsequence $\{g_{n_k}\}$ and $g \in \mathcal{S}$ s.t. $g(0) = g_0$ and $g_{n_k}(t) \to g(t)$ for all $t \geq 0$.

Generalized Semiflows: dynamical system notions

Within this framework:

- **orbit** of a solution/set
- ω-**limit** of a solution/set
- **invariance under the semiflow** of a set
- **attracting set** (w.r.t. the Hausdorff semidistance of $\mathcal{X}$)

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Definition

A set $\mathcal{A} \subset \mathcal{X}$ is a **global attractor** for a **generalized semiflow** $\mathcal{S}$ if:

- ♣ $\mathcal{A}$ is **compact**
- ♣ $\mathcal{A}$ is **invariant** under the semiflow
- ♣ $\mathcal{A}$ **attracts** the **bounded** sets of $\mathcal{X}$ (w.r.t. the Hausdorff semidistance of $\mathcal{X}$)

Long-time behaviour for p-curves of maximal slope

$$\frac{d}{dt}\phi(u(t)) = -\frac{1}{p}|u'|^{p}(t) - \frac{1}{q}|\partial\phi|^{q}(u(t)) \quad \text{for a.e. } t \in (0, T),$$

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Choice of the phase space:

$$\mathcal{X} = D(\phi) \subset X,$$
$$d_{\mathcal{X}}(u,v) := d(u,v) + |\phi(u) - \phi(v)| \quad \forall\, u, v \in \mathcal{X}.$$

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Choice of the solution notion: We consider the set $\mathcal{S}$ of the locally absolutely continuous p-curves of maximal slope for ϕ $u : [0,+\infty) \to \mathcal{X}$.

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- ¿ Is $\mathcal{S}$ a **generalized semiflow**?
- ¿ Does $\mathcal{S}$ **possess a global attractor**?

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Theorem 1 [R., Segatti, Stefanelli '06]

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Suppose that

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$$\mathcal{S} \text{ admits a global attractor } \mathcal{A}.$$

The transport equation

$$\partial_t \rho - \mathrm{div}\left(\rho j_q(\nabla V)\right) = 0 \text{ in } \mathbb{R}^n \times (0, +\infty) \qquad \begin{cases} p > 2, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$$

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- $V : \mathbb{R}^n \to (-\infty, +\infty]$ proper and **lower semicontinuous**
- $\exists\, \varepsilon, M_1, M_2 > 0 : \; V(x) \geq M_1 |x|^{p+\varepsilon} - M_2$ for all $x \in \mathbb{R}^n$
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