# A vanishing viscosity approach to rate-independent modelling in metric spaces

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**joint work (in progress) with**
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**Rate-independence, Homogenization and Multiscaling**

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# The subdifferential formulation of rate-independent problems

Rate-independent systems can be modelled by

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B', \qquad t \in (0,T) \qquad \text{(DNE)}$$

- $B$ Banach space;
- $\Psi : B \to [0,+\infty]$, $\Psi(0)=0$, l.s.c. and **convex**
- $\partial$ subdifferential in the sense of **convex analysis**
- $\Psi$ **positively 1-homogeneous** $\leftrightarrow$ (DNE) invariant under time rescaling
- $\mathcal{E} : [0,T] \times B \to (-\infty,+\infty]$ **smooth** w.r.t. $t \in (0,T)$
- $\partial_u$ "subdifferential" of $\mathcal{E}$ **w.r.t. the second variable**

## Drawbacks

The subdifferential formulation works well in a smooth/convex setting.
Standard regularity of solutions is $u \in \mathrm{BV}(0,T;B)$ ($u$ may have **jumps!!!**):
how to handle derivatives?

# Looking for a derivative-free formulation

## Applications [... most of the people here....]

1. **quasistatic solid-solid phase transformations (in SMA)**
2. **linearized & finite-strain elastoplasticity**
3. **quasistatic propagation of fractures**
4. **damage**
5. **delamination problems**
6. **ferromagnetism, ferroelectricity**
7. **shape evolution of debonding membranes**......

## In applications

- $\mathcal{E}$ may be **neither smooth nor convex!**
- $B$ may be **non reflexive** (e.g. $L^1$ in SMA)
- $B$ may **lack a linear structure** (e.g. in fractures)

Need of a **derivative-free** formulation (without $u'$ and $\partial_u \mathcal{E}$!)

# The energetic formulation

**In this spirit:** generalized formulations for doubly nonlinear problems based on **global variational principles**: [Visintin'01], [Mielke-Ortiz07], [Stefanelli06-07]

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### Energetic solutions [Mielke-Theil'99,'04], [Mielke-Theil-Levitas'02]

$u : [0, T] \to B$ satisfying **global stability condition** & **energy balance**

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, z) + \mathcal{D}(u(t), z) \qquad \forall\, z \in B\,,$$

$$\mathcal{E}(t, u(t)) + \mathrm{Diss}_{\mathcal{D}}(u, [0, t]) = \mathcal{E}(t, u(0)) + \int_0^t \partial_t \mathcal{E}(r, u(r))\, \mathrm{d}r\,.$$

**Pro's**

- ✓ Completely **derivative-free** $\rightsquigarrow$ adaptable to more general ambient spaces (general topological spaces **[Mainik-Mielke'05], [Bucur-Buttazzo'07] (Minimizing movements approach)**)
- ✓ **equivalence** with the **subdifferential** formulation (DNE) if $\mathcal{E}$ **convex**

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**BUT** (in the **non convex** case), **global stability** forces energetic solutions to **jump early** to avoid energy losses (Maxwell rule)

# Viscous regularizations & jumps & local, rather than global, stability

## Aims

- **model** ("natural") **jumps** (due to $u \in \mathrm{BV}(0, T; B)$)
- obtain **solutions jumping later** (than via the Maxwell rule)

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## Vanishing viscosity in the applications

- quasistatic evolution of fractures: [Toader-Zanini'06], [Cagnetti'07], [Cagnetti-Toader'07], [Knees-Mielke-Zanini'07], leading to **local stability**-oriented formulations: [Dal Maso-Toader'02], [Negri-Ortner'07], [Garroni-Larsen07] **(threshold evolutions in damage)**..
- plasticity with softening: [Dal Maso-DeSimone-Mora-Morini'06]
- Kurzweil formulation of rate-independent processes with **convex energies & discontinuous inputs**: [Krejčí-Liero'07]

# The vanishing viscosity analysis by Efendiev & Mielke

## Problem

In the vanishing viscosity limit:

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## The approach by Efendiev-Mielke

- Jumps in the vanishing viscosity limit correspond to **viscous transitions** between stable states
- To capture the viscous transition path: **NOT SHRINK** jumps at a point, look at curves with their **arc length parametrization**
- Asymptotic analysis of (reparametrized) trajectories in an extended phase space

# The vanishing viscosity analysis by Efendiev & Mielke

[Efendiev-Mielke, J. Convex Anal.'06]

## Setting:

- $B$ **finite dimensional** space
- $\mathcal{E} \in \mathrm{C}^1([0,T] \times B; \mathbb{R}^+)$
- $C_1\|u\| \leq \Psi(u) \leq C_2\|u\| \quad \forall\, u \in B$

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The **viscous regularization** of $\Psi$:

$$\Phi_\varepsilon(u) := \Psi(u) + \frac{\varepsilon}{2}\|u\|^2 \quad \forall\, \varepsilon > 0.$$

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be the family of solutions of the Cauchy problem

$$\begin{cases} \partial\Phi_\varepsilon(u_\varepsilon'(t)) + \mathrm{D}\mathcal{E}(t, u_\varepsilon(t)) \ni 0 \quad t \in (0,T), \\ u_\varepsilon(0) = u_0. \end{cases}$$

**Problem:** limit behaviour of $\{u_\varepsilon\}$ as $\varepsilon \searrow 0$

# A rescaling technique

- **Arc length parametrization** of the graph $\{(t, u_\varepsilon(t)) : t \in [0, T]\}$:

$$s_\varepsilon(t) := t + \int_0^t \|u_\varepsilon{}'(s)\| \mathrm{d}s$$

$\{s_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(0, T)$: up to a subseq. $s_\varepsilon(T) \to \widehat{T}$.

- Introduce the rescaled functions

$$\widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad \forall s \in [0, s_\varepsilon(T)]$$

- From the **normalization condition**

$$\widehat{t}_\varepsilon'(s) + \|\widehat{u}_\varepsilon'(s)\| = 1 \quad \text{for a.e. } s \in (0, s_\varepsilon(T))$$

$\Rightarrow$ a priori estimates for $\{\widehat{t}_\varepsilon\}$, $\{\widehat{u}_\varepsilon\}$

- Ascoli-Arzelà + finite dimension

$$\widehat{t}_\varepsilon \to \widehat{t}, \quad \widehat{u}_\varepsilon \to \widehat{u} \quad \text{uniformly on } [0, \widehat{T}]$$

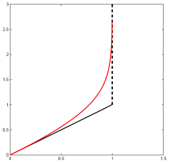
## A rescaling technique

The **limit problem** solved by $(\widehat{t}, \widehat{u})$

$$\begin{cases} \partial\widehat{\Psi}(\widehat{u}'(s)) + \mathrm{D}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) \ni 0 & s \in (0, \widehat{T}) \\ \widehat{u}(0) = u_0, \quad \widehat{t}(0) = 0, \quad \widehat{t}(\widehat{T}) = T, \\ \widehat{t}'(s) + \|\widehat{u}'(s)\| = 1 & s \in (0, \widehat{T}) \end{cases}$$

where

$$\widehat{\Psi}(u') := \begin{cases} \Psi(u') & \|u'\| \leq 1, \\ +\infty & \|u'\| > 1 \end{cases}$$



# Vanishing viscosity limit: dry friction vs. viscous slips

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$\widehat{\Psi}$ is NOT 1-homogeneous $\Rightarrow$ the problem is NOT rate-independent!

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### "Dry friction vs. viscous slips"

Three regimes

| | |
|---|---|
| $\|\widehat{u}'(s)\| = 0 \Leftrightarrow \widehat{t}'(s) = 1$ | Sticking |
| $\|\widehat{u}'(s)\| < 1 \Leftrightarrow \widehat{t}'(s) \in (0,1)$ | Dry friction motion |
| $\|\widehat{u}'(s)\| = 1 \Leftrightarrow \widehat{t}'(s) = 0$ | Viscous slip |

# Vanishing viscosity limit: dry friction vs. viscous slips

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## "Dry friction vs. viscous slips"

Three regimes

1. for $\|\widehat{u}'(s)\| = 0$ the system is stationary
2. for $\|\widehat{u}'(s)\| < 1$ the system is driven by rate-independent dissipation ($\sim$ dry friction): reparametrizing $\widehat{u}$ leads to a standard rate-independent problem
3. $\|\widehat{u}'(s)\| = 1$ corresponds to viscous transition between stable states ("instantaneous" w.r.t. the slow time scale, whence $\widehat{t}'(s) = 0$); viscous path described by a gradient flow

# Towards metric spaces

## In applications

- $\mathcal{E}$ may be **neither smooth nor convex!**
- $B$ may be **non reflexive** (e.g. $L^1$ in SMA)
- $B$ may **lack a linear structure** (e.g. in fractures)

$\Rightarrow$ extend the Efendiev-Mielke analysis to a **metric setting**

The metric framework will lead to local, rather than global, stability!

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## Outline

In a **metric framework**:

1. **Approximate** rate-independent evolutions with **viscous evolutions** **[Mielke, R., Savaré, work in progress]**
2. Analysis of doubly nonlinear evolution equations where **dissipation with superlinear growth**: existence & approximation of solutions **[Mielke, R., Savaré, preprint'07]**

# Doubly nonlinear evolutions in metric spaces

Analysis of

$$\partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) \ni 0 \quad t \in (0,T) \tag{DNE}$$

$\Psi$ with superlinear growth

in the framework of a **metric space** $(X,d)$.

**Relying on**: theory of **gradient flows** in metric spaces (i.e. **quadratic** $\Psi$):

- **De Giorgi, Marino, Saccon, Tosques, Degiovanni, Ambrosio** '80 $\sim$ '90 $\rightsquigarrow$ theory of **Curves of Maximal Slope** and **Minimizing Movements**
- [ *Gradient flows in metric spaces*, **Ambrosio-Gigli-Savaré** 2005] $\rightsquigarrow$ systematic theory of existence, approximation & uniqueness of solutions of metric gradient flows, with applications to gradient flows in Wasserstein spaces.

## Towards the metric formulation

**Problem:**
How to formulate

$$\text{“}\ \partial\Psi(u'(t)) + \partial_u\mathcal{E}(t,u(t)) = 0, \quad t \in (0,T)\ \text{”}$$

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**Heuristics:**

If the **chain rule** holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = \langle \partial_u\mathcal{E}(t,u(t)), u'(t)\rangle$$

then (DNE) is equivalent to

$$\Psi(u'(t))) + \Psi^*(-\partial_u\mathcal{E}(t,u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = 0 \quad t \in (0,T)$$

(abuse of notation: $\partial_u\mathcal{E}(t,u(t)) \sim$ singleton...)

# Towards the metric formulation

In the particular case

$$\Psi(x) := \frac{|x|^p}{p}, \quad 1 < p < \infty, \qquad \Psi^*(x) := \frac{|x|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\Psi(u'(t)) + \Psi^*(-\partial_u \mathcal{E}(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0 \quad t \in (0, T)$$

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New formulation features the **modulus of derivatives,** rather than derivatives!

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New formulation features the **modulus of derivatives,** rather than derivatives!

Adaptable to metric spaces upon introducing suitable **"metric surrogates"** of **"modulus of derivatives"**.

# The metric derivative

- **Setting:** $(X, d)$ complete metric space

## Metric derivative

- We say that a curve $u : [0, T] \to X$ is **absolutely continuous** if

$$\exists m \in L^1(0, T) \ : \quad d(u(t), u(s)) \leq \int_s^t m(r)\,\mathrm{d}r \quad \forall\, 0 \leq s \leq t \leq T.$$

- Given $u \in \mathrm{AC}(0, T; X)$, its **metric derivative**

$$|u'|(t) := \lim_{h \to 0} \frac{d(u(t), u(t+h))}{|h|} \quad \text{for a.e. } t \in (0, T)$$

$$\|u'(t)\| \rightsquigarrow |u'|(t)$$

# Slope & Chain rule

- **Setting:** $(X, d)$ complete metric space

## Local slope & Chain rule

- Given $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ and $u \in D(\mathcal{E}(t, \cdot))$, the **local slope** of $\mathcal{E}(t, \cdot)$ at $u$ is

$$|\partial\mathcal{E}|(t, u) := \limsup_{v \to u} \frac{(\mathcal{E}(t, u) - \mathcal{E}(t, v))^+}{d(u, v)}$$

$$\| - \partial_u \mathcal{E}(t, u)\| \rightsquigarrow |\partial\mathcal{E}|(t, u)$$

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- $\mathcal{E}$ complies with the **chain rule** w.r.t. $|\partial \mathcal{E}|$ if $\forall v \in \mathrm{AC}(0, T; D(\mathcal{E}))$ the map $t \mapsto \mathcal{E}(t, v(t))$ is **absolutely continuous** and

$$\partial_t \mathcal{E}(t, v(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t, v(t)) \leq |v'|(t)\, |\partial \mathcal{E}|(t, v(t)) \quad \text{for a.e. } t \in (0, T).$$

# The metric formulation

- **Basic setting**:
  - $(X, d)$ complete metric space
  - **Energy** $\rightsquigarrow$ $\mathcal{E} : [0,T] \times X \to (-\infty, +\infty]$ l.s.c., complying with the **chain rule** w.r.t. $|\partial \mathcal{E}|$
  - **Dissipation** $\rightsquigarrow$ $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ l.s.c., convex, $\psi(0) = 0$, with

$$\lim_{x \to +\infty} \frac{\psi(x)}{x} = +\infty$$

## Metric formulation

A curve $u \in \mathrm{AC}(0,T;X)$ satisfies the **metric formulation** of

$$\text{“}\partial\Psi(u'(t)) + \partial_u \mathcal{E}(t, u(t)) = 0 \quad t \in (0,T)\text{”}$$

if for a.e. $t \in (0,T)$

$$\psi(|u'|(t)) + \psi^*(|\partial \mathcal{E}|(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t \mathcal{E}(t, u(t)) = 0$$

## An existence result

**Theorem [Mielke, R., Savaré, preprint'07]**

- $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ convex, l.s.c., $\psi(0) = 0$, superlinear growth
- $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ smooth w.r.t. $t \in [0, T]$
- $\mathcal{E}$ l.s.c. and coercive w.r.t. $u \in X$, **chain rule** w.r.t. $|\partial\mathcal{E}|$
- $u \mapsto |\partial\mathcal{E}|(t, u)$ is l.s.c. (along bounded energy sequences)

Then, for all $u_0 \in D(\mathcal{E})$ **there exists** a curve $u \in \mathrm{AC}(0, T; X)$ such that $u(0) = u_0$ and

$$\psi(|u'|(t)) + \psi^* \left(|\partial\mathcal{E}|(t, u(t))\right) = \partial_t\mathcal{E}(t, u(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) \quad \text{for a.e. } t \in (0, T)\,.$$

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**Proof**: approximation via time discretization (**incremental minimization** problems), based on De Giorgi's **Minimizing Movements** theory.
**Applications**: existence results for doubly nonlinear evolution equations in (possibly non reflexive) spaces

# Approximation of rate-independent problems with viscous evolutions

**Second step**

In the metric space $(X, d)$, approximate

$$\partial\Psi(u'(t)) + \partial_u \mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \quad t \in (0, T), \qquad \text{(DNE)}$$

$\Psi$ **1-positively homogeneous**

with the viscous evolution

$$\varepsilon u'(t) + \partial\Psi(u'(t)) + \partial_u \mathcal{E}(t, u(t)) \ni 0 \quad t \in (0, T), \qquad \text{as } \varepsilon \searrow 0$$

# Approximation of rate-independent problems with viscous evolutions

## In the metric setting

- $(X, d)$ metric space
- $\mathcal{E} : [0, T] \times X \to \mathbb{R} \cup \{+\infty\}$: assumptions for $\exists$ + **Chain rule**
- $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ convex **1-positively homogeneous** ($\psi(r) = r \ \forall\, r \in \mathbb{R}^+$)
- Viscous regularization of $\psi$: $\psi_\varepsilon(x) := x + \frac{\varepsilon}{2}x^2 \ \forall\, x \geq 0 \ \forall\, \varepsilon > 0$.
- $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathrm{AC}(0, T; X)$: **metric solutions** of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u_\varepsilon(t)) - \partial_t\mathcal{E}(t, u_\varepsilon(t)) = \\ \qquad - \psi_\varepsilon(|u_\varepsilon{}'|(t)) - \psi_\varepsilon{}^*(|\partial\mathcal{E}|\,(t, u_\varepsilon(t))) \ \text{ for a.e. } t \in (0, T) \\ u_\varepsilon(0) = u_0. \end{cases}$$

- **Problema:** ¿ limit of $\{u_\varepsilon\}$ as $\varepsilon \searrow 0$?

## Vanishing viscosity revisited

Extend the Mielke-Efendiev technique to the metric setting:

♣ reparametrize by the **arc length** of $\{(t, u_\varepsilon(t)) : t \in [0,T]\}$:

$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

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Extend the Mielke-Efendiev technique to the metric setting:

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$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon{}'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

♣ you pass from

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u_\varepsilon(t)) - \partial_t \mathcal{E}(t, u_\varepsilon(t)) = \\ \qquad - \psi_\varepsilon(|u_\varepsilon{}'|(t)) - \psi_\varepsilon{}^*(|\partial\mathcal{E}|(t, u_\varepsilon(t))) \qquad t \in (0, T) \\ u_\varepsilon(0) = u_0. \end{cases}$$

## Vanishing viscosity revisited

Extend the Mielke-Efendiev technique to the metric setting:

♣ reparametrize by the **arc length** of $\{(t, u_\varepsilon(t)) : t \in [0,T]\}$:

$$\begin{cases} s_\varepsilon(t) := t + \int_0^t |u_\varepsilon{}'|(r)\,\mathrm{d}r \\ \widehat{t}_\varepsilon(s) := s_\varepsilon^{-1}(s), \quad \widehat{u}_\varepsilon(s) := u_\varepsilon(\widehat{t}_\varepsilon(s)) \quad s \in [0, s_\varepsilon(T)] \end{cases}$$

♣ to

$$\begin{cases} \widehat{t}_\varepsilon(0) = 0 \qquad \widehat{t}_\varepsilon(s_\varepsilon(T)) = T \\ \widehat{t}_\varepsilon'(s) + |\widehat{u}_\varepsilon'|(s) = 1 \quad \text{for a.e. } s \in (0, s_\varepsilon(T)) \\ \text{rescaled metric formulation of (DNE) } (\psi_\varepsilon, \mathcal{E}) \end{cases}$$

**Problem:** ¿ asymptotic analysis of $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ as $\varepsilon \searrow 0$?

## The asymptotic analysis result

Let

$$\widehat{\psi}(r) := \begin{cases} r & r \in [0,1]\,, \\ +\infty & r > 1\,, \end{cases} \qquad \widehat{T} := \lim_{\varepsilon \downarrow 0} s_\varepsilon(T)\,.$$

**Theorem [Mielke, R., Savaré, in preparation '07]**

**Assumptions:** like for $\exists$ of metric solution (in particular, **chain rule**).

Then, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ converges as $\varepsilon \searrow 0$ to $(\widehat{t}, \widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0,T] \times X)$, which satisfies

$$\begin{cases} \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T \\ \widehat{t}'(s) + |\widehat{u}'|(s) = 1 \quad \text{for a.e. } s \in (0, \widehat{T}) \end{cases}$$

and the **"rescaled metric formulation"**

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s)$$
$$= -\widehat{\psi}(|\widehat{u}'|(s)) - \widehat{\psi}^*\left(|\partial\mathcal{E}|\left(\widehat{t}(s), \widehat{u}(s)\right)\right) \quad s \in (0, \widehat{T})\,.$$

# More insight into the vanishing viscosity limit

$$(\widehat{t}, \widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0, T] \times X) \qquad \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T$$

$$\widehat{t}'(s) + \widehat{u}'(s) = 1 \quad \text{for a.e. } s \in (0, \widehat{T})$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) = -\widehat{\psi}(|\widehat{u}'|(s)) - \widehat{\psi}^* \left(|\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s))\right) \quad \text{for a.e. } s \in (0, \widehat{T})$$

# More insight into the vanishing viscosity limit

$$(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s)=1 \quad \text{for a.e. } s\in(0,\widehat{T})$$

$$\begin{cases} \textbf{En. id.:} & \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \\ \textbf{Constraint:} & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \ \text{ for a.e. } s\in(0,\widehat{T}) \end{cases}$$

# More insight into the vanishing viscosity limit

$$(\widehat{t}, \widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0, \widehat{T}]; [0, T] \times X) \qquad \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T$$

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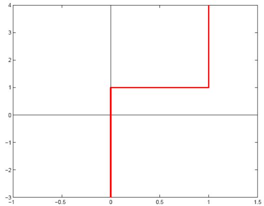
$$\begin{cases} \textbf{En. id.:} & \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\; |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \\ \textbf{Constraint:} & |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \quad \text{for a.e. } s \in (0, \widehat{T}) \end{cases}$$

$$|\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \quad \text{for a.e. } s \in (0, \widehat{T})$$



# More insight into the vanishing viscosity limit

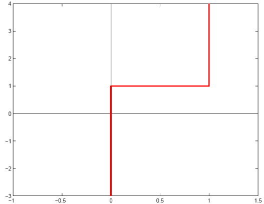
$$(\widehat{t},\widehat{u}) \in \mathrm{C}^0_{\mathrm{Lip}}([0,\widehat{T}];[0,T]\times X) \qquad \widehat{t}(0)=0 \qquad \widehat{t}(\widehat{T})=T$$

$$\widehat{t}'(s)+\widehat{u}'(s)=1 \quad \text{for a.e. } s\in(0,\widehat{T})$$

**En. id.:** $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$$

**Three regimes:**
$$\begin{cases} |\widehat{u}'|(s)=1 \ (\Leftrightarrow \widehat{t}'(s)=0) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\geq 1 \\ |\widehat{u}'|(s)\in(0,1) \ (\Leftrightarrow \widehat{t}'(s)\in(0,1)) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))= 1 \\ |\widehat{u}'|(s)=0 \ (\Leftrightarrow \widehat{t}'(s)=1) & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\leq 1 \end{cases}$$

$$|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \in \partial\widehat{\psi}(|\widehat{u}'|(s)) \quad \text{for a.e. } s\in(0,\widehat{T})$$



# Towards Metric Parametrized Rate-Independent Flows

These are the properties to retain:

$$(\widehat{t}, \widehat{u}) \in \mathrm{AC}(0, \widehat{T}; [0, T] \times X) \qquad \widehat{t}(0) = 0 \qquad \widehat{t}(\widehat{T}) = T$$

$$\widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$

**En. id.:** $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s), \widehat{u}(s)) - \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s))$

**Three regimes:** $\begin{cases} \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \end{cases}$

## Metric Parametrized Rate-Independent Flows

### Definition

A pair $(\widehat{t},\widehat{u}) \in \mathrm{AC}(0,\widehat{T};[0,T]\times X)$ is a **metric parametrized rate-independent flow** if

1. $\widehat{t}$ is non-decreasing, with $\widehat{t}(0)=0$ and $\widehat{t}(\widehat{T})=T$
2. there holds
$$\widehat{t}'(s)+\widehat{u}'(s)>0 \quad \text{for a.e. } s\in(0,\widehat{T})$$
3. the map $s\in[0,\widehat{T}]\mapsto\mathcal{E}(\widehat{t}(s),\widehat{u}(s))$ is absolutely continuous and

**En. id.:** $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s))-\partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s)=-|\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$$

**Three regimes:** $$\begin{cases}\widehat{t}'(s)=0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\geq 1\\ \widehat{t}'(s)\,|\widehat{u}'|(s)>0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))=1\\ |\widehat{u}'|(s)=0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))\leq 1\end{cases}$$

# Metric Parametrized Rate-Independent Flows: rate-invariance

## Metric Parametrized Rate-Independent Flow

**En. id.:** $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\;|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$$

**Three regimes:** $$\begin{cases} \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \end{cases}$$

- Approximable (via vanishing viscosity) solutions are MPRIFs.
- The class of MPRIF is **invariant** for (strictly increasing) reparametrizations $\Rightarrow$ MPRIF is a **truly rate-independent** notion.
- Notion **compatible with other vanishing viscosity approximations**, based on different reparametrizations (not necessarily by arc-length)!
- Existence can also be proved via time discretization (**Minimizing Movements**)

# Metric Parametrized Rate-Independent Flows: flow regimes

## Metric Parametrized Rate-Independent Flows

**En. id.:** $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$$

**Three regimes:** $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \end{cases}$$

## Mechanical interpretation

- **sticking** $\leftrightarrow$ $|\widehat{u}'|(s) = 0$
- **sliding** (dry friction motion) $\leftrightarrow$ $\widehat{t}'(s)|\widehat{u}'|(s) > 0$
- **viscous slip** $\leftrightarrow$ $\widehat{t}'(s) = 0.$

# Flow regimes

## Metric Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$

**En. id:**
$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big( \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\, \widehat{t}'(s) - |\widehat{u}'|(s)\, |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

**Differential conditions:**
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

**Sticking:** stationary set

$$\mathcal{S} := \Big\{ s_0 \in (0, \widehat{T}) : |\widehat{u}'|(s) = 0 \text{ in a neighb. of } s_0 \Big\}$$

- $|\widehat{u}'|(s) = 0 \ \Rightarrow\ \widehat{t}'(s) > 0$ and $|\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1$ (**local stability**)
- in a neighb. $I(s_0)$ we have $\widehat{u}(s) \equiv \widehat{u}(s_0)$ and the **energy identity**

$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_0)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_0)) = \int_{s_1}^{s_2} \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s_0))\, \widehat{t}'(s)\, \mathrm{d}s \ \forall\, s_1 \leq s_2 \in I(s_0)\,.$$

# Flow regimes

**Metric Parametrized Rate-Independent Flows**

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$

**En. id:** $$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) - |\widehat{u}'|(s)\, |\partial\mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s))\Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

**Differential conditions:** $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

**Sliding:**

- $\widehat{t}'(s_0) > 0$ & $|\widehat{u}'|(s_0) > 0 \Rightarrow |\partial\mathcal{E}|\,(\widehat{t}(s_0), \widehat{u}(s_0)) = 1$ (**local stability**)
- in a neighb. $I(s_0)$ the **energy identity** reads $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big(\partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s))\,\widehat{t}'(s) - |\widehat{u}'|(s)\Big)\, \mathrm{d}s$$

# Flow regimes

## Metric Parametrized Rate-Independent Flows

$$\widehat{t}'(s) \geq 0\,, \qquad \widehat{t}'(s) + \widehat{u}'(s) > 0 \quad \text{for a.e. } s \in (0, \widehat{T})$$

**En. id:** 
$$\mathcal{E}(\widehat{t}(s_2), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_1), \widehat{u}(s_1)) = \int_{s_1}^{s_2} \Big( \partial_t \mathcal{E}(\widehat{t}(s), \widehat{u}(s)) \widehat{t}'(s) - |\widehat{u}'|(s)\, |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \Big)\, \mathrm{d}s \quad \forall\, 0 \leq s_1 \leq s_2 \leq \widehat{T}$$

**Differential conditions:**
$$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\, |\widehat{u}'|(s) > 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1 \end{cases}$$

**Viscous slip:** jump set

$$\mathcal{J} := \Big\{ s_0 \in (0, \widehat{T}) \,:\, |\widehat{t}'|(s) = 0 \text{ in a neighb. of } s_0 \Big\}$$

- $\widehat{t}'(s) = 0 \ \Rightarrow \ |\widehat{u}'|(s) > 0 \ \Rightarrow \ |\partial \mathcal{E}|\, (\widehat{t}(s), \widehat{u}(s)) \geq 1$
- in a neighb. $I(s_0)$ $\widehat{t}(s) \equiv \widehat{t}(s_0)$ & **energy identity** $\forall\, s_1 \leq s_2 \in I(s_0)$

$$\mathcal{E}(\widehat{t}(s_0), \widehat{u}(s_2)) - \mathcal{E}(\widehat{t}(s_0), \widehat{u}(s_1)) = -\int_{s_1}^{s_2} |\partial \mathcal{E}|\, (\widehat{t}(s_0), \widehat{u}(s))\, |\widehat{u}'|(s) \Big)\, \mathrm{d}s$$

# From virtual to real jumps

With a suitable transformation,

**Metric Parametrized Rate-Independent Flows**

**En. id.:** $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\widehat{t}(s),\widehat{u}(s)) - \partial_t\mathcal{E}(\widehat{t}(s),\widehat{u}(s))\,\widehat{t}'(s) = -|\widehat{u}'|(s)\,|\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s))$$

**Three regimes:** $$\begin{cases} |\widehat{u}'|(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1 \\ \widehat{t}'(s)\,|\widehat{u}'|(s) > 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1 \\ \widehat{t}'(s) = 0 & \Rightarrow\ |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1 \end{cases}$$

$$\Downarrow \quad \Uparrow$$

**BV (non parametrized) Rate-Independent Flows**

**En. id.:** $$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = -\Sigma(t,u(t-),u(t+))\cdot\mu_u \qquad \text{in } \mathcal{D}'(0,T)$$

**Three regimes:** $$\begin{cases} t\in[0,T]\setminus J_u & \Rightarrow\ |\partial\mathcal{E}|\,(t,u(t)) \leq 1 \\ t\in[0,T]\in \mathrm{supp}(\mu_u)\setminus J_u & \Rightarrow\ |\partial\mathcal{E}|\,(t,u(t)) = 1 \\ t\in J_u & \Rightarrow \begin{cases} \exists y\in \mathrm{AC}([0,1],X) \text{ and } \theta\in[0,1] \text{ s. t.} \\ y(0)=u(t-),\ u(\theta)=u(t),\ y(1)=u(t+), \\ |\partial\mathcal{E}|\,(t,y(r)) \geq 1 \quad \text{for a.e. } r\in(0,1), \\ \mathcal{E}(t,u(t+)) - \mathcal{E}(t,u(t-)) = -\int_0^1 |\partial\mathcal{E}|\,(t,y(r))|y'|(r)\mathrm{d}r. \end{cases} \end{cases}$$

# Local vs. Global Slope

- **Setting:** $(X, d)$ complete metric space

## Global slope

Given $\mathcal{E} : [0, T] \times X \to (-\infty, +\infty]$ and $u \in D(\mathcal{E}(t, \cdot))$, the **global slope** of $\mathcal{E}(t, \cdot)$ at $u$ is

$$|\mathcal{G}\ell(\mathcal{E})|\,(t, u) := \sup_{v \neq u} \frac{(\mathcal{E}(t, u) - \mathcal{E}(t, v))^+}{d(u, v)}$$

Suppose that $\mathcal{E}(t, \cdot)$ is $\lambda$–(geodesically) convex, $\lambda \geq 0$. Then

$$|\partial\mathcal{E}|\,(t, u) = \mathcal{G}\ell(\mathcal{E})(t, u)$$

# Comparison with the energetic formulation

During **sliding regime** (rate-independent) we have

$$|\partial\mathcal{E}|\,(t,u(t)) = 1 \qquad \textbf{local stability}$$

For **global energetic solutions** we would have

$$|\mathcal{G}\ell(\mathcal{E})|(t,u(t)) = 1 \qquad \textbf{global stability}$$

**BUT** global energetic solutions jump too early!

# Local vs. Global stability

## A one-dimensional example

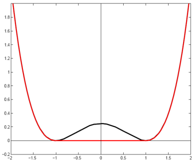
- Metric setting: $(\mathbb{R}, d_\eta)$, $d_\eta(u,v) := \eta|v-u|$, $\eta > 0$.
- Energy:

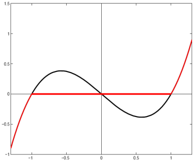
$$\mathcal{E}(t,u) = \overbrace{\frac{1}{4}(u^2-1)^2}^{\mathcal{E}(u)\text{ double-well potential}} - \overbrace{f(t)}^{\text{(pcw.-)monotone input}} u$$

  to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.
- Initial state: $u_0 \in [-\sqrt{2}, -1)$

## Global Energetic evolution

**Convexified** energy $\mathcal{E}^{**}_{u_0}$ on $(u_0, +\infty)$ with derivative $\mathrm{D}\mathcal{E}^{**}_{u_0}$



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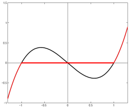
  to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.
- Initial state: $u_0 \in [-\sqrt{2}, -1)$

## Global Energetic evolution

the evolution of $u$ is

$$u(t) = \mathrm{D}\mathcal{E}_{u_0}^{**\,-1}\left(\max\{(f(t)-\eta), \partial\mathcal{E}_{u_0}^{**}(u_0)\}\right) \qquad t > 0$$

i.e. **Maxwell rule** ($\leftrightarrow$ convexification) with a delay of $\eta$ and $u$ jumps early



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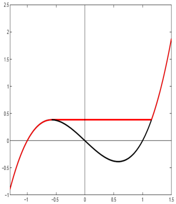
to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.

- Initial state: $u_0 \in [-\sqrt{2}, -1)$

## Metric Parametrized Rate-Independent Evolution

**Minimal non decreasing** graph $M(\mathcal{E}')_{u_0}$ s. t. $M(\mathcal{E}')_{u_0}(u) \geq \mathcal{E}'(u)$ for $u \geq u_0$



# Local vs. Global stability

## A one-dimensional example

- Metric setting: $(\mathbb{R}, d_\eta)$, $d_\eta(u,v) := \eta|v-u|$, $\eta > 0$.
- Energy:

$$\mathcal{E}(t,u) = \overbrace{\frac{1}{4}(u^2-1)^2}^{\mathcal{E}(u)\text{ double-well potential}} - \overbrace{f(t)}^{\text{(pcw.-)monotone input}} \; u$$

  to fix ideas: $f(t) := t_0 + t$, $t \in [0,T]$.
- Initial state: $u_0 \in [-\sqrt{2}, -1)$

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the evolution of $u$ is

$$u(t) = M(\mathcal{E}')_{u_0}{}^{-1} \left( \max\{(f(t)-\eta), \mathcal{E}'(u_0)\}\right) \qquad t > 0$$

i.e. **Hysteresis behaviour** with a delay of $\eta$ and $u$ jumps later