Analysis of a rate-independent model for adhesive contact with thermal effects

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joint work with
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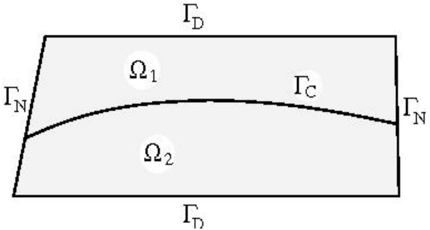
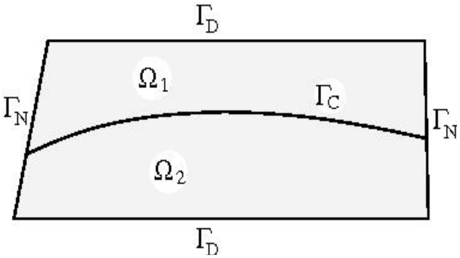
Univerzita Karlova v Praze, 22.03.2010

Geometrical setting



- Ω_1, Ω_2: viscoelastic bulk domains, $\Omega \doteq \Omega_1 \cup \Omega_2$
- on Γ_C, Ω_1 and Ω_2 are in **adhesive contact**. Denote by ν unit normal on Γ_C, oriented from Ω_2 to Ω_1
- $\partial\Omega \doteq \Gamma_D \cup \Gamma_N$:
  - Γ_D with Dirichlet boundary conditions
  - Γ_N with Neumann boundary conditions
- evolution problem in $[0, T]$

Adhesive contact versus brittle delamination



Two different models

- delamination as **an inelastic process** $\Rightarrow$ **brittle delamination** models
- **elastic response** of the adhesive $\Rightarrow$ **adhesive contact** models

Outline

- Modeling **adhesive contact**
- PDE system
- Weak formulation
- **Existence** results
- Sketch of the proof
- Outlook to **delamination**

Modeling approaches: fracture mechanics versus damage mechanics

Fracture mechanics approach:

- **brittle delamination** $\sim$ evolution during $[0, T]$ of a **single crack** along **prescribed path** $= \Gamma_{\rm C}$

$$\forall t \in [0, T] \qquad \Gamma_{\rm C} = \underbrace{\Gamma_A(t)}_{\text{perfect adhesion}} \cup \underbrace{\Gamma_{\rm C} \backslash \Gamma_A(t)}_{\text{complete delamination}}$$

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- **irreversibility** enforced by dissipation distance

$$\mathcal{D}(\Gamma_{A,1},\Gamma_{A,2}) = \begin{cases} \int_{\Gamma_{A,1}\backslash\Gamma_{A,2}} a(x)\mathrm{d}S & \text{if } \Gamma_{A,1} \supset \Gamma_{A,2}, \\ +\infty & \text{otherwise,} \end{cases}$$

$a = a(x) \geq 0$ activation energy for delamination: $t \mapsto \Gamma_A(t)$ “decreasing”

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- **activated, rate-independent** phenomenon

See [Dal Maso–Zanini'07, Thomas–Sändig'06, Toader–Zanini'09, Negri–Ortner'08, Cagnetti'09, Knees–Mielke–Zanini'08,'09..]

Modeling approaches: fracture mechanics versus damage mechanics

Damage mechanics approach

Proposed by **[M. Frémond'82,'87]**

- delamination/adhesive contact described by a **damage variable** z
- $z \sim$ volume fraction of debonded molecular links
- evolution of z:

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  - or **rate-independent**. Results in the **isothermal case**:
    - **adhesive contact** problem **[Kočvara–Mielke–Roubíček'06]**
    - **delamination** problem **[Roubíček–Scardia–Zanini'09]**

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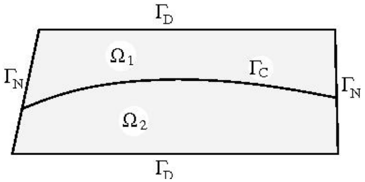
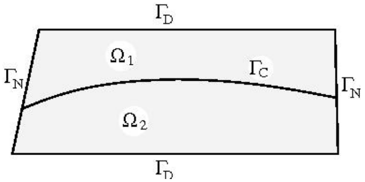
◇ Approach based on **hemivariational inequalities [Panagiotopoulos..]**

State variables



- in the **volume domain Ω**:
  - **displacement $\mathbf{u}$** $\rightsquigarrow$ $e(\mathbf{u})$ symm. linear. strain tensor **(small strains)**
  - thermal effects $\rightsquigarrow$ θ **absolute temperature**

State variables



- in the **volume domain** Ω:
  - **displacement u** $\rightsquigarrow$ $e(\mathbf{u})$ symm. linear. strain tensor **(small strains)**
  - thermal effects $\rightsquigarrow$ θ **absolute temperature**
- on the **contact surface** Γ_C:
  - **adhesion** variable z $\rightsquigarrow$ "damage parameter"

On Γ_C we also consider

$$[\![\mathbf{u}]\!] = \underbrace{\mathbf{u}^+|_{\Gamma_C}}_{\substack{\text{trace on } \Gamma_C \\ \text{of } \mathbf{u}|_{\Omega_1}}} - \underbrace{\mathbf{u}^-|_{\Gamma_C}}_{\substack{\text{trace on } \Gamma_C \\ \text{of } \mathbf{u}|_{\Omega_2}}} = \text{the jump of } \mathbf{u} \text{ across } \Gamma_C.$$

Constraints: irreversibility, unilateral contact..

- Admissible values for z: $\rightsquigarrow$ $z \in [0,1]$
  - $z(x) = 1$: at $x \in \Gamma_{\mathrm{C}}$ adhesive completely sound & fully effective
  - $0 < z(x) < 1$: at $x \in \Gamma_{\mathrm{C}}$ a fraction of the molecular links is broken
  - $z(x) = 0$: at $x \in \Gamma_{\mathrm{C}}$ surface is completely debonded

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- θ is the **absolute** temperature: $\rightsquigarrow$ $\theta > 0$
- Damaging of the glue is a unidirectional process:

$$\dot{z} \leq 0 \quad \textbf{(irreversibility)}$$

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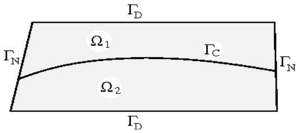
- **No interpenetration** between Ω_1 and Ω_2: $\rightsquigarrow$ unilateral contact conditions

Unilateral (frictionless) Signorini contact



- **Signorini conditions** in complementarity form:

$$[\![\mathbf{u}]\!] \cdot \nu \geq 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T) \quad \textbf{(no interpenetration)} \qquad (\mathrm{Sign}_1)$$

$$\underbrace{\sigma|_{\Gamma_{\mathrm{C}}}}_{\text{traction stress on } \Gamma_{\mathrm{C}}} \nu \cdot \nu \geq 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T) \qquad (\mathrm{Sign}_2)$$

$$\sigma|_{\Gamma_{\mathrm{C}}}\nu \cdot [\![\mathbf{u}]\!] = 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T) \qquad (\mathrm{Sign}_3)$$

$$\sigma|_{\Gamma_{\mathrm{C}}}\nu \cdot \mathrm{t} = 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T) \quad \forall \mathrm{t} \text{ s.t. } \nu \cdot \mathrm{t} = 0 \qquad (\mathrm{Sign}_4)$$

(Sign_2) & (Sign_3) & (Sign_4) yield
- $[\![\mathbf{u}]\!] \cdot \nu > 0 \Rightarrow \sigma|_{\Gamma_{\mathrm{C}}}\nu = 0$ **(no reaction)**
- $[\![\mathbf{u}]\!] \cdot \nu = 0 \Rightarrow \sigma|_{\Gamma_{\mathrm{C}}}\nu = \lambda\nu,\ \lambda \geq 0$ **(reaction is triggered)**

Equation for u

Momentum equilibrium equation

$$\varrho\ddot{\mathbf{u}} - \mathrm{div}(\sigma) = F \quad \text{in } \Omega \times (0, T).$$

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Ansatz of generalized standard solids:

- inertial effects

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$$\varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}(e(\mathbf{u}) - \mathbb{E}\theta)\big) = F \quad \text{in } \Omega \times (0,T). \qquad (\mathrm{eq}_u)$$

Ansatz of generalized standard solids:

- inertial effects
- stress σ features **viscous** response of material (Kelvin-Voigt rheology)

$$\sigma = \underbrace{\mathbb{D}e(\dot{\mathbf{u}})}_{\text{viscosity}} + \mathbb{C}(e(\mathbf{u}) - \mathbb{E}\theta)$$

$$\begin{cases} \mathbb{C},\ \mathbb{D} & \text{4th-order positive definite and symmetric tensors} \\ \mathbb{E} & \text{matrix of thermal expansion coefficients} \end{cases}$$

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\+ boundary conditions on $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$:

$$\begin{cases} \mathbf{u} = 0 & \text{on } \Gamma_{\mathrm{D}} \times (0, T) \\ \sigma\mathbf{n} = f & \text{on } \Gamma_{\mathrm{N}} \times (0, T) \end{cases}$$

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+ complementarity problem on Γ_{C} encompassing **adhesion variable** z in **Signorini** contact

Equation for θ

Heat equation

$$c_{\rm v}(\theta)\dot{\theta} + \mathrm{div}(j) = \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \theta\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}}) + G \quad \text{in } \Omega \times (0, T)$$

It balances heat flux & rate of heat production due to dissipation:

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- $c_{\mathrm{v}}(\theta)$ heat capacity

Equation for θ

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It balances heat flux & rate of heat production due to dissipation:

- $c_{\mathrm{v}}(\theta)$ heat capacity
- j heat flux, given by Fourier's law in an anisotropic medium

$$j = -\mathbb{K}(e(\mathbf{u}),\theta)\nabla\theta,$$

$\mathbb{K}(e(\mathbf{u}),\theta)$ pos. def. matrix heat conduction coefficients

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- $\mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}})$ **viscous** dissipation potential in the bulk

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- G external heat source

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+ Neumann boundary conditions on $\partial\Omega$:

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$+$ conditions $\Gamma_{\rm C}$ featuring **dissipation rate on $\Gamma_{\rm C}$**

$$\frac{1}{2}\big(\mathbb{K}(e(\mathbf{u}),\theta)\nabla\theta|^{+}_{\Gamma_{\rm C}} + \mathbb{K}(e(\mathbf{u}),\theta)\nabla\theta|^{-}_{\Gamma_{\rm C}}\big)\cdot\nu + \eta(\llbracket\mathbf{u}\rrbracket, z)\llbracket\theta\rrbracket = 0 \quad \text{on } \Gamma_{\rm C}\times(0,T), \quad (\mathrm{T1})$$

$$\llbracket\mathbb{K}(e(\mathbf{u}),\theta)\nabla\theta\rrbracket\cdot\nu = \zeta(\dot{z}) \quad \text{on } \Gamma_{\rm C}\times(0,T) \qquad (\mathrm{T2})$$

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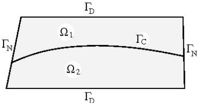
$$\llbracket \mathbb{K}(e(\mathbf{u}),\theta)\nabla\theta \rrbracket \cdot \nu = \zeta(\dot{z}) \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T) \quad (\mathrm{T2})$$

- (T1): **transient condition** on Γ_{C} with

$$\begin{cases} \eta(\llbracket \mathbf{u} \rrbracket, z) \text{ heat transfer coefficient} \rightsquigarrow \textbf{heat convection} \\ \llbracket \theta \rrbracket \text{ jump of temperature across } \Gamma_{\mathrm{C}} \end{cases}$$

- (T2) balances normal jump of heat flux $j = -\mathbb{K}\nabla\theta$ with **dissipation rate $\zeta(\dot{z})$ on Γ_{C}**

(Frictionless) unilateral contact in the adhesive case



- Complementarity problem with $\boxed{\sigma = \mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}(e(\mathbf{u}) - \mathbb{E}\theta)}$ and z

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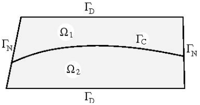

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$$[\![\mathbf{u}]\!] \cdot \nu \geq 0 \quad \text{on } \Gamma_C \times (0, T) \quad \textbf{(no interpenetration)} \qquad (\text{Sign}_1)$$

$$\left(\sigma|_{\Gamma_C}\nu + \kappa z [\![\mathbf{u}]\!]\right) \cdot \nu \geq 0 \quad \text{on } \Gamma_C \times (0, T) \qquad (\text{Sign}_2^{\text{new}})$$

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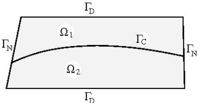

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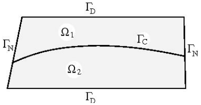
When $z = 0$ $(\text{Sign}_2^{\text{new}})$ & $(\text{Sign}_3^{\text{new}})$ & $(\text{Sign}_4^{\text{new}})$ reduce to **Signorini conditions**

- $[\![\mathbf{u}]\!] \cdot \nu > 0 \Rightarrow \sigma|_{\Gamma_C}\nu = 0$ **(no reaction)**
- $[\![\mathbf{u}]\!] \cdot \nu = 0 \Rightarrow \sigma|_{\Gamma_C}\nu = \lambda\nu,\ \lambda \geq 0$ **(reaction is triggered)**

(Frictionless) unilateral contact in the adhesive case



- Complementarity problem with $\boxed{\sigma = \mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}(e(\mathbf{u}) - \mathbb{E}\theta)}$ and z

$$[\![\mathbf{u}]\!] \cdot \nu \geq 0 \quad \text{on } \Gamma_C \times (0,T) \quad \textbf{(no interpenetration)} \qquad (\text{Sign}_1)$$

$$\left(\sigma|_{\Gamma_C}\nu + \kappa z [\![\mathbf{u}]\!]\right) \cdot \nu \geq 0 \quad \text{on } \Gamma_C \times (0,T) \qquad (\text{Sign}_2^{\text{new}})$$

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When $z > 0$ (adhesion active) $(\text{Sign}_2^{\text{new}})$ & $(\text{Sign}_3^{\text{new}})$ & $(\text{Sign}_4^{\text{new}})$ yield

$$\sigma|_{\Gamma_C}\nu = \lambda\nu - \kappa z [\![\mathbf{u}]\!]\, \nu, \quad \lambda \geq 0$$

even for $\lambda = 0$ there's a reaction $\sim \kappa z [\![\mathbf{u}]\!]$ counteracting separation $\rightsquigarrow$ this is the **elastic response of the adhesive**

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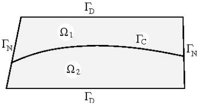

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Equivalently formulated as differential inclusion

$$\sigma|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}\left([\![\mathbf{u}]\!]\right) \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T), \quad \text{with}$$

$$\mathcal{C} = \mathcal{C}(x) = \{v \in \mathbb{R}^d \ : \ v \cdot \nu(x) \geq 0\} \quad \text{for a.a. } x \in \Gamma_{\mathrm{C}}$$

and $\partial I_{\mathcal{C}}$ convex analysis subdifferential of the indicator function $I_{\mathcal{C}}$.

General contact conditions on Γ_{C}

Signorini contact can be replaced by

$$\sigma|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}([\![\mathbf{u}]\!]) \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T)$$

with

$$\mathcal{C} = \mathcal{C}(x) \quad \text{closed cone} \quad \text{for a.a. } x \in \Gamma_{\mathrm{C}}\,.$$

Examples

- (Signorini) unilateral contact, **no interpenetration**

$$\mathcal{C} = \mathcal{C}(x) = \{v \in \mathbb{R}^d;\ v \cdot \nu(x) \geq 0\} \quad \text{for a.a. } x \in \Gamma_{\mathrm{C}}$$

- **tangential slip** along Γ_{C}

$$\mathcal{C} = \mathcal{C}(x) = \{v \in \mathbb{R}^d;\ v \cdot \nu(x) = 0\} \quad \text{for a.a. } x \in \Gamma_{\mathrm{C}}$$

- very simplified model: $\mathcal{C} = \mathcal{C}(x)$ **linear subspace** of $\mathbb{R}^d$

Equation for z

Flow rule for z

$$\partial\zeta(\dot{z}) + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa\left|[\![u]\!]\right|^2 - a_0 \ni 0 \quad \text{on } \Gamma_C \times (0,T) \qquad (\text{eq}_z)$$

It's a balance law between **dissipation** and **stored energy**

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It's a balance law between **dissipation** and **stored energy**

- $\zeta(\dot{z})$ **dissipation** potential on Γ_{C}, enforces irreversibility
- $I_{[0,1]}(z) \rightsquigarrow$ **constraint** $z \in [0,1]$
- $\frac{1}{2}\kappa\left|[\![u]\!]\right|^2 \sim$ **elastic response** of the adhesive
- a_0 (phenomenological specific) **stored** energy by disintegrating the adhesive.

Rate-dependent vs. rate-independent evolution for z

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Viscous models

$\zeta = \zeta(\dot{z})$ has superlinear growth at infinity. In particular,

$$\zeta(\dot{z}) = \frac{1}{2}|\dot{z}|^2 + I_{(-\infty,0]}(\dot{z}) \quad \text{(gradient flow case)}$$

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Rate-independent models, our choice

$\zeta = \zeta(\dot z)$ has linear growth at infinity: 1-positively homogeneous

$$\zeta(\lambda v) = \lambda\zeta(v) \quad \forall\, \lambda \geq 0$$

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$$\zeta(\dot z) = a_1|\dot z| + I_{(-\infty,0]}(\dot z)$$

with a_1 (phenomenological specific) **dissipated** energy by disintegrating the adhesive.

Rate-dependent vs. rate-independent evolution for z

$$\partial I_{(-\infty,0]}(\dot{z}) - a_1 + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa\left|\llbracket u \rrbracket\right|^2 - a_0 \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T) \qquad (\mathrm{eq}_z)$$

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Features

- **invariance** under **time-rescaling**:

$$\zeta \text{ is 1-homogeneous} \quad \Rightarrow \quad \partial\zeta \text{ is 0-homogeneous}$$

  Hence ***z* is solution of** (eq_z) **if and only if** $z \circ \alpha$ **is solution of** (eq_z) for every strictly increasing reparametrization α.
- Typical of **activated** systems: z responds to the activation energy in a rate-independent way possibly with **hysteresis effects**

Rate-independent evolutions

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Mathematical difficulties

- ζ does **NOT** grow **superlinearly** at ∞ $\rightsquigarrow$ no "good" estimates for $\dot{z}$

  standard regularity of $t \mapsto z(t)$ is ONLY BV

- z may have **jumps!!!** $\rightsquigarrow$ weak formulations

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Theory of energetic solutions [Mielke et al.]

Weak, derivative-free formulations, based on

- **energetic balance** (energy identity)
- **stability** conditions
- enforcing **irreversibility**

The complete PDE system: viscous vs. rate-independent behaviour

$$\varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\theta\big)\big) = F \qquad \text{in } \Omega \times (0, T),$$

$$+ \text{ Dir. b.c. on } \Gamma_{\mathrm{D}} \times (0, T) \;+\; \text{Neu. b.c. on } \Gamma_{\mathrm{N}} \times (0, T),$$

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Not fully rate-independent model: **viscosity**-driven equations for $\mathbf{u}$ and ϑ coupled with **rate-independent** evolution for z.

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Not fully rate-independent model: **viscosity**-driven equations for $\mathbf{u}$ and ϑ coupled with **rate-independent** evolution for z. **General theory** for rate-independent evolutions coupled with viscous evolutions: **[Roubíček'09,'10]**

Mathematical difficulties (I)

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♢ **(quadratic) coupling** terms between (eq_u) and (eq_θ) $\rightsquigarrow$ only L^1-estimates for r.h.s. of (eq_θ)

Enthalpy reformulation

Only L^1 estimates for the r.h.s. of

$$c_{\mathrm{v}}(\theta)\dot{\theta} - \mathrm{div}\big(\mathbb{K}(e(\mathbf{u}),\theta)\nabla\theta\big) = \mathbb{D}e(\dot{\mathbf{u}})\colon e(\dot{\mathbf{u}}) + \theta\mathbb{C}\mathbb{E}\colon e(\dot{\mathbf{u}}) + G \quad \text{in } \Omega\times(0,T)$$

$\Rightarrow$ **Boccardo-Gallouët** techniques + suitable growth conditions on c_{v}, e.g.

$$c_0(\theta+1)^{\omega_0} \leq c_{\mathrm{v}}(\theta) \leq c_1(\theta+1)^{\omega_1}$$

Enthalpy reformulation

Only L^1 estimates for the r.h.s. of

$$c_{\mathrm{v}}(\theta)\dot{\theta} - \mathrm{div}\big(\mathbb{K}(e(\mathbf{u}),\theta)\nabla\theta\big) = \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \theta\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}}) + G \quad \text{in } \Omega\times(0,T)$$

$\Rightarrow$ **Boccardo-Gallouët** techniques + suitable growth conditions on c_{v}, e.g.

$$c_0(\theta+1)^{\omega_0} \leq c_{\mathrm{v}}(\theta) \leq c_1(\theta+1)^{\omega_1}$$

To combine this with time-discretization, **enthalpy re-formulation**

$$\begin{cases} w = w(\theta) = \int_0^{\vartheta} c_{\mathrm{v}}(r)\,\mathrm{d}r, \\ \theta = \Theta(w) \end{cases}$$

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$$\begin{cases} w = w(\theta) = \int_0^{\vartheta} c_{\rm v}(r)\,\mathrm{d}r, \\ \theta = \Theta(w) \end{cases}$$

hence

$$\dot{w} - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}),w)\nabla w\big) = \mathbb{D}e(\dot{\mathbf{u}}):e(\dot{\mathbf{u}}) + \Theta(w)\mathbb{C}\mathbb{E}:e(\dot{\mathbf{u}}) + G \quad \text{in } \Omega\times(0,T) \quad (\mathrm{eq}_w)$$

with

$$C_\theta^1(w^{1/\omega_1} - 1) \leq \Theta(w) \leq C_\theta^2(w^{1/\omega_0} - 1)$$

Mathematical difficulties (II)

$$\varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}(e(\mathbf{u}) - \mathbb{E}\Theta(w))\big) = F \qquad \Omega \times (0,T),$$

$+$ Dir. b.c. on $\Gamma_{\mathrm{D}} \times (0,T)$ $+$ Neu. b.c. on $\Gamma_{\mathrm{N}} \times (0,T)$,

$$\dot{w} - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w\big) = \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \Theta(w)\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}}) + G \qquad \Omega \times (0,T),$$

$+$ Neu. b.c. on $\partial\Omega \times (0,T)$,

$$\big[\!\big[\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w))\big]\!\big]\nu = 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] + \partial I_C([\![\mathbf{u}]\!]) \ni 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$\partial\zeta(\dot{z}) + \partial I_{[0,1]}(z) + \tfrac{1}{2}\kappa\big|[\![u]\!]\big|^2 - a_0 \ni 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$\frac{1}{2}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w|^{+}_{\Gamma_{\mathrm{C}}} + \mathcal{K}(e(\mathbf{u}), w)\nabla w|^{-}_{\Gamma_{\mathrm{C}}}\big)\cdot\nu + \eta([\![\mathbf{u}]\!], z)[\![\Theta(w)]\!] = 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$[\![\mathcal{K}(e(\mathbf{u}), w)\nabla w]\!] \cdot \nu = \zeta(\dot{z}) \qquad \Gamma_{\mathrm{C}} \times (0,T)$$

Mathematical difficulties (II)

$$
\begin{aligned}
&\varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}(e(\mathbf{u}) - \mathbb{E}\Theta(w))\big) = F && \Omega\times(0,T),\\
&+\ \text{Dir. b.c. on } \Gamma_{\mathrm{D}}\times(0,T)\ +\ \text{Neu. b.c. on } \Gamma_{\mathrm{N}}\times(0,T),\\
&\dot{w} - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}),w)\nabla w\big) = \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \Theta(w)\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}}) + G && \Omega\times(0,T)\\
&+\ \text{Neu. b.c. on } \partial\Omega\times(0,T),\\
&\big[\!\!\big[\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w))\big]\!\!\big]\nu = 0 && \Gamma_{\mathrm{C}}\times(0,T),\\
&(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z[\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}([\![\mathbf{u}]\!]) \ni 0 && \Gamma_{\mathrm{C}}\times(0,T),\\
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&\big[\!\!\big[\mathcal{K}(e(\mathbf{u}),w)\nabla w\big]\!\!\big]\cdot\nu = \zeta(\dot{z}) && \Gamma_{\mathrm{C}}\times(0,T)
\end{aligned}
$$

♢ **coupling terms** between (eq_u), (eq_w) and (eq_z) involve traces of $\mathbf{u}$ and w on Γ_{C} $\rightsquigarrow$ need of sufficient regularity of $\mathbf{u}$ and w to control $\mathbf{u}|_{\Gamma_{\mathrm{C}}}$ and $w|_{\Gamma_{\mathrm{C}}}$

Mathematical difficulties (III)

$$\varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\Theta(w)\big)\big) = F \qquad \Omega \times (0,T),$$

$+$ Dir. b.c. on $\Gamma_{\mathrm{D}} \times (0,T)$ $+$ Neu. b.c. on $\Gamma_{\mathrm{N}} \times (0,T)$,

$$\dot{w} - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w\big) = \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \Theta(w)\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}}) + G \qquad \Omega \times (0,T),$$

$+$ Neu. b.c. on $\partial\Omega \times (0,T)$,

$$[\![\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w))]\!]\,\nu = 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}([\![\mathbf{u}]\!]) \ni 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$\partial\zeta(\dot{z}) + \partial I_{[0,1]}(z) + \tfrac{1}{2}\kappa\big|[\![u]\!]\big|^2 - a_0 \ni 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$\frac{1}{2}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w|^{+}_{\Gamma_{\mathrm{C}}} + \mathcal{K}(e(\mathbf{u}), w)\nabla w|^{-}_{\Gamma_{\mathrm{C}}}\big)\cdot\nu + \eta([\![\mathbf{u}]\!], z)[\![\Theta(w)]\!] = 0 \qquad \Gamma_{\mathrm{C}} \times (0,T),$$

$$[\![\mathcal{K}(e(\mathbf{u}), w)\nabla w]\!]\cdot\nu = \zeta(\dot{z}) \qquad \Gamma_{\mathrm{C}} \times (0,T)$$

♢ **rate-independent** evolution for z $\rightsquigarrow$ lack of regularity of $t \mapsto z(t)$, z may have jumps, $\dot{z}$ need not be well-defined!

Weak formulation of the equation for z (I)

$$\partial\zeta(\dot{z}) + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa\left|[\![u]\!]\right|^2 - a_0 \ni 0$$

Weak, derivative-free formulation $\rightsquigarrow$ **semi-stability condition**:

$$\forall \tilde{z} \in L^\infty(\Gamma_C): \qquad \Phi(\mathbf{u}(t), z(t)) \le \Phi(\mathbf{u}(t), \tilde{z}) + \mathcal{R}(\tilde{z} - z(t)) \ \text{ for a.a. } t \in (0,T) \quad (S)$$

with

- dissipation potential

$$\mathcal{R}(\tilde{z}-z) := \int_{\Gamma_C} \zeta(\tilde{z}-z)\,\mathrm{d}S = \begin{cases} \int_{\Gamma_C} a_1|\tilde{z}-z|\,\mathrm{d}S & \text{if } \tilde{z} \le z \text{ a.e. in } \Gamma_C, \\ +\infty & \text{otherwise.} \end{cases}$$

- stored energy functional

$$\Phi(\mathbf{u}, z) := \int_\Omega \frac{1}{2}\mathbb{C}e(\mathbf{u}) : e(\mathbf{u})\,\mathrm{d}x + I_{\mathcal{C}}([\![\mathbf{u}]\!]) + \int_{\Gamma_C}\left(\frac{\kappa}{2} z\left|[\![\mathbf{u}]\!]\right|^2 + I_{[0,1]}(z) - a_0 z\right)\mathrm{d}S$$

Weak formulation of the equation for z (I)

$$\partial\zeta(\dot{z}) + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa\left|\,[\![u]\!]\,\right|^2 - a_0 \ni 0$$

Weak, derivative-free formulation $\rightsquigarrow$ **semi-stability condition**:

$$\forall \tilde{z} \in L^\infty(\Gamma_{\mathrm{C}}): \qquad \Phi\big(\mathbf{u}(t), z(t)\big) \leq \Phi\big(\mathbf{u}(t), \tilde{z}\big) + \mathcal{R}\big(\tilde{z} - z(t)\big) \text{ for a.a. } t \in (0,T) \quad (\mathrm{S})$$

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Remark: (S) only a **semi**-stability condition ($\mathbf{u}(t)$ is fixed)! This reflects the fact that PDE system **NOT FULLY RATE-INDEPENDENT**, $\mathbf{u}$ has viscosity-driven evolution!

Weak formulation of the equation for z (II)

$$\partial\zeta(\dot{z}) + \underbrace{\partial I_{[0,1]}(z) + \tfrac{1}{2}\kappa \left| [\![u]\!] \right|^2 - a_0}_{= \partial_z \Phi(\mathbf{u}, z)} \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T) \qquad (\mathrm{eq}_z)$$

implies

$$\forall \tilde{z} \in L^\infty(\Gamma_{\mathrm{C}}): \quad \Phi\big(\mathbf{u}(t), z(t)\big) \leq \Phi\big(\mathbf{u}(t), \tilde{z}\big) + \mathcal{R}\big(\tilde{z} - z(t)\big) \text{ for a.a. } t \in (0,T) \quad (\mathrm{S})$$

Weak formulation of the equation for z (II)

Proof:

- use that $\partial\zeta(\dot{z}) \subset \partial\zeta(0)$ (1-homogeneity of ζ)
- fix $\tilde{z} \in L^\infty(\Gamma_{\mathrm{C}})$ and test

$$\underbrace{-\left(\partial I_{[0,1]}(z(t)) + \tfrac{1}{2}\kappa\left|\llbracket u(t)\rrbracket\right|^2 - a_0\right)}_{= -\partial_z\Phi(\mathbf{u}(t),z(t))} \in \partial\zeta(\dot{z}(t)) \subset \partial\zeta(0) \quad \text{by } \tilde{z} - z(t)$$

Weak formulation of the equation for z (II)

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- Hence

$$\underbrace{\int_{\Gamma_{\mathrm{C}}} \zeta(\tilde{z} - z(t))}_{= \mathcal{R}(\tilde{z} - z(t))} - \underbrace{\int_{\Gamma_{\mathrm{C}}} \zeta(0)}_{= 0} \geq \langle -\partial_z \Phi(\mathbf{u}, z(t)), \tilde{z} - z(t) \rangle$$

Weak formulation of the equation for z (II)

Proof:

- use that $\partial\zeta(\dot{z}) \subset \partial\zeta(0)$ (1-homogeneity of ζ)
- fix $\tilde{z} \in L^\infty(\Gamma_C)$ and test

$$\underbrace{-\left(\partial I_{[0,1]}(z(t)) + \tfrac{1}{2}\kappa\left|\,[\![u(t)]\!]\,\right|^2 - a_0\right)}_{= -\partial_z\Phi(\mathbf{u}(t),z(t))} \in \partial\zeta(\dot{z}(t)) \subset \partial\zeta(0) \quad \text{by } \tilde{z} - z(t)$$

- Hence

$$\underbrace{\int_{\Gamma_C} \zeta(\tilde{z}-z(t))}_{=\mathcal{R}(\tilde{z}-z(t))} - \underbrace{\int_{\Gamma_C} \zeta(0)}_{=0} \geq \langle -\partial_z\Phi(\mathbf{u}, z(t)), \tilde{z} - z(t)\rangle$$

$$\overset{\Phi(\mathbf{u},\cdot)\text{ convex}}{\geq} \Phi(\mathbf{u}(t), z(t)) - \Phi(\mathbf{u}(t), \tilde{z})$$

Weak formulation of the equation for z (II)

Proof:

- use that $\partial\zeta(\dot{z}) \subset \partial\zeta(0)$ (1-homogeneity of ζ)
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$$\underbrace{-\left(\partial I_{[0,1]}(z(t)) + \tfrac{1}{2}\kappa\left|[\![u(t)]\!]\right|^2 - a_0\right)}_{= -\partial_z\Phi(\mathbf{u}(t),z(t))} \in \partial\zeta(\dot{z}(t)) \subset \partial\zeta(0) \quad \text{by } \tilde{z} - z(t)$$

- Hence

$$\underbrace{\int_{\Gamma_C} \zeta(\tilde{z}-z(t))}_{=\mathcal{R}(\tilde{z}-z(t))} - \underbrace{\int_{\Gamma_C} \zeta(0)}_{=0} \geq \langle -\partial_z\Phi(\mathbf{u}, z(t)), \tilde{z} - z(t)\rangle$$

$$\overset{\Phi(\mathbf{u},\cdot)\text{ convex}}{\geq} \Phi(\mathbf{u}(t), z(t)) - \Phi(\mathbf{u}(t), \tilde{z})$$

Remark: if $t \mapsto z(t)$ absolutely continuous (**no jumps**):

semi-stability condition (S) (+ energy identity) $\Rightarrow$ (eq_z)

Weak formulation of the equation for u

From

$$\begin{cases} \varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\Theta(w)\big)\big) = F & \text{in } \Omega \times (0,T), \\ + \text{ Dir. b.c. on } \Gamma_{\mathrm{D}} \times (0,T) \ + \text{ Neu. b.c. on } \Gamma_{\mathrm{N}} \times (0,T), & \\ [\![\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w))]\!]\nu = 0 & \text{on } \Gamma_{\mathrm{C}} \times (0,T), \\ (\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z[\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}\big([\![\mathbf{u}]\!]\big) \ni 0 & \text{on } \Gamma_{\mathrm{C}} \times (0,T), \end{cases}$$

Weak formulation of the equation for u

From

$$\begin{cases} \varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\Theta(w)\big)\big) = F & \text{in } \Omega \times (0,T), \\ + \text{ Dir. b.c. on } \Gamma_{\mathrm{D}} \times (0,T) \; + \; \text{Neu. b.c. on } \Gamma_{\mathrm{N}} \times (0,T), & \\ [\![\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w))]\!]\nu = 0 & \text{on } \Gamma_{\mathrm{C}} \times (0,T), \\ (\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z[\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}\big([\![\mathbf{u}]\!]\big) \ni 0 & \text{on } \Gamma_{\mathrm{C}} \times (0,T), \end{cases}$$

to

$$\begin{cases} [\![\mathbf{u}]\!] \in \mathcal{C} \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T), \end{cases}$$

Weak formulation of the equation for u

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$$\begin{cases} \varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\Theta(w)\big)\big) = F & \text{in } \Omega \times (0,T), \\ + \text{ Dir. b.c. on } \Gamma_{\mathrm{D}} \times (0,T) \ + \ \text{Neu. b.c. on } \Gamma_{\mathrm{N}} \times (0,T), & \\ [\![\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w))]\!]\nu = 0 & \text{on } \Gamma_{\mathrm{C}} \times (0,T), \\ (\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z[\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}([\![\mathbf{u}]\!]) \ni 0 & \text{on } \Gamma_{\mathrm{C}} \times (0,T), \end{cases}$$

to

$$\begin{cases} [\![\mathbf{u}]\!] \in \mathcal{C} \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T), \\ \varrho \int_\Omega \dot{\mathbf{u}}(T) \cdot \big(\mathbf{v}(T) - \dot{\mathbf{u}}(T)\big)\,\mathrm{d}x \\ \quad + \int_0^T \int_\Omega \big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\Theta(w)\big) : e(\mathbf{v} - \mathbf{u})\,\mathrm{d}x\mathrm{d}t \\ \quad - \int_0^T \int_\Omega \varrho\dot{\mathbf{u}} \cdot (\dot{\mathbf{v}} - \dot{\mathbf{u}})\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_{\mathrm{C}}} \kappa z [\![\mathbf{u}]\!] \cdot [\![\mathbf{v} - \mathbf{u}]\!]\,\mathrm{d}S\mathrm{d}t \\ \quad \geq \varrho \int_\Omega \dot{\mathbf{u}}_0 \cdot \big(\mathbf{v}(0) - \mathbf{u}(0)\big)\mathrm{d}x + \int_0^T \int_\Omega F \cdot (\mathbf{v} - \mathbf{u})\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_{\mathrm{N}}} f \cdot (\mathbf{v} - \mathbf{u})\,\mathrm{d}S\mathrm{d}t \\ \text{for all test func. } \mathbf{v} \text{ s.t. } \mathbf{v} = 0 \text{ on } \Gamma_{\mathrm{D}} \times (0,T) \text{ and } [\![\mathbf{v}]\!] \in \mathcal{C} \text{ on } \Gamma_{\mathrm{C}} \times (0,T) \end{cases}$$

Weak formulation of the enthalpy equation

From

$$\begin{cases} \dot{w} - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w\big) = \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \Theta(w)\mathbb{B} : e(\dot{\mathbf{u}}) + G & \Omega \times (0,T), \\ + \text{ Neu. b.c. on } \partial\Omega \times (0,T) & \\ \frac{1}{2}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w|^{+}_{\Gamma_{\mathrm{C}}} + \mathcal{K}(e(\mathbf{u}), w)\nabla w|^{-}_{\Gamma_{\mathrm{C}}}\big)\cdot\nu + \eta([\![\mathbf{u}]\!], z)[\![\Theta(w)]\!] = 0 & \Gamma_{\mathrm{C}} \times (0,T), \\ [\![\mathcal{K}(e(\mathbf{u}), w)\nabla w]\!] \cdot \nu = \zeta(\dot{z}) & \Gamma_{\mathrm{C}} \times (0,T) \end{cases}$$

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From

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to

$$\begin{cases} \int_\Omega w(T)v(T)\,\mathrm{d}x + \int_0^T\int_\Omega \mathcal{K}(e(\mathbf{u}), w)\nabla w \cdot \nabla v - w\dot{v}\,\mathrm{d}x\mathrm{d}t \\ \qquad + \int_0^T\int_{\Gamma_{\mathrm{C}}} \eta([\![\mathbf{u}]\!], z)[\![\Theta(w)]\!]\,[\![v]\!]\,\mathrm{d}S\mathrm{d}t \\ = \int_0^T\int_\Omega \big(\mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \Theta(w)\mathbb{C} : \mathbb{E}e(\dot{\mathbf{u}})\big) v\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\Gamma_{\mathrm{C}}} \frac{v|^{+}_{\Gamma_{\mathrm{C}}} + v|^{-}_{\Gamma_{\mathrm{C}}}}{2}\,\mathfrak{h}_z(\mathrm{d}S\mathrm{d}t) \\ \quad + \int_0^T\int_\Omega Gv\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\partial\Omega} gv\,\mathrm{d}S\mathrm{d}t + \int_\Omega w_0 v(0)\,\mathrm{d}x \\ \text{for all test functions } v, \\ \text{with } \mathfrak{h}_z \text{ measure induced by dissipation } \zeta \end{cases}$$

Weak formulation of the adhesive contact PDE system

Find a triple $(\mathbf{u}, z, w)$ with

$$\mathbf{u} \in W^{1,2}(0,T; W^{1,2}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T; L^2(\Omega;\mathbb{R}^d)),$$

$$z \in L^{\infty}(\Gamma_{\mathrm{C}} \times (0,T)) \cap \mathrm{BV}([0,T]; L^1(\Gamma_{\mathrm{C}})),$$

$$w \in L^r(0,T; W^{1,r}(\Omega\backslash\Gamma_{\mathrm{C}})) \cap L^{\infty}(0,T; L^1(\Omega)) \cap \mathrm{BV}([0,T]; W^{1,r'}(\Omega\backslash\Gamma_{\mathrm{C}})^*)$$

$\forall\, 1 \leq r < \frac{d+2}{d+1}$,

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$\forall\, 1 \leq r < \frac{d+2}{d+1}$, fulfilling

- **semi-stability**
- **weak formulation of the momentum equation**
- **weak formulation of the enthalpy equation**

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- **semi-stability**
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- **weak formulation of the enthalpy equation**
- **total energy inequality**

$$\frac{1}{2}\int_\Omega \varrho\,|\dot{\mathbf{u}}(T)|^2\,\mathrm{d}x + \Phi\big(\mathbf{u}(T), z(T)\big) + \int_\Omega w(T)\,\mathrm{d}x$$
$$\leq \frac{1}{2}\int_\Omega \varrho\,|\dot{\mathbf{u}}(0)|^2\,\mathrm{d}x + \Phi\big(\mathbf{u}(0), z(0)\big) + \int_\Omega w(0)\,\mathrm{d}x$$
$$+ \int_0^T\int_\Omega F\cdot\dot{\mathbf{u}}\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Gamma_N} f\cdot\dot{\mathbf{u}}\,\mathrm{d}S\mathrm{d}t + \int_0^T\int_\Omega G\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\partial\Omega} g\,\mathrm{d}S\mathrm{d}t$$

Existence theorem (I)

Under

conditions on the data c_{v}, $\mathbb{K}$, η +

conditions on the initial data $(\mathbf{u}_0, \dot{\mathbf{u}}_0, z_0, w_0)$

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$$
\begin{aligned}
&\Phi\big(\mathbf{u}(T), z(T)\big) + \int_\Omega w(T)\,\mathrm{d}x \\
&\leq \Phi\big(\mathbf{u}(0), z(0)\big) + \int_\Omega w(0)\,\mathrm{d}x + \int_0^T \int_\Omega F\cdot\dot{\mathbf{u}}\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\Gamma_{\mathrm{N}}} f\cdot\dot{\mathbf{u}}\,\mathrm{d}S\mathrm{d}t + \int_0^T \int_\Omega G\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\partial\Omega} g\,\mathrm{d}S\mathrm{d}t
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then the Cauchy problem for the weak formulation has a solution $(\mathbf{u}, w, z)$.

Existence theorem (II)

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conditions on the data $c_{\rm v}$, $\mathbb{K}$, η +

conditions on the initial data $(\mathbf{u}_0, \dot{\mathbf{u}}_0, z_0, w_0)$

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Outline of the proof (I)

◇ Approximation via ε-**Yosida regularization** of the constraint $\boxed{[\![\mathbf{u}]\!] \in \mathcal{C} \text{ on } \Gamma_{\mathrm{C}}}$, i.e.

$$(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}([\![\mathbf{u}]\!]) \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0, T)$$

replaced by

$$(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] + (\partial I_{\mathcal{C}})_{\varepsilon}([\![\mathbf{u}]\!]) \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0, T).$$

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Penalization

In the case of (frictionless) Signorini contact

$$\mathcal{C} = \mathcal{C}(x) = \{v \in \mathbb{R}^d;\ v \cdot \nu(x) \geq 0\} \quad \text{for a.a. } x \in \Gamma_{\mathrm{C}}$$

then $(\partial I_{\mathcal{C}})_{\varepsilon}([\![\mathbf{u}]\!]) = -\frac{1}{\varepsilon}([\![\mathbf{u}]\!] \cdot \nu)^{-}\nu$, hence approximation reduces to **penalization**

$$(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] - \frac{1}{\varepsilon}([\![\mathbf{u}]\!] \cdot \nu)^{-}\nu \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T)$$

Outline of the proof (II)

♢ Approximation of the ε-**Yosida regularized** problem via **semi-implicit time discretization**:

$$\tau > 0 \text{ time-step } \rightsquigarrow \text{ partition } \{t_0 = 0 < t_1 < \ldots < t_k < \ldots < t_{K_\tau} = T\}$$

Time-discrete problem: find $\{(u^k_{\varepsilon\tau}, w^k_{\varepsilon\tau}, z^k_{\varepsilon\tau})\}_{k=1}^{K_\tau}$ fulfilling

$$\varrho \mathrm{D}_t^2 u^k_{\varepsilon\tau} - \mathrm{div}\Big(\mathbb{D}e(\mathrm{D}_t u^k_{\varepsilon\tau}) + \mathbb{C}(e(u^k_{\varepsilon\tau}) - \mathbb{E}\Theta(w^k_{\varepsilon\tau})) + \tau|e(u^k_{\varepsilon\tau})|^{\gamma-2}e(u^k_{\varepsilon\tau})\Big) = F^k_\tau \quad \text{in } \Omega,$$

$+$ Dir. b.c. on Γ_{D} $+$ Neu. b.c. on Γ_N

$$\mathrm{D}_t w^k_{\varepsilon\tau} - \mathrm{div}(\mathcal{K}(w^k_{\varepsilon\tau}, e(u^k_{\varepsilon\tau}))\nabla w^k_{\varepsilon\tau}) \frac{1}{2}(2-\sqrt{\tau})\mathbb{D}e(\mathrm{D}_t u^k_{\varepsilon\tau}) : e(\mathrm{D}_t u^k_{\varepsilon\tau}) + \Theta(w^k_{\varepsilon\tau})\mathbb{E} : \mathbb{C}e(\mathrm{D}_t u^k_{\varepsilon\tau}) + G^k_\tau \quad \text{in } \Omega,$$

$+$ Neu. b.c. on $\partial\Omega$,

$$\partial\zeta(\mathrm{D}_t z^k_{\varepsilon\tau}) + \partial I_{[0,1]}(z^k_{\varepsilon\tau}) + \frac{\kappa}{2}|[\![u^k_{\varepsilon\tau}]\!]|^2 - a_0 + \tau^\alpha z^k_{\varepsilon\tau} \ni 0 \quad \text{on } \Gamma_C,$$

$$[\![\mathbb{D}e(\mathrm{D}_t u^k_{\varepsilon\tau}) + \mathbb{C}(e(u^k_{\varepsilon\tau}) - \Theta(w^k_{\varepsilon\tau})\mathbb{E}) + \tau|e(u^k_{\varepsilon\tau})|^{\gamma-2}e(u^k_{\varepsilon\tau})]\!]\,\nu = 0 \quad \text{on } \Gamma_C,$$

$$\kappa z^k_{\varepsilon\tau}[\![u^k_{\varepsilon\tau}]\!] + (\partial I_{\mathcal{C}})_\varepsilon([\![u^k_{\varepsilon\tau}]\!]) + \Big[\mathbb{D}e(\mathrm{D}_t u^k_{\varepsilon\tau}) + \mathbb{C}(e(u^k_{\varepsilon\tau}) - \Theta(w^k_{\varepsilon\tau})\mathbb{E}) + \tau|e(u^k_{\varepsilon\tau})|^{\gamma-2}e(u^k_{\varepsilon\tau})\Big]\nu$$
$$+ \tau^\beta(1+|[\![u^k_{\varepsilon\tau}]\!]|^2)^{\frac{\mu}{2}-1}[\![u^k_{\varepsilon\tau}]\!] = 0 \quad \text{on } \Gamma_C,$$

$$\frac{1}{2}(\mathcal{K}(w^k_{\varepsilon\tau}, e(u^k_{\varepsilon\tau}))\nabla w^k_{\varepsilon\tau}|^+_{\Gamma_C} + \mathcal{K}(w^k_{\varepsilon\tau}, e(u^k_{\varepsilon\tau}))\nabla w^k_{\varepsilon\tau}|^-_{\Gamma_C}) \cdot \nu + \eta([\![\mathbf{u}^{k-1}_{\varepsilon\tau}]\!], z^k_{\varepsilon\tau})[\![\Theta(w^k_{\varepsilon\tau})]\!] = 0 \quad \text{on } \Gamma_C,$$

$$[\![\mathcal{K}(w^k_{\varepsilon\tau}, e(u^k_{\varepsilon\tau}))\nabla w^k_{\varepsilon\tau}]\!]\,\nu = -\zeta(\mathrm{D}_t z^k_{\varepsilon\tau}) \quad \text{on } \Gamma_C.$$

Outline of the proof (III)

- ♢ A priori estimates
- ♢ Passage to the limit as $\tau \downarrow 0$
- ♢ Passage to the limit as $\varepsilon \downarrow 0$
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formally shown on the PDE system

$$\varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\Theta(w)\big)\big) = F \qquad \text{in } \Omega \times (0,T),$$

$+$ Dir. b.c. on $\Gamma_{\mathrm{D}} \times (0,T)$ $+$ Neu. b.c. on $\Gamma_{\mathrm{N}} \times (0,T)$,

$$\dot{w} - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w\big) = \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) + \Theta(w)\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}}) + G \qquad \text{in } \Omega \times (0,T)$$

$+$ Neu. b.c. on $\partial\Omega \times (0,T)$,

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$$(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z[\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}([\![\mathbf{u}]\!]) \ni 0 \qquad \text{on } \Gamma_{\mathrm{C}} \times (0,T),$$

$$\partial\zeta(\dot{z}) + \partial I_{[0,1]}(z) + \tfrac{1}{2}\kappa\big|[\![u]\!]\big|^2 - a_0 \ni 0 \qquad \text{on } \Gamma_{\mathrm{C}} \times (0,T),$$

$$\frac{1}{2}\big(\mathcal{K}(e(\mathbf{u}), w)\nabla w|^{+}_{\Gamma_{\mathrm{C}}} + \mathcal{K}(e(\mathbf{u}), w)\nabla w|^{-}_{\Gamma_{\mathrm{C}}}\big)\cdot\nu + \eta([\![\mathbf{u}]\!], z)[\![\Theta(w)]\!] = 0 \qquad \text{on } \Gamma_{\mathrm{C}} \times (0,T),$$

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Basic a priori estimates (I)

First estimate

$$\varrho\ddot{\mathbf{u}} - \mathrm{div}\big(\mathbb{D}e(\dot{\mathbf{u}}) + \mathbb{C}\big(e(\mathbf{u}) - \mathbb{E}\Theta(w)\big)\big) = F \quad \times\dot{\mathbf{u}}$$

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$\Rightarrow$ **"Energy" a-priori estimates** on $\mathbf{u}$, w, z

Basic a priori estimates (II)

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$$\left| \int_0^T \int_\Omega \varrho \ddot{\mathbf{u}} \mathbf{v} + \int_0^T \int_{\Gamma_C} \partial I_{\mathcal{C}}(\llbracket \mathbf{u} \rrbracket) \mathbf{v} \right| \leq C \quad \text{for all test functions } \mathbf{v}$$

$\Rightarrow$ inertial term and subdifferential term **CANNOT be estimated separately**. Hence we distinguish cases

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Compactness theorems: strongly/weakly converging (sub)sequences of approx. solutions $(\mathbf{u}_n, w_n, z_n) \to (\mathbf{u}, w, z)$

Passage to the limit: step (I)

Momentum equation & Semi-stability condition & Total energy inequality

- By strong-weak convergences we deduce that $(\mathbf{u}, w, z)$ fulfils **weak momentum equation**.

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  we obtain that $(\mathbf{u}, w, z)$ fulfils **semi-stability condition**.
- Lower semicontinuity argument $\Rightarrow$ **total energy inequality**.

Passage to the limit: step (II)

Enthalpy equation

♢ To pass to the limit in

$$
\begin{cases}
\dot{w}_n - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n\big) = \mathbb{D}e(\dot{\mathbf{u}}_n) \colon e(\dot{\mathbf{u}}_n) + \Theta(w_n)\mathbb{C}\mathbb{E} \colon e(\dot{\mathbf{u}}_n) + G & \Omega, \\
+ \text{ Neu. b.c. on } \partial\Omega, & \\
\frac{1}{2}\big(\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n|^{+}_{\Gamma_{\mathrm{C}}} + \mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n|^{-}_{\Gamma_{\mathrm{C}}}\big)\cdot\nu + \eta([\![\mathbf{u}_n]\!], z_n)[\![\Theta(w_n)]\!] = 0 & \Gamma_{\mathrm{C}}, \\
[\![\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n]\!] \cdot \nu = \zeta(\dot{z}_n) & \Gamma_{\mathrm{C}}
\end{cases}
$$

we need

$$
\begin{cases}
\mathbb{D}e(\dot{\mathbf{u}}_n) \colon e(\dot{\mathbf{u}}_n) \to \mathbb{D}e(\dot{\mathbf{u}}) \colon e(\dot{\mathbf{u}}) & \text{in } L^1(0, T; L^1(\Omega)) \\
\zeta(\dot{z}_n) \to \mathfrak{h}_z \text{ (measure induced by dissipation } \zeta) & \text{in the sense of measures}
\end{cases}
$$

Passage to the limit: step (II)

Enthalpy equation

♢ To pass to the limit in

$$\begin{cases} \dot{w_n} - \operatorname{div}\big(\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n\big) = \mathbb{D}e(\dot{\mathbf{u}_n}) : e(\dot{\mathbf{u}_n}) + \Theta(w_n)\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}_n}) + G & \Omega, \\ + \text{ Neu. b.c. on } \partial\Omega, & \\ \frac{1}{2}\big(\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n|^{+}_{\Gamma_C} + \mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n|^{-}_{\Gamma_C}\big)\cdot\nu + \eta([\![\mathbf{u}_n]\!], z_n)[\![\Theta(w_n)]\!] = 0 & \Gamma_C, \\ [\![\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n]\!] \cdot \nu = \zeta(\dot{z}_n) & \Gamma_C \end{cases}$$

we need

$$\begin{cases} \mathbb{D}e(\dot{\mathbf{u}_n}) : e(\dot{\mathbf{u}_n}) \to \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) & \text{in } L^1(0,T;L^1(\Omega)) \\ \zeta(\dot{z}_n) \to \mathfrak{h}_z \text{ (measure induced by dissipation } \zeta) & \text{in the sense of measures} \end{cases}$$

♢ **Technical trick:** from an **additional energy equality** deduce convergences for $\{\mathbb{D}e(\dot{\mathbf{u}_n}) : e(\dot{\mathbf{u}_n})\}$ and $\{\zeta(\dot{z}_n)\}$ via a lim inf / lim sup **argument**..

Passage to the limit: step (II)

Enthalpy equation

◇ To pass to the limit in

$$\begin{cases} \dot{w_n} - \mathrm{div}\big(\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n\big) = \mathbb{D}e(\dot{\mathbf{u}_n}) : e(\dot{\mathbf{u}_n}) + \Theta(w_n)\mathbb{C}\mathbb{E} : e(\dot{\mathbf{u}_n}) + G & \Omega, \\ + \text{ Neu. b.c. on } \partial\Omega, & \\ \frac{1}{2}\big(\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n|^{+}_{\Gamma_\mathrm{C}} + \mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n|^{-}_{\Gamma_\mathrm{C}}\big)\cdot\nu + \eta([\![\mathbf{u}_n]\!], z_n)[\![\Theta(w_n)]\!] = 0 & \Gamma_\mathrm{C}, \\ [\![\mathcal{K}(e(\mathbf{u}_n), w_n)\nabla w_n]\!]\cdot\nu = \zeta(\dot{z}_n) & \Gamma_\mathrm{C} \end{cases}$$

we need

$$\begin{cases} \mathbb{D}e(\dot{\mathbf{u}_n}) : e(\dot{\mathbf{u}_n}) \to \mathbb{D}e(\dot{\mathbf{u}}) : e(\dot{\mathbf{u}}) & \text{in } L^1(0,T;L^1(\Omega)) \\ \zeta(\dot{z}_n) \to \mathfrak{h}_z \text{ (measure induced by dissipation } \zeta) & \text{in the sense of measures} \end{cases}$$

◇ **Technical trick:** from an **additional energy equality** deduce convergences for $\{\mathbb{D}e(\dot{\mathbf{u}_n}) : e(\dot{\mathbf{u}_n})\}$ and $\{\zeta(\dot{z}_n)\}$ via a $\liminf / \limsup$ **argument**..

⇒ Conclusion of the proof of existence!

Passage to the limit: step (II)

Enthalpy equation

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⇒ Conclusion of the proof of existence!

♢ **Uniqueness** not expected due to

- nonlinear character of PDE system
- rate-independent character of equation for z

Modeling and analysis of delamination & thermal effects

i.e., anisothermal extension of **[Roubíček–Scardia–Zanini'09]**

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Delamination model: letting $\kappa \to +\infty$ in

$$(\mathbb{D}e(\dot{\mathbf{u}}) + (\mathbb{C}e(\mathbf{u}) - \mathbb{E}\Theta(w)))|_{\Gamma_{\mathrm{C}}}\nu + \kappa z [\![\mathbf{u}]\!] + \partial I_{\mathcal{C}}([\![\mathbf{u}]\!]) \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T),$$

$$\partial\zeta(\dot{z}) + \partial I_{[0,1]}(z) + \frac{1}{2}\kappa \left| [\![u]\!] \right|^2 - a_0 \ni 0 \quad \text{on } \Gamma_{\mathrm{C}} \times (0,T).$$

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Main difficulties:

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How to pass to the limit in approximate problem????
Work with **an even weaker** formulation of PDE system????