

Analysis of doubly nonlinear evolution equations driven by nonconvex energies

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in collaboration with

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Doubly nonlinear evolution equations

$$\partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \quad (\text{DNE})$$

- ▶ B is a **reflexive** Banach space;
- ▶ $\Psi : B \rightarrow [0, +\infty]$, with $\Psi(0) = 0$, l.s.c. and **convex**
- ▶ ∂ **convex analysis** subdifferential;
- ▶ $\mathcal{E} : [0, T] \times B \rightarrow (-\infty, +\infty]$, such that $z \mapsto \mathcal{E}(t, z)$ is l.s.c.
- ▶ ∂_z is the “subdifferential” of \mathcal{E} **w.r.t. the second variable**:

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Main focus on:

Nonsmooth, nonconvex energies

Physical interpretation

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is a generalized **balance relation** in **Thermomechanics**:

- ▶ $\Psi_z \sim$ **dissipation** potential
- ▶ $\mathcal{E} \sim$ **energy** functional ($t \mapsto \mathcal{E}(t, z) \sim$ (power of) **external forces**)

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- ▶ Ψ has **superlinear growth** \leftrightarrow dissipation with **viscosity** effects

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- ▶ Ψ has **linear growth** and is **positively 1-homogeneous**

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- \leftrightarrow **rate-independent** models

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- From now ON, we take

Ψ with **superlinear growth**

The convex & autonomous case

$$\boxed{\mathcal{E}(t, z) = \mathcal{E}(z)} \quad \& \quad \mathcal{E} \text{ convex}$$

Then,

$$\partial\Psi(z'(t)) + \partial_z^{\text{co}}\mathcal{E}(z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \quad (\text{DNE})$$

Applications

- ▶ phase transitions
- ▶ elasto-visco-plasticity
- ▶ gas flow through porous media....

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Existence results

- ▶ existence & approximation of solutions: [Barbu '75], [Arai '79], [Semba '86], [Colli-Visintin '90], [Colli '92]
- ▶ existence can be extended to C^1 - (or lower order) **perturbations** of convex energies [Segatti '06], [Schimperna-Segatti-Stefanelli '07]

Reduced energy functionals

♣ Consider **marginal energy functionals**

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- ◇ For a **suitable** subdifferential notion $\partial_z \mathcal{E}$,

$$\partial \Psi(z'(t)) + \partial_z \mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \quad (\text{DNE})$$

yields solutions to:

The associated PDE system

$$\begin{cases} \partial \Psi(z'(t)) + D_z I(t, \eta(t), z(t)) = 0, \\ D_\eta I(t, \eta(t), z(t)) = 0 \end{cases} \quad \text{in } B^*, \quad \text{for a.a. } t \in (0, T),$$

where

- ▶ z is the **internal variable**, through which **dissipation** occurs
- ▶ η fulfils **stationary** equation

Nonconvexity & nonsmoothness

$$\mathcal{E}(t, z) := \min_{\eta \in \mathcal{F}} I(t, \eta, z)$$

- ▶ $z \mapsto \mathcal{E}(t, z)$ is (highly) **nonconvex**

Which notion for the subdifferential $\partial_z \mathcal{E}???$

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- ▶ $t \mapsto \mathcal{E}(t, z)$ is **nonsmooth**

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Main goal

$$\partial \Psi(z'(t)) + \partial_z \mathcal{E}(t, z(t)) \ni 0 \quad \text{in } B', \quad t \in (0, T), \quad (\text{DNE})$$

- ▶ deal with energies s.t. $z \mapsto \mathcal{E}(t, z)$ is **nonsmooth**, **nonconvex** & $t \mapsto \mathcal{E}(t, z)$ is **nonsmooth**
- ▶ use suitable **generalized** notions for $\partial_z \mathcal{E}$ and $\partial_t \mathcal{E}$
- ▶ find **sufficient conditions on $(\Psi, \mathcal{E}, \partial_z \mathcal{E})$ for existence** of solutions to (DNE)

Overview

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Overview

$$z'(t) + \partial_z \mathcal{E}(z(t)) \ni 0 \quad \text{in } \mathcal{H}, \quad t \in (0, T), \quad (\text{GF})$$

Outline

1. consider

gradient flow case $\Psi(v) = \frac{1}{2} \|v\|^2$, $B = \mathcal{H}$ Hilbert space
autonomous case $\mathcal{E}(t, z) = \mathcal{E}(z)$

but with $z \mapsto \mathcal{E}(z)$ **nonconvex, nonsmooth**

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doubly nonlinear case Ψ general, superlinear growth
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4. **application**

Gradient flows of convex functionals: results

♣ Suppose that $\mathcal{E}(t, z) = \mathcal{E}(z)$, with \mathcal{E} **convex**:

$$\begin{cases} z'(t) + \partial_z^{\text{co}} \mathcal{E}(z(t)) \ni 0 & \text{in } \mathcal{H}, \quad t \in (0, T) \\ z(0) = z_0, \end{cases}$$

- ▶ **Well-established literature on:** existence, uniqueness, approximation of solutions [Kōmura'67, Crandall-Pazy'69, Brézis'73..]

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- ▶ **Well-established literature on:** existence, uniqueness, approximation of solutions [Kōmura'67, Crandall-Pazy'69, Brézis'73..]
- ▶ How extend (some of) these results to the **nonconvex** case?
- ▶ **Idea:** Let's have a deeper look at the **convex** case!

Gradient flows of convex functionals: approximation

- ▶ Fix time step $\tau > 0 \rightsquigarrow$ partition of $(0, T)$
- ▶ **Discrete solutions** $Z_\tau^0, Z_\tau^1, \dots, Z_\tau^N$: solve recursively

$$Z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

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- ▶ **Approximate solutions**: interpolants on $(0, T)$ of $\{Z_\tau^k\}_{k=1}^n$:

$$\begin{aligned} \{\bar{Z}_\tau\} &\text{ piecewise constant;} \\ \{\hat{Z}_\tau\} &\text{ piecewise linear.} \end{aligned}$$

- ▶ **Approximate equation**:

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- ▶ **Approximate energy inequality**:

$$\int_0^t \|\widehat{Z}_\tau'(s)\|^2 ds + \mathcal{E}(\bar{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

\Rightarrow **A priori estimates** for $\{\bar{Z}_\tau\}, \{\widehat{Z}_\tau\}$

Gradient flows of convex functionals: existence

- **Compactness:** $\exists z \in H^1(0, T; \mathcal{H})$

$$\begin{cases} \bar{Z}_\tau, \hat{Z}_\tau \rightarrow z & \text{in } L^\infty(0, T; \mathcal{H}), \\ \hat{Z}'_\tau \rightarrow z' & \text{in } H^1(0, T; \mathcal{H}) \end{cases}$$

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- **Passage to the limit based on :**

convexity: the set $\partial_z^{\text{co}} \mathcal{E}(z)$ is **convex** and closed

closedness: graph of $\partial_z^{\text{co}} \mathcal{E}$ is seq. closed w.r.t. **strong-weak** top. in $\mathcal{H} \times \mathcal{H}$:

$$z_n \rightarrow z, \quad \xi_n \rightarrow \xi, \quad \xi_n \in \partial_z^{\text{co}} \mathcal{E}(z_n) \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad \xi \in \partial_z^{\text{co}} \mathcal{E}(z)$$

- **Conclusion:**

$$-\widehat{Z}'_\tau(t) \in \partial_z^{\text{co}} \mathcal{E}(\bar{Z}_\tau(t)), \quad t \in (0, T)$$

$$\lim_{\tau \downarrow 0}$$

$$-z'(t) \in \partial_z^{\text{co}} \mathcal{E}(z(t)), \quad t \in (0, T)$$

Gradient flows of convex functionals: energy identity

Since $\partial_z^{\text{co}}\mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0, T; \mathcal{H}), \\ \xi \in L^2(0, T; \mathcal{H}), \\ \xi(t) \in \partial_z^{\text{co}}\mathcal{E}(z(t)) \text{ a.e. in } (0, T) \end{cases} \Rightarrow \frac{d}{dt}\mathcal{E}(z(t)) = \langle \xi(t), z'(t) \rangle \text{ a.e. in } (0, T)$$

\Rightarrow the solution z satisfies the **energy identity**:

$$\int_s^t \|z'(r)\|^2 dr + \mathcal{E}(z(t)) = \mathcal{E}(z(s)) \quad \text{for all } 0 \leq s \leq t \leq T.$$

From convex to non convex: heuristics

♠ Now, $\mathcal{E} = \mathcal{E}(z)$ is **nonconvex**

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Tentative approximation

- ▶ Fix time step $\tau > 0 \rightsquigarrow$ partition of $(0, T)$
- ▶ **Discrete solutions:** the recursive minimization problem

$$\begin{aligned} & \text{Find } Z_\tau^0 := z_0, \quad Z_\tau^1, \dots, Z_\tau^N: \\ & Z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\} \end{aligned}$$

has a solution if \mathcal{E} is **coercive** (for instance, $\mathcal{E}(\cdot) + \|\cdot\|^2$ has **compact** sublevels)

- ▶ Discrete solutions fulfil the **Euler equation**

$$\frac{Z_\tau^n - Z_\tau^{n-1}}{\tau} + \partial_z^{\text{fr}} \mathcal{E}(Z_\tau^n) \ni 0,$$

with $\partial_z^{\text{fr}} \mathcal{E}$ the **Fréchet subdifferential** of \mathcal{E}

The Fréchet subdifferential

First idea: “localize” the convex subdifferential

The convex subdifferential

Given $z \in D(\mathcal{E})$,

$$\xi \in \partial_z^{\text{co}} \mathcal{E}(z) \Leftrightarrow \mathcal{E}(w) - \mathcal{E}(z) \geq \langle \xi, w - z \rangle \quad \text{for all } w \in \mathcal{H}$$

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♣ If \mathcal{E} is **convex**, then $\partial_z^{\text{fr}} \mathcal{E} \equiv \partial_z^{\text{co}} \mathcal{E}$

From convex to non convex: heuristics

- ▶ **Discrete solutions:** $Z_T^0 = z_0, \quad Z_T^1, \dots, Z_T^N \Rightarrow$ approximate solutions

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$$\int_0^t \|\widehat{Z}'_\tau(s)\|^2 ds + \mathcal{E}(\bar{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

To obtain *some* energy inequality, need to work with a **suitable interpolant** of $Z_\tau^0 = z_0, Z_\tau^1, \dots, Z_\tau^N$.

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The variational interpolant [E. DeGiorgi, theory of Minimizing Movements, Gradient Flows in Metric Spaces]

Defined by

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Compare with the **time-incremental minimization**

$$\operatorname{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}$$

From convex to non convex: heuristics

- ▶ **Discrete solutions:** $Z_\tau^0 = z_0, \quad Z_\tau^1, \dots, Z_\tau^N \Rightarrow$ approximate solutions
- ▶ **Approximate energy inequality:** in the nonconvex case it is not clear how to prove

$$\int_0^t \|\tilde{Z}'_\tau(s)\|^2 ds + \mathcal{E}(\bar{Z}_\tau(t)) \leq \mathcal{E}(z_0).$$

To obtain *some* energy inequality, need to work with a **suitable interpolant** of $Z_\tau^0 = z_0, Z_\tau^1, \dots, Z_\tau^N$.

The variational interpolant [E. DeGiorgi, theory of Minimizing Movements, Gradient Flows in Metric Spaces]

Defined by

$$\tilde{Z}_\tau(0) = Z_\tau^0 = z_0,$$

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Euler equation

$$\frac{\tilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t - t_{n-1})} + \partial_z \operatorname{fr} \mathcal{E}(\tilde{Z}_\tau(t)) \ni 0, \quad t \in (0, T),$$

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Euler equation

$$\exists \tilde{\xi}_\tau(t) \in \partial_z^{\text{fr}} \mathcal{E}(\tilde{Z}_\tau(t)) \quad \text{s.t.} \quad \frac{\tilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t - t_{n-1})} + \tilde{\xi}_\tau(t) = 0, \quad t \in (0, T).$$

First step: approximate energy inequality

Approximate energy inequality in the **convex case**:

$$\int_0^t \|\widehat{\mathbf{Z}}'_\tau(s)\|^2 ds + \mathcal{E}(\overline{\mathbf{Z}}_\tau(t)) \leq \mathcal{E}(z_0).$$

In the **nonconvex case**:

$$\frac{1}{2} \int_0^t \|\widehat{\mathbf{Z}}'_\tau(s)\|^2 ds + \frac{1}{2} \int_0^t \|\widetilde{\xi}_\tau(s)\|^2 ds + \mathcal{E}(\widetilde{\mathbf{Z}}_\tau(t)) \leq \mathcal{E}(z_0)$$

⇒ **A priori estimates** for $(\overline{\mathbf{Z}}_\tau)_\tau$, $(\widehat{\mathbf{Z}}_\tau)_\tau$, $(\widetilde{\mathbf{Z}}_\tau)_\tau$, $(\widetilde{\xi}_\tau)_\tau$, and **compactness**:

$$\begin{aligned} \overline{\mathbf{Z}}_\tau, \widehat{\mathbf{Z}}_\tau, \widetilde{\mathbf{Z}}_\tau &\rightarrow z && \text{in } L^\infty(0, T; \mathcal{H}), \\ \widehat{\mathbf{Z}}'_\tau &\rightarrow z' && \text{in } L^2(0, T; \mathcal{H}), \\ \widetilde{\xi}_\tau &\rightarrow \xi && \text{in } L^2(0, T; \mathcal{H}). \end{aligned}$$

Second step: upper energy estimate

Second idea:

Instead of passing to the limit in the **pointwise** equation

$$-\widehat{Z}'_\tau(t) \in \partial_z^{\text{fr}} \mathcal{E}(\overline{Z}_\tau(t)) \quad t \in (0, T)$$

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$$-\widehat{Z}'_\tau(t) \in \partial_z^{\text{fr}} \mathcal{E}(\bar{Z}_\tau(t)) \quad t \in (0, T)$$

pass to the limit in the **approximate energy inequality**

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$$\liminf_{\tau \downarrow 0} \frac{1}{2} \int_0^t \|\widehat{Z}'_\tau(s)\|^2 ds + \liminf_{\tau \downarrow 0} \frac{1}{2} \int_0^t \|\widetilde{\xi}_\tau(s)\|^2 ds + \liminf_{\tau \downarrow 0} \mathcal{E}(\widetilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

↓ (via **LOWER SEMICONTINUITY**)

$$\frac{1}{2} \int_0^t \|z'(s)\|^2 ds + \frac{1}{2} \int_0^t \|\xi(s)\|^2 ds + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

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BUT: $\dot{\xi}(s) \in \partial_z^{\text{fr}} \mathcal{E}(z(s))??$

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BUT: $\dot{\xi}(s) \in \partial_z^{\text{fr}} \mathcal{E}(z(s))??$

♣ **YES, if $\partial_z^{\text{fr}} \mathcal{E}$ IS strongly-weakly closed** in the sense of graphs, i.e.

$$z_n \rightarrow z, \quad \xi_n \rightharpoonup \xi, \quad \sup_n \mathcal{E}(z_n) < +\infty, \quad \xi_n \in \partial_z^{\text{fr}} \mathcal{E}(z_n) \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad \xi \in \partial_z^{\text{fr}} \mathcal{E}(z)$$

Third step: LOWER energy estimate and conclusion

UPPER energy estimate

If $\partial_z^{\text{fr}} \mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

$$\frac{1}{2} \int_0^t \|z'(s)\|^2 ds + \frac{1}{2} \int_0^t \|\xi(s)\|^2 ds + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0), \quad \xi(s) \in \partial_z^{\text{fr}} \mathcal{E}(z(s))$$

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♣ **IF** $\partial_z^{\text{fr}} \mathcal{E}$ fulfils the **chain rule**

$$\begin{cases} z \in H^1(0, T; \mathcal{H}), \\ \xi \in L^2(0, T; \mathcal{H}), \\ \xi(t) \in \partial_z^{\text{fr}} \mathcal{E}(z(t)) \text{ a.e. in } (0, T) \end{cases} \Rightarrow \frac{d}{dt} \mathcal{E}(z(t)) = \langle \xi(t), z'(t) \rangle \text{ a.e. in } (0, T)$$

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whence

$$\implies z'(t) + \xi(t) \ni 0, \quad \xi(t) \in \partial_z^{\text{fr}} \mathcal{E}(z(t)) \quad \text{for a.a. } t \in (0, T).$$

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If $\partial_z^{\text{fr}} \mathcal{E}$ **strongly-weakly closed**, then we have **energy inequality**

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then z solves the gradient flow eq. and fulfils the **energy identity**

$$\frac{1}{2} \int_0^t \|z'(s)\|^2 ds + \frac{1}{2} \int_0^t \|\xi(s)\|^2 ds + \mathcal{E}(z(t)) = \mathcal{E}(z_0)$$

An existence and approximation result (I)

Theorem 1 [R., Savaré ESAIM-COCV'06]

Assume

- ▶ \mathcal{E} is **coercive**,
- ▶ $\partial_z^{\text{fr}} \mathcal{E}$ is **strongly-weakly closed**,
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Then, the approximate solutions converge to a function $z \in H^1(0, T; \mathcal{H})$ which **solves** the Cauchy problem

$$z'(t) + \partial_z^{\text{fr}}(z(t)) \ni 0 \quad \text{for a.a. } t \in (0, T), \quad z(0) = z_0.$$

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$$z'(t) + \partial_z^{\text{fr}} \mathcal{E}(z(t)) \ni 0 \quad \text{for a.a. } t \in (0, T), \quad z(0) = z_0.$$

Moreover, z fulfils the **energy identity** for all $0 \leq s \leq t \leq T$

$$\frac{1}{2} \int_s^t \|z'(r)\|^2 dr + \underbrace{\frac{1}{2} \int_s^t \|\xi(r)\|^2 dr}_{= \frac{1}{2} \int_s^t \|z'(r)\|^2 dr} + \mathcal{E}(z(t)) = \mathcal{E}(z(s))$$

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Closedness and chain rule of $\partial_z^{\text{fr}} \mathcal{E}$

satisfied for **dominated concave perturbations of convex** functionals, viz.

$$\mathcal{E}(z) = \mathcal{E}_1(z) + \mathcal{E}_2(z) \quad \text{with} \quad \left\{ \begin{array}{l} \mathcal{E}_1 \text{ convex} \\ \mathcal{E}_2 \text{ concave} \\ \|\partial \mathcal{E}_2(z)\| \leq c \|\partial \mathcal{E}_1(z)\| + C \end{array} \right.$$

A more general viewpoint

Replace $\partial_z^{\text{fr}}\mathcal{E}$ with a GENERAL subdifferential notion $\partial_z\mathcal{E}\dots$ which properties do we need on $\partial_z\mathcal{E}$?

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization

$$Z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if \mathcal{E} is **coercive**

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Step 1: approximate energy inequality: depends on **Euler equation** $\frac{\tilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t - t_{n-1})} + \partial_z^{\text{fr}} \mathcal{E}(\tilde{Z}_\tau(t)) \ni 0$

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Step 2: UPPER energy estimate

$$\frac{1}{2} \int_0^t \|z'(s)\|^2 ds + \frac{1}{2} \int_0^t \|\partial_z \operatorname{fr} \mathcal{E}(s)\|^2 ds + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

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Step 2: UPPER energy estimate: holds if $\partial_z^{\text{fr}} \mathcal{E}$ **strongly-weakly closed**

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A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if \mathcal{E} is **coercive**

$$Z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

Step 1: approximate energy inequality: depends on **Euler equation** $\frac{\tilde{Z}_\tau(t) - Z_\tau^{n-1}}{(t - t_{n-1})} + \partial_z^{\text{fr}} \mathcal{E}(\tilde{Z}_\tau(t)) \ni 0$

$$\frac{1}{2} \int_0^t \|\tilde{Z}'_\tau(s)\|^2 ds + \frac{1}{2} \int_0^t \|\partial_z^{\text{fr}} \mathcal{E}(\tilde{Z}_\tau(s))\|^2 ds + \mathcal{E}(\tilde{Z}_\tau(t)) \leq \mathcal{E}(z_0)$$

Step 2: UPPER energy estimate: holds if $\partial_z^{\text{fr}} \mathcal{E}$ **strongly-weakly closed**

$$\frac{1}{2} \int_0^t \|z'(s)\|^2 ds + \frac{1}{2} \int_0^t \|\partial_z^{\text{fr}} \mathcal{E}(s)\|^2 ds + \mathcal{E}(z(t)) \leq \mathcal{E}(z_0)$$

Step 3: LOWER energy estimate

$$\mathcal{E}(z_0) = \mathcal{E}(z(t)) - \int_0^t \langle \partial_z^{\text{fr}} \mathcal{E}(s), z'(s) \rangle ds \leq \frac{1}{2} \int_0^t \|z'(s)\|^2 ds + \frac{1}{2} \int_0^t \|\partial_z^{\text{fr}} \mathcal{E}(s)\|^2 ds + \mathcal{E}(z(t))$$

A more general viewpoint

Proof of existence – revisited

Step 0: incremental minimization: OK if \mathcal{E} is **coercive**

$$Z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{H}} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\}, \quad Z_\tau^0 := z_0$$

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Step 3: LOWER energy estimate:

$$\mathcal{E}(z_0) = \mathcal{E}(z(t)) - \int_0^t \langle \partial_z^{\text{fr}} \mathcal{E}(s), z'(s) \rangle ds \leq \frac{1}{2} \int_0^t \|z'(s)\|^2 ds + \frac{1}{2} \int_0^t \|\partial_z^{\text{fr}} \mathcal{E}(s)\|^2 ds + \mathcal{E}(z(t))$$

holds if $\partial_z^{\text{fr}} \mathcal{E}$ **fulfils chain rule** $\frac{d}{dt} \mathcal{E}(z(t)) = \langle \partial_z^{\text{fr}} \mathcal{E}(z(t)), z'(t) \rangle$

A general subdifferential notion

From now on, **back to Banach** spaces:

$\mathcal{E} : B \rightarrow (-\infty, +\infty]$ such that $z \mapsto \mathcal{E}(z)$ is **coercive**

- We work with $\partial_z \mathcal{E} : B \rightrightarrows B^*$ fulfilling

Euler equation for minimum problems

$$\bar{z} \in \operatorname{Argmin}_{z \in B} \left\{ \frac{1}{2\tau} \|z - z_0\|^2 + \mathcal{E}(z) \right\} \Rightarrow \frac{\bar{z} - z_0}{\tau} + \partial_z \mathcal{E}(\bar{z}) \ni 0$$

viz.

$$\exists \xi \in \partial_z \mathcal{E}(\bar{z}) \text{ s.t. } \frac{\bar{z} - z_0}{\tau} + \xi = 0$$

strong-weak closedness

$$z_n \rightarrow z, \quad \xi_n \rightarrow \xi, \quad \sup_n \mathcal{E}(z_n) < +\infty, \quad \xi_n \in \partial_z \mathcal{E}(z_n) \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad \xi \in \partial_z \mathcal{E}(z)$$

chain rule inequality

$$-\frac{d}{dt} \mathcal{E}(z(t)) \leq -\langle \xi(t), z'(t) \rangle \quad \text{for all } \xi(t) \in \partial_z \mathcal{E}(z(t)).$$

A general subdifferential notion

Back to Banach spaces:

$$\mathcal{E} : B \rightarrow (-\infty, +\infty] \text{ such that } z \mapsto \mathcal{E}(z) \text{ coercive}$$

Conditions on $\partial_z \mathcal{E} : B \rightrightarrows B^*$

Euler equation for minimum problems

strong-weak closedness

chain rule inequality

Examples

$\mathcal{E} \in C^1(B)$	\Rightarrow	$\partial_z \mathcal{E}(z) = D\mathcal{E}(z)$	Fréchet diff.	OK
$\mathcal{E} : B \rightarrow (-\infty, +\infty]$ convex	\Rightarrow	$\partial_z \mathcal{E}(z) = \partial_z^{\text{co}} \mathcal{E}(z)$	cvx subdif.	OK
$\mathcal{E} : B \rightarrow (-\infty, +\infty]$ λ - convex	\Rightarrow	$\partial_z \mathcal{E}(z) = \partial_z^{\text{fr}} \mathcal{E}(z)$	Fréchet subdif.	OK

A general subdifferential notion

Back to Banach spaces:

$$\mathcal{E} : B \rightarrow (-\infty, +\infty] \text{ such that } z \mapsto \mathcal{E}(z) \text{ **coercive**}$$

Conditions on $\partial_z \mathcal{E} : B \rightrightarrows B^*$

Euler equation for minimum problems

strong-weak closedness

chain rule inequality

Examples

♣ If

$$\mathcal{E}(z) := \min_{\eta \in \mathcal{F}} I(\eta, z) \text{ **marginal functional**}$$

then a suitable subdifferential notion is

$$\partial_z \mathcal{E}(z) = \partial_z^{\text{ma}} \mathcal{E}(z) = \{D_z I(t, \eta, z) : \eta \in \text{Argmin} I(t, \cdot, z)\} \text{ **Marginal subdifferential**}$$

General dissipation, smooth time-dependence

Dissipation potential

From $\Psi(v) = \frac{1}{2}\|v\|^2$, we pass to a **general**

$$\Psi : B \rightarrow [0, +\infty) \text{ l.s.c. \& convex, } \Psi(0) = \min_{v \in B} \Psi(v) = 0,$$

$$\lim_{\|v\| \rightarrow \infty} \frac{\Psi(v)}{\|v\|} = +\infty, \quad \lim_{\|\xi\|_* \rightarrow \infty} \frac{\Psi^*(\xi)}{\|\xi\|_*} = +\infty$$

with $\Psi^* : B^* \rightarrow [0, +\infty)$ conjugate functional to Ψ

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Back to a *time-dependent* energy $\mathcal{E} : [0, T] \times B \rightarrow (-\infty, +\infty+]$, such that

$z \mapsto \mathcal{E}(t, z)$ is lower semicontinuous and **coercive**

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Work with $\partial_z \mathcal{E}(t, \cdot) : B \rightrightarrows B^*$ fulfilling

Euler equation for minimum problems
 strong-weak closedness
 chain rule inequality

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Euler equation for minimum problems
strong-weak closedness
chain rule inequality

in the **doubly nonlinear** setting

Step 0

Cauchy Problem:

$$\begin{cases} \partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 & \text{in } B', \quad t \in (0, T), \\ z(0) = z_0 \end{cases}$$

Step 0: time-incremental minimization

Fix time-step $\tau > 0$ and recursively find $Z_\tau^0 := z_0, Z_\tau^1, \dots, Z_\tau^N$:

$$Z_\tau^n \in \operatorname{Argmin}_{z \in B} \left\{ \frac{1}{2\tau} \|z - Z_\tau^{n-1}\|^2 + \mathcal{E}(z) \right\},$$

Step 0

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Fix time-step $\tau > 0$ and recursively find $Z_\tau^0 := z_0, Z_\tau^1, \dots, Z_\tau^N$:

$$Z_\tau^n \in \operatorname{Argmin}_{z \in B} \left\{ \tau\Psi\left(\frac{z - Z_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, z) \right\},$$

♣ OK, because $\mathcal{E}(t, \cdot)$ is **coercive**

Step 0

Cauchy Problem:

$$\begin{cases} \partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 & \text{in } B', \quad t \in (0, T), \\ z(0) = z_0 \end{cases}$$

Step 0: time-incremental minimization

Fix time-step $\tau > 0$ and recursively find $Z_\tau^0 := z_0, Z_\tau^1, \dots, Z_\tau^N$:

$$Z_\tau^n \in \operatorname{Argmin}_{z \in B} \left\{ \tau\Psi\left(\frac{z - Z_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, z) \right\},$$

♣ **OK**, because $\mathcal{E}(t, \cdot)$ is **coercive**

$\partial_z\mathcal{E}$ fulfils **Euler equation for minimum problems** \Rightarrow

$$\partial\Psi\left(\frac{Z_\tau^n - Z_\tau^{n-1}}{\tau}\right) + \partial_z\mathcal{E}(t_n, Z_\tau^n) \ni 0$$

Step 1

Approximate solutions:

$\{\bar{Z}_\tau\}$ piecewise constant interpolant;

$\{\widehat{Z}_\tau\}$ piecewise linear interpolant;

$\{\tilde{Z}_\tau\}$ *variational* interpolant:

$$\tilde{Z}_\tau(t) \in \operatorname{Argmin}_{Z \in B} \left\{ (t - t_{n-1}) \Psi \left(\frac{z - Z_\tau^{n-1}}{t - t_{n-1}} \right) + \mathcal{E}(t, z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

Step 1: approximate energy inequality

$$\int_0^t \Psi(\tilde{Z}'_\tau(s)) \, ds + \int_0^t \Psi^*(-\tilde{\xi}_\tau(s)) \, ds + \mathcal{E}(t, \tilde{Z}_\tau(t)) \leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, \tilde{Z}_\tau(s)) \, ds$$

with for a.a. $s \in (0, t)$:

$$\tilde{\xi}_\tau(s) \in \partial_z \mathcal{E}(s, \tilde{Z}_\tau(s)), \quad \partial \Psi \left(\frac{\tilde{Z}_\tau(s) - Z_\tau^{n-1}}{t - t_{n-1}} \right) + \tilde{\xi}_\tau(s) \ni 0.$$

Step 1

Step 1: approximate energy inequality

$$\int_0^t \Psi(\tilde{Z}'_\tau(s)) ds + \int_0^t \Psi^*(-\tilde{\xi}_\tau(s)) ds + \mathcal{E}(t, \tilde{Z}_\tau(t)) \leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, \tilde{Z}_\tau(s)) ds$$

with for a.a. $s \in (0, t)$:

$$\tilde{\xi}_\tau(s) \in \partial_z \mathcal{E}(s, \tilde{Z}_\tau(s)), \quad \partial \Psi \left(\frac{\tilde{Z}_\tau(s) - Z_\tau^{n-1}}{t - t_{n-1}} \right) + \tilde{\xi}_\tau(s) \ni 0.$$

\Rightarrow **A priori estimates** for $(\bar{Z}_\tau)_\tau$, $(\hat{Z}_\tau)_\tau$, $(\tilde{Z}_\tau)_\tau$, $(\tilde{\xi}_\tau)_\tau$. Since Ψ & Ψ^* have **superlinear growth**, we have compactness:

$$\begin{aligned} \bar{Z}_\tau, \hat{Z}_\tau, \tilde{Z}_\tau &\rightarrow z && \text{in } L^\infty(0, T; B), \\ \tilde{Z}'_\tau &\rightarrow z' && \text{in } L^1(0, T; B), \\ \tilde{\xi}_\tau &\rightarrow \xi && \text{in } L^1(0, T; B^*). \end{aligned}$$

Step 2

Step 2: UPPER energy estimate

Passage to the limit

$$\begin{aligned}
 & \liminf_{\tau \downarrow 0} \int_0^t \Psi(\tilde{Z}'_\tau(s)) \, ds + \liminf_{\tau \downarrow 0} \int_0^t \Psi^*(-\tilde{\xi}_\tau(s)) \, ds + \liminf_{\tau \downarrow 0} \mathcal{E}(t, \tilde{Z}_\tau(t)) \\
 & \leq \mathcal{E}(0, z_0) + \limsup_{\tau \downarrow 0} \int_0^t \partial_t \mathcal{E}(s, \tilde{Z}_\tau(s)) \, ds \\
 & \quad \downarrow \text{(via LOWER SEMICONTINUITY)} \\
 & \int_0^t \Psi(z'(s)) \, ds + \int_0^t \Psi^*(-\xi(s)) \, ds + \mathcal{E}(t, z(t)) \leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, z(s)) \, ds
 \end{aligned}$$

Step 2

Step 2: UPPER energy estimate

Passage to the limit

$$\begin{aligned} \liminf_{\tau \downarrow 0} \int_0^t \Psi(\tilde{Z}'_\tau(s)) \, ds + \liminf_{\tau \downarrow 0} \int_0^t \Psi^*(-\tilde{\xi}_\tau(s)) \, ds + \liminf_{\tau \downarrow 0} \mathcal{E}(t, \tilde{Z}_\tau(t)) \\ \leq \mathcal{E}(0, z_0) + \limsup_{\tau \downarrow 0} \int_0^t \partial_t \mathcal{E}(s, \tilde{Z}_\tau(s)) \, ds \end{aligned}$$

↓ (via **LOWER SEMICONTINUITY**)

$$\int_0^t \Psi(z'(s)) \, ds + \int_0^t \Psi^*(-\xi(s)) \, ds + \mathcal{E}(t, z(t)) \leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, z(s)) \, ds$$

with $\xi(s) \in \partial_z \mathcal{E}(s, z(s))$ for a.a. $s \in (0, t)$ **IF** $(\partial_z \mathcal{E}, \partial_t \mathcal{E})$ fulfils the **strong-weak closedness** property

$$z_n \rightarrow z, \quad \xi_n \rightarrow \xi, \quad \sup_n \mathcal{E}(t_n, z_n) < +\infty, \quad \xi_n \in \partial_z \mathcal{E}(t, z_n) \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad \xi \in \partial_z \mathcal{E}(t, z), \quad \limsup_{n \rightarrow \infty} \partial_t \mathcal{E}(t, z_n) \leq \partial_t \mathcal{E}(t, z)$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z \mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0, T; B), & \int_0^T \Psi(z'(t)) dt < +\infty, \\ \xi \in L^1(0, T; B^*), & \int_0^T \Psi^*(\xi(t)) dt < +\infty, \\ \xi(t) \in \partial_z \mathcal{E}(t, z(t)) \text{ a.e. in } (0, T) \end{cases}$$
$$\Rightarrow -\frac{d}{dt} \mathcal{E}(t, z(t)) + \partial_t \mathcal{E}(t, z(t)) \leq -\langle \xi(t), z'(t) \rangle \text{ a.e. in } (0, T)$$

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then

$$\begin{aligned} \int_0^t \Psi(z'(s)) ds &+ \int_0^t \Psi^*(-\xi(s)) ds + \mathcal{E}(t, z(t)) \\ &\leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, z(s)) ds \\ &\leq \mathcal{E}(t, z(t)) - \int_0^t \langle \xi(s), z'(s) \rangle ds \\ &\leq \int_0^t \Psi(z'(s)) ds + \int_0^t \Psi^*(-\xi(s)) ds + \mathcal{E}(t, z(t)) \end{aligned}$$

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Hence,

$$\int_0^t \Psi(z'(s)) ds + \int_0^t \Psi^*(-\xi(s)) ds - \int_0^t \langle -\xi(s), z'(s) \rangle ds \leq 0$$

Step 3

Step 3: LOWER energy estimate and conclusion

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Step 3

Step 3: LOWER energy estimate and conclusion

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$$\Rightarrow -\frac{d}{dt} \mathcal{E}(t, z(t)) + \partial_t \mathcal{E}(t, z(t)) \leq -\langle \xi(t), z'(t) \rangle \text{ a.e. in } (0, T)$$

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$$\begin{aligned} \int_0^t \Psi(z'(s)) ds &+ \int_0^t \Psi^*(-\xi(s)) ds + \mathcal{E}(t, z(t)) \\ &\leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, z(s)) ds \\ &\leq \mathcal{E}(t, z(t)) - \int_0^t \langle \xi(s), z'(s) \rangle ds \\ &\leq \int_0^t \Psi(z'(s)) ds + \int_0^t \Psi^*(-\xi(s)) ds + \mathcal{E}(t, z(t)) \end{aligned}$$

Hence,

$$\Psi(z'(t)) + \Psi^*(-\xi(t)) - \langle -\xi(t), z'(t) \rangle = 0 \quad \text{for a.a. } t \in (0, T).$$

Step 3

Step 3: LOWER energy estimate and conclusion

IF $\partial_z \mathcal{E}$ fulfils the **chain rule inequality**

$$\begin{cases} z \in W^{1,1}(0, T; B), & \int_0^T \Psi(z'(t)) dt < +\infty, \\ \xi \in L^1(0, T; B^*), & \int_0^T \Psi^*(\xi(t)) dt < +\infty, \\ \xi(t) \in \partial_z \mathcal{E}(t, z(t)) \text{ a.e. in } (0, T) \end{cases}$$

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then

$$\begin{aligned} \int_0^t \Psi(z'(s)) ds &+ \int_0^t \Psi^*(-\xi(s)) ds + \mathcal{E}(t, z(t)) \\ &\leq \mathcal{E}(0, z_0) + \int_0^t \partial_t \mathcal{E}(s, z(s)) ds \\ &\leq \mathcal{E}(t, z(t)) - \int_0^t \langle \xi(s), z'(s) \rangle ds \\ &\leq \int_0^t \Psi(z'(s)) ds + \int_0^t \Psi^*(-\xi(s)) ds + \mathcal{E}(t, z(t)) \end{aligned}$$

Hence,

$$-\xi(t) \in \partial \Psi(z'(t)), \quad \xi(t) \in \partial_z \mathcal{E}(t, z(t))$$

i.e. z is a solution of the Cauchy problem.

An existence and approximation result (II)

Theorem 2 [Mielke, R., Savaré2011]

Assume

- ▶ Ψ convex with **superlinear growth**,
- ▶ $\mathcal{E}(\cdot, z)$ **differentiable**
- ▶ $\mathcal{E}(t, \cdot)$ **coercive**,

and suppose that $\partial_z \mathcal{E}(t, \cdot)$ satisfies

Euler equation for minimum problems
strong-weak closedness
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} in the doubly nonlinear setting

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} in the doubly nonlinear setting

Then, the approximate solutions converge to a function $z \in H^1(0, T; \mathcal{H})$ which **solves** the Cauchy problem for (DNE), and fulfils the **energy identity** for all $0 \leq s \leq t \leq T$

$$\int_s^t \Psi(z'(r)) \, dr + \int_s^t \Psi^*(-\xi(r)) \, dr + \mathcal{E}(t, z(t)) = \mathcal{E}(s, z(s)) + \int_s^t \partial_t \mathcal{E}(r, z(r)) \, dr.$$

Nonsmooth time-dependence: a surrogate for $\partial_t \mathcal{E}$

♠ And what, if $t \mapsto \mathcal{E}(t, z)$ is **no longer smooth**? How to replace $\partial_t \mathcal{E}(t, z)$?

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- ♣ Use a **generalized** partial time derivative $\boxed{P_t \mathcal{E}}$, which must satisfy the **chain rule**
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- ◇ For the existence theorem, impose that $P_t \mathcal{E}$ complies with
 - strong-weak closedness**
 - chain rule inequality**

A material model with finite-strain elasticity

Variables

Elastic body in a bded domain $\Omega \subset \mathbb{R}^d$, with

- ▶ $\phi : \Omega \rightarrow \mathbb{R}^d$ elastic deformation field
- ▶ $z : \Omega \rightarrow \mathbb{R}^m$ internal variable: a *mesoscopic averaged phase variable*, e.g.
.....

The energy functional $I(t, \phi, z) = \mathcal{E}_1(z) + I_2(t, \phi, z)$

$$\mathcal{E}_1(z) = \int_{\Omega} \frac{1}{q} |\nabla z|^q + I_K(z) \quad q > d, K \text{ convex set,}$$

$$I_2(t, \phi, z) = \int_{\Omega} W(\nabla \phi(x), z(x)) - \langle \ell(t), \phi \rangle_{W^{1,p}}$$

$W(\cdot, z)$ polycvx, p -coerc., ℓ smooth ext. loading

The dissipation potential

$$\Psi(v) = \int_{\Omega} \psi(v(x)) \, dx \quad \psi : \mathbb{R} \rightarrow \mathbb{R} \text{ cvx, with superlinear growth}$$

The PDE system

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$$I(t, \phi, z) = \mathcal{E}_1(z) + I_2(t, \phi, z) \quad \text{with} \quad \begin{aligned} \mathcal{E}_1(z) &= \int_{\Omega} \frac{1}{q} |\nabla z|^q + I_{\mathbb{K}}(z) \\ I_2(t, \phi, z) &= \int_{\Omega} W(\nabla \phi(x), z(x)) - \langle \ell(t), \phi \rangle_{W^{1,p}} \end{aligned}$$

♣ Consider the PDE system with unknowns (z, ϕ)

$$\partial_t \psi(\dot{z}) - \Delta_q z + \partial I_{\mathbb{K}}(z) + D_z W(\nabla \phi, z) \ni 0 \quad \text{a.e. in } (0, T) \times \Omega,$$

$$I(t, \phi, z) \leq I(t, \eta, z) \quad \text{for all } \eta \in \underbrace{\{\phi \in W^{1,p}(\Omega; \mathbb{R}^d) : \phi = \phi_{\text{Dir}} \text{ on } \Gamma_{\text{Dir}}\}}_{\doteq \mathcal{F}}$$

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An abstract approach

Consider the **marginal** energy

$$\mathcal{E}(t, z) := \min_{\phi \in \mathcal{F}} I(t, \phi, z) = \mathcal{E}_1(z) + \min_{\phi \in \mathcal{F}} I_2(t, \phi, z) \doteq \mathcal{E}_1(z) + \mathcal{E}_2(t, z)$$

and the doubly nonlinear equation (in $B = L^2(\Omega; \mathbb{R}^m)$)

$$\partial \Psi(z'(t)) + \partial_z \mathcal{E}(t, z(t)) \ni 0 \quad \text{a.e. in } (0, T)$$

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$$\begin{aligned} \partial \Psi(z'(t)) + \partial_z \mathcal{E}(t, z(t)) \ni 0 \quad \text{a.e. in } (0, T) \\ \text{with } \partial_z \mathcal{E}(t, z) = \partial_z^{\text{co}} \mathcal{E}_1(z) + \partial_z^{\text{ma}} \mathcal{E}_2(t, z) \\ \partial_z^{\text{ma}} \mathcal{E}_2(t, z) = \{D_z I_2(t, \phi, z) : \phi \in \text{Argmin} I_2(t, \cdot, z)\} \end{aligned} \quad (\text{DNE})$$

Solutions to (DNE) solve the PDE system!

An abstract approach

$$\begin{aligned}
 \partial\Psi(z'(t)) + \partial_z\mathcal{E}(t, z(t)) \ni 0 \quad \text{a.e. in } (0, T) \\
 \partial_z\mathcal{E}(t, z) = \partial_z^{\text{co}}\mathcal{E}_1(z) + \partial_z^{\text{ma}}\mathcal{E}_2(t, z) \\
 \partial_z^{\text{ma}}\mathcal{E}_2(t, z) = \{D_z l_2(t, \phi, z) : \phi \in \text{Argmin} l_2(t, \cdot, z)\}
 \end{aligned}
 \tag{DNE}$$

♣ The general existence theorem applies to (DNE):

- ▶ $\partial_z\mathcal{E}$ complies with

Euler equation for minimum problems
strong-weak closedness
chain rule inequality

- ▶ use as surrogate for time derivative $\partial_t\mathcal{E}$:

$$\begin{aligned}
 P_t(t, z, \xi) := \sup \{ \partial_t l_2(t, \phi, z) : \phi \in \text{Argmin} l_2(t, \cdot, z), \xi = D_z l_2(t, \phi, z) \} \\
 \text{for } \xi \in \partial_z^{\text{ma}}\mathcal{E}_2(t, z).
 \end{aligned}$$