Well-posedness results for two classes of generalized viscous Cahn-Hilliard equations

Dissipative models in phase transitions

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The equation

We consider the following fourth-order nonlinear parabolic equation

$$\chi_t - \Delta(\alpha(\delta\chi_t - \Delta\chi + \chi^3 - \chi)) = 0 \quad \text{in } \Omega \times (0, T),$$

- $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, a bounded smooth domain, with boundary Γ, $(0, T)$ a time interval,
- $\alpha : D(\alpha) \subset \mathbb{R} \to \mathbb{R}$ is strictly increasing and differentiable,
- $\delta \geq 0$ a constant.

Links with the Cahn-Hilliard equations

If α is linear,

$$\alpha(r) := \kappa r \qquad \forall r \in \mathbb{R},$$

(where $\kappa > 0$ is the **mobility**), then

$$\chi_t - \Delta(\alpha(\delta\chi_t - \Delta\chi + \chi^3 - \chi) = 0 \quad \text{in } \Omega \times (0,T),$$

reduces with $\delta = 1$ to the standard viscous Cahn-Hilliard equation

$$\chi_t - \kappa\Delta(\chi_t - \Delta\chi + \chi^3 - \chi) = 0 \quad \text{in } \Omega \times (0,T),$$

and with $\delta = 0$ to the standard Cahn-Hilliard equation

$$\chi_t - \kappa\Delta(-\Delta\chi + \chi^3 - \chi) = 0 \quad \text{in } \Omega \times (0,T).$$

Derivation of the Cahn-Hilliard equation

- Given the total free energy functional $\mathcal{F}$

$$\mathcal{F}(\chi) := \int_\Omega \left(\frac{1}{2} |\nabla \chi|^2 + \frac{(\chi^2 - 1)^2}{4} \right)$$

($=$ interfacial energy $+$ free energy density), define the chemical potential w as the first variation of $\mathcal{F}$

$$w = -\Delta \chi + \mathcal{W}'(\chi).$$

- Couple the mass balance law

$$\partial_t \chi + \operatorname{div} \mathbf{h} = 0,$$

with the *constitutive law* for the mass flux $\mathbf{h}$

$$\mathbf{h} = -\kappa \mathbf{I} \, \nabla w, \quad (\mathbf{I} \text{ is the } N \times N \text{ identity matrix}).$$

The Cahn-Hilliard and the viscous Cahn-Hilliard equations

- Supplement $\chi_t - \kappa\Delta(-\Delta\chi + \chi^3 - \chi) = 0$ in $\Omega \times (0,T)$ with $\chi(\cdot,0) = \chi_0$ and the no-flux boundary conditions

$$\partial_n \chi = 0, \quad -\mathbf{h}\cdot\mathbf{n} = \partial_n w = 0 \qquad \text{on } \Gamma \times (0,T).$$

Then, χ is **a conserved parameter**, i.e.,

$$m(\chi(t)) := \frac{1}{|\Omega|}\int_\Omega \chi(x,t)\,dx = m(\chi_0) \quad \forall t \in [0,T].$$

- [**Novick-Cohen '88**] proposed the viscous Cahn-Hilliard equation

$$\chi_t - \kappa\Delta(\chi_t - \Delta\chi + \chi^3 - \chi) = 0 \quad \text{in } \Omega \times (0,T)$$

to account for viscosity effects in the phase separation in polymers.

- Wide literature on **well-posedness** (for various variants of the model), **long-time behaviour**, dynamics of pattern formation..

Gurtin's generalized viscous Cahn-Hilliard equation

- **M.E. Gurtin** [PHYS. D '96] proposed a novel derivation of the Cahn-Hilliard equations, thus obtaining the

generalized viscous Cahn-Hilliard equation

$$\chi_t - \mathrm{div}(\mathbf{M}(Z)\nabla(\delta\chi_t - \Delta\chi + \chi^3 - \chi)) = 0 \qquad \text{(GVCHE)}$$

where the mobility tensor $\mathbf{M}(Z)$ is symmetric positive definite, and

constitutive variables: $Z{=}(\chi,\ \nabla\chi, \chi_t, w, \nabla w)$!

- Equation rigorously derived for the chemical potential:

$$w = \delta\chi_t - \Delta\chi + \chi^3 - \chi.$$

- **Several results [Miranville]:** well-posedness and long-time behaviour with periodic and Neumann b.c., and $\mathbf{M}(Z) = \mathbf{M},\ \mathbf{M}(\chi)$.

Back to the equation

$$\chi_t - \Delta(\alpha(\chi_t - \Delta\chi + \chi^3 - \chi)) = 0 \quad \text{in } \Omega \times (0,T),$$

(we have set $\delta = 1$) is a particular case of (GVCHE):

$$\chi_t - \mathrm{div}(\mathbf{M}(Z)\nabla(\chi_t - \Delta\chi + \chi^3 - \chi)) = 0 \quad \text{in } \Omega \times (0,T)$$
$$\mathbf{M}(Z) = \mathbf{M}(w) := \alpha'(w)\mathbf{I},$$

(admissible, $\alpha' > 0!$), i.e., a **chemical potential**-dependent mobility tensor!!!!

- Several well-posedness for a **concentration**-dependent mobility tensor in the viscous and non viscous cases: **[Cahn, Elliott, Novick-Cohen, Garcke, Bonetti, Colli, Gilardi, Dreyer, Schimperna, Sprekels]**.
- We prove well-posedness results for two different I.B.V. corresponding to two choices of the **mobility law** α.

Back to the equation

- Insert a **"source term"** $f : \Omega \times (0,T) \to \mathbb{R}$ in the equation:

$$\chi_t - \Delta(\alpha(\chi_t - \Delta\chi + \chi^3 - \chi)) = f \quad \text{in } \Omega \times (0,T).$$

- Introduce

$$\rho := \alpha^{-1} \quad \text{and the variable} \quad u := \alpha(w).$$

- Rephrase the (GVCHE) with **source term** f as

$$\chi_t - \Delta u = f \quad \text{in } \Omega \times (0,T),$$
$$\chi_t - \Delta\chi + \chi^3 - \chi = \rho(u) \quad \text{in } \Omega \times (0,T),$$

supplemented with an initial condition on χ and **suitable boundary conditions** on χ and u, **depending on the properties of** ρ.

Idea of the proof of existence

We introduce the **approximate** system of **phase field** type

$$\nu u_t + \chi_t - \Delta u = f \quad \text{in } \Omega \times (0,T),$$
$$\chi_t - \Delta\chi + \chi^3 - \chi = \rho(u) \quad \text{in } \Omega \times (0,T),$$
$$+\text{initial and boundary conditions on } u \text{ and } \chi$$

Main steps:

- well-posedness for the approximate system;
- a priori estimates on the solutions $\{(u_\nu, \chi_\nu)\}_\nu$ **independent** of ν;
- extraction of subsequences converging as $\nu \downarrow 0$ to the unique solution to the i.b.v. problem.

A bilipschitz mobility law

Suppose that $\alpha = \alpha_{\mathbf{1}}$, with

$\alpha_{\mathbf{1}} : \mathbb{R} \longrightarrow \mathbb{R}$ strictly increasing, differentiable, and **bi-Lipschitz**.

$$\exists m_1, M_1 > 0 \quad \text{s.t.} \quad m_1 \leq {\alpha_{\mathbf{1}}}'(r) \leq M_1 \quad \forall r \in \mathbb{R}.$$

Then, $\rho_{\mathbf{1}}$ is strictly incr., diff., **bi-Lipschitz**; $\widehat{\rho_{\mathbf{1}}}$: (convex) primitive.

- We supplement

$$\chi_t - \Delta u = f \quad \text{in } \Omega \times (0, T),$$

$$\chi_t - \Delta \chi + \chi^3 - \chi = \rho_{\mathbf{1}}(u) \quad \text{in } \Omega \times (0, T),$$

with the initial condition $\chi(\cdot, 0) = \chi_0$ and the **Neumann** b.c.

$$\partial_n \chi = 0 \quad \text{and} \quad \partial_{\mathbf{n}} \mathbf{u} = \mathbf{g} \quad \text{in } \Gamma \times (0, T),$$

for some $g : \Gamma \times (0, T) \to \mathbb{R}$.

A bilipschitz mobility law

- If f and g fulfil the **compatibility condition**

$$\int_\Omega f(x,t)dx + \int_\Gamma g(s,t)ds = 0, \qquad t \in (0,T),$$

then any solution χ of

$$\chi_t - \Delta u = f \quad \text{in } \Omega \times (0,T),$$

$$\chi_t - \Delta\chi + \chi^3 - \chi = \rho_{\mathbf{1}}(u) \quad \text{in } \Omega \times (0,T),$$

$$\chi(0) = \chi_0 \quad \text{on } \Omega, \quad \partial_n \chi = 0, \quad \partial_{\mathbf{n}}\mathbf{u} = \mathbf{g} \quad \text{in } \Gamma \times (0,T),$$

enjoys the **physically significant** **conservation property**

$$m(\chi(t)) = m(\chi_0) \quad \forall t \in [0,T]$$

(multiply the first equation by 1).

Problem formulation

- Consider the spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \left\{ v \in H^2(\Omega) \ : \ \partial_n v = 0 \right\},$$

with the dense and compact embeddings $W \subset V \subset H \cong H' \subset V' \subset W'$.

- Consider the realization of the Laplace operator with homog. N.B.C.,

$$A : V \to V' \qquad \langle Au, v \rangle := \int_\Omega \nabla u \nabla v \, dx \quad \forall u, v \in V.$$

- The inverse operator A^{-1} is defined for the elements $v \in V'$ of *zero* mean value $m(v)$. Take on V and V' the equivalent norms:

$$\|u\|_V^2 := \langle Au, u \rangle + (u, m(u)) \quad \forall u \in V$$
$$\|v\|_{V'}^2 := \langle v, A^{-1}(v - m(v)) \rangle + (v, m(v)) \quad \forall v \in V'.$$

A first well-posedness result

Problem $\mathbf{P_1}$. Given the data $\chi_0 \in V$, $f \in L^2(0,T;V')$, $g \in L^2(0,T;H^{-1/2}(\Gamma))$, define $F \in L^2(0,T;V')$ by

$$\langle F(t), v\rangle := \langle f(t), v\rangle + \langle g(t), v\rangle_\Gamma \quad \forall v \in V, \text{ for a.e. } t \in (0,T).$$

Find $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$, $u \in L^2(0,T;V)$ s. t. $\chi(0) = \chi_0$ and

$$\partial_t \chi + Au = F \quad \text{in } V' \text{ for a.e. } t \in (0,T),$$
$$\partial_t \chi + A\chi + \chi^3 - \chi = \rho_1(u) \quad \text{in } H \text{ for a.e. } t \in (0,T).$$

- **Neumann boundary conditions** for χ and u in the **formulation**!

Thm. **[R., CPAA '04]** Problem $\mathbf{P_1}$ has a **unique solution.**

Proof of existence: approximation

Existence for the approximate problem For every $\nu > 0$ and data $\chi_{0\nu} \in V$ and $u_{0\nu} \in H$ $\forall \nu > 0$, $\exists!(\chi_\nu,\ u_\nu)$ with $u_\nu \in L^2(0,T;V) \cap H^1(0,T;V')$, $\chi_\nu \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;H)$ and

$$\nu \partial_t u_\nu + \partial_t \chi_\nu + A u_\nu = F \quad \text{in } V' \text{ for a.e. } t \in (0,T), \qquad (1_\nu)$$

$$\partial_t \chi_\nu + A\chi_\nu + \chi_\nu^3 - \chi_\nu = \rho_{\mathbf{1}}(u_\nu) \quad \text{in } H \text{ for a.e. } t \in (0,T), \qquad (2_\nu)$$

$$u_\nu(\cdot,0) = u_{0\nu}, \quad \chi_\nu(\cdot,0) = \chi_{0\nu}.$$

Idea of the proof:

- examine (1_ν) and (2_ν) separately: existence and continuous dependence on the data for each;
- construction of a solution operator T_ν for the system;
- fixed point argument for $T_\nu \Rightarrow \exists!(\chi_\nu,\ u_\nu)$.

Proof of existence: a priori estimates

Suppose $\chi_{0\nu} \rightharpoonup \chi_0$ in V, and $\nu^{1/2}|u_{0\nu}|_H \to 0$ as $\nu \downarrow 0$:

$\longrightarrow$ find estimates on $\{(\chi_\nu, u_\nu)\}_\nu \ldots$

- **First a priori estimate:** test

$$\int_0^t \left(\nu \partial_t u_\nu + \partial_t \chi_\nu + \boxed{A u_\nu} = F \qquad \times \boldsymbol{\rho_1}(u_\nu)\right),$$

$$\int_0^t \left(\partial_t \chi_\nu + A\chi_\nu + \chi_\nu^3 - \chi_\nu = \boldsymbol{\rho_1}(u_\nu) \qquad \times \partial_t \chi_\nu\right),$$

whence $\nu \int_\Omega \widehat{\boldsymbol{\rho_1}}(u_\nu) + \boxed{\int_0^t (\nabla(u_\nu), \nabla(\boldsymbol{\rho_1}(u_\nu)))_H} + \int_0^t |\partial_t \chi_\nu|_H^2$

$$+ \frac{1}{2}|\nabla(\chi_\nu(t))|_H^2 + \int_\Omega \left(\frac{1}{4}\chi_\nu^4 - \frac{1}{2}\chi_\nu^2\right) = (\text{initial data}) + \int_0^t \langle F, \boldsymbol{\rho_1}(u_\nu)\rangle.$$

- **Estimates for** χ_ν: $\|\chi_\nu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C$
- **Crucial:** by $\boxed{{\alpha_{\mathbf{1}}}' \geq m_1 > 0}$ ($\rho_{\mathbf{1}}$ is Lipschitz),

$$\boxed{\int_0^t (\nabla(u_\nu), \nabla(\rho_{\mathbf{1}}(u_\nu)))_H} \geq m_1 \int_0^t |\nabla(\rho_{\mathbf{1}}(u_\nu))|_H^2$$

$\Longrightarrow \|\rho_{\mathbf{1}}(u_\nu) - m(\rho_{\mathbf{1}}(u_\nu))\|_{L^2(0,T;V)} \leq C$;

$m(\rho_{\mathbf{1}}(u_\nu))$ is estimated from (2_ν);

$\Longrightarrow$ estimate for $\rho_{\mathbf{1}}(u_\nu)$ in $L^2(0,T;V)$!!

- **Estimate for** u_ν (WLOG, $\rho_{\mathbf{1}}(0) = 0$): $\boxed{\text{by } {\alpha_{\mathbf{1}}}' \leq M_1,}$

$$|m(u_\nu)| \leq \frac{1}{|\Omega|} \int_\Omega |\alpha_{\mathbf{1}}(\rho_{\mathbf{1}}(u_\nu))| \leq M_1 m(|\rho_{\mathbf{1}}(u_\nu)|),$$

Second estimate: $$\int_0^t \left(\nu \partial_t u_\nu + \partial_t \chi_\nu + \boxed{A u_\nu} = F \qquad \times u_\nu \right),$$

$$\Longrightarrow \nu^{1/2} \|u_\nu\|_{L^\infty(0,T;H)} + \|u_\nu\|_{L^2(0,T;V)} + \nu \|\partial_t u_\nu\|_{L^2(0,T;V')} \leq C.$$

Proof of existence: passage to the limit

Despite the **lack of coercicity** in u_ν due to the b.c.,
by the **Lipschitz continuity** of α_1 we estimate u_ν in terms of $\rho_1(u_\nu)$!!

- By the priori estimates and weak compactness results $\exists(u,\chi)$:

$$\chi_\nu \rightharpoonup^* \chi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$$

$$\chi_\nu \to \chi \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V),$$

$$u_\nu \rightharpoonup u \quad \text{in } L^2(0,T;V), \quad \nu u_\nu \to 0 \quad \text{in } L^\infty(0,T;H)$$

$$\nu u_\nu \rightharpoonup 0 \quad \text{in } H^1(0,T;V').$$

- Passage to the limit (in the nonlinear term $\rho_1(u_\nu)$) by monotonicity.

The pair (u,χ) **solves** (the variational formulation) of **Problem P_1!**

Proof of uniqueness

- Let χ_1 (u_1) and χ_2 (u_2) be two solutions to Problem $\mathbf{P_1}$, set

$$\boxed{\underline{\chi}:=\chi_1 - \chi_2 \quad \underline{u} := u_1 - u_2.}$$

- **Aim:** proving that $\underline{\chi} = 0$, $\underline{u} = 0$.

- **Technique:** deduce Gronwall estimates from the equations for $(\underline{\chi}, \underline{u})$:

$$\partial_t \underline{\chi} + A\underline{u} = 0 \quad \text{in } V' \quad \text{a.e. in } (0,T),$$

$$\partial_t \underline{\chi} + A\underline{\chi} + \chi_1^3 - \chi_2^3 - \underline{\chi} = \rho_{\mathbf{1}}(u_1) - \rho_{\mathbf{1}}(u_2) \quad \text{in } V' \quad \text{a.e. in } (0,T),$$

Main estimate (**same technique for continuous dependence!!!**):

$$\int_0^t \Big(\partial_t \underline{\chi} + A\underline{u} = 0 \quad \times A^{-1}\left(\rho_{\mathbf{1}}(u_1) - \rho_{\mathbf{1}}(u_2) - m(\rho_{\mathbf{1}}(u_1) - \rho_{\mathbf{1}}(u_2))\right)\Big)$$

$$\int_0^t \Big(\partial_t \underline{\chi} + A\underline{\chi} + \chi_1^3 - \chi_2^3 - \underline{\chi} = \rho_{\mathbf{1}}(u_1) - \rho_{\mathbf{1}}(u_2) \quad \times A^{-1}(\partial_t \underline{\chi}(t))\Big)$$

- **Technical:** interplay $A/\ A^{-1}$ to deal with coupling terms;
- **Substantial:** use the bi-Lipschitzianity of $\alpha_{\mathbf{1}}$ to estimate $\underline{u}$:

$$\Longrightarrow \quad m_1 \int_0^t |\rho_{\mathbf{1}}(u_1) - \rho_{\mathbf{1}}(u_2)|_H^2 + \int_0^t \|\partial_t \underline{\chi}\|_{V'}^2 + |\underline{\chi}(t)|_H^2 \leq$$

$$C\left(\int_0^t |\rho_{\mathbf{1}}(u_1) - \rho_{\mathbf{1}}(u_2)|_H^2 + \int_0^t |\underline{\chi}|_H^2\right)$$

$$\Longrightarrow \quad \underline{\chi} = 0, \quad \underline{u} = 0.$$

A singular mobility law

Suppose that $\alpha = \alpha_{\mathbf{2}}$, with

$\alpha_{\mathbf{2}} : (a, +\infty) \to \mathbb{R}$ strictly increasing, differentiable, $\lim_{r \downarrow a} \alpha_{\mathbf{2}}(r) = -\infty$,

$$\exists m_2 > 0 \quad \text{s.t.} \quad m_2 \leq {\alpha_{\mathbf{2}}}'(r) \quad \forall r > a.$$

Then, $\rho_{\mathbf{2}}$ is strictly incr., diff., **Lipschitz;** but **not bi-Lipschitz!!**

$$\chi_t - \Delta u = f \quad \text{in } \Omega \times (0, T),$$
$$\chi_t - \Delta \chi + \chi^3 - \chi = \rho_{\mathbf{2}}(u) \quad \text{in } \Omega \times (0, T),$$

supplemented with the initial cond. $\chi(\cdot, 0) = \chi_0$ and the **Robin** b.c.

$$\partial_n \chi = 0 \quad \text{and} \quad \partial_n u = -\omega u + h \quad \text{in } \Gamma \times (0, T),$$

for some $\omega > 0$ and $h : \Gamma \times (0, T) \to \mathbb{R}$.

A singular mobility law

• To fix ideas, think of $\boxed{\alpha_2(r) := r - \frac{1}{r},\ r > 0}$: (GVCHE) is the **formal limit** as $\varepsilon \downarrow 0$ of

$$\varepsilon\vartheta_t + \chi_t - \Delta(\vartheta - \frac{1}{\vartheta}) = f,$$
$$\chi_t - \Delta\chi + \chi^3 - \chi = -\frac{1}{\vartheta},$$

i.e., the **Penrose-Fife system with special heat flux law**!

• Third type boundary conditions on u are **unusual for Cahn-Hilliard** equations (no more conservation property for χ, different phys. interpretation), but here **compensate for the singularity** of α_2 by providing **coercivity** for u.

Problem formulation

• Introduce the operator $J : V \to V'$

$$\langle Ju, v \rangle := \int_\Omega \nabla u \cdot \nabla v + \omega \int_\Gamma uv \quad \forall u, v \in V.$$

By Poincaré's inequality, the operator J is **coercive on V**

estimate for ${}_{V'}\langle Jv, v\rangle_V \Rightarrow$ estimate for $\|v\|_V$!

• A is **coercive only** on the subspace of V of elements of **zero mean value**:

estimate for ${}_{V'}\langle Av, v\rangle_V \not\Rightarrow$ estimate for $\|v\|_V$,

need for an estimate of $m(v)$, too!!

A second well-posedness result

Problem $\mathbf{P_2}$. Given the data $\chi_0 \in V$, $f \in L^2(0,T;V')$, $h \in L^2(0,T;H^{-1/2}(\Gamma))$, define $G \in L^2(0,T;V')$ by

$$\langle G(t), v\rangle := \langle f(t), v\rangle + \langle h(t), v\rangle_\Gamma \quad \forall v \in V, \text{ for a.e. } t \in (0,T).$$

Find $\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)$, $u \in L^2(0,T;V)$ s. t. $\chi(0) = \chi_0$ and

$$\partial_t \chi + Ju = G \quad \text{in } V' \text{ for a.e. } t \in (0,T),$$
$$\partial_t \chi + A\chi + \chi^3 - \chi = \rho_{\mathbf{2}}(u) \quad \text{in } H \text{ for a.e. } t \in (0,T).$$

• Third type boundary conditions for u in the formulation!

Thm. **[R., CPAA '04]** Problem $\mathbf{P_2}$ has a **unique solution**.

Proof of existence

Well-posedness for the approximate problem Given the approximate initial data $\{(\chi_{0\mu}, u_{0\mu})\} \in V \times H$, we construct (same fixed point technique!!) (χ_μ, u_μ) fulfilling $u_\mu(\cdot, 0) = u_{0\mu}$, $\chi_\mu(\cdot, 0) = \chi_{0\mu}$,

$$\mu \partial_t u_\mu + \partial_t \chi_\mu + \boxed{\ J u_\mu} = G \quad \text{in } V' \text{ for a.e. } t \in (0,T), \qquad (1_\mu)$$

$$\partial_t \chi_\mu + A\chi_\mu + \chi_\mu^3 - \chi_\mu = \rho_{\mathbf{2}}(u_\mu) \quad \text{in } H \text{ for a.e. } t \in (0,T). \qquad (2_\mu)$$

A priori estimates:

$$\int_0^t ((1_\mu) \times [u_\mu + \rho_{\mathbf{2}}(u_\mu)])$$

$$\int_0^t ((2_\mu) \times \partial_t \chi_\mu) \Rightarrow \|\chi_\mu\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \le C$$

Proof of existence

- **Note** that

$$\int_0^t {}_{V'}\langle J u_\mu, u_\mu \rangle_V \le C \quad \Rightarrow \quad \|u_\mu\|_{L^2(0,T;V)} \le C,$$

$$\int_0^t {}_{V'}\langle J u_\mu, \boldsymbol{\rho_2}(u_\mu) \rangle_V \le C \quad \Rightarrow \quad \|\boldsymbol{\rho_2}(u_\mu)\|_{L^2(0,T;V)} \le C \quad (\boldsymbol{\rho_2} \text{ lipschitz!})$$

- No need to estimate $m(u_\mu)$ by $m(\boldsymbol{\rho_2}(u_\mu))$ (which is no more possible, as $\alpha_{\mathbf{2}}$ is **not** Lipschitz!)

- **Conclusion of the proof:** extraction of converging subsequences; passage to the limit by monotonicity arguments.

Uniqueness and regularity

• Analogous estimates as in the bi-lipschitz case yield **continuous dependence on the data** of the solutions!

• **A regularity result:** [**R., CPAA '04**] **Further regularity** for

$$f \in H^1(0,T;V'), \quad h \in H^1(0,T;H^{-1/2}(\Gamma)), \quad \chi_0 \in H^3(\Omega) \cap W.$$

Then, the unique solution χ to Problem $\mathbf{P_2}$ fulfils

$$\boxed{\chi \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W) \cap L^\infty(0,T;H^3(\Omega)).}$$

Idea of the proof: prove further regularity for the approximate solutions, then pass to the limit.

• Here, third type boundary conditions are **crucial** in the estimates (argument not adaptable to Neumann boundary conditions!)