SINGULAR VANISHING-VISCOSITY LIMITS OF GRADIENT FLOWS: THE FINITE-DIMENSIONAL CASE

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Abstract. In this note we study the singular vanishing-viscosity limit of a gradient flow set in a finite-dimensional Hilbert space and driven by a smooth, but possibly *nonconvex*, time-dependent energy functional. We resort to ideas and techniques from the variational approach to gradient flows and rate-independent evolution to show that, under suitable assumptions, the solutions to the singularly perturbed problem converge to a curve of stationary points of the energy, whose behavior at jump points is characterized in terms of the notion of *Dissipative Viscosity* solution. We also provide sufficient conditions under which Dissipative Viscosity solutions enjoy better properties, which turn them into *Balanced Viscosity* solutions. Finally, we discuss the *generic character* of our assumptions.

1. Introduction

We address the singular limit, as $\varepsilon \downarrow 0$, of the gradient flow equation

$$\varepsilon u'(t) + \mathrm{D}\mathcal{E}_t(u(t)) = 0 \qquad \text{in } X \quad \text{for a.a. } t \in (0,T). \tag{1.1}$$

Here, $(X, \|\cdot\|)$ is a finite-dimensional Hilbert space, and the driving energy functional $\mathcal{E}$ is *smooth*, i.e.

$$\mathcal{E} \in \mathrm{C}^1([0,T] \times X) \tag{E$_0$}$$

($\mathrm{D}\mathcal{E}$ denoting the differential with respect to the variable u), but we allow for the mapping $u \mapsto \mathcal{E}_t(u)$ to be *nonconvex*. In this paper we aim to present the basic ideas underlying a *novel, variational* approach to this singular perturbation problem.

Our motivation. We are driven to studying the singular limit of (1.1) by a twofold reason. On the one hand, we aim to gain further insight into yet another aspect of the theory of *abstract* gradient flows. On the other hand, we want to understand the analogies, and the differences, between this singular perturbation problem and that related to the vanishing-viscosity analysis of generalized gradient systems, cf. (1.2) below.

Indeed, the study of gradient flows, after the seminal works [18, 11, 8, 9] in Hilbert spaces, has flourished over the last two decades as a consequence of the novel interpretation [17, 27] of a wide class of evolution equations and systems as gradient flows in the Wasserstein spaces of probability measures, in close connection with Optimal Transport [32]. The theory of gradient flows in metric spaces [4] has provided the theoretical basis for this interpretation, and set the foundations for a *variational* approach to the evolution of gradient systems. Despite this, to our knowledge the singular perturbation problem has been addressed in the abstract setting of (1.1) only recently, in [33], with tools rather based on the theory of dynamical systems, and unrelated to the variational theory of gradient flows.

In turn, techniques borrowed from this theory have been employed for investigating the vanishing-viscosity limit in the viscous regularization of rate-independent systems. In the abstract setup of (1.1), this singular

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perturbation problem reads

$$\partial\psi(u'(t)) + \varepsilon u'(t) + \mathrm{D}\mathcal{E}_t(u(t)) \ni 0 \qquad \text{in } X \quad \text{for a.a. } t \in (0,T), \tag{1.2}$$

with $\psi : X \to [0,+\infty]$ a positively 1-homogeneous (convex) dissipation potential and $\partial\psi : X \rightrightarrows X$ its subdifferential in the sense of convex analysis. The application of the variational approach for gradient systems to the singular limit as $\varepsilon \downarrow 0$ of (1.2) has led [14, 22, 23, 25] to the concept of *Balanced Viscosity* solution of a rate-independent system [26], that encodes a significant description of the system behavior at jump points.

In this paper we aim to address (1.1) from the *variational perspective* adopted in [22, 23, 25] for rate-independent systems. We will prove the convergence as $\varepsilon \downarrow 0$ of (sequences of) solutions to (the Cauchy problem for) (1.1), to a curve $u : [0,T] \to X$ of critical points for $\mathcal{E}$, i.e. fulfilling the stationary problem

$$\mathrm{D}\mathcal{E}_t(u(t)) = 0 \quad \text{in } X \quad \text{for a.a. } t \in (0,T) \tag{1.3}$$

and a suitable energy balance, encompassing information on the behavior of the system governed by (1.3) at its jump points. The properties of u will be codified by the two different concepts of *Dissipative Viscosity* and *Balanced Viscosity* solution to (1.3). In fact, our convergence results will demonstrate that, in analogy with rate-indepenent systems, the vanishing-viscosity approximation provides a selection criterion for mechanically feasible solutions to (1.3).

The finite-dimentional setting, with a smooth energy, considered here will enable us to illustrate the cornerstones of our analysis, unhampered by the technical issues related to nonsmoothness and infinite dimensionality. Nonetheless, our variational approach will be adapted, and refined, to address the asymptotic analysis of (1.1) as $\varepsilon \downarrow 0$ in an infinite-dimensional Hilbertian setting, and with a possibly *nonsmooth*, as well as *nonconvex*, driving energy functional $\mathcal{E}$, in the forthcoming [3].

Before illustrating our results, let us hint at the main analytical difficulties attached to this problem, as well as at the (few) results available in the literature. In particular, in the following lines we will focus on the case of *(uniformly) convex* energies, and of energy functionals $\mathcal{E}_t(\cdot)$ complying with the *transversality conditions*, as required in [33]. Let us however mention that new results have emerged in the recent [5] for the *linearly constrained* evolution of critical points, based on a constructive approach instead of the vanishing-viscosity analysis of (1.1), and applied to a cohesive fracture model.

Preliminary considerations. Under suitable conditions, for every fixed $\varepsilon > 0$ and for every $u_0 \in X$ there exists at least a solution $u_\varepsilon \in H^1(0,T;X)$ to the gradient flow (1.1), fulfilling the Cauchy condition $u_\varepsilon(0) = u_0$. Multiplying (1.1) by u_ε', integrating in time, and exploiting the chain rule for $\mathcal{E}$, it is immediate to check that u_ε complies with the *energy identity*

$$\int_s^t \varepsilon\|u_\varepsilon'(r)\|^2\,\mathrm{d}r + \mathcal{E}_t(u_\varepsilon(t)) = \mathcal{E}_s(u_\varepsilon(s)) + \int_s^t \partial_t\mathcal{E}_r(u_\varepsilon(r))\,\mathrm{d}r \quad \text{for all } 0 \le s \le t \le T, \tag{1.4}$$

balancing the dissipated energy $\int_s^t \varepsilon\|u_\varepsilon'(r)\|^2\,\mathrm{d}r$ with the stored energy and with the work of the external forces $\int_s^t \partial_t\mathcal{E}_r(u_\varepsilon(r))\,\mathrm{d}r$. From (1.4) all the a priori estimates on a family $(u_\varepsilon)_\varepsilon$ of solutions can be deduced. More specifically, using the *power-control* condition $|\partial_t\mathcal{E}_t(u)| \le C_1\mathcal{E}_t(u) + C_2$ for some C_1, $C_2 > 0$, via the Gronwall Lemma one obtains

$$\begin{aligned} &\textit{(i)} \qquad \text{The energy bound } \sup_{t\in(0,T)} \mathcal{E}_t(u_\varepsilon(t)) \le C;\\ &\textit{(ii)} \qquad \text{The estimate } \int_0^T \varepsilon\|u_\varepsilon'(t)\|^2\,\mathrm{d}t \le C', \end{aligned} \tag{1.5}$$

for positive constants C, $C' > 0$ independent of $\varepsilon > 0$. While *(i)*, joint with a suitable *coercivity* condition on $\mathcal{E}$ (typically, compactness of the energy sublevels), yields that there exists a *compact* set $K \subset X$ s.t. $u_\varepsilon(t) \in K$ for all $t \in [0,T]$ and $\varepsilon > 0$, the *equicontinuity* estimate provided by *(ii) degenerates* as $\varepsilon \downarrow 0$. Thus, no Arzelà-Ascoli type result applies to deduce compactness for $(u_\varepsilon)_\varepsilon$. This is the major difficulty in the asymptotic analysis of (1.1). This also represents the main difference between the singular limits (1.1) and (1.2): in the latter case, the presence of the residual dissipation term $\partial\psi(u'(t))$ provides a bound on the total variation of the curves $(u_\varepsilon)_\varepsilon$, uniform w.r.t. ε, so that it is possible to resort to a Helly-type compactness argument.

Let us point out that this obstruction can be circumvented by *convexity* arguments. Indeed, suppose that $\mathcal{E} \in \mathrm{C}^2([0,T] \times X)$ with the mapping $u \mapsto \mathcal{E}_t(u)$ uniformly convex. Then, starting from any $u_0 \in X$ with $\mathrm{D}\mathcal{E}_0(u_0) = 0$ and $\mathrm{D}^2\mathcal{E}_0(u_0)$ positive definite (with $\mathrm{D}^2\mathcal{E}$ the second order derivative of $\mathcal{E}$ w.r.t. u), so that u_0 is a *non-degenerate* critical point of $\mathcal{E}_0(\cdot)$, it can be shown there exists a *unique* curve $u \in \mathrm{C}^1([0,T];X)$ of stationary points, to which the *whole* family $(u_\varepsilon)_\varepsilon$ converge as $\varepsilon \downarrow 0$, uniformly on $[0,T]$.

Therefore, it is indeed significant to focus on the case in which the energy $u \mapsto \mathcal{E}_t(u)$ is allowed to be *nonconvex*. In this context, two problems arise:

(1) Prove that, up to the extraction of a subsequence, the gradient flows $(u_\varepsilon)_\varepsilon$ converge as $\varepsilon \downarrow 0$ to some limit curve u, pointwise in $[0,T]$;
(2) Describe the evolution of u. Namely, one expects u to be a curve of critical points, jumping at *degenerate* critical points for $\mathcal{E}_t(\cdot)$. In this connection, one aims to provide a thorough description of the energetic behavior of u at jump points.

Results for *smooth* energies in finite dimension: the approach via the *transversality conditions*. For the singular perturbation limit (1.1), a first answer to problems (1)&(2) was provided, still in finite dimension, in [33], whose results were later extended to second-order systems in [1]. The key assumptions are that the energy $\mathcal{E} \in \mathrm{C}^3([0,T] \times X)$

(i) has a *finite* number of degenerate critical points,
(ii) the vector field $\mathrm{F} := \mathrm{D}\mathcal{E}$ complies with the so-called *transversality conditions* at every degenerate critical point,

and a further, quite technical condition. While postponing to Section 6 a discussion on the *transversality conditions*, well-known in the realm of bifurcation theory (see, e.g., [15, 16, 31]) we may mention here that, essentially, they prevent degenerate critical points from being "too singular".

Then, in [33, Thm. 3.7] it was shown that, starting from a "well-prepared" datum u_0, there exists a unique piecewise C^2-curve $u : [0,T] \to X$ with a finite jump set $J = \{t_1, \ldots, t_k\}$, such that:

(1) $\mathrm{D}\mathcal{E}_t(u(t)) = 0$ with $\mathrm{D}^2\mathcal{E}_t(u(t))$ positive definite for all $t \in [t_i, t_{i-1})$ and $i = 1, \ldots, k-1$;
(2) at every jump point $t_i \in J$, the left limit $u_-(t_i)$ is a degenerate critical point for $\mathcal{E}_{t_i}(\cdot)$ and there exists a unique curve $v \in \mathrm{C}^2(\mathbb{R};X)$ connecting $u_-(t_i)$ to the right limit $u_+(t_i)$, in the sense that $\lim_{s\to-\infty} v(s) = u_-(t_i)$, $\lim_{s\to+\infty} v(s) = u_+(t_i)$, and fulfilling
$$v'(s) + \mathrm{D}\mathcal{E}_{t_i}(v(s)) = 0 \quad \text{for all } s \in \mathbb{R}; \tag{1.6}$$
(3) the *whole* sequence $(u_\varepsilon)_\varepsilon$ converge to u uniformly on the compact sets of $[0,T]\backslash J$, and suitable rescalings of u_ε converge to v.

The fact that at each jump point t_i the unique heterocline v connecting the left and the right limits $u_-(t_i)$ and $u_+(t_i)$ is a gradient flow of the energy $\mathcal{E}_{t_i}(\cdot)$, does bear a mechanical interpretation, akin to the one for solutions to rate-independent processes obtained in the vanishing-viscosity limit of *viscous* gradient systems, cf. [14, 22, 23, 25]. Namely, one observes that the internal scale of the system, neglected in the singular limit $\varepsilon \downarrow 0$, "takes over" and governs the dynamics in the jump regime, which can be in fact viewed as a fast transition between two metastable states.

The structure of the statement in [33] reflects the line of its proof. First, the unique limit curve is *a priori* constructed via the Implicit Function Theorem, also resorting to the *transversality* conditions. Secondly, the convergence of $(u_\varepsilon)_\varepsilon$ is proved.

Our results. In this paper, we aim to extend the result from [33] to a wider class of energy functionals, still smooth in the sense of (E_0) but not necessarily of class C^3, and not necessarily complying with the transversality conditions. To this end, we will address the singular perturbation problem from a different perspective. Combining ideas from the *variational approach* to gradient flows, possibly driven by nonsmooth and nonconvex energies, cf. [4, 24, 28], with the techniques for the vanishing-viscosity approximation of rate-independent systems from [23, 25], we will prove the existence of a limit curve by refined *compactness* tools. *Variational* arguments will lead to a suitable energetic characterization of its fast dynamics at jumps. Indeed, the flexibility of this approach will allow us to extend the results obtained in this paper, to the infinite-dimensional setting, and to nonsmooth energies, in the forthcoming [3].

A central role in our vanishing-viscosity analysis is played by the study of the limit of the energy identity (1.4) as $\varepsilon \downarrow 0$. Indeed, in analogy with [23, 25],

(1) from (1.4) we will extract all the compactness information needed to prove convergence, up to a subsequence, for the curves $(u_\varepsilon)_\varepsilon$;
(2) passing to the limit in (1.4) we will in fact select a notion of solution to the limit problem (1.3) featuring an energy balance with information on the energetic behavior of the system at jumps.

In particular, the starting point is the key observation that, using equation (1.1) to the rewrite the contribution $\int_s^t \varepsilon \|u_\varepsilon'(r)\|^2 \,\mathrm{d}r$ of the dissipated energy, the energy identity (1.4) can be reformulated as

$$\int_s^t \left(\frac{\varepsilon}{2}\|u_\varepsilon'(r)\|^2+\frac{1}{2\varepsilon}\|\mathrm{D}\mathcal{E}_r(u_\varepsilon(r))\|^2\right) \mathrm{d}r + \mathcal{E}_t(u_\varepsilon(t)) = \mathcal{E}_s(u_\varepsilon(s)) + \int_s^t \partial_t \mathcal{E}_r(u_\varepsilon(r))\,\mathrm{d}r \tag{1.7}$$

for all $0 \le s \le t \le T$. In addition to estimates (1.5), from (1.7) it is possible to deduce that

$$\int_0^T \|\mathrm{D}\mathcal{E}_r(u_\varepsilon(r))\|\|u_\varepsilon'(r)\|\,\mathrm{d}r \le C. \tag{1.8}$$

Thus, while no (uniform w.r.t. $\varepsilon > 0$) bounds are available on $\|u_\varepsilon'\|$, estimate (1.8) suggests that:

(i) The limit of the *energy-dissipation integral* $\int_s^t \|\mathrm{D}\mathcal{E}_r(u_\varepsilon(r))\|\|u_\varepsilon'(r)\|\,\mathrm{d}r$ will describe the dissipation of energy (at jumps) in the limit $\varepsilon \downarrow 0$;
(ii) To extract compactness information from the integral (1.8), with the *degenerating* weight $\|\mathrm{D}\mathcal{E}_r(u_\varepsilon(r))\|$, it will be expedient to suppose that the (degenerate) critical points of $\mathcal{E}$, in whose neighborhood this weight tends to zero, are somehow "well separated" one from each other.

In fact, in addition to the aforementioned coercivity and power-control conditions on $\mathcal{E}$, typical of the variational approach to existence for non-autonomous gradient systems [24], in order to prove our results for the singular limit (1.1) we will resort to the condition that for every $t \in [0,T]$ the critical set

$$\mathcal{C}(t) := \{u \in X \,:\, \mathrm{D}\mathcal{E}_t(u) = 0\} \text{ consists of } \textit{isolated points}. \tag{1.9}$$

This condition will allow us to overcome the aforementioned lack of compactness of the sequence $(u_\varepsilon)_\varepsilon$, and to prove in **Theorem 1**, that, up to a subsequence, the gradient flows u_ε pointwise converge to a solution u of the limit problem (1.3), defined at *every* $t \in [0,T]$, enjoying the following properties:

(1) $u : [0,T] \to X$ is *regulated*, i.e. the left and right limits $u_-(t)$ and $u_+(t)$ exist at every $t \in (0,T)$, and so do the limits $u_+(0)$ and $u_-(T)$;
(2) u fulfills the energy balance

$$\mu([s,t]) + \mathcal{E}_t(u_+(t)) = \mathcal{E}_s(u_-(s)) + \int_s^t \partial_t \mathcal{E}_r(u(r))\,\mathrm{d}r \quad \text{for all } 0 \le s \le t \le T, \tag{1.10a}$$

with μ a positive Radon measure with an at most countable set J of atoms;
(3) u is continuous on $[0,T] \setminus J$, and solves

$$\mathrm{D}\mathcal{E}_t(u(t)) = 0 \quad \text{in } X \text{ for all } t \in [0,T] \setminus J;$$

(4) J coincides with the jump set of u, and there hold the jump relations

$$\mu(\{t\}) = \mathcal{E}_t(u_-(t)) - \mathcal{E}_t(u_+(t)) = \mathsf{c}(t; u_-(t), u_+(t)) \quad \text{for all } t \in J\,, \tag{1.10b}$$

which encode the information on the behavior of the system at jumps.

Indeed, in (1.10b), the cost function $\mathsf{c} : [0,T]\times X\times X \to [0,+\infty)$ is defined by minimizing the energy-dissipation integrals, namely

$$\mathsf{c}(t; u_-, u_+) := \inf\left\{ \int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\|\vartheta'(s)\|\,\mathrm{d}s \ :\ \vartheta \in \mathscr{A}^t_{u_-,u_+} \right\} \qquad \text{for } t\in[0,T],\ u_-,\, u_+ \in X,$$

over a suitable class $\mathscr{A}^t_{u_-,u_+}$ of *admissible curves* connecting u_- and u_+. These curves somehow capture the asymptotic behavior of the gradient flows $(u_\varepsilon)_\varepsilon$ on intervals shrinking to the jump point t. Now, from (1.10b) it is possible to deduce that any curve ϑ attaining the infimum in the definition of $\mathsf{c}(t; u_-(t), u_+(t))$, hereafter referred to as *optimal jump transition*, can be reparameterized to a curve $\tilde\vartheta$ solving the gradient flow of $\mathcal{E}$ at fixed process time t, namely

$$\tilde\vartheta'(\sigma) + \mathrm{D}\mathcal{E}_t(\tilde\vartheta(\sigma)) = 0 \quad \text{in } X, \tag{1.11}$$

in analogy with (1.6). Thus, the notion of solution to (1.3) given by (1)–(4), and hereafter referred to as *Dissipative Viscosity* solution, bears the same mechanical interpretation as the solution concept in [33], although it has been obtained via a completely different approach, and, in particular, independently of the transversality conditions assumed in [33].

Nonetheless, Using the results of [2], we also show that our condition (1.9) on the critical points can be deduced from the *transversality conditions*, cf. Proposition 6.2 ahead. In turn, as we will see, these conditions have a generic character, see Theorem 6.3.

Our second main result, **Theorem 2**, shows that if $\mathcal{E}$ fulfills the following condition

$$\limsup_{v\to u} \frac{\mathcal{E}_t(v) - \mathcal{E}_t(u)}{\|\mathrm{D}\mathcal{E}_t(v)\|} \geq 0 \qquad \text{at every } u \in \mathcal{C}(t) \qquad \text{for all } t\in[0,T], \tag{1.12}$$

then for every Dissipative Viscosity solution the absolutely continuous and the Cantor part of the associated defect measure μ are zero. This results in a more transparent form of the energy balance (2.8): in fact, (2.11) ahead only involves the jump contributions to μ, which are fully described by (1.10b). Since (2.11) is akin to the energy balance featuring in the concept of Balanced Viscosity solution to a rate-independent system, cf. [23, 25], we will refer to this second solvability notion for (1.3) as *Balanced Viscosity* solution. Observe (cf. Remark 2.5), that a sufficient condition for (1.12) is that $\mathcal{E}$ complies with the celebrated Łojasiewicz inequality, cf., e.g., [7, 20, 30], as well as the recent survey paper [10].

Plan of the paper. In Section 2 we list our conditions on the energy functional $\mathcal{E}$, and then give the definition of *admissible curve* connecting two points and the induced notion of *energy-dissipation cost* c. We then introduce the two notions of *Dissipative Viscosity* and *Balanced Viscosity* solutions to (1.3) and finally state **Theorems 1 & 2**. In Section 3 we gain further insight into the properties of optimal jump transitions. Section 4 is devoted to the preliminary analysis of the asymptotic behavior of the energy-dissipation integrals in the vanishing-viscosity limit, and to the properties of the cost c. These results lie at the core of the proof of Theorem 1, developed in Section 5 together with the proof of Theorem 2. In Section 6 we present examples of energies complying with our set of assumptions. In particular, on the one hand we show that (1.9) is guaranteed by the *transversality conditions*, whose genericity is discussed. On the other hand, we introduce the class of subanalytic functions, which comply with the Lojasiewicz inequality, hence with (1.12).

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2. Main results

Preliminarily, let us fix some **general notation** that will be used throughout. As already mentioned in the introduction, X is a finite-dimensional Hilbert space (although all of the results of this paper could be trivially extended to the, still finite-dimensional, Banach framework), with inner product $\langle\cdot,\cdot\rangle$. Given $x \in X$ and $\rho > 0$, we will denote by $B(x,\rho)$ the open ball centered at x with radius ρ.

We will denote by $\mathrm{B}([0,T];X)$ the class of measurable, everywhere defined, and bounded functions from $[0,T]$ to X, whereas $\mathrm{M}(0,T)$ stands for the set of Radon measures on $[0,T]$.

Finally, the symbols c, C, C', ... will be used to denote a positive constant depending on given quantities, and possibly varying from line to line.

Basic conditions on $\mathcal{E}$. In addition to (E_0), we will require

Coercivity: the map $u \mapsto \mathcal{G}(u) := \sup_{t\in[0,T]} |\mathcal{E}_t(u)|$ fulfills

$$\forall\,\rho > 0 \;\; \text{the sublevel set } \mathcal{S}_\rho := \{u \in X \,:\, \mathcal{G}(u) \le \rho\} \text{ is bounded.} \tag{E_1}$$

Power control: the partial time derivative $\partial_t\mathcal{E}$ fulfills

$$\exists\, C_1,\, C_2 > 0 \;\; \forall\,(t,u) \in [0,T] \,:\;\; |\partial_t\mathcal{E}_t(u)| \le C_1\mathcal{E}_t(u) + C_2. \tag{E_2}$$

Observe that (E_2) in particular yields that $\mathcal{E}$ is bounded from below. In what follows, without loss of generality we will suppose that $\mathcal{E}$ is nonnegative. A simple argument based on the Gronwall Lemma ensures that

$$\mathcal{G}(u) \,\le\, \exp(C_1 T)\left(\inf_{t\in[0,T]} \mathcal{E}_t(u) + C_2\,T\right) \qquad \text{for all } u \in X. \tag{2.1}$$

Under these conditions, the existence of solutions to the gradient flow (1.1) is classical. Multiplying (1.1) by u' and using the chain rule fulfilled by the (smooth) energy $\mathcal{E}$ leads to the energy identity (2.2) below, which will be the starting point in the derivation of all our estimates for the singular perturbation limit as $\varepsilon \downarrow 0$.

Theorem 2.1. *Let $\mathcal{E} : [0,T]\times X \to [0,+\infty)$ comply with (E_0), (E_1), and (E_2). Then, for every $u_0 \in X$ there exists $u \in H^1(0,T;X)$, with $u(0) = u_0$, solving (1.1) and fulfilling for every $0 \le s \le t \le T$ the energy identity*

$$\int_s^t \left(\frac{\varepsilon}{2}\|u'(r)\|^2 + \frac{1}{2\varepsilon}\|\mathrm{D}\mathcal{E}_r(u(r))\|^2\right)\mathrm{d}r + \mathcal{E}_t(u(t)) = \mathcal{E}_s(u(s)) + \int_s^t \partial_t\mathcal{E}_r(u(r))\,\mathrm{d}r. \tag{2.2}$$

A condition on the critical points of $\mathcal{E}$. In what follows, we will denote the set of the critical points of $\mathcal{E}_t(\cdot)$, for fixed $t \in [0,T]$, by

$$\mathcal{C}(t) := \big\{u \in X \,:\, \mathrm{D}\mathcal{E}_t(u) = 0\big\}.$$

It is immediate to deduce from (E_1), the fact that X is finite-dimensional, and the lower semicontinuity of $\mathcal{E}_t(\cdot)$ that $\mathrm{Argmin}\{\mathcal{E}_t(u) \,:\, u \in X\} \neq \emptyset$, so that $\mathcal{C}(t) \neq \emptyset$, for every $t \in [0,T]$. We assume that

$$\text{for every } t \in [0,T] \quad \text{the set } \mathcal{C}(t) \text{ consists of isolated points.} \tag{E_3}$$

We postpone to Section 6 a discussion on sufficient conditions for (E_3), as well as on its *generic* character.

Solution concepts. We now illustrate the two notions of evolution of curves of critical points that we will obtain in the limit passage as $\varepsilon \downarrow 0$. Preliminarily, we need to give the definitions of *admissible curve* and of *energy-dissipation cost*, obtained by minimizing the energy-dissipation integrals along admissible curves. The latter notion somehow encodes the asymptotic properties of (the energy-dissipation integrals along) sequences of absolutely continuous curves (in fact, the solutions of our gradient flow equation), considered on intervals shrinking to a point $t \in [0,T]$, cf. Proposition 4.1 ahead. Basically, admissible curves are piecewise locally Lipschitz continuous curves joining critical points. Note however that we do not impose that their end-points be critical. That is why, we choose to confine our definition to the case the end-points are different: otherwise, we should have to allow for curves degenerating to a single, possibly non-critical, point, which would not be consistent with (2.3) below.

Definition 2.2. *Let $t \in [0,T]$ and u_1, $u_2 \in X$ be fixed.*

(1) *In the case $u_1 \neq u_2$, we call a curve $\vartheta \in \mathrm{C}([0,1];X)$ with $\vartheta(0)=u_1$ and $\vartheta(1)=u_2$* admissible *if there exists a partition $0=\mathsf{t}_0<\mathsf{t}_1<\ldots<\mathsf{t}_j=1$ such that*

$$\begin{aligned}&\vartheta|_{(\mathsf{t}_i,\mathsf{t}_{i+1})} \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}((\mathsf{t}_i,\mathsf{t}_{i+1});X) \text{ for all } i=0,\ldots,j-1,\\ &\vartheta(\mathsf{t}_i)\in\mathcal{C}(t) \text{ for all } i\in\{1,\ldots,j-1\}, \quad \vartheta(r)\notin\mathcal{C}(t)\ \forall\, r\in(\mathsf{t}_i,\mathsf{t}_{i+1}) \text{ for all } i=0,\ldots,j-1.\end{aligned} \tag{2.3}$$

We will denote by $\mathscr{A}^t_{u_1,u_2}$ the class of admissible curves connecting u_1 and u_2 at time t. Furthermore, for a given $\rho>0$ we will use the notation

$$\mathscr{A}^{t,\rho}_{u_1,u_2} := \Big\{\vartheta\in\mathscr{A}^t_{u_1,u_2}\ :\ \vartheta(s)\in\mathcal{S}_\rho \text{ for all } s\in[0,1]\Big\}.$$

(2) *We define the* energy-dissipation cost

$$\mathsf{c}_t(u_1;u_2) := \begin{cases} \inf\Big\{\int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\|\vartheta'(s)\|\,\mathrm{d}s\ :\ \vartheta\in\mathscr{A}^t_{u_1,u_2}\Big\} & \text{if} \quad u_1\neq u_2,\\ 0 & \text{if} \quad u_1=u_2.\end{cases} \tag{2.4}$$

We call the integral $\int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\|\vartheta'(s)\|\,\mathrm{d}s$, for some $\vartheta\in\mathscr{A}^t_{u_1,u_2}$, an *energy-dissipation integral.* Observe that, up to a reparameterization, every absolutely continuous curve $\vartheta\in\mathrm{AC}([a,b];X)$ such that $\exists\ a=\mathsf{t}_0<\mathsf{t}_1<\ldots<\mathsf{t}_j=b$ with

$$\vartheta(\mathsf{t}_i)\in\mathcal{C}(t), \qquad \vartheta(\mathsf{t}_i)\neq\vartheta(\mathsf{t}_j), \qquad \vartheta(r)\notin\mathcal{C}(t) \text{ for all } r\in(\mathsf{t}_i,\mathsf{t}_{i+1})$$

and all i, $k\in\{1,\ldots,j-1\}$, $i\neq k$, is an admissible curve. Note that the chain rule holds along *admissible* curves ϑ with finite energy-dissipation integral at time t. This is the content of the following lemma, which can be easily proved.

Lemma 2.3. *Let $t\in[0,T]$ be fixed, and $\vartheta\in\mathscr{A}^t_{u_1,u_2}$ be an admissible curve connecting u_1 and u_2 such that*

$$\int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\|\vartheta'(s)\|\,\mathrm{d}s<\infty.$$

Then, the map $s\mapsto\mathcal{E}_t(\vartheta(s))$ belongs to $\mathrm{AC}([0,1])$ and there holds the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_t(\vartheta(s)) = \langle\mathrm{D}\mathcal{E}_t(\vartheta(s)),\vartheta'(s)\rangle \quad \text{for a.a. } s\in(0,1)\,. \tag{2.5}$$

The following result, whose proof is postponed to Section 4, collects the properties of the cost c.

Theorem 2.4. *Assume* (E_0)–(E_3). *Then, for every $t\in[0,T]$ and u_1, $u_2\in X$ we have:*

(1) *$\mathsf{c}_t(u_1;u_2)=0$ if and only if $u_1=u_2$;*
(2) *c_t is symmetric;*
(3) *if $\mathsf{c}_t(u_1;u_2)>0$, there exists an optimal curve $\vartheta\in\mathscr{A}^t_{u_1,u_2}$ attaining the* inf *in* (2.4)*;*
(4) *for every $u_3\in\mathcal{C}(t)$, the triangle inequality holds*

$$\mathsf{c}_t(u_1;u_2)\le\mathsf{c}_t(u_1;u_3)+\mathsf{c}_t(u_3;u_2); \tag{2.6}$$

(5) *there holds*

$$\begin{aligned}\mathsf{c}_t(u_1;u_2)\le\inf\Big\{&\liminf_{n\to\infty}\int_{t_1^n}^{t_2^n}\|\mathrm{D}\mathcal{E}_s(\vartheta_n(s))\|\|\vartheta_n'(s)\|\,\mathrm{d}s\ :\\ &\vartheta_n\in\mathrm{AC}([t_1^n,t_2^n];X),\ t_i^n\to t,\ \vartheta_n(t_i^n)\to u_i \text{ for } i=1,2\Big\};\end{aligned} \tag{2.7}$$

(6) *the following lower semicontinuity property holds*

$$(u_1^k,u_2^k)\to(u_1,u_2) \text{ as } k\to\infty \quad\Rightarrow\quad \liminf_{k\to\infty}\mathsf{c}_t(u_1^k;u_2^k)\ \ge\ \mathsf{c}_t(u_1;u_2).$$

We are now in the position to give the definition of *Dissipative Viscosity* solution to equation (1.3).

Definition 1 (Dissipative Viscosity solution). *We call* Dissipative Viscosity solution *to* (1.3) *a curve* $u \in \mathrm{B}([0,T];X)$ *such that*

(1) for every $0 \le t < T$ *and every* $0 < s \le T$*, the left and right limits* $u_-(s) := \lim_{\tau\uparrow s} u(\tau)$ *and* $u_+(t) := \lim_{\tau\downarrow t} u(\tau)$ *exist, there exists a positive Radon measure* $\mu \in \mathrm{M}(0,T)$ *such that the set* J *of its atoms is countable, and* (u,μ) *fulfill the energy identity*

$$\mu([s,t]) + \mathcal{E}_t(u_+(t)) = \mathcal{E}_s(u_-(s)) + \int_s^t \partial_t \mathcal{E}_r(u(r))\,\mathrm{d}r \quad \text{for all } 0 \le s \le t \le T, \tag{2.8}$$

where we understand $u_-(0) := u(0)$ *and* $u_+(T) := u(T)$*;*

(2) u *is continuous on the set* $[0,T]\setminus J$*, and*

$$u(t) \in \mathcal{C}(t) \qquad \text{for all } t \in (0,T]\setminus J; \tag{2.9}$$

(3) The left and right limits fulfill

$$u_-(s) \in \mathcal{C}(s) \text{ and } u_+(t) \in \mathcal{C}(t) \quad \text{at every } 0 < s \le T \text{ and } 0 \le t < T, \tag{2.10a}$$

$$J = \{t \in [0,T] \,:\, u_-(t) \ne u_+(t)\}, \tag{2.10b}$$

$$0 < \mathsf{c}_t(u_-(t);u_+(t)) = \mu(\{t\}) = \mathcal{E}_t(u_-(t)) - \mathcal{E}_t(u_+(t)), \quad \text{for every } t \in J. \tag{2.10c}$$

A straightforward consequence of (2.9) and (2.10a) is that, if $u(0)$ is not a critical point for $\mathcal{E}_0(\cdot)$, then $u(0) \ne u_+(0)$, i.e. u immediately jumps at the initial time $t = 0$. A comparison between the energy balances (2.2) and (2.8) highlights the fact that the contribution to (2.8) given by the measure $\mu([s,t])$ surrogates the role of the energy-dissipation integral $\int_s^t \|\mathrm{D}\mathcal{E}_r(u(r))\|\|u'(r)\|\,\mathrm{d}r$. That is why, in what follows we will refer to μ as the *defect energy-dissipation measure* (for short, *defect measure*), associated with u. Let us highlight that, by (2.10b), u jumps at the atoms of μ, and that the jump conditions (2.10c) provide a description of its energetic behavior in the jump regime (see Propositions 3.1 and 3.2).

The notion of *Balanced Viscosity* solution below brings the additional information that the measure μ is purely atomic. Then, taking into account conditions (2.10c), we obtain

$$\mu([s,t]) = \sum_{r\in J\cap[s,t]} \mu(\{r\}) = \sum_{r\in J\cap[s,t]} \mathsf{c}_r(u_-(r);u_+(r))\,.$$

Definition 2 (Balanced Viscosity solution). *We call a Dissipative Viscosity solution* u *to* (1.3) Balanced Viscosity *solution if the absolutely continuous and the Cantor part of the defect measure* μ *are zero. Therefore,* (2.8) *reduces to*

$$\sum_{r\in J\cap[s,t]} \mathsf{c}_r(u_-(r);u_+(r)) + \mathcal{E}_t(u_+(t)) = \mathcal{E}_s(u_-(s)) + \int_s^t \partial_t\mathcal{E}_r(u(r))\,\mathrm{d}r \quad \text{for every } 0 \le s \le t \le T. \tag{2.11}$$

Convergence to Dissipative Viscosity solutions. Our **first main result**, whose proof will be given throughout Sections 4 & 5, ensures the convergence, up to a subsequence, of any family of solutions to (the Cauchy problem for) (1.1), to a Dissipative Viscosity solution.

Theorem 1. *Assume* (E$_0$)–(E$_3$). *Let* $(\varepsilon_n)_n$ *be a vanishing sequence, and consider a sequence* $(u^0_{\varepsilon_n})_n$ *of initial conditions for* (1.1) *such that*

$$u^0_{\varepsilon_n} \to u_0 \quad \text{as } n \to \infty. \tag{2.12}$$

Then there exist a (not relabeled) subsequence and a curve $u \in \mathrm{B}([0,T];X)$ *such that*

(1) the following convergences hold

$$u_{\varepsilon_n}(t) \to u(t) \quad \text{for all } t \in [0,T], \tag{2.13}$$

$$u_{\varepsilon_n} \rightharpoonup^* u \quad \text{in } L^\infty(0,T;X), \qquad u_{\varepsilon_n} \to u \quad \text{in } L^p(0,T;X) \text{ for all } 1 \le p < \infty; \tag{2.14}$$

(2) $u(0) = u_0$ *and* u *is Dissipative Viscosity solution to* (1.3).

Observe that, we do not need to require that $u_0 \in \mathcal{C}(0)$. Therefore, the limiting Dissipative Viscosity solution u might well have a jump already at the initial time.

In our **second main result** we address the improvement of Dissipative Viscosity to Balanced Viscosity solutions, under the condition that at every $t \in [0,T]$ there holds

$$\limsup_{v \to u} \frac{\mathcal{E}_t(v) - \mathcal{E}_t(u)}{\|\mathrm{D}\mathcal{E}_t(v)\|} \geq 0 \quad \text{for all } u \in \mathcal{C}(t). \tag{E$_4$}$$

The proof of Theorem 2 is also postponed to Section 5.

Theorem 2. *In the setting of* (E_0)–(E_2)*, assume in addition* (E_4)*. Let u be a* Dissipative Viscosity *solution to* (1.3) *and let μ be its associated defect measure. Then, the absolutely continuous part μ_{AC} and the Cantor part μ_{Ca} of the measure μ are zero, i.e. u is a* Balanced Viscosity *solution to* (1.3).

Remark 2.5 (A discussion of (E_4))**.** Observe that (E_4) is trivially satisfied in the case the functional $u \mapsto \mathcal{E}_t(u)$ is *convex*. Indeed, if $\mathrm{D}\mathcal{E}_t(u) = 0$, then $\mathcal{E}_t(v) \geq \mathcal{E}_t(u)$ for all $v \in X$.

Another sufficient condition for (E_4) is that $\mathcal{E}$ complies with the celebrated Łojasiewicz inequality, namely

$$\forall t \in [0,T]\ \forall u \in \mathcal{C}(t)\ \exists \theta \in (0,1)\ \exists C > 0\ \exists R > 0\ \forall v \in B_R(u)\ :\ |\mathcal{E}_t(v) - \mathcal{E}_t(u)|^\theta \leq C\|\mathrm{D}\mathcal{E}_t(v)\|. \tag{2.15}$$

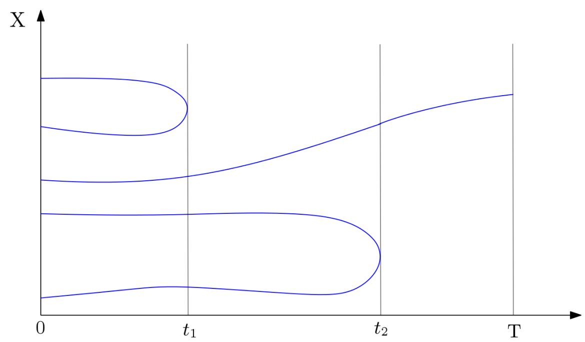
In this case, we even have

$$\lim_{v \to u} \frac{|\mathcal{E}_t(v) - \mathcal{E}_t(u)|}{\|\mathrm{D}\mathcal{E}_t(v)\|} \leq C \lim_{v \to u} |\mathcal{E}_t(v) - \mathcal{E}_t(u)|^{1-\theta} = 0$$

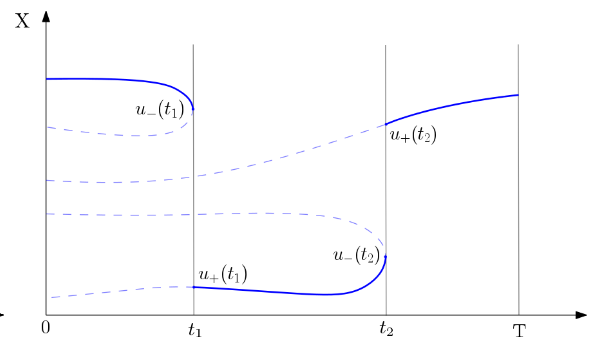
by continuity of $u \mapsto \mathcal{E}_t(u)$.

We conclude this section with an illustration of the type of evolution described by the concepts of Dissipative Viscosity and Balanced Viscosity solution. First of all, it follows from condition (E_3) that, for every $t \in [0,T]$ and every compact subset C of X, $\mathcal{C}(t) \cap C$ consists of finitely many points. In particular, $\mathcal{C}(t) \cap \mathcal{S}_\rho$ has finite cardinality for every $\rho > 0$. Picture (A) in Figure 1 depicts a feasible critical set $\mathcal{C} = \cup_{t\in[0,T]}\mathcal{C}(t)$: in this particular case, $\mathcal{C}(t)$ consists of finitely many points for every $t \in [0,T]$, and $\mathcal{C}$ results in the union of finitely many smooth curves. More in general, it has been shown in [2, Thm. 2.6] that, if the energy functional $\mathcal{E}$ is analytic and satisfies the first two transversality conditions (T1) and (T2) (cf. Definition 6.1 ahead), then for every fixed $t \in [0,T]$ the set $\mathcal{C}(t)$ is the disjoint union of a discrete set and of an analytic manifold of dimension 1, whose connected components are compact curves. Moreover, for every connected curve $C \subset \mathcal{C}(t)$, there exist $\delta > 0$ and an open neighborhood V such that $\mathcal{C}(s) \cap V = \emptyset$ for every $s \in (0,T)$ with $0 < |t-s| < \delta$.

Cartoon (B) in Figure 1 shows a Dissipative Viscosity solution which has *finitely many* jump points and the following, quite expectable, property: if $u_-(t)$ is a *degenerate* critical point for $\mathcal{E}_t(\cdot)$ at some $t \in (0,T)$, then t



(A)



(B)

FIGURE 1. Sketch of a critical set and of a Dissipative Viscosity solution.

is a jump point. In this paper we do not conclude this result. Nonetheless, in the forthcoming [3] we will show that, if the energy functional $\mathcal{E}$ is in $\mathrm{C}^3([0,T]\times X)$ and complies with all the transversality conditions (T1), (T2), and (T3), then any Dissipative Viscosity is a Balanced Viscosity solution, it has finitely many points, and has the above mentioned property at degenerate critical points. Still, observe from identity (2.10c) in Definition 1 that the solutions to (1.3) that we are able to select fulfill the following property: when they jump at t from a point x_1, they jump to another critical point x_2 of $\mathcal{E}_t(\cdot)$ only if $\mathcal{E}_t(x_2)<\mathcal{E}_t(x_1)$ (see the illustrative Example 2.6 below).

Example 2.6. In this example, we take $X=\mathbb{R}$ and the energy functional $\mathcal{E}$ defined by

$$\mathcal{E}_t(u) := \big(u^2-t^2\big)^2 \quad \text{for all } (t,u)\in[-T,T]\times\mathbb{R},$$

for some $T>0$. Since $\mathrm{D}\mathcal{E}_t(u)=4u\big(u^2-t^2\big)$, we have that $\mathcal{E}_t(u)=0$ iff $u\in\{0,\pm t\}$. Let us consider the three distinguished curves of critical points

$$u_1(t):=t, \quad u_2(t):=-t, \quad u_3(t):=0 \qquad \text{for all } t\in[-T,T]$$

and see how a Dissipative Viscosity solution relates to them.

First of all, any Dissipative Viscosity solution does not jump at the process time $t=0$, since the critical set $\mathcal{C}(0)$ is reduced to the singleton $\{0\}$. Nonetheless, $(0,0)\in[-T,T]\times\mathbb{R}$ is a bifurcation point, in the sense that a Dissipative Viscosity solution, when passing through $(0,0)$ upon coming from negative times, can choose to follow any of the branches u_i. Observe also that

$$\mathcal{E}_t(0)=t^4>0=\mathcal{E}_t(\pm t) \quad \text{ for every } t\in[-T,T]\setminus\{0\}. \tag{2.16}$$

Therefore, from property (2.10c) of Definition 1, a Dissipative Viscosity solution u to (1.3) is such that if $u(t)=\pm t$ for some $t\in(0,T)$, then $u(s)=\pm s$ for every $s\in[t,T]$. In particular, if u jumps at some $t\in(0,T)$ (and it can do it only if $u(t)=0$ due to (2.16)), then it cannot jump at any other following time. For example, the curve u defined by

$$u(t):=\begin{cases} 0 & \text{for all } t\in[-T,T/2),\\ t & \text{for all } t\in[T/2,T]\end{cases}$$

is a Dissipative Viscosity solution, with only one jump at $t=T/2$. Note from this discussion that if our solutions cannot *jump* from branches u_1 and u_2 to the branch u_3, they can *switch* from branches u_1 and u_2 to the branch u_3 when passing through $(0,0)$. Hence, the curve u defined by

$$u(t):=\begin{cases} t & \text{for all } t\in[-T,0),\\ 0 & \text{for all } t\in[0,T/2),\\ t & \text{for all } t\in[T/2,T]\end{cases}$$

is another possible Dissipative Viscosity solution (with only one jump at $t=T/2$).

3. Optimal jump transitions

In this section, we get further insight into the jump conditions (2.10c). Due to Theorem 2.4 (3), for every $t\in J$ the left and right limits $u_-(t)$ and $u_+(t)$ are connected by a curve $\vartheta\in\mathscr{A}^t_{u_-(t),u_+(t)}$ minimizing the cost $\mathsf{c}_t(u_-(t);u_+(t))$, which will be hereafter referred to as an *optimal jump transition* between $u_-(t)$ and $u_+(t)$. The following result states that every $\mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}$-piece of an optimal jump transition can be reparameterized to a curve solving the gradient flow equation (3.1) below.

Proposition 3.1. *Let $u\in\mathrm{B}([0,T];X)$ be a* Dissipative Viscosity solution *to* (1.3)*. Also, let $t\in J$ be fixed, and let $\vartheta\in\mathscr{A}^t_{u_-(t),u_+(t)}$ be an optimal jump transition between $u_-(t)$ and $u_+(t)$; let $(\mathsf{a},\mathsf{b})\subset[0,1]$ be such that $\vartheta|_{(\mathsf{a},\mathsf{b})}\in\mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}((\mathsf{a},\mathsf{b});X)$ and $\vartheta(s)\notin\mathcal{C}(t)$ for all $s\in(\mathsf{a},\mathsf{b})$. Then, there exists a reparametrization $\sigma\mapsto$*

$\mathfrak{s}(\sigma)$ mapping a (possibly unbounded) interval $(\tilde{\mathsf{a}},\tilde{\mathsf{b}})$ into (a,b), such that the curve $\tilde{\vartheta}(\sigma) := \vartheta(\mathfrak{s}(\sigma))$ is locally absolutely continuous and fulfills the gradient flow equation

$$\tilde{\vartheta}'(\sigma) + \mathrm{D}\mathcal{E}_t(\tilde{\vartheta}(\sigma)) = 0 \qquad \text{in } X \quad \text{for a.a. } \sigma \in (\tilde{\mathsf{a}},\tilde{\mathsf{b}})\,. \tag{3.1}$$

Proof. Any optimal jump transition $\vartheta \in \mathscr{A}^t_{u_-(t),u_+(t)}$ fulfills the jump condition (2.10c) with $u_-(t) = \vartheta(0)$ and $u_+(t) = \vartheta(1)$. Combining this with the chain rule (see Lemma 2.3), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_t(\vartheta(s)) = \langle \mathrm{D}\mathcal{E}_t(\vartheta(s)), \vartheta'(s)\rangle = -\|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\|\vartheta'(s)\| \quad \text{for a.a. } s \in (\mathsf{a},\mathsf{b}), \tag{3.2}$$

hence

$$\text{for a.a. } s \in (\mathsf{a},\mathsf{b}) \quad \exists\,\lambda(s) > 0 : \quad \lambda(s)\vartheta'(s) + \mathrm{D}\mathcal{E}_t(\vartheta(s)) = 0 \qquad \text{in } X. \tag{3.3}$$

In order to find $\mathfrak{s} = \mathfrak{s}(\sigma)$, we fix $\bar{s} \in (\mathsf{a},\mathsf{b})$ and set

$$\sigma(s) := \int_{\bar{s}}^{s} \frac{1}{\lambda(r)}\,\mathrm{d}r\,. \tag{3.4}$$

Indeed, it follows from (3.3) that $\lambda(s) = \frac{\|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|}{\|\vartheta'(s)\|}$ for almost all $s \in (\mathsf{a},\mathsf{b})$. Since $s \in (\mathsf{a},\mathsf{b}) \mapsto \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|$ is strictly positive and continuous, and since ϑ is locally Lipschitz continuous on (a,b), it is immediate to deduce that for every closed interval $[\mathsf{a}+\rho,\mathsf{b}-\rho] \subset (\mathsf{a},\mathsf{b})$ there exists $\lambda_\rho > 0$ such that $\lambda(s) \geq \lambda_\rho$ for all $s \in [\mathsf{a}+\rho,\mathsf{b}-\rho]$. Therefore, σ is a well-defined, locally Lipschitz continuous map with $\sigma'(s) > 0$ for almost all $s \in (\mathsf{a},\mathsf{b})$. We let $\tilde{\mathsf{a}} := \sigma(\mathsf{a})$, $\tilde{\mathsf{b}} := \sigma(\mathsf{b})$, and set $\mathfrak{s} : (\tilde{\mathsf{a}},\tilde{\mathsf{b}}) \to (\mathsf{a},\mathsf{b})$ to be the inverse map of σ: it satisfies

$$\mathfrak{s}'(\sigma) = \lambda(\mathfrak{s}(\sigma)) \quad \text{for a.a. } \sigma \in (\tilde{\mathsf{a}},\tilde{\mathsf{b}}), \tag{3.5}$$

and it is an absolutely continuous map, being $\int_{\tilde{a}}^{\tilde{b}} \mathfrak{s}'(\sigma)d\sigma = \int_{\tilde{a}}^{\tilde{b}} \lambda(\mathfrak{s}(\sigma))d\sigma = b - a$. Using the definition of $\tilde{\vartheta}$, (3.5), and (3.3), we conclude that $\tilde{\vartheta}$ fulfills (3.1). Since $\mathfrak{s}$ is absolutely continuous and ϑ locally Lipschitz continuous, the curve $\tilde{\vartheta}$ turns out to be locally absolutely continuous. □

The symmetry property of the cost proved in Theorem 2.4 (2) gives some information about the number of the optimal jump transitions. This is the content of the following proposition.

Proposition 3.2. *Let $u \in \mathrm{B}([0,T];X)$ be a* Dissipative Viscosity solution *to* (1.3)*, and let $t \in J$. There exists a finite number of optimal jump transitions between $u_-(t)$ and $u_+(t)$.*

Proof. Suppose by contradiction that there exists an infinite number of optimal jump transitions connecting $x := u_-(t)$ and $y := u_+(t)$ and, for an arbitrary natural number N, choose $2N+1$ of them: $\vartheta_1, ..., \vartheta_{2N+1}$. Let us fix an arbitrary partition $0 = t_0 < \ldots < t_{2N+1} = 1$. We can suppose that, up to reparametrizations, $\vartheta_{2i+1} : [t_{2i}, t_{2i+1}] \to X$ is such that $\vartheta_{2i+1}(t_{2i}) = x$, $\vartheta_{2i+1}(t_{2i+1}) = y$ for $i = 0, ..., N$, and $\vartheta_{2i} : [t_{2i-1}, t_{2i}] \to X$ is such that $\vartheta_{2i}(t_{2i-1}) = y$, $\vartheta_{2i}(t_{2i}) = x$ for $i = 1, ..., N$. Consider the function $\vartheta : [0,1] \to X$ defined as

$$\vartheta := \begin{cases} \vartheta_{2i+1} & \text{on } [t_{2i}, t_{2i+1}] \text{ for } i = 0, ..., N, \\ \vartheta_{2i} & \text{on } [t_{2i-1}, t_{2i}] \text{ for } i = 1, ..., N, \end{cases}$$

and note that $\vartheta(0) = x$ and $\vartheta(1) = y$. Therefore, by the chain rule we have

$$\begin{aligned} +\infty > \mathcal{E}_t(x) - \mathcal{E}_t(y) &= -\sum_{i=0}^{2N} \int_{t_i}^{t_{i+1}} \langle \mathrm{D}\mathcal{E}_t(\vartheta_{i+1}(s)), \vartheta'_{i+1}(s)\rangle\,\mathrm{d}s \\ &= \sum_{i=0}^{2N} \int_{t_i}^{t_{i+1}} \|\mathrm{D}\mathcal{E}_t(\vartheta_{i+1}(s))\|\|\vartheta'_{i+1}(s)\|\,\mathrm{d}s = (2N+1)\mathsf{c}_t(x;y), \end{aligned}$$

where the second equality is due to the fact that ϑ_{i+1} is an optimal jump transition on $[t_i, t_{i+1}]$ for $i = 0, ..., 2N$ (cf. (3.2)), and the third descends from the symmetry of the cost. Since N can be chosen arbitrarily large, the above equalities give a contradiction. □

In what follows we show that, if the energy $\mathcal{E}$ complies with the Łojasiewicz inequality (2.15) (which implies (E_4), as observed in Remark 2.5), the optimal jump transitions connecting jump points of Balanced Viscosity solutions (see Definition 2 and Theorem 2) have a further property. In fact, they have finite length.

Theorem 3.3. *In the setting of* (E_0)–(E_2), *assume in addition* (2.15), *and let* u *be a* Balanced Viscosity *solution to* (1.3). *For* $t \in [0,T]$ *fixed, let* $\vartheta \in \mathscr{A}^t_{u_-(t),u_+(t)}$ *be an optimal jump transition between* $u_-(t)$ *and* $u_+(t)$, *and let* $(\mathsf{a},\mathsf{b}) \subset [0,1]$ *be such that* $\vartheta|_{(\mathsf{a},\mathsf{b})} \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}((\mathsf{a},\mathsf{b});X)$, $\vartheta(s) \notin \mathcal{C}(t)$ *for all* $s \in (\mathsf{a},\mathsf{b})$, *and* $\vartheta(a)$, $\vartheta(b) \in \mathcal{C}(t)$. *Then the curve* $\vartheta|_{(\mathsf{a},\mathsf{b})}$ *has finite length.*

For the proof of Theorem 3.3 we shall exploit the crucial fact that, since ϑ is an *optimal jump transition*, (a reparameterization of) $\vartheta|_{(\mathsf{a},\mathsf{b})}$ is a gradient flow of the energy $\mathcal{E}_t$, cf. Proposition 3.1. This will allow us to develop arguments for gradient systems driven by energies satisfying the Łojasiewicz inequality, showing that the related trajectories have finite length.

Proof. Recall from Proposition 3.1 that $\vartheta|_{(\mathsf{a},\mathsf{b})}$ can be reparameterized to a curve $\tilde\vartheta$ on a (possibly unbounded) interval $(\tilde{\mathsf{a}},\tilde{\mathsf{b}})$ such that $\tilde\vartheta$ fulfills the gradient flow equation

$$\tilde\vartheta'(\sigma) + \mathrm{D}\mathcal{E}_t(\tilde\vartheta(\sigma)) = 0 \qquad \text{in } X \qquad \text{for a.a. } \sigma \in (\tilde{\mathsf{a}},\tilde{\mathsf{b}}). \tag{3.6}$$

Observe that for every $R>0$ there exists $\sigma_R > \tilde{\mathsf{a}}$ such that

$$\tilde\vartheta(\sigma) \in B_R(\vartheta(b)) \qquad \text{for every} \quad \sigma > \sigma_R. \tag{3.7}$$

Supposing for simplicity that $\mathcal{E}_t(\vartheta(b)) = 0$, observe that (2.15) reads

$$\big(\mathcal{E}_t(v)\big)^\theta \le C\|\mathrm{D}\mathcal{E}_t(v)\| \qquad \text{for all} \quad v \in B_R(\vartheta(b)). \tag{3.8}$$

From (3.6), (3.7) and (3.8) we deduce that for every $\tilde\sigma \in (\sigma_R,\tilde{\mathsf{b}})$ it holds

$$\begin{aligned}\int_{\sigma_R}^{\tilde\sigma} \|\tilde\vartheta'(\sigma)\|\,\mathrm{d}\sigma = \int_{\sigma_R}^{\tilde\sigma} \|\mathrm{D}\mathcal{E}_t(\tilde\vartheta(\sigma))\|\,\mathrm{d}\sigma &\le C\int_{\sigma_R}^{\tilde\sigma} \frac{\|\mathrm{D}\mathcal{E}_t(\tilde\vartheta(\sigma))\|^2}{\big(\mathcal{E}_t(\tilde\vartheta(\sigma))\big)^\theta}\,\mathrm{d}\sigma \\ &= -\frac{C}{1-\theta}\Big[\big(\mathcal{E}_t(\tilde\vartheta(\tilde\sigma))\big)^{1-\theta} - \big(\mathcal{E}_t(\tilde\vartheta(\sigma_R))\big)^{1-\theta}\Big] \le C'.\end{aligned}$$

Note that in the second equality we have used the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\big(\mathcal{E}_t(\tilde\vartheta(\sigma))\big)^{1-\theta} = -(1-\theta)\big(\mathcal{E}_t(\tilde\vartheta(\sigma))\big)^{-\theta}\|\mathrm{D}\mathcal{E}_t(\tilde\vartheta(\sigma))\|^2,$$

cf. also (3.2). In particular, we have obtained that $\int_{s_{\mathsf{b},R}}^{\mathsf{b}} \|\vartheta'(s)\|\,\mathrm{d}s < \infty$, for some $s_{\mathsf{b},R} \in (\mathsf{a},\mathsf{b})$. Arguing in a similar way, one can obtain that $\int_{\mathsf{a}}^{s_{\mathsf{a},R}} \|\vartheta'(s)\|\,\mathrm{d}s < \infty$ as well, for some $s_{\mathsf{a},R} \in (\mathsf{a},\mathsf{b})$, and this finishes the proof. □

4. Properties of the energy-dissipation integrals and cost

In order to prove Theorem 2.4 on properties of the cost function c, it is necessary to investigate the limit of the energy-dissipation integrals $\int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\|\vartheta'(s)\|\,\mathrm{d}s$, which enter in the definition (2.4) of c_t, along sequences of (admissible) curves. This section collects all the technical results underlying the proof of convergence to Dissipative Viscosity solutions.

With our first result, Proposition 4.1, we gain insight into the asymptotic behavior of the energy-dissipation integrals $\int_{t_1^n}^{t_2^n} \|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r$, where the curves $(\vartheta_n)_n$ are defined on intervals $[t_1^n,t_2^n]$ shrinking to a singleton $\{t\}$, whereas in the integrand $\|\mathrm{D}\mathcal{E}_\cdot(\vartheta_n(\cdot))\|\,\|\vartheta_n'(\cdot)\|$ the time variable is not fixed. Both Proposition 4.1 and a variant of it, Proposition 4.5, will be at the core of the proof of Theorem 2.4. Their proof is based on a reparameterization technique, combined with careful compactness arguments for the reparameterized curves.

Proposition 4.1. *Assume* (E_0)–(E_3). *Let* $t \in [0,T]$, $\rho>0$, *and* u_1, $u_2 \in X$ *be fixed and let* $(t_1^n)_n$, $(t_2^n)_n$, *with* $0 \le t_1^n \le t_2^n \le T$ *for every* $n \in \mathbb{N}$, *and* $(\vartheta_n)_n \subset \mathrm{AC}([t_1^n, t_2^n]; X)$ *fulfill*

$$t_1^n,\, t_2^n \to t, \quad \vartheta_n(t_1^n) \to u_1,\ \vartheta_n(t_2^n) \to u_2\,. \tag{4.1}$$

Then, the following implications hold:

(1) If

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r = 0, \tag{4.2}$$

then $u_1 = u_2$;

(2) In the case $u_1 \neq u_2$, *so that*

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r > 0, \tag{4.3}$$

there exists $\vartheta \in \mathscr{A}_{u_1,u_2}^t$ *such that*

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r \ \ge\ \int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\,\|\vartheta'(s)\|\,\mathrm{d}s. \tag{4.4}$$

Preliminarily, we need the following result.

Lemma 4.2. *Let* K *be a closed subset of some sublevel set* $\mathcal{S}_\rho$, *with* $\rho > 0$, *and suppose that for some* $t \in (0,T)$

$$\inf_{u\in K} \|\mathrm{D}\mathcal{E}_t(u)\| > 0. \tag{4.5}$$

Then, the inf *in* (4.5) *is attained, and there exists* $\alpha = \alpha(t) > 0$ *such that*

$$\min_{u\in K, s\in[t-\alpha,t+\alpha]} \|\mathrm{D}\mathcal{E}_s(u)\| > 0\,. \tag{4.6}$$

Proof. It follows from (E_1) that K is compact, therefore $\inf_{u\in K}\|\mathrm{D}\mathcal{E}_s(u)\|$ is attained for every $s\in[0,T]$, and the function $s \mapsto \min_{u\in K}\|\mathrm{D}\mathcal{E}_s(u)\|$ is continuous since $\mathcal{E}\in \mathrm{C}^1([0,T]\times X)$. Combining this fact with (4.5), we conclude (4.6). □

We are now in the position to develop the **proof of Proposition 4.1**: preliminarily, we observe that there exists $\rho>0$ such that the curves $(\vartheta_n)_n$ in (4.1) fulfill

$$\vartheta_n([t_1^n, t_2^n]) \subset \mathcal{S}_\rho \quad \text{for every } n\in\mathbb{N}\,. \tag{4.7}$$

Indeed, since $\mathcal{E}_0(\cdot)$ is continuous, from $\vartheta_n(t_i^n)\to u_i$ for $i=1,2$ we deduce that $\sup_n |\mathcal{E}_0(\vartheta_n(t_1^n))| + |\mathcal{E}_0(\vartheta_n(t_2^n))| \le C$. Hence, $\sup_n \mathcal{G}(\vartheta_n(t_1^n)) + \mathcal{G}(\vartheta_n(t_2^n)) \le C'$ in view of (2.1). We now apply the chain rule along the curve ϑ_n to conclude that

$$\mathcal{E}_t(\vartheta_n(t)) \le \mathcal{E}_{t_1^n}(\vartheta_n(t_1^n)) + \int_{t_1^n}^t \partial_t \mathcal{E}_r(\vartheta_n(r))\,\mathrm{d}r + \int_{t_1^n}^{t_2^n} \|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\|\vartheta_n'(r)\|\,\mathrm{d}r.$$

From the above estimate, we immediately conclude via the power estimate (E_2), the Gronwall Lemma, and condition (4.2) in the case $u_1=u_2$ (estimate (4.12) ahead in the case $u_1\neq u_2$, respectively), that $\sup_n \sup_{t\in[t_1^n,t_2^n]} \mathcal{E}_t(\vartheta_n(t)) \le C$. Then, (4.7) ensues.

***Ad Claim (1)*:** By contradiction, suppose that $u_1 \neq u_2$. Thanks to (E_3) the set $\mathcal{S}_\rho \cap \mathcal{C}(t)$ is finite (since the energy sublevel $\mathcal{S}_\rho$ is compact in X by (E_1)), hence there exists $\overline{\delta} = \overline{\delta}(t,u_1,u_2)$ such that

$$B(x,2\delta)\cap B(y,2\delta) = \emptyset, \quad \text{for every } x,y\in (\mathcal{C}(t)\cap\mathcal{S}_\rho)\cup\{u_1,u_2\} \text{ with } x\neq y \qquad \text{for all } 0<\delta\le\overline{\delta}. \tag{4.8}$$

Observe that u_1 may well belong to $\mathcal{C}(t)$ as well as not, and the same for u_2. Let us introduce the compact set K_δ defined by

$$K_\delta := \mathcal{S}_\rho \setminus \bigcup_{x\in(\mathcal{C}(t)\cap\mathcal{S}_\rho)\cup\{u_1,u_2\}} B(x,\delta) \tag{4.9}$$

and remark that $\min_{u\in K_\delta}\|\mathrm{D}\mathcal{E}_t(u)\|>0$. It follows from Lemma 4.2 that for some $\alpha=\alpha(t,u_1,u_2)>0$

$$e_\delta := \min_{u\in K_\delta, r\in[t-\alpha,t+\alpha]}\|\mathrm{D}\mathcal{E}_r(u)\|>0. \tag{4.10}$$

Note that $[t_1^n,t_2^n]\subset[t-\alpha,t+\alpha]$ for every n sufficiently large. Moreover, from (4.1) and from the definition of K_δ we obtain that $\{r\in[t_1^n,t_2^n]\,:\,\vartheta_n(r)\in K_\delta\}\neq\emptyset$ for every n large enough, and that $\vartheta_n(r_1)\in\partial B(u_1,\delta)$, $\vartheta_n(r_2)\in\partial B(u_2,\delta)$, for some r_1, $r_2\in\{r\in[t_1^n,t_2^n]\,:\,\vartheta_n(r)\in K_\delta\}$ with $r_1\neq r_2$. Thus, by (4.10),

$$\begin{aligned}\int_{t_1^n}^{t_2^n}\|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\|\vartheta_n'(r)\|dr &\geq \int_{\{r\in[t_1^n,t_2^n]\,:\,\vartheta_n(r)\in K_\delta\}}\|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\|\vartheta_n'(r)\|dr\\ &\geq e_\delta\int_{\{r\in[t_1^n,t_2^n]\,:\,\vartheta_n(r)\in K_\delta\}}\|\vartheta_n'(r)\|dr,\\ &\geq e_\delta \min_{x,y\in(\mathcal{C}(t)\cap K_\delta)\cup\{u_1,u_2\}}(\|x-y\|-2\delta)\doteq\eta. \end{aligned}\tag{4.11}$$

Observe that η is positive in view of (4.10) and of the definition of δ from (4.8). Thus we have a contradiction with (4.2).

***Ad Claim (2)*:** Suppose that $u_1\neq u_2$. Up to a subsequence we can suppose that there exists

$$\lim_{n\to\infty}\int_{t_1^n}^{t_2^n}\|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r=:L_t>0,$$

hence

$$\sup_{n\in\mathbb{N}}\int_{t_1^n}^{t_2^n}\|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r\leq C<\infty. \tag{4.12}$$

We split the proof of (4.4) in several steps.

Step 1: reparameterization. Let us define, for every $r\in[t_1^n,t_2^n]$,

$$s_n(r):=r+\int_{t_1^n}^{r}\|\mathrm{D}\mathcal{E}_\tau(\vartheta_n(\tau))\|\,\|\vartheta_n'(\tau)\|\,\mathrm{d}\tau.$$

Also, we set

$$s_1^n:=s_n(t_1^n)=t_1^n,\qquad s_2^n:=s_n(t_2^n),$$

and note that

$$s_1^n\to t,\qquad s_2^n\to(t+L_t)>t.$$

Since $s_n'>0$, we define

$$r_n(s):=s_n^{-1}(s)\ \text{ and }\ \tilde\vartheta_n(s):=\vartheta_n(r_n(s))\ \text{ for every }\ s\in[s_1^n,s_2^n].$$

Observe that

$$\tilde\vartheta_n(s_1^n)\to u_1,\qquad \tilde\vartheta_n(s_2^n)\to u_2, \tag{4.13}$$

that $(r_n)_n$ is equi-Lipschitz continuous and that

$$\|\mathrm{D}\mathcal{E}_{r_n(s)}(\tilde\vartheta_n(s))\|\,\|\tilde\vartheta_n'(s)\|=1-\frac{1}{1+\|\mathrm{D}\mathcal{E}_{r_n(s)}(\tilde\vartheta_n(s))\|\,\|\tilde\vartheta_n'(s)\|}\leq 1\ \text{ for a.a. }\ s\in(s_1^n,s_2^n). \tag{4.14}$$

The change of variable formula yields

$$\int_{t_1^n}^{t_2^n}\|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r=\int_{s_1^n}^{s_2^n}\|\mathrm{D}\mathcal{E}_{r_n(s)}(\tilde\vartheta_n(s))\|\,\|\tilde\vartheta_n'(s)\|\,\mathrm{d}s. \tag{4.15}$$

Step 2: localization and equicontinuity estimates. Let $\overline{\delta}$, $\delta>0$, K_δ, and e_δ be as in (4.8), (4.9), (4.10), and define the open set

$$A_n^\delta:=\left\{s\in(s_1^n,s_2^n)\,:\,\tilde\vartheta_n(s)\in\mathrm{int}(K_\delta)\right\}.$$

Observe that $A_n^\delta \neq \emptyset$ for every n sufficiently large, in view of the definition of K_δ and of (4.13). We write A_n^δ as the countable union of its connected components

$$A_n^\delta = \bigcup_{k=1}^{\infty} (a_{n,k}^\delta, b_{n,k}^\delta) \quad \text{with} \quad b_{n,k}^\delta \le a_{n,k+1}^\delta \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.16}$$

Inequality (4.14), the definition (4.10) of e_δ, and the definition of $a_{n,k}^\delta$ and $b_{n,k}^\delta$ imply that

$$e_\delta \|\tilde{\vartheta}_n'(s)\| \le \|\mathrm{D}\mathcal{E}_{r_n(s)}(\tilde{\vartheta}_n(s))\| \, \|\tilde{\vartheta}_n'(s)\| \le 1 \quad \text{for a.a.} \quad s \in \left(a_{n,k}^\delta, b_{n,k}^\delta\right). \tag{4.17}$$

Furthermore, it is clear that

$$\tilde{\vartheta}_n(a_{n,k}^\delta) \in \partial B(x,\delta), \quad \tilde{\vartheta}_n(b_{n,k}^\delta) \in \partial B(y,\delta) \quad \text{for some} \quad x,\, y \in (\mathcal{C}(t) \cap \mathcal{S}_\rho) \cup \{u_1, u_2\}. \tag{4.18}$$

Note that it may happen $x = y$. Nonetheless, from now on we will just focus on the case where $x \neq y$ in (4.18) and we will show that there is a *finite* number of intervals $(a_{n,k}^\delta, b_{n,k}^\delta)$ on which $\tilde{\vartheta}_n$ travels from one ball to another, centered at a different point in $(\mathcal{C}(t) \cap \mathcal{S}_\rho) \cup \{u_1, u_2\}$. In this way, we will conclude that the function ϑ of the statement consists of a finite number of $\mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}$-pieces. To this aim, let us introduce the set

$$\begin{gathered} B_n^\delta := \bigcup_{(a_{n,k}^\delta, b_{n,k}^\delta) \in \mathfrak{B}_n^\delta} (a_{n,k}^\delta, b_{n,k}^\delta) \quad \text{with} \\ \mathfrak{B}_n^\delta = \left\{ (a_{n,k}^\delta, b_{n,k}^\delta) \subset A_n^\delta \,:\, \tilde{\vartheta}_n(a_{n,k}^\delta) \in \partial B(x,\delta),\ \tilde{\vartheta}_n(b_{n,k}^\delta) \in \partial B(y,\delta) \text{ for } x,\, y \in \mathcal{C}(t) \cap \mathcal{S}_\rho,\ x \neq y \right\}. \end{gathered} \tag{4.19}$$

From (4.15), (4.17), and the definition of A_n^δ and B_n^δ, we obtain

$$\begin{aligned} \int_{t_1^n}^{t_2^n} \|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\| \, \|\vartheta_n'(r)\| \, \mathrm{d}r \quad &\ge \quad \int_{A_n^\delta} \|\mathrm{D}\mathcal{E}_{r_n(s)}(\tilde{\vartheta}_n(s))\| \, \|\tilde{\vartheta}_n'(s)\| \, \mathrm{d}s \\ &\ge \quad e_\delta \int_{B_n^\delta} \|\tilde{\vartheta}_n'(s)\| \, \mathrm{d}s \\ &\ge \quad e_\delta \sum_{(a_{n,k}^\delta, b_{n,k}^\delta) \in \mathfrak{B}_n^\delta} \overline{m}, \end{aligned} \tag{4.20}$$

where $0 < \overline{m} := \min_{\substack{x,y \in \mathcal{C}(t) \cup \{u_1,u_2\} \\ \pi_t(x) \neq \pi_t(y)}} (\|x-y\| - 2\overline{\delta})$. Inequality (4.20), together with estimate (4.12), implies that B_n^δ has a finite number $N(n,\delta)$ of components, more precisely

$$N(n,\delta) \le \frac{C}{e_\delta \overline{m}} \quad \text{for every } 0 < \delta \le \overline{\delta},\ n \in \mathbb{N}, \tag{4.21}$$

with C from (4.12). In what follows, we will show that we may take $N(n,\delta)$ to be bounded uniformly w.r.t. $n \in \mathbb{N}$ and $\delta > 0$ (cf. (4.24) ahead).

For this, we need to fix some preliminary remarks. In view of the ordering assumed in (4.16), we have that

$$\tilde{\vartheta}_n(a_{n,1}^\delta) \in \partial B(u_1,\delta), \qquad \tilde{\vartheta}_n(b_{n,N(n,\delta)}^\delta) \in \partial B(u_2,\delta). \tag{4.22}$$

Also, observe that, up to throwing some of the intervals $(a_{n,k}^\delta, b_{n,k}^\delta) \in \mathfrak{B}_n^\delta$ away, we may suppose that for every fixed $x \in (\mathcal{C}(t) \cap \mathcal{S}_\rho) \cup \{u_1, u_2\}$, if $\tilde{\vartheta}(a_{n,k}^\delta) \in \partial B(x,\delta)$ for some k, then $\tilde{\vartheta}(b_{n,m}^\delta) \notin \partial B(x,\delta)$ for every $m > k$. Finally, note that for all $(a_{n,k}^\delta, b_{n,k}^\delta) \in \mathfrak{B}_n^\delta$ there holds

$$\frac{1}{e_\delta}(b_{n,k}^\delta - a_{n,k}^\delta) \ge \|\tilde{\vartheta}_n(b_{n,k}^\delta) - \tilde{\vartheta}_n(a_{n,k}^\delta)\| \ge \overline{m}, \tag{4.23}$$

where the first inequality ensues from (4.17), and the second one is due to the definition of $\overline{m}$.

Remark 4.3. Observe that a bound for $N(n,\delta)$, uniform with respect to $n \in \mathbb{N}$ *and* $\delta > 0$, cannot be directly deduced from (4.21) since the constant $C/(e_\delta \overline{m})$ grows as δ decreases. Indeed, e_δ goes to zero as $\delta \to 0$.

Step 3: compactness. We will prove the following
Claim: there exist a sequence $(n_j, \delta_{m_j})_j$, such that

$$N(n_j, \delta_{m_j}) = N \qquad \textit{for every } j \in \mathbb{N}, \tag{4.24}$$

a partition

$$\{t \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le ... \le \alpha_N < \beta_N \le t + L_t\} \quad \textit{of} \quad [t, t+L_t], \quad \textit{and} \tag{4.25}$$

a curve $\vartheta \in \mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}\left(\bigcup_{k=1}^N (\alpha_k, \beta_k); X\right)$ such that, in the limit $j \to \infty$,

$$\begin{aligned} &\tilde{\vartheta}_{n_j} \to \vartheta \quad \textit{uniformly on compact subsets of } \bigcup_{k=1}^N (\alpha_k, \beta_k), \\ &\tilde{\vartheta}'_{n_j} \rightharpoonup^* \vartheta' \quad \textit{in } L^\infty(\alpha_k + \eta, \beta_k - \eta; X) \quad \textit{for every } \eta > 0 \textit{ and } k = 1, \ldots, N. \end{aligned} \tag{4.26}$$

Therefore, $\vartheta(s) \in \mathcal{S}_\rho$ for every $s \in \bigcup_{k=1}^N (\alpha_k, \beta_k)$.

First of all, let us observe that, since $(N(n,\delta))_n$ is a bounded sequence by (4.21), there exists a subsequence $(n_l^\delta)_l$ and an integer $N(\delta)$ such that

$$N(n_l^\delta, \delta) \to N(\delta) \quad \text{as } l \to \infty. \tag{4.27}$$

Clearly, since $u_1 \neq u_2$, taking (4.13) into account we see that $N(\delta) \ge 1$ for every $0 < \delta \le \overline{\delta}$. Also, for every fixed $n \in \mathbb{N}$, we have that

$$N(n, \delta) \ \text{ decreases as } \delta \text{ decreases.} \tag{4.28}$$

Indeed, if $\delta_1 > \delta_2$, then $K_{\delta_1} \subset K_{\delta_2}$ and in turn $B_n^{\delta_1} \subset B_n^{\delta_2}$. This means that, for every $k \in \{1, ..., N(n, \delta_1)\}$, we have

$$(a^{\delta_1}_{n,k}, b^{\delta_1}_{n,k}) \subset (a^{\delta_2}_{n,j_k}, b^{\delta_2}_{n,j_k}), \qquad \text{for some} \quad j_k \in \{1, ..., N(n, \delta_2)\}. \tag{4.29}$$

At the same time, $(a^\delta_{n,k}, b^\delta_{n,k}) \subset [s_1^n, s_2^n]$ for every $0 < \delta \le \overline{\delta}$, for every n, and for every $k = 1, ..., N(n,\delta)$, and there holds $[s_1^n, s_2^n] \to [t, t+L_t]$ as $n \to \infty$. Therefore we may suppose that there exists $\eta > 0$ such that for every $n \in \mathbb{N}$ and $k = 1, ..., N(n,\delta)$ there holds $(a^\delta_{n,k}, b^\delta_{n,k}) \subset [t - \eta, t + L_t + \eta]$. This fact, coupled with (4.29), gives (4.28).

In order to prove (4.24), we develop the following diagonal argument. Consider a sequence $(\delta_m)_m \subset (0, \overline{\delta}]$ such that $\delta_m \to 0$, as $m \to \infty$. Using (4.27) and (4.28), it is possible to construct for each $m \in \mathbb{N}$ a subsequence $(n_l^m)_l$, where n_l^m is a short-hand notation for $n_l^{\delta_m}$, such that

$$N(n_l^m, \delta_m) \to N(\delta_m) \qquad \text{as} \qquad l \to \infty \quad \text{ for every } m \in \mathbb{N}, \tag{4.30}$$

$$N(\delta_1) \ge N(\delta_2) \ge ... \ge N(\delta_m) \ge ... \ge 1, \tag{4.31}$$

and

$$a^{\delta_m}_{n_l^m, k} \to \alpha_k^m, \quad b^{\delta_m}_{n_l^m, k} \to \beta_k^m \quad \text{as } l \to \infty \quad \text{for all } k = 1, ..., N(\delta_m) \text{ and all } m \in \mathbb{N}. \tag{4.32}$$

Due to (4.23), one has $\alpha_k^m < \beta_k^m$ for all $k = 1, ..., N(\delta_m)$ and all $m \in \mathbb{N}$. Now, since $(N(\delta_m))_m$ is a decreasing sequence of integers, we may suppose that

$$\text{there exists} \ \ N \in \mathbb{N} \text{ such that } \ N(\delta_m) = N \ \text{ for every } m \in \mathbb{N}, \tag{4.33}$$

and that there exist the limits

$$\alpha_k := \lim_{m \to \infty} \alpha_k^m, \qquad \beta_k := \lim_{m \to \infty} \beta_k^m \qquad \text{for every } k = 1, ..., N. \tag{4.34}$$

Note that the points α^k, β^k satisfy (4.25) with N from (4.33). Hence, choose $k \in \{1, ..., N\}$ and observe that for every $j \in \mathbb{N}$ arbitrarily large there exists m_j and l_j such that

$$\left[\alpha_k + \frac{1}{j}, \beta_k - \frac{1}{j}\right] \subset \left(a^{\delta_{m_j}}_{n_l^{m_j}, k}, b^{\delta_{m_j}}_{n_l^{m_j}, k}\right) \qquad \text{for every } l \ge l_j, \tag{4.35}$$

and

$$\left|a^{\delta_{m_j}}_{n^{m_j}_l,k}-\alpha^{m_j}_k\right|+\left|b^{\delta_{m_j}}_{n^{m_j}_l,k}-\beta^{m_j}_k\right|\le\frac{1}{m_j}\qquad\text{for every } l\ge l_j. \tag{4.36}$$

Moreover, we can suppose that m_j, $l_j\to\infty$, as $j\to\infty$. We combine (4.35) with estimate (4.17) for $(\tilde{\vartheta}'_n)_n$ and the fact that $\tilde{\vartheta}_n(s)\in\mathcal{S}_\rho$ for every $s\in[s^n_1,s^n_2]$. The Arzelà-Ascoli Theorem ensures that, up to a subsequence,

$$\tilde{\vartheta}_{n^{m_j}_l}\to\vartheta\quad\text{in}\quad \mathrm{C}^0\left(\left[\alpha_k+\frac{1}{j},\beta_k-\frac{1}{j}\right];X\right)\quad\text{as } l\to\infty, \tag{4.37a}$$

$$\tilde{\vartheta}'_{n^{m_j}_l}\rightharpoonup^*\vartheta'\quad\text{in } L^\infty\left(\alpha_k+\frac{1}{j},\beta_k-\frac{1}{j};X\right)\quad\text{as } l\to\infty \tag{4.37b}$$

for some $\vartheta\in\mathrm{C}^{\mathrm{lip}}\left(\left[\alpha_k+\frac{1}{j},\beta_k-\frac{1}{j}\right];X\right)$.

If at each step j we extract a subsequence from the previous one, we may obtain a sequence $(n^{m_j}_{l_j})_j$, which we relabel by $(n_j)_j$, and a unique $\vartheta\in\mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}\left(\bigcup_{k=1}^N(\alpha_k,\beta_k);X\right)$, such that for all $j\in\mathbb{N}$ and $k\in\{1,...,N\}$ there holds

$$\left[\alpha_k+\frac{1}{j},\beta_k-\frac{1}{j}\right]\subset\left(\tilde{a}_{j,k},\tilde{b}_{j,k}\right),\quad\text{where}\quad\tilde{a}_{j,k}:=a^{\delta_{m_j}}_{n_j,k},\quad\tilde{b}_{j,k}:=b^{\delta_{m_j}}_{n_j,k}, \tag{4.38}$$

$$\|\tilde{\vartheta}_{n_j}(s)-\vartheta(s)\|<\frac{1}{j}\quad\text{for every } s\in\left[\alpha_k+\frac{1}{j},\beta_k-\frac{1}{j}\right]. \tag{4.39}$$

Therefore, we have proved (4.24)–(4.26). From (4.34) and (4.36) we obtain also that

$$\tilde{a}_{j,k}\to\alpha_k,\qquad\tilde{b}_{j,k}\to\beta_k\quad\text{as } j\to\infty, \tag{4.40}$$

where $\tilde{a}_{j,k}$, $\tilde{b}_{j,k}$ are defined in (4.38). These observations will be useful in Step 5.

Remark 4.4. Let us recall (cf. (4.24)), that N is the number of the pieces of the trajectory of $\tilde{\vartheta}_{n_j}$ which go from $\partial B(x,\delta_{m_j})$ to $\partial B(y,\delta_{m_j})$, for some x, $y\in(\mathcal{C}(t)\cap\mathcal{S}_\rho)\cup\{u_1,u_2\}$ with $x\neq y$. Thus, we have so far excluded that, for example, on some interval $(\tilde{a}_{j,k},\tilde{b}_{j,k})$ the trajectory of $\tilde{\vartheta}_{n_j}$ runs from $\partial B(x,\delta_{m_j})$ to $\partial B(x,\delta_{m_j})$. Moreover, so far we have overlooked what happens to the trajectory of $\tilde{\vartheta}_{n_j}$ on the interval $[\tilde{b}_{j,k},\tilde{a}_{j,k+1}]$. It is not difficult to imagine that, if $\beta_k<\alpha_{k+1}$ some "loops" around a certain connected component of $(\mathcal{C}(t)\cap\mathcal{S}_\rho)\cup\{u_1,u_2\}$ may have been created by the trajectories of $\tilde{\vartheta}_{n_j}$ on $[\tilde{b}_{j,k},\tilde{a}_{j,k+1}]$ as $j\to\infty$. Note that we cannot deduce that the number of these loops is definitely bounded, as we have done for $N(n_j,\delta_{m_j})$.

Step 4: passage to the limit. In order to take the limit of the integral term in (4.4), we observe that

$$\begin{aligned}\int_{t_1^{n_j}}^{t_2^{n_j}}\|\mathrm{D}\mathcal{E}_r(\vartheta_{n_j}(r))\|\,\|\vartheta'_{n_j}(r)\|\,\mathrm{d}r&\ge\sum_{k=1}^N\int_{\tilde{a}_{j,k}}^{\tilde{b}_{j,k}}\|\mathrm{D}\mathcal{E}_{r_{n_j}(s)}(\tilde{\vartheta}_{n_j}(s))\|\,\|\tilde{\vartheta}'_{n_j}(s)\|\,\mathrm{d}s\\&\ge\sum_{k=1}^N\int_{\alpha_k+1/j}^{\beta_k-1/j}\|\mathrm{D}\mathcal{E}_{r_{n_j}(s)}(\tilde{\vartheta}_{n_j}(s))\|\,\|\tilde{\vartheta}'_{n_j}(s)\|\,\mathrm{d}s,\end{aligned} \tag{4.41}$$

where we have used (4.24) and (4.38). We now pass to the limit as $j\to\infty$ in (4.41). Observe that, since $(r_{n_j}(s))_j\subset[t_1^{n_j},t_2^{n_j}]$ for every $s\in[s_1^{n_j},s_2^{n_j}]$, then $r_{n_j}(s)\to t$ as $j\to\infty$. Hence, the first convergence in (4.26) yields

$$\lim_{j\to\infty}\|\mathrm{D}\mathcal{E}_{r_{n_j}(s)}(\tilde{\vartheta}_{n_j}(s))\|=\|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\text{ for every } s\in[\alpha_k+\eta,\beta_k-\eta],\ \eta>0,\ k=1,\dots,N. \tag{4.42}$$

Combining (4.42) with the second of (4.26) and applying Ioffe's Theorem [19], we have that

$$\liminf_{j\to\infty}\int_{\alpha_k+1/j}^{\beta_k-1/j}\|\mathrm{D}\mathcal{E}_{r_{n_j}(s)}(\tilde{\vartheta}_{n_j}(s))\|\,\|\tilde{\vartheta}'_{n_j}(s)\|\,\mathrm{d}s\ge\int_{\alpha_k+\eta}^{\beta_k-\eta}\|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\,\|\vartheta'(s)\|\,\mathrm{d}s \tag{4.43}$$

for all $\eta > 0,\ k = 1, \dots, N$. From (4.43) and (4.12) it follows that the map $s \mapsto \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\| \, \|\vartheta'(s)\|$ is integrable on (α_k, β_k) for all $k = 1, \dots, N$. Summing up, we conclude that

$$\liminf_{j\to\infty} \int_{t_1^{n_j}}^{t_2^{n_j}} \|\mathrm{D}\mathcal{E}_r(\vartheta_{n_j}(r))\| \, \|\vartheta'_{n_j}(r)\| \, \mathrm{d}r \geq \sum_{k=1}^{N} \int_{\alpha_k}^{\beta_k} \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\| \, \|\vartheta'(s)\| \, \mathrm{d}s. \tag{4.44}$$

Step 5: conclusion of the proof of Proposition 4.1. Relying on the previously proved part (1) of the statement, the first of (4.26), and the inclusion in (4.38), we will now show that

$$\lim_{s\to\alpha_1^+} \vartheta(s) = u_1, \qquad \lim_{s\to\beta_N^-} \vartheta(s) = u_2 \tag{4.45}$$

and that

$$\lim_{s\to\beta_k^-} \vartheta(s) = \lim_{s\to\alpha_{k+1}^+} \vartheta(s) = x \qquad \text{for some } x \in (\mathcal{C}(t) \cap \mathcal{S}_\rho) \cup \{u_1, u_2\}, \tag{4.46}$$

for every $k = 1, ..., N-1$. Let us only check the first limit in (4.45), since the other limits can be verified in a similar way. Let $(s_i)_i \subset (\alpha_1, \beta_1)$ be a sequence such that $s_i \to \alpha_1^+$ as $i \to \infty$. We want to prove that

$$\lim_{i\to\infty} \vartheta(s_i) = u_1. \tag{4.47}$$

Now, let us fix $i \in \mathbb{N}$: the first of (4.26) gives that $\tilde{\vartheta}_{n_j}(s_i) \to \vartheta(s_i)$ as $j \to \infty$. In particular, there exists a strictly increasing sequence $(j_i)_i$ such that

$$\|\tilde{\vartheta}_{n_{j_i}}(s_i) - \vartheta(s_i)\| \leq \frac{1}{i} \qquad \text{for every } i \in \mathbb{N}. \tag{4.48}$$

Note that $\tilde{a}_{j_i,1} \to \alpha_1$ as $i \to \infty$, in view of (4.40). Moreover, from the definition (4.38) of $\tilde{a}_{j_i,1}$ and (4.22) it follows that

$$\lim_{i\to\infty} \tilde{\vartheta}_{n_{j_i}}(\tilde{a}_{j_i,1}) = u_1 \,. \tag{4.49}$$

Next, observe that from (4.14) and from the fact that $s_i,\ \tilde{a}_{j_i,1} \to \alpha_1$ as $i \to \infty$, we have that

$$\int_{s_i}^{\tilde{a}_{j_i,1}} \|\mathrm{D}\mathcal{E}_{r_{n_{j_i}}(s)}(\tilde{\vartheta}_{n_{j_i}}(s))\| \, \|\tilde{\vartheta}'_{n_{j_i}}(s)\| \, \mathrm{d}s \leq |s_i - \tilde{a}_{j_i,1}| \to 0 \quad \text{as } i \to \infty. \tag{4.50}$$

Also, we have that

$$\int_{s_i}^{\tilde{a}_{j_i,1}} \|\mathrm{D}\mathcal{E}_{r_{n_{j_i}}(s)}(\tilde{\vartheta}_{n_{j_i}}(s))\| \, \|\tilde{\vartheta}'_{n_{j_i}}(s)\| \, \mathrm{d}s = \int_{r_i}^{\tilde{r}_i} \|\mathrm{D}\mathcal{E}_r(\vartheta_{n_{j_i}}(r))\| \, \|\vartheta'_{n_{j_i}}(r)\| \, \mathrm{d}r, \tag{4.51}$$

for some $(r_i)_i,\ (\tilde{r}_i)_i \subset \left[t_1^{n_{j_i}}, t_2^{n_{j_i}}\right]$, where

$$\vartheta_{n_{j_i}}(r_i) = \tilde{\vartheta}_{n_{j_i}}(s_i), \quad \vartheta_{n_{j_i}}(\tilde{r}_i) = \tilde{\vartheta}_{n_{j_i}}(\tilde{a}_{j_i,1}) \qquad \text{for every } i \in \mathbb{N}. \tag{4.52}$$

Furthermore, we can suppose that, up to a subsequence, $r_i \leq \tilde{r}_i$ for every i, and that

$$\vartheta_{n_{j_i}}(r_i) \to \hat{x} \qquad \text{for some } \hat{x} \in X. \tag{4.53}$$

We combine this fact with the limit

$$\vartheta_{n_{j_i}}(\tilde{r}_i) \to u_1, \qquad \text{as } i \to \infty$$

which comes from (4.49) and the second of (4.52), and apply Proposition 4.1 (1) to the sequence $(\vartheta_{n_{j_i}})_i$ on the shrinking interval $[r_i, \tilde{r}_i]$, using that $\int_{r_i}^{\tilde{r}_i} \|\mathrm{D}\mathcal{E}_r(\vartheta_{n_{j_i}}(r))\| \, \|\vartheta'_{n_{j_i}}(r)\| \, \mathrm{d}r \to 0$ by (4.51). Therefore,

$$\tilde{\vartheta}_{n_{j_i}}(s_i) = \vartheta_{n_{j_i}}(r_i) \to u_1 \qquad \text{as } i \to \infty, \tag{4.54}$$

Inequality (4.48) and convergence (4.54) imply (4.47).

By the limits in (4.45) and (4.46) we can trivially extend ϑ to the whole interval $[\alpha_1, \beta_N]$ and obtain, from (4.44), that

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\| \, \|\vartheta'_n(r)\| \, \mathrm{d}r \geq \int_{\alpha_1}^{\beta_N} \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\| \, \|\vartheta'(s)\| \, \mathrm{d}s.$$

Thus, we have deduced the lim inf-inequality (4.4), with a curve defined on the interval $[a,b]=[\alpha_1,\beta_N]$.

Finally, by the scaling invariance of the integral on the right-hand side of (4.4), we can reparameterize the limiting curve ϑ in such a way as to have it defined on the interval $[0,1]$, in accord with the definition (2.3) of admissible curves. This concludes the proof. ∎

We now give a variant of Proposition 4.1, in which the curves ϑ_n belong to the class $\mathscr{A}^t_{u_1^n,u_2^n}$, for $t\in[0,T]$ fixed, and with (u_1^n), $(u_2^n)\subset X$ given sequences, and the integrands $\|\mathrm{D}\mathcal{E}_r(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|$ in (4.4) are replaced by $\|\mathrm{D}\mathcal{E}_t(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|$.

Proposition 4.5. *Assume* (E$_0$)–(E$_3$). *Given $t\in[0,T]$, $\rho>0$, and u_1, $u_2\in X$, let $(u_1^n)_n$, $(u_2^n)_n$ fulfill $u_1^n\to u_1$, $u_2^n\to u_2$, and let $\vartheta_n\in\mathscr{A}^t_{u_1^n,u_2^n}$ for every n. The following two implications hold:*

(1) If $\liminf_{n\to\infty}\int_0^1\|\mathrm{D}\mathcal{E}_t(\vartheta_n(s))\|\,\|\vartheta_n'(s)\|\,\mathrm{d}s=0$, then $u_1=u_2$;

(2) If $u_1\neq u_2$, so that

$$\liminf_{n\to\infty}\int_0^1\|\mathrm{D}\mathcal{E}_t(\vartheta_n(s))\|\,\|\vartheta_n'(s)\|\,\mathrm{d}s>0,$$

then there exists $\vartheta\in\mathscr{A}^t_{u_1,u_2}$ such that

$$\liminf_{n\to\infty}\int_0^1\|\mathrm{D}\mathcal{E}_t(\vartheta_n(s))\|\,\|\vartheta_n'(s)\|\,\mathrm{d}s\ \geq\ \int_0^1\|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\,\|\vartheta'(s)\|\,\mathrm{d}s. \tag{4.55}$$

Proof. We will only sketch the proof, dwelling on the differences with the argument for Proposition 4.1.

By the very same arguments developed at the beginning of Prop. 4.1, we conclude that the (images of) all the curves ϑ_n in fact lie in some energy sublevel.

The proof of ***Claim (1)*** follows the very same lines as for Prop. 4.1.

Ad Claim (2): By definition of $\mathscr{A}^{t,\tilde\rho}_{u_1^n,u_2^n}$, we have that there exists a partition $0=\tau_0^n<\tau_1^n<...<\tau_{M_n}^n=1$ such that $\vartheta_n(0)=u_1^n$, $\vartheta_n(1)=u_2^n$, $\vartheta_n|_{(\tau_i^n,\tau_{i+1}^n)}\in\mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}((\tau_i^n,\tau_{i+1}^n);X)$ for all $i=0,\dots,M_n-1$, and the curves $\vartheta_n|_{(\tau_i^n,\tau_{i+1}^n)}$ connect different connected components of $\mathcal{C}(t)\cap\mathcal{S}_{\tilde\rho}$. In analogy with the proof of Prop. 4.1, we use the rescaling r_n defined as the inverse of the function $s_n(r):=\int_0^r\|\mathrm{D}\mathcal{E}_t(\vartheta_n(s))\|\,\|\vartheta_n'(s)\|\,\mathrm{d}s$, for $r\in[a,b]$, and set $\tilde\vartheta_n(s):=\vartheta_n(r_n(s))$ for every $s\in\left[\tilde a_n,\tilde b_n\right]$, where $\tilde a_n:=r_n^{-1}(0)$ and $\tilde b_n:=r_n^{-1}(1)$. Then, there exists a partition $\tilde a_n=\sigma_0^n<\sigma_1^n<...<\sigma_{M_n}^n=\tilde b_n$, with $\tilde\vartheta_n(\tilde a_n)=u_1^n$, $\tilde\vartheta_n(\tilde b_n)=u_2^n$, such that $\tilde\vartheta_n|_{(\sigma_i^n,\sigma_{i+1}^n)}\in\mathrm{C}^{\mathrm{lip}}_{\mathrm{loc}}((\sigma_i^n,\sigma_{i+1}^n);X)$ for all $i=0,\dots,M_n-1$. Moreover,

$$\|\mathrm{D}\mathcal{E}_t(\tilde\vartheta_n(s))\|\,\|\tilde\vartheta_n'(s)\|\leq 1\quad\text{for a.a.}\quad s\in(\tilde a_n,\tilde b_n),$$

and

$$\int_0^1\|\mathrm{D}\mathcal{E}_t(\vartheta_n(r))\|\,\|\vartheta_n'(r)\|\,\mathrm{d}r=\int_{\tilde a_n}^{\tilde b_n}\|\mathrm{D}\mathcal{E}_t(\tilde\vartheta_n(s))\|\,\|\tilde\vartheta_n'(s)\|\,\mathrm{d}s.$$

We now define for every $i=0,...,M_n-1$ the sets $A_n^{i,\delta}:=\left\{s\in(\sigma_i^n,\sigma_{i+1}^n)\,:\,\tilde\vartheta_n(s)\in\mathrm{int}(K_\delta)\right\}$, where K_δ is defined as in (4.9), and write $A_n^{i,\delta}$ as the countable union of its connected components, i.e. $A_n^{i,\delta}=\bigcup_{k=1}^\infty(a_{n,k}^{i,\delta},b_{n,k}^{i,\delta})$. Similarly, we consider the analogues of the sets B_n^δ (4.19), viz.

$$B_n^{i,\delta}:=\bigcup_{(a_{n,k}^{i,\delta},b_{n,k}^{i,\delta})\in\mathfrak{B}_n^{i,\delta}}(a_{n,k}^{i,\delta},b_{n,k}^{i,\delta})\quad\text{with}$$

$$\mathfrak{B}_n^{i,\delta}=\left\{(a_{n,k}^{i,\delta},b_{n,k}^{i,\delta})\subset A_n^{i,\delta}\ :\tilde\vartheta(a_{n,k}^{i,\delta})\in\partial B(x,\delta),\ \tilde\vartheta(b_{n,k}^{i,\delta})\in\partial B(y,\delta)\text{ for }x,\,y\in(\mathcal{C}(t)\cap\mathcal{S}_\rho)\text{ with }x\neq y\right\}, \tag{4.56}$$

for $i = 1, .., M_n - 1$. We denote by $N(i,n,\delta)$ the cardinality of the set $\mathfrak{B}_n^{i,\delta}$. Then, we have

$$C \geq \int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta_n(r))\| \, \|\vartheta_n'(r)\| \, \mathrm{d}r \quad \geq \quad e_\delta M_n \sum_{(a_{n,k}^{i,\delta}, b_{n,k}^{i,\delta}) \in \mathfrak{B}_n^{i,\delta}} \overline{m},$$

where $0 < \overline{m} := \min_{n\in\mathbb{N}} \min_{\substack{x,y \in (\mathcal{C}(t)\cap \mathcal{S}_\rho)\cup\{u_1,u_2\} \\ x\neq y}} (\|x-y\| - 2\overline{\delta})$, with $\overline{\delta}$ as in (4.8) and C is as in (4.12). Therefore, we conclude the estimate

$$M_n N(i,n,\delta) \leq \frac{C}{e_\delta \overline{m}} \quad \text{for every } 0 < \delta \leq \overline{\delta}, \quad n \in \mathbb{N}.$$

Observing that we may suppose $M_n,\ N(i,n,\delta) \geq 1$, we conclude a bound for both $(M_n)_n$ and $((N(i,n,\delta))_{i=1}^{M_n})_n$.

The proof can be then carried out by suitably adapting the argument for Proposition 4.1. □

We are now in the position to develop the

Proof of Theorem 2.4. Ad (1): Suppose $\mathsf{c}_t(u_1;u_2) = 0$. Then, by definition of $\mathsf{c}_t(u_1;u_2)$, there exists a sequence $(\vartheta_n)_n \subset \mathscr{A}_{u_1,u_2}^t$ such that

$$0 = \lim_{n\to\infty} \int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta_n(s))\| \, \|\vartheta_n'(s)\| \, \mathrm{d}s.$$

Then, it follows from Prop. 4.5 *(1)* that $u_1 = u_2$.

Ad (2): consider the nontrivial case $u_1 \neq u_2$ and a curve $\vartheta \in \mathscr{A}_{u_1,u_2}^t$. Define $\zeta : [0,1] \to X$ by $\zeta(s) := \vartheta(1-s)$. Then $\zeta \in \mathscr{A}_{u_2,u_1}^t$ and

$$\mathsf{c}_t(u_1;u_2) \leq \int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(r))\| \|\vartheta'(r)\| \, \mathrm{d}r = \int_0^1 \|\mathrm{D}\mathcal{E}_t(\zeta(r))\| \|\zeta'(r)\| \, \mathrm{d}r\,.$$

With this argument we easily conclude that $\mathsf{c}_t(u_1;u_2) \leq \mathsf{c}_t(u_2;u_1)$. Interchanging the role of u_1 and u_2 we conclude the symmetry of the cost.

Ad (3): We use the direct method of the calculus of variations: let $(\vartheta_n)_n \subset \mathscr{A}_{u_1,u_2}^t$ be a minimizing sequence for $\mathsf{c}_t(u_1;u_2)(< \infty)$. Applying Proposition 4.5 *(2)* to the curves ϑ_n (in fact, we are in the case $u_1 \neq u_2$), we conclude.

Ad (4): We confine the discussion to the case in which $\mathsf{c}_t(u_1;u_3) > 0$ and $\mathsf{c}_t(u_3;u_2) > 0$, as the other cases can be treated with simpler arguments. Let $\vartheta_{1,3}$ and $\vartheta_{3,2}$ be two optimal curves for $\mathsf{c}_t(u_1;u_3)$ and $\mathsf{c}_t(u_3;u_2)$, respectively. Set

$$\vartheta_{1,2}(s) := \begin{cases} \vartheta_{1,3}(2s) & \text{for } s \in \left[0, \frac{1}{2}\right], \\ \vartheta_{3,2}(2s-1) & \text{for } s \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Since $u_3 \in \mathcal{C}(t)$, it is immediate to check that $\vartheta_{1,2} \in \mathscr{A}_{u_1,u_2}^t$, and by the definition of $\mathsf{c}_t(u_1;u_2)$ we obtain

$$\begin{aligned} \mathsf{c}_t(u_1;u_2) \leq \int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta_{1,2}(s))\| \, \|\vartheta'(s)\| \, \mathrm{d}s &= 2\int_0^{1/2} \|\mathrm{D}\mathcal{E}_t(\vartheta_{1,3}(2s))\| \, \|\vartheta_{1,3}'(2s)\| \, \mathrm{d}s \\ &\quad + 2\int_{1/2}^1 \|\mathrm{D}\mathcal{E}_t(\vartheta_{3,2}(2s-1))\| \, \|\vartheta_{3,2}'(2s-1)\| \, \mathrm{d}s \\ &= \mathsf{c}_t(u_1;u_3) + \mathsf{c}_t(u_3;u_2), \end{aligned}$$

and conclude (2.6).

Ad (5): (2.7) is a direct consequence of Proposition 4.1.

Ad (6): We may suppose that $u_1 \neq u_2$ (otherwise, $\mathsf{c}_t(u_1;u_2) = 0$ and the desired inequality trivially follows), and that $\liminf_{k\to\infty} \mathsf{c}_t(u_1^k;u_2^k) < \infty$. By definition of $\mathsf{c}_t(u_1^k;u_2^k)$, we have that for every $k \geq 1$ there exists $\vartheta_k \in \mathscr{A}^t_{u_1^k,u_2^k}$ such that

$$\int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta_k(s))\|\|\vartheta_k'(s)\|\,\mathrm{d}s \;\le\; \mathsf{c}_t(u_1^k;u_2^k)+\frac{1}{k}.$$

By Prop. 4.5 (2), there exists $\vartheta \in \mathscr{A}^t_{u_1,u_2}$ such that

$$\int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\|\|\vartheta'(s)\|\,\mathrm{d}s \;\le\; \liminf_{k\to\infty}\int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta_k(s))\|\|\vartheta_k'(s)\|\,\mathrm{d}s \;\le\; \liminf_{k\to\infty}\mathsf{c}_t(u_1^k;u_2^k).$$

This concludes the proof, in view of the definition of $\mathsf{c}_t(u_1;u_2)$. ∎

5. Proof of the main results

5.1. **Proof of Theorem 1.** Let $(u_{\varepsilon_n})_n \subset H^1(0,T;X)$ be a sequence of solutions to the Cauchy problem for (1.1), supplemented with initial conditions $(u^0_{\varepsilon_n})_n$ fulfilling (2.12).

In the upcoming result we derive from the energy identity (2.2) for the sequence family $(u_{\varepsilon_n})_n$, namely

$$\int_s^t \left(\frac{\varepsilon_n}{2}\|u_{\varepsilon_n}'(r)\|^2+\frac{1}{2\varepsilon_n}\|\mathrm{D}\mathcal{E}_r(u_{\varepsilon_n}(r))\|^2\right)\mathrm{d}r+\mathcal{E}_t(u_{\varepsilon_n}(t))=\mathcal{E}_s(u_{\varepsilon_n}(s))+\int_s^t\partial_t\mathcal{E}_r(u_{\varepsilon_n}(r))\,\mathrm{d}r, \tag{5.1}$$

a series of a priori estimates, which will allow us to prove a preliminary compactness result, Proposition 5.2 below.

Proposition 5.1 (A priori estimates)**.** *Assume* (E_0)–(E_2)*. Then, there exists a constant* $C>0$ *such that for every* $n\in\mathbb{N}$ *the following estimates hold*

$$\sup_{t\in[0,T]}\mathcal{G}(u_{\varepsilon_n}(t))+\sup_{t\in[0,T]}|\partial_t\mathcal{E}_t(u_{\varepsilon_n}(t))|\le C, \tag{5.2}$$

$$\int_s^t \left(\frac{\varepsilon_n}{2}\|u_{\varepsilon_n}'(r)\|^2+\frac{1}{2\varepsilon_n}\|\mathrm{D}\mathcal{E}_r(u_{\varepsilon_n}(r))\|^2\right)\mathrm{d}r\le C \quad \text{for all } 0\le s\le t\le T. \tag{5.3}$$

Proof. Combining (2.2) with estimate (E_2) for the power function $\partial_t\mathcal{E}$, we find that

$$\int_0^t \left(\frac{\varepsilon_n}{2}\|u_{\varepsilon_n}'(s)\|^2+\frac{1}{2\varepsilon_n}\|\mathrm{D}\mathcal{E}_s(u_{\varepsilon_n}(s))\|^2\right)\mathrm{d}s+\mathcal{E}_t(u_{\varepsilon_n}(t))\le\mathcal{E}_0(u^0_{\varepsilon_n})+C_1\int_0^t\mathcal{E}_s(u_{\varepsilon_n}(s))\,\mathrm{d}s+C_2T. \tag{5.4}$$

Now, in view of (2.12) we have $\sup_n\mathcal{E}_0(u^0_{\varepsilon_n})\le C$. Hence, with the Gronwall Lemma we conclude that $\sup_{t\in[0,T]}\mathcal{G}(u_{\varepsilon_n}(t))\le C$, which in turn implies (5.2), in view of (E_2). Therefore, we also conclude (5.3). □

The ensuing compactness result provides what will reveal to be the *defect measure* μ (cf. Definition (1)) associated with the limiting curve u that shall be constructed later on. In what follows we will also show that the limiting energy and power functions $\mathscr{E}$ and $\mathscr{P}$, cf. (5.7) and (5.8) below, coincide with the energy and power evaluated along u.

Proposition 5.2. *Assume* (E_0)–(E_2)*. Consider the sequence of measures*

$$\mu_n:=\left(\frac{\varepsilon_n}{2}\|u_{\varepsilon_n}'(\cdot)\|^2+\frac{1}{2\varepsilon_n}\|\mathrm{D}\mathcal{E}_{(\cdot)}(u_{\varepsilon_n}(\cdot))\|^2\right)\mathscr{L}^1, \tag{5.5}$$

with $\mathscr{L}^1$ *the Lebesgue measure on* $(0,T)$*. Then, there exist a positive Radon measure* $\mu\in\mathrm{M}(0,T)$ *and functions* $\mathscr{E}\in\mathrm{BV}([0,T])$ *and* $\mathscr{P}\in L^\infty(0,T)$ *such that, along a not relabeled subsequence, there hold as* $n\to\infty$

$$\mu_n \rightharpoonup^* \mu \quad \text{in } \mathrm{M}(0,T), \tag{5.6}$$

$$\lim_{n\to\infty}\mathcal{E}_t(u_{\varepsilon_n}(t)) = \mathscr{E}(t) \quad \text{for all } t\in[0,T], \tag{5.7}$$

$$\partial_t\mathcal{E}_t(u_{\varepsilon_n}(t)) \rightharpoonup^* \mathscr{P} \quad \text{in } L^\infty(0,T). \tag{5.8}$$

Moreover, denoting by $\mathscr{E}_-(t)$ *and* $\mathscr{E}_+(t)$ *the left and right limits of* $\mathscr{E}$ *at* $t \in [0,T]$*, with the convention that* $\mathscr{E}_-(0) := \mathscr{E}(0)$ *and* $\mathscr{E}_+(T) := \mathscr{E}(T)$*, we have that*

$$\mu([s,t]) + \mathscr{E}_+(t) = \mathscr{E}_-(s) + \int_s^t \mathscr{P}(r)\,\mathrm{d}r \qquad \text{for every } 0 \le s \le t \le T. \tag{5.9}$$

Furthermore, denoting by $\mathrm{d}\mathscr{E}$ *the distributional derivative of* $\mathscr{E}$*, we get from the previous identitities that*

$$\mathscr{E}_-(t) - \mathscr{E}_+(t) = \mu(\{t\}) \qquad \text{for every } 0 \le t \le T; \tag{5.10}$$

$$\mathrm{d}\mathscr{E} + \mu = \mathscr{P}\mathscr{L}^1\,. \tag{5.11}$$

Finally, let J *be the set where the measure* μ *is atomic. Then*

$$J = \{t \in [0,T] \,:\, \mu(\{t\}) > 0\} \text{is countable.} \tag{5.12}$$

Proof. It follows from estimate (5.3) in Proposition 5.1 that the measures $(\mu_n)_n$ have uniformly bounded variation, therefore (5.6) follows. As for (5.7), we observe that, by (5.4), the maps $t \mapsto \mathscr{F}_n(t) := \mathcal{E}_t(u_{\varepsilon_n}(t)) - \int_0^t \partial_t \mathcal{E}_s(u_{\varepsilon_n}(s))\,\mathrm{d}s$ are nonincreasing on $[0,T]$. Therefore, by Helly's Compactness Theorem there exist $\mathscr{F} \in \mathrm{BV}([0,T])$ such that, up to a subsequence, $\mathscr{F}_n(t) \to \mathscr{F}(t)$ for all $t \in [0,T]$. On the other hand, (5.2) also yields (5.8), up to a subsequence. Therefore, (5.7) follows with $\mathscr{E}(t) := \mathscr{F}(t) + \int_0^t \mathscr{P}(s)\,\mathrm{d}s$.

To prove identity (5.9), let us first suppose, for simplicity, that $0 < s \le t < T$. We note on the one hand that $[s,t] = \bigcap_m (s - 1/m, t + 1/m)$. Hence, from the fact that $\mu([s,t]) = \lim_{m\to\infty} \mu\big((s-1/m, t+1/m)\big)$ and from convergence (5.6), we get

$$\begin{aligned}
\mu([s,t]) &= \lim_{m\to\infty} \mu\big((s-1/m,t+1/m)\big) \le \lim_{m\to\infty} \liminf_{n\to\infty} \mu_n\big((s-1/m,t+1/m)\big) \\
&= \lim_{m\to\infty} \liminf_{n\to\infty} \mu_n\big([s-1/m,t+1/m]\big) \\
&= \lim_{m\to\infty} \left\{ \mathscr{E}(s-1/m) - \mathscr{E}(t+1/m) + \int_{s-1/m}^{t+1/m} \mathscr{P}(r)\,\mathrm{d}r \right\} = \mathscr{E}_-(s) - \mathscr{E}_+(t).
\end{aligned} \tag{5.13}$$

Note that in the last equality we have used the fact that the limits $\mathscr{E}_-(s)$ and $\mathscr{E}_+(t)$ always exist, since $\mathscr{E} \in \mathrm{BV}([0,T])$. On the other hand, since the identity $[s,t] = \bigcap_m [s-1/m, +1/m]$ holds as well, we have at the same time that

$$\begin{aligned}
\mu([s,t]) &= \lim_{m\to\infty} \mu\big([s-1/m,t+1/m]\big) \ge \lim_{m\to\infty} \limsup_{n\to\infty} \mu_n\big([s-1/m,t+1/m]\big) \\
&= \lim_{m\to\infty} \left\{ \mathscr{E}(s-1/m) - \mathscr{E}(t+1/m) + \int_{s-1/m}^{t+1/m} \mathscr{P}(r)\,\mathrm{d}r \right\} = \mathscr{E}_-(s) - \mathscr{E}_+(t).
\end{aligned} \tag{5.14}$$

From inequalities (5.13) and (5.14) we obtain (5.9). With obvious modifications we can handle the cases $s = 0$ and $t = T$. Identity (5.11) trivially follows from (5.9).

Finally, let us denote by $(\mathrm{d}\mathscr{E})_{\mathrm{jump}}$ the jump part of the measure $\mathrm{d}\mathscr{E}$: it follows from (5.11) that

$$\mathrm{supp}((\mathrm{d}\mathscr{E})_{\mathrm{jump}}) = J. \tag{5.15}$$

Then, (5.12) follows from recalling that $\mathscr{E} \in \mathrm{BV}([0,T])$ has countably many jump points. □

Notation 5.3. Hereafter, we will denote by B the set

$$B = \{t \in (0,T) \,:\, \|\mathrm{D}\mathcal{E}_t(u_{\varepsilon_n}(t))\| \to 0 \text{ as } n \to \infty\} \tag{5.16}$$

where $(u_{\varepsilon_n})_n$ is (a suitable subsequence of) the sequence for which convergences (5.6)–(5.8) hold. Due to (5.3), we have that

$$\lim_{n\to\infty} \int_0^T \|\mathrm{D}\mathcal{E}_r(u_{\varepsilon_n}(r))\|^2\,\mathrm{d}r = 0,$$

hence the set B (defined for a suitable subsequence), has full Lebesgue measure.

While the compactness statements in Proposition 5.2 only relied on assumptions (E_0)–(E_2), for the next result, which will play a key role in the compactness argument within the proof of Theorem 1, we additionally need condition (E_3) on the critical points of $\mathcal{E}$.

Lemma 5.4. *Assume* (E_0)–(E_3)*. For every* $t \in [0,T]$ *and for all sequences* $(t_1^n)_n$, $(t_2^n)_n$ *fulfilling* $0 \le t_1^n \le t_2^n \le T$ *for every* $n \in \mathbb{N}$ *and*

$$t_1^n \to t, \quad t_2^n \to t, \quad u_{\varepsilon_n}(t_1^n) \to u_1, \quad u_{\varepsilon_n}(t_2^n) \to u_2 \text{ for some } u_1,\, u_2 \in X, \tag{5.17}$$

there holds

$$\mu(\{t\}) \ge \mathsf{c}_t(u_1; u_2)\,. \tag{5.18}$$

In particular, for every $t \in [0,T] \setminus J$ *we have that* $u_1 = u_2$*.*

Proof. Observe that for every $\eta > 0$ there holds

$$\begin{aligned}\mu([t-\eta, t+\eta]) &\ge \limsup_{n\to\infty} \mu_n([t_1^n, t_2^n]) \\ &= \limsup_{n\to\infty} \int_{t_1^n}^{t_2^n} \left(\frac{\varepsilon_n}{2} \|u'_{\varepsilon_n(s)}\|^2 + \frac{1}{2\varepsilon_n} \|\mathrm{D}\mathcal{E}_s(u_{\varepsilon_n}(s))\|^2 \right) \mathrm{d}s \\ &\ge \limsup_{n\to\infty} \int_{t_1^n}^{t_2^n} \|u'_{\varepsilon_n(s)}\| \|\mathrm{D}\mathcal{E}_s(u_{\varepsilon_n}(s))\| \,\mathrm{d}s \ge \mathsf{c}_t(u_1; u_2),\end{aligned} \tag{5.19}$$

where the first inequality is due to (5.6), the second one to the definition (5.5) of μ_n, the third one to the Young inequality, and the last one to (2.7) in Proposition 2.4. Since $\eta > 0$ is arbitrary, we conclude (5.18). In particular, if $\mu(\{t\}) = 0$ then by (1) in Proposition 2.4 we deduce that $u_1 = u_2$. □

We are now in the position to perform the **proof of Theorem 1:** we will split the arguments in several points.

Ad (2.13)**:** Let us consider the set (cf. Notation 5.3)

$$I := J \cup A \cup \{0\} \text{ with } A \subset (B \backslash J) \text{ dense in } [0,T] \text{ and consisting of countably many points.} \tag{5.20}$$

From (5.2) we gather that

$$\exists\, C > 0 \ \ \forall\, n \in \mathbb{N}\ \forall\, t \in [0,T] : \quad u_{\varepsilon_n}(t) \in \mathcal{S}_C, \text{ with } \mathcal{S}_C \Subset X \text{ by } (E_1), \tag{5.21}$$

where the symbol $\Subset$ stands for *compact* inclusion. Since I has countably many points, with a diagonal procedure it is possible to extract from $(u_{\varepsilon_n})_n$ a (not relabeled) subsequence such that there exists $\hat{u} : I \to X$ with

$$u_{\varepsilon_n}(t) \to \hat{u}(t) \quad \text{for all } t \in I, \tag{5.22}$$

with $\hat{u}(0) = u_0$ thanks to the convergence (2.12) of the initial conditions. Moreover, since $A \subset B$ from (5.16), we also have

$$\hat{u}(t) \in \mathcal{C}(t) \qquad \text{for every } t \in A. \tag{5.23}$$

We now extend $\hat{u}$ to a function defined on the whole interval $[0,T]$, by showing that

$$\begin{aligned}&\forall\, t \in (0,T] \setminus I \quad \tilde{u}(t) := \lim_{k\to\infty} \hat{u}(t_k) \text{ is uniquely defined for every } (t_k)_k \in \mathfrak{S}(t) \text{ and fulfills } \tilde{u}(t) \in \mathcal{C}(t), \\ &\qquad\qquad \text{with} \qquad \mathfrak{S}(t) = \left\{ (t_k)_k \subset A \,:\, t_k \to t \text{ and } \exists \lim_{k\to\infty} \hat{u}(t_k) \right\}\end{aligned} \tag{5.24}$$

(in the case $t = T$, the sequence $(t_k)_k$ is to be understood as $t_k \uparrow t$). Observe that $\mathfrak{S}(t) \neq \emptyset$ since $\hat{u}(I)$ is contained in the compact set K from (5.21), To check (5.24), let $(t_1^k)_k,\ (t_2^k)_k \in \mathfrak{S}(t)$ be such that

$$\lim_{k\to\infty} \hat{u}(t_1^k) =: u_1 \qquad \text{and} \qquad \lim_{k\to\infty} \hat{u}(t_2^k) =: u_2.$$

We want to show that $u_1 = u_2$. Note that $\hat{u}(t_1^k) = \lim_{n\to\infty} u_{\varepsilon_n}(t_1^k)$ and $\hat{u}(t_2^k) = \lim_{n\to\infty} u_{\varepsilon_n}(t_2^k)$ for every $k \in \mathbb{N}$, because of (5.22). Since $t_1^k,\, t_2^k \in A \subset B$ for every $k \in \mathbb{N}$, there holds $\mathrm{D}\mathcal{E}_{t_1^k}(\hat{u}(t_1^k)) = \mathrm{D}\mathcal{E}_{t_2^k}(\hat{u}(t_2^k)) = 0$ for every $k \in \mathbb{N}$. Therefore, we get that $u_1, u_2 \in \mathcal{C}(t)$. Furthermore, with a diagonal procedure we can extract a subsequence $(n_k)_k$ such that

$$u_1 = \lim_{k\to\infty} u_{n_k}(t_1^k) \qquad \text{and} \qquad u_2 = \lim_{k\to\infty} u_{n_k}(t_2^k).$$

Therefore, we are in the position to apply Lemma 5.4 to u_1 and u_2. Since $t \notin J$, we have that $u_1 = u_2$. This concludes the proof of (5.24). Therefore, we can define the (candidate) limit function u everywhere on $[0,T]$ by setting

$$u(t) := \begin{cases} \hat{u}(t) & \text{if } t \in I \\ \tilde{u}(t) & \text{if } t \in (0,T] \setminus I. \end{cases} \tag{5.25}$$

By construction, u complies with (2.9).

We now address the pointwise convergence (2.13): in view of (5.22), we have to show it at $t \in (0,T] \setminus I$. We will prove that at any such point t, any subsequence of $(u_{\varepsilon_n}(t))_n$ admits a further subsequence converging to $u(t)$. Let us fix a (not relabeled) subsequence $(u_{\varepsilon_n}(t))_n$ and consider a sequence $(t_k)_k \subset A$ such that $t_k \uparrow t$ and $u(t) = \tilde{u}(t) = \lim_{k\to\infty} \hat{u}(t_k)$. With a diagonal procedure as in the above lines, we find a subsequence $(\varepsilon_{n_k})_k$ such that

$$u(t) = \lim_{k\to\infty} u_{\varepsilon_{n_k}}(t_k),$$

whereas, again using that $u_{\varepsilon_{n_k}}([0,T]) \subset \mathcal{S}_C$ for every $k \in \mathbb{N}$, we extract a further (not relabeled) subsequence from $(u_{\varepsilon_{n_k}}(t))_k$, such that

$$u_{\varepsilon_{n_k}}(t) \to \tilde{u} \quad \text{for some } \tilde{u} \in X.$$

Since $t \notin J$, an application of Lemma 5.4 with t_k, t, $u_{\varepsilon_{n_k}}$, $u(t)$, and $\tilde{u}$ in place of t_1^n, t_2^n, u_{ε_n}, u_1, and u_2, respectively, gives that $\tilde{u} = u(t)$. Therefore, convergence (2.13) holds at $t \in (0,T] \setminus J$, and at $t \in J$ due to (5.22) and definition (5.25).

Ad (2.14)**:** Since the sequence $(u_{\varepsilon_n})_n$ is bounded in $L^\infty(0,T;X)$ by (5.21), (2.14) follows from (2.13).

It now remains to verify that $u \in \mathrm{B}([0,T];X)$ complies with the properties (2.8)—-(2.10) defining the notion of Dissipative Viscosity solution.

Ad (2.8)**:** To prove (2.8), we first need to prove that the left and the right limits of u always exist. We now show that for every $0 \le t < T$ the right limit $u_+(t)$ exists. The same argument can be trivially adapted to prove the existence of the left limit $u_-(s)$ for every $s \in (0,T]$. Consider $(t_1^k)_k$, $(t_2^k)_k \subset [0,T]$ such that $t_1^k \downarrow t$, $t_2^k \downarrow t$, and the limits

$$\lim_{k\to\infty} u(t_1^k) =: u_1 \qquad\qquad \lim_{k\to\infty} u(t_2^k) =: u_2 \tag{5.26}$$

exist. Note that, up to subsequences, we have that either $t_1^k \le t_2^k$ or $t_2^k \le t_1^k$ for every $k \in \mathbb{N}$. Suppose for simplicity that we are in the first case. Observe that from (E_0) and from (5.7), we have that $\mathcal{E}_t(u(t)) = \mathscr{E}(t)$ for every $t \in [0,T]$, due to convergence (2.13). In particular, since $\mathscr{E} \in \mathrm{BV}([0,T])$, there exist

$$\lim_{k\to\infty} \mathcal{E}_{t_1^k}(u(t_1^k)) \; = \; \lim_{k\to\infty} \mathcal{E}_{t_2^k}(u(t_2^k)) = \mathscr{E}_+(t). \tag{5.27}$$

Now, (5.7) gives that $\mathcal{E}_{t_i^k}(u(t_i^k)) = \lim_{n\to\infty} \mathcal{E}_{t_i^k}(u_{\varepsilon_n}(t_i^k))$ for every $k \in \mathbb{N}$ and for $i = 1, 2$. Hence, there exists $(\varepsilon_{n_k})_k$ such that

$$\left| \mathcal{E}_{t_1^k}(u_{\varepsilon_{n_k}}(t_1^k)) - \mathcal{E}_{t_1^k}(u(t_1^k)) \right| \le \frac{1}{k}, \qquad \left| \mathcal{E}_{t_2^k}(u_{\varepsilon_{n_k}}(t_2^k)) - \mathcal{E}_{t_2^k}(u(t_2^k)) \right| \le \frac{1}{k}$$

for every $k \in \mathbb{N}$, so that

$$\lim_{k\to\infty} \mathcal{E}_{t_1^k}(u_{\varepsilon_{n_k}}(t_1^k)) = \lim_{k\to\infty} \mathcal{E}_{t_1^k}(u(t_1^k)), \qquad \lim_{k\to\infty} \mathcal{E}_{t_2^k}(u_{\varepsilon_{n_k}}(t_2^k)) = \lim_{k\to\infty} \mathcal{E}_{t_2^k}(u(t_2^k))\,. \tag{5.28}$$

Arguing as previously done, we can also suppose that, up to a subsequence,

$$u_1 = \lim_{k\to\infty} u_{\varepsilon_{n_k}}(t_1^k), \qquad\qquad u_2 = \lim_{k\to\infty} u_{\varepsilon_{n_k}}(t_2^k). \tag{5.29}$$

Now, recalling definition (5.5) of μ_n, the energy identity (2.2) with t_1^k, t_2^k, $u_{\varepsilon_{n_k}}$ in place of s, t, and u_ε, respectively, gives

$$\mathcal{E}_{t_1^k}(u_{\varepsilon_{n_k}}(t_1^k)) - \mathcal{E}_{t_2^k}(u_{\varepsilon_{n_k}}(t_2^k)) + \int_{t_1^k}^{t_2^k} \partial_t \mathcal{E}_r(u_{\varepsilon_{n_k}}(r))\,\mathrm{d}r = \mu_{n_k}([t_1^k, t_2^k]).$$

This equality, together with (5.28), (5.29), and with (2.7) in Theorem 2.4, implies that

$$0 = \lim_{k\to\infty} \mathcal{E}_{t_1^k}(u(t_1^k)) - \lim_{k\to\infty} \mathcal{E}_{t_2^k}(u(t_2^k)) \geq \liminf_{k\to\infty} \mu_{\varepsilon_{n_k}}([t_1^k, t_2^k]) \geq \mathsf{c}_t(u_1; u_2) \tag{5.30}$$

(note that we have also used (5.8) and (5.27)). Hence, we have obtained that $\mathsf{c}_t(u_1; u_2) = 0$ and in turn that $u_1 = u_2$, in view of Proposition 2.4 (1), whence we conclude that the right limit $u_+(t)$ exists.

Combining (2.13) with (5.7), and taking into account that $\mathcal{E} \in \mathrm{C}^1([0,T] \times X)$, we gather that

$$\mathscr{E}_+(t) = \mathcal{E}_t(u_+(t)) \quad \text{for all } 0 \leq t < T, \qquad \mathscr{E}_-(s) = \mathcal{E}_s(u_-(s)) \quad \text{for all } 0 < s \leq T,$$

and

$$\partial_t \mathcal{E}_t(u_{\varepsilon_n}(t)) \to \partial_t \mathcal{E}_t(u(t)) \qquad \text{for all } t \in [0,T].$$

In view of (5.2) and the Lebesgue theorem, we then have $\partial_t \mathcal{E}_t(u_{\varepsilon_n}(t)) \to \partial_t \mathcal{E}_t(u(t))$ in $L^p(0,T)$ for every $1 \leq p < \infty$. Therefore,

$$\mathscr{P}(t) = \partial_t \mathcal{E}_t(u(t)) \quad \text{for a.a. } t \in (0,T),$$

and the energy balance (2.8) follows from (5.9).

Ad (2.10a)**:** To prove that $u_+(t) \in \mathcal{C}(t)$ for every $t \in [0,T)$ (the argument for $u_-(t)$, with $t \in (0,T]$, is perfectly analogous), it is sufficient to observe that there always exists $t^k \downarrow t$ such that $(t^k)_k \subset (0,T] \setminus J$, so that in particular

$$u_+(t) = \lim_{k\to\infty} u(t^k),$$

and $u(t^k) \in \mathcal{C}(t^k)$ for every $k \in \mathbb{N}$. Therefore, by this limit and by (E_0), $u_+(t) \in \mathcal{C}(t)$.

Ad (2.10b)**&**(2.10c)**:** preliminarily, we show that

$$\mathcal{E}_t(u_-(t)) - \mathcal{E}_t(u_+(t)) \geq \mathsf{c}_t(u_-(t); u_+(t)) \quad \text{for every } t \in (0,T) \tag{5.31}$$

(suitable analogues hold at the points $t = 0$ and $t = T$). Indeed, fix $t_1^k \uparrow t$ and $t_2^k \downarrow t$, so that (cf. (5.26)) $u_1 = \lim_{k\to\infty} u(t_1^k) = u_-(t)$ and $u_2 = \lim_{k\to\infty} u(t_2^k) = u_+(t)$. The very same arguments leading to (5.30) show that

$$\lim_{k\to\infty} \mathcal{E}_{t_1^k}(u(t_1^k)) - \lim_{k\to\infty} \mathcal{E}_{t_2^k}(u(t_2^k)) \geq \liminf_{k\to\infty} \mu_{\varepsilon_{n_k}}([t_1^k, t_2^k]) \geq \mathsf{c}_t(u_1; u_2)\,.$$

Then, (5.31) ensues. On account of identity (5.10), we deduce

$$\mu(\{t\}) \geq \mathsf{c}_t(u_-(t); u_+(t)) \quad \text{for every } t \in [0,T]\,. \tag{5.32}$$

In particular, if $t \notin J$, we have $\mathsf{c}_t(u_-(t); u_+(t)) = 0$, hence $u_-(t) = u_+(t)$. Thus, we have proved the one-sided inclusion $\supset$ in (2.10b).

Let us now prove the converse of inequality (5.31), namely

$$\mathcal{E}_t(u_-(t)) - \mathcal{E}_t(u_+(t)) \leq \mathsf{c}_t(u_-(t); u_+(t)).$$

We may confine the discussion to the case $t \in J$ for, otherwise, we have $u_-(t) = u_+(t)$ and the above inequality trivially holds. Let $\vartheta \in \mathscr{A}^t_{u_-(t),u_+(t)}$ be a minimizing curve for the cost $\mathsf{c}_t(u_-(t); u_+(t))$: its existence is guaranteed by Theorem 2.4 (3). Then, by the chain rule

$$\begin{aligned} \mathsf{c}_t(u_-(t); u_+(t)) &= \int_0^1 \|\mathrm{D}\mathcal{E}_t(\vartheta(s))\| \|\vartheta'(s)\|\,\mathrm{d}s \\ &\geq -\left(\mathcal{E}_t(\vartheta(1)) - \mathcal{E}_t(\vartheta(0))\right) = \mathcal{E}_t(u_-(t)) - \mathcal{E}_t(u_+(t)). \end{aligned} \tag{5.33}$$

All in all, again taking into account (5.10), we have proved that

$$\mathsf{c}_t(u_-(t); u_+(t)) = \mu(\{t\}) = \mathcal{E}_t(u_-(t)) - \mathcal{E}_t(u_+(t)) \quad \text{for all } t \in [0,T],$$

whence (2.10b)&(2.10c) also in view of Theorem 2.4(1).

This concludes the proof of Theorem 1. ■

5.2. **Proof of Theorem 2.** Let us denote by $\mu_{\rm AC}$, $\mu_{\rm J}$, and $\mu_{\rm CA}$, the absolutely continuous, jump, and Cantor parts of the defect measure μ. Recall that $\mu_{\rm J}([s,t]) = \sum_{r\in J\cap[s,t]} \mathsf{c}_r(u_-(r); u_+(r))$ in view of (2.10c). It follows from (2.8) that, for every $0 \le t \le T$,

$$\mu_{\rm J}([0,t]) + \mathcal{E}_t(u_+(t)) \le \mu_{\rm AC}([0,t]) + \mu_{\rm J}([0,t]) + \mu_{\rm CA}([0,t]) + \mathcal{E}_t(u_+(t)) = \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_r(u(r))\,\mathrm{d}r\,.$$

We will now show that

$$\mu_{\rm J}([0,t]) + \mathcal{E}_t(u_+(t)) \ge \mathcal{E}_0(u(0)) + \int_0^t \partial_t\mathcal{E}_r(u(r))\,\mathrm{d}r \qquad \text{for every } \ 0 \le t \le T, \tag{5.34}$$

and therefore conclude that $\mu_{\rm AC}([0,t]) = \mu_{\rm CA}([0,t]) = 0$ at every $t \in [0,T]$, whence the thesis.

We will deduce (5.34) by applying the following result, which is a variant of [29, Lemma 6.2].

Lemma 5.5. *Let $g : [a,b] \to \mathbb{R}$ be a strictly increasing function, and $f : [a,b] \to \mathbb{R}$ be right continuous and such that its restriction to the set $[a,b] \setminus J_g$ is lower semicontinuous. Suppose that*

$$\liminf_{r\uparrow t} f(r) - f(t) \ge g_-(t) - g_+(t) \qquad \textit{for all } t \in J_g, \tag{5.35}$$

$$\limsup_{s\downarrow t} \frac{f(t)-f(s)}{g_+(s)-g_+(t)} \ge -1 \qquad \textit{for all } t \in [a,b]\,. \tag{5.36}$$

Then, the map $f - g$ is non-increasing on $[a,b]$.

In fact, [29, Lemma 6.2] has the same thesis as the result above, but ‘specular’ conditions on f and g, involving left continuity of f, the lim sup from the right, in place of the lim inf from the left, in (5.35), and analogously for (5.36), etc. For our purposes, though, it is more convenient to apply the present version of the result. Its proof can be deduced from that of [29, Lemma 6.2] by observing that, for $f - g$ to be non-increasing on $[a,b]$, it is sufficient to have

$$f(b) - g(b) \le f(a) - g(a) \;\Leftrightarrow\; f^{\#}(a) - g^{\#}(a) \le f^{\#}(b) - g^{\#}(b) \text{ with } f^{\#}(t) := f(b+a-t),\ g^{\#}(t) := g(b+a-t)\,.$$

Therefore, we are led to prove that the function $f^{\#} - g^{\#}$ is non-decreasing, which follows from applying [29, Lemma 6.2] to the functions $-f^{\#}$ and $-g^{\#}$. Rewriting the conditions on $-f^{\#}$ and $-g^{\#}$ from [29, Lemma 6.2] in terms of the original functions f and g, one obtains the statement of Lemma 5.5.

We are now in the position to conclude the **proof of Theorem 2**. Mimicking the argument from the proof of [29, Thm. 6.5], in order to conclude the lower energy estimate (5.34) we shall apply Lemma 5.5 to the functions f, $g : [0,T] \to \mathbb{R}$ defined by

$$f(t) := \int_0^t \partial_t\mathcal{E}_r(u(r))\,\mathrm{d}r - \mathcal{E}_t(u_+(t)) \qquad \text{and} \qquad g(t) := \sum_{r\in J\cap[0,t]} \mathsf{c}_r(u_-(r); u_+(r)) + \eta t \tag{5.37}$$

with $\eta > 0$ arbitrary, so that g is strictly increasing. By construction f is right continuous. Since u is continuous on $[0,T] \setminus J_g$, and since $\mathcal{E} \in \mathrm{C}^1([0,T] \times X)$, we deduce that f is even continuous on $[0,T] \setminus J_g$. To check (5.35), we observe that

$$\begin{aligned}\liminf_{r\uparrow t} f(r) - f(t) &= \lim_{r\uparrow t} \int_t^r \partial_t\mathcal{E}_\tau(u(\tau))\,\mathrm{d}\tau + \liminf_{r\uparrow t} \mathcal{E}_t(u_+(t)) - \mathcal{E}_r(u_+(r)) \\ &= \mathcal{E}_t(u_+(t)) - \mathcal{E}_t(u_-(t)) = -\mathsf{c}_t(u_-(t); u_+(t)) = g_-(t) - g_+(t)\,.\end{aligned}$$

Note that in the last equality we have used the fact that

$$g_+(t) - g_-(t) = \lim_{\tau\downarrow t} \lim_{s\uparrow t} \sum_{r\in J\cap[s,\tau]} \mathsf{c}_r(u_-(r); u_+(r)) = \mathsf{c}_t(u_-(t); u_+(t)).$$

Finally, in order to verify (5.36), we preliminarily calculate

$$\begin{aligned} f(t) - f(s) &= \int_s^t \partial_t \mathcal{E}_r(u(r))\,\mathrm{d}r + \mathcal{E}_s(u_+(s)) - \mathcal{E}_t(u_+(t)) \\ &= \int_s^t (\partial_t\mathcal{E}_r(u(r)) - \partial_t\mathcal{E}_r(u_+(s)))\,\mathrm{d}r + \mathcal{E}_t(u_+(s)) - \mathcal{E}_t(u_+(t)) \doteq I_1 + I_2\,. \end{aligned}$$

Observing that

$$g_+(s) - g_+(t) \geq \eta(s-t), \tag{5.38}$$

we find that

$$\left|\frac{I_1}{g_+(s)-g_+(t)}\right| = \frac{|I_1|}{g_+(s)-g_+(t)} \leq \frac{1}{\eta} \sup_{r\in[s,t]} |\partial_t\mathcal{E}_r(u(r)) - \partial_t\mathcal{E}_r(u_+(s))| \to 0 \quad \text{ as } s\downarrow t\,,$$

due to the continuity of the map $(t,u)\mapsto \partial_t\mathcal{E}_t(u)$. Therefore,

$$\limsup_{s\downarrow t} \frac{f(t)-f(s)}{g_+(s)-g_+(t)} = \limsup_{s\downarrow t} \frac{\mathcal{E}_t(u_+(s)) - \mathcal{E}_t(u_+(t))}{g_+(s)-g_+(t)} = \limsup_{s\downarrow t} \frac{\mathcal{E}_t(u_+(s)) - \mathcal{E}_t(u_+(t))}{\|\mathrm{D}\mathcal{E}_t(u_+(s))\|} \frac{\|\mathrm{D}\mathcal{E}_t(u_+(s))\|}{g_+(s)-g_+(t)} \geq 0\,,$$

which follows from condition (E_4), and from the fact that $\frac{\|\mathrm{D}\mathcal{E}_t(u_+(s))\|}{g_+(s)-g_+(t)} \geq 0$ for all $s \geq t$ since g_+ is strictly increasing.

All in all, (5.34) ensues from writing $f(t) - g(t) \leq f(0) - g(0)$ with f and g from (5.37), and letting $\eta \downarrow 0$. ■

6. Applications

In this section, we discuss two classes of conditions which guarantee the validity for $\mathcal{E}_t(\cdot)$, $t\in[0,T]$, of hypothesis (E_3) on the set of its critical points, and of the Łojasiewicz inequality (2.15), respectively.

We start by introducing the *transversality conditions*, concerning the properties of the energy $\mathcal{E}$ at points (t,u) where u is a *degenerate* critical point, i.e. on the set

$$\mathcal{S} := \left\{(t,u)\in[0,T]\times X :\ u\in\mathcal{C}(t) \text{ and } \mathrm{D}^2\mathcal{E}_t(u) \text{ is non-invertible}\right\} \tag{6.1}$$

Definition 6.1. *We say that the functional $\mathcal{E}$ satisfies the transversality conditions if each point $(t_0,u_0)\in\mathcal{S}$ fulfills*

(T1) $\dim(\mathrm{N}(\mathrm{D}^2\mathcal{E}_{t_0}(u_0))) = 1$*;*
(T2) *If* $0\neq v\in \mathrm{N}(\mathrm{D}^2\mathcal{E}_{t_0}(u_0))$ *then* $\langle\partial_t\mathrm{D}\mathcal{E}_{t_0}(u_0), v\rangle \neq 0$*;*
(T3) *If* $0\neq v\in \mathrm{N}(\mathrm{D}^2\mathcal{E}_{t_0}(u_0))$ *then* $\mathrm{D}^3\mathcal{E}_{t_0}(u_0)[v,v,v]\neq 0$,

where $\mathrm{N}(\mathrm{D}^2\mathcal{E}_{t_0}(u_0))$ *denotes the kernel of the mapping* $\mathrm{D}^2\mathcal{E}_{t_0}(u_0)$.

Under the transversality conditions we have the following result, proved in [2, Corollary 3.6], ensuring that the critical set $\mathcal{C}(t)$ is discrete at every $t\in[0,T]$.

Proposition 6.2 ([2])**.** *Let $\mathcal{E}\in \mathrm{C}^3([0,T]\times X)$ comply with the transversality conditions. Then, for every $t\in[0,T]$, the set $\mathcal{C}(t)$ consists of isolated points. Hence,* (E_3) *holds.*

With the following *genericity* result, proved in [2, Theorem 1.3. and Corollary 3.7], we see that suitable second-order perturbations of an arbitrary energy functional lead to an energy fulfilling the transversality conditions. In order to state it, we need to introduce the set $\mathrm{Sym}(X)$ of the symmetric bilinear forms on $X\times X$. Moreover, for a further technical reason that we do not detail here, in the following theorem we have to require $\mathcal{E}\in \mathrm{C}^4([0,T]\times X)$.

Theorem 6.3 ([2]). *Let $\mathcal{E}$ be in $\mathrm{C}^4([0,T]\times X)$. Then, every open neighborhood U of the origin in $X\times\mathrm{Sym}(X)$ contains a set U_r of full Lebesgue measure such that, for every $(y,\mathscr{K})\in U_r$, the functionals*

$$(t,u) \longmapsto \mathcal{E}_t(u) + \langle y,u\rangle + \frac{1}{2}\mathscr{K}(u,u) \tag{6.2}$$

satisfy the transversality conditions.

Let us mention that in [2] a similar result (cf. [2, Cor. 3.7]) is proved in a more general, infinite-dimensional setting, with perturbations of the same form as (6.2), fulfilling an infinite-dimensional version of the transversality conditions. Such perturbations are constructed by means of elements $(y,\mathscr{K})\in(X\times\mathrm{Sym}(X))\setminus N$, where N is in general only a *meagre* subset of $X\times\mathrm{Sym}(X)$. In the present finite-dimensional context, N meagre improves to an N with zero Lebesgue measure, due to the classical Sard's Theorem.

Concerning the Łojasiewicz inequality, we are now going to point out its connections with the concept of *subanaliticity*. For the reader's convenience, let us first recall the definition of *subanalytic* function, referring to [6, 13, 21] for all details, and to the recent [7] for the proof of the result that will be used in what follows.

Definition 6.4. *(1) A subset $A\subset\mathbb{R}^d$ is called* semianalytic *if for every $x\in\mathbb{R}^d$ there exists a neighborhood V such that*

$$A\cap V = \cup_{i=1}^{p}\cap_{j=1}^{q}\{x\in V\ :\ f_{ij}(x)=0,\ g_{ij}(x)>0\}, \tag{6.3}$$

where for every $1\le i\le p$ and $1\le j\le q$ the functions f_{ij}, $g_{ij}: V\to\mathbb{R}$ are analytic.

(2) We call a set $A\subset\mathbb{R}^d$ subanalytic *if every $x\in\mathbb{R}^d$ admits a neighborhood V such that there exists $B\subset\mathbb{R}^d\times\mathbb{R}^m$, for some $m\ge 1$, with*

$$A\cap V = \pi_1(B) \text{ and } B \text{ is a bounded semianalytic subset of } \mathbb{R}^d\times\mathbb{R}^m, \tag{6.4}$$

$\pi_1:\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}^d$ denoting the projection on the first component.

(3) We say that a function $E:\mathbb{R}^d\to(-\infty,+\infty]$ is subanalytic *if its graph is a subanalytic subset of $\mathbb{R}^d\times\mathbb{R}$.*

As the above definition shows, the concept of *subanalytic function* has a clear geometric character. Without entering into details, let us recall that a fundamental example of subanalytic sets (hence of subanalytic functions) is provided by *semialgebraic sets*, i.e. sets $A\subset\mathbb{R}^d$ of the form

$$A = \cup_{i=1}^{p}\cap_{j=1}^{q}\{x\in V\ :\ f_{ij}(x)=0,\ g_{ij}(x)>0\} \qquad \text{with } f_{ij},\ g_{ij}:\mathbb{R}^d\to\mathbb{R} \textit{ polynomial} \text{ functions} \tag{6.5}$$

for all $1\le i\le p$ and $1\le j\le q$.

We now consider for the functional $\mathcal{E}$ the condition

$$\text{for every } \ t\in[0,T] \ \text{ the functional } \ u\mapsto\mathcal{E}_t(u) \ \text{ is subanalytic.} \tag{6.6}$$

To fix ideas, we may think of the case in which $\mathcal{E}_t(u)=E(u)-\langle\ell(t),u\rangle$, with $\ell\in\mathrm{C}^1([0,T];X)$ and $E:X\to\mathbb{R}$ of class C^1 and subanalytic. Thanks to [7, Thm. 3.1], for every $t\in[0,T]$, $\mathcal{E}_t(\cdot)$ complies with the Łojasiewicz inequality (2.15). All in all, also in view of this result, we can state the following theorem.

Theorem 6.5. *In the setting of* (E_0)–(E_2)*, assume in addition the subanalyticity* (6.6)*, and that $\mathcal{E}\in\mathrm{C}^3([0,T]\times X)$ fulfills the transversality conditions. Consider a sequence $(u^0_{\varepsilon_n})_n$ of initial conditions for* (1.1) *such that*

$$u^0_{\varepsilon_n}\to u_0 \quad \textit{as } n\to\infty.$$

Then there exist a (not relabeled) subsequence and a curve $u\in\mathrm{B}([0,T];X)$ such that convergences (2.13)–(2.14) *hold, $u(0)=u_0$, and u is a Balanced Viscosity solution to* (1.3).

This result is a consequence of the fact that, thanks to Proposition 6.2, all the hypotheses of Theorem 1 are in force, and therefore the statement holds true with u being a Dissipative Viscosity solution to (1.3). Moreover, due to the Łojasiewicz inequality (2.15), which is implied by (6.6), the Dissipative Viscosity solution u improves to a Balanced Viscosity solution in view of Theorem 2 (cf. also Remark 2.5).

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