Mathematical Models and Methods in Applied Sciences

FROM ADHESIVE TO BRITTLE DELAMINATION IN VISCO-ELASTODYNAMICS

RICCARDA ROSSI

DIMI (Department of Mechanical and Industrial Engineering), Università degli studi di Brescia
via Branze 38
Brescia, I–25133, Italy
riccarda.rossi@unibs.it

MARITA THOMAS

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39,
Berlin, D–10117, Germany
marita.thomas@wias-berlin.de

Received (Day Month Year)
Revised (Day Month Year)
Communicated by (xxxxxxxxxx)

In this paper we analyze a system for *brittle delamination* between two visco-elastic bodies, also subject to inertia, which can be interpreted as a model for dynamic fracture. The rate-independent flow rule for the delamination parameter is coupled with the momentum balance for the displacement, including inertia. This model features a nonsmooth constraint ensuring the continuity of the displacements outside the crack set, which is marked by the support of the delamination parameter. A weak solvability concept, generalizing the notion of energetic solution for rate-independent systems to the present mixed rate-dependent/rate-independent frame, is proposed. Via refined variational convergence techniques, existence of solutions is proved by passing to the limit in approximating systems which regularize the nonsmooth constraint by conditions for adhesive contact. The presence of the inertial term requires the design of suitable recovery spaces small enough to provide compactness but large enough to recover the information on the crack set in the limit.

Keywords: Adhesive contact; brittle delamination; Kelvin-Voigt visco-elasticity; inertia; non-smooth brittle constraint; coupled rate-dependent/rate-independent evolution; energetic solutions.

AMS Subject Classification: 49J53, 49J45, 74H20, 74C05, 74C10, 74M15, 74R10

1. Introduction

Over the last two decades, crack propagation has been intensively studied from a mathematical viewpoint, starting from the seminal paper[23]. This article proposed a variational scheme for the dissipative, rate-independent evolution of fracture, coupled with the 'static' momentum balance for the *purely elastic* displacement variable. Several papers, cf. e.g. references[10,9,22,11,13] (see also the survey[4]), have ever since consolidated the existence theory, and the study of the fine properties, for the notion of *quasistatic evolution* of fracture due to G. Dal Maso and coworkers. Also alternative solution notions have been advanced[34]. In this realm, great generality as far as the modeling of the crack set has been achieved thanks to the toolbox of Geometric Measure Theory.

The study of *dynamic fracture*, with the displacement variable subject to viscosity and inertia within Kelvin-Voigt rheology, is at a less refined stage. Indeed, phase-field models for (rate-independent) fracture, coupled with elasto-viscodynamics, have been extensively studied in the works[5,33], where the evolution of a volume, damage-like variable approximating the fracture is governed by the so-called Ambrosio-Tortorelli functional[2] of Mumford-Shah type. However, the convergence of the solutions of this regularized system, to solutions of a model for brittle fracture, has been proved only in the case of *purely* rate-independent evolution (i.e., with the static momentum balance), see reference[27]. While the asymptotic analysis to the Mumford-Shah fracture regime has also been carried out for the *gradient flow* of the Ambrosio-Tortorelli functional[3], the passage to the limit in the case where the displacements are subject to the equation of visco-elastodynamics remains open. So is the study of dynamic fracture without strong geometric assumptions on the cracks, due to the challenges posed by the coupling between the rate-independent propagation of the fracture, and the rate-dependent evolution of the displacement.

The basics for the study of the dynamic case with arbitrary cracks have been established in the work[14], focusing on the analysis of the equations of elastodynamics for the displacement out of the (arbitrarily growing) crack set, whose evolution is assumed to be *given*. The existence and uniqueness results from article[14] have been recently extended to the case of mixed Dirichlet/Neumann boundary conditions in reference[15]. In both works the crack evolution is preassigned. To our knowledge, existence results for models on dynamic fracture without this restriction have only recently been obtained in the works[16,17], and these results are strongly based on the 1 or 2-dimensional geometry of the problem. The work[16] tackles a 2D-model for dynamic fracture with prescribed, sufficiently smooth, connected crack path, but evolving with unknown speed. In this setting, the evolution of the crack is fully described by that of the crack-tip. Restricting the problem to a class of sufficiently smooth crack-tip evolutions, the evolution criterion for the crack is given by a maximal dissipation condition, selecting, within this class, the crack-tip evolution that runs as fast as possible consistently with the energy balance, and thus preventing stationary cracks from always being solutions. In the article[17] an existence result for dynamic fracture without pre-assigned crack evolution has been proved in the case of a *dynamic peeling test* for a thin film. This model (cf. reference[35]) describes the debonding of a film initially attached on a rigid substrate. Its special 1-dimensional geometry is crucial for the analysis in the work[17].

In this paper we aim to contribute to the investigation of dynamic fracture from a yet different perspective. We will consider a model describing the evolution, during a finite time interval $(0,T)$, of brittle delamination between two elastic bodies Ω_+

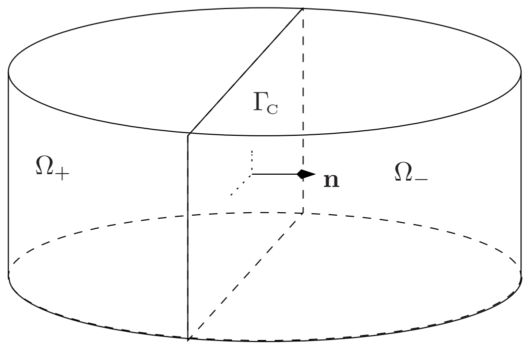

Fig. 1. A feasible domain Ω with convex interface Γ_C.

and Ω_-, subject to viscosity and inertia, along a prescribed contact surface Γ_C. On Γ_C the crack evolution is not prescribed, but falls into the class of rate-independent evolutions, as it is governed by a unidirectional, positively 1-homogeneous dissipation potential, cf. (1.7). In our setup the crack set as a subset of Γ_C need not be connected and it may even jump with respect to time. Moreover, also differently to references[16,17], our results will be obtained in general space dimension $d \geq 1$. We refer to (2.6) ahead for the precise statement of our conditions on Ω and Γ_C. A prototype of feasible domain is the one depicted in Fig. 1.

Following the approach by M. Frémond[24,25], within the theory of generalized standard materials[29], delamination is described in terms of an internal variable $z : (0,T) \times \Gamma_C \to [0,1]$, which has in fact the meaning of a damage variable as it describes the fraction of fully effective molecular links in the bonding. Namely,

$$z(t,x) = \begin{cases} 1 \\ 0 \end{cases} \quad \text{means that the bonding is} \quad \begin{cases} \text{fully intact} \\ \text{completely broken} \end{cases} \tag{1.1}$$

at the time $t \in (0,T)$, at the material point $x \in \Gamma_C$. The rate-independent flow rule for the delamination parameter z is coupled to the *dynamic momentum balance* for the displacement field $u : (0,T) \times (\Omega_- \cup \Omega_+) \to \mathbb{R}^d$. Our model enforces the

$$\text{brittle constraint:} \qquad [\![u(t)]\!] = 0 \quad \text{a.e. on } (0,T) \times \operatorname{supp} z(t) \tag{1.2}$$

where $[\![u]\!] = u^+|_{\Gamma_C} - u^-|_{\Gamma_C}$ is the jump of u across Γ_C, $u^\pm|_{\Gamma_C}$ denoting the traces on Γ_C of the restrictions $u^\pm$ of u to $\Omega_\pm$. Moreover, $\operatorname{supp} z$ denotes the support of $z \in L^\infty(\Gamma_C)$. The brittle constraint (1.2) ensures the continuity of the displacements ($[\![u(t,x)]\!] = 0$) in the (closure of the) set of points where (a portion of) the bonding is still active ($z(t,x) > 0$), and it allows for displacement jumps only in points $x \in \Gamma_C$ where the bonding is completely broken ($z(t,x) = 0$). In other words, (1.2) distinguishes between the crack set $\Gamma_C \backslash \operatorname{supp} z(t)$, where the displacements may jump, and the complementary set with active bonding, where it imposes a transmission condition on the displacements. That is why, the brittle delamination system can be understood as a model for *dynamic fracture*, albeit in a special setting:

the crack occurs along a prescribed surface, but with *unknown evolution*. The main result of this paper states the existence of *energetic*-type solutions, obtained by approximation via a model for adhesive contact.

Let us now have a closer look at the adhesive contact and brittle delamination systems, discuss the analytical difficulties attached to the adhesive-to-brittle limit, and illustrate our arguments and results.

The adhesive contact system

The *classical formulation* of the adhesive contact model we will at first consider consists of the momentum equation, with viscosity and inertia, for the displacement u in the bulk domain, namely

$$\varrho\ddot{u} - \mathrm{div}\,(\mathbb{D}\dot{e} + \mathbb{C}e) = F \quad \text{in } (0,T)\times(\Omega_+ \cup \Omega_-), \tag{1.3a}$$

with $\varrho > 0$ the (assumed constant, for simplicity) mass density of the body, $e = e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ the linearized strain tensor (throughout the paper, we shall often write $\dot{e}$ as a short-hand for $e(\dot{u})$), and F a time-dependent applied volume force. Equation (1.3a) is supplemented with homogeneous Dirichlet boundary conditions on the Dirichlet part Γ_{D} of the boundary $\partial\Omega$, where $\Omega := \Omega_+ \cup \Gamma_{\mathrm{C}} \cup \Omega_-$, and subject to an applied traction f on the Neumann part $\Gamma_{\mathrm{N}} = \partial\Omega \setminus \Gamma_{\mathrm{D}}$, i.e.

$$u = 0 \quad \text{on } (0,T)\times\Gamma_{\mathrm{D}}, \qquad (\mathbb{D}\dot{e} + \mathbb{C}e)\,|_{\Gamma_{\mathrm{N}}}\nu = f \quad \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \tag{1.3b}$$

with ν the outward unit normal to $\partial\Omega$. The evolutions of u and of the delamination parameter z from (1.1) are coupled through the following boundary condition on the contact surface Γ_{C}

$$(\mathbb{D}\dot{e} + \mathbb{C}e)\,|_{\Gamma_{\mathrm{C}}}\mathbf{n} + kz[\![u]\!] = 0 \quad \text{on } (0,T)\times\Gamma_{\mathrm{C}}, \tag{1.3c}$$

with $\mathbf{n}$ the unit normal to Γ_{C} oriented from Ω_+ to Ω_- and k a positive constant: The adhesive-to-brittle limit passage results from letting $k \to \infty$. In the adhesive contact model, the flow rule for z reads

$$\partial I_{(-\infty,0]}(\dot{z}) + \partial\mathcal{G}_k(z) - a_k^0 - a_k^1 \ni -\tfrac{1}{2}k\left|[\![u]\!]\right|^2 \quad \text{on } (0,T)\times\Gamma_{\mathrm{C}}. \tag{1.3d}$$

In (1.3d), $I_{(-\infty,0]}$ is the indicator function of the half-line $(-\infty,0]$, by means of which unidirectionality $\dot{z} \le 0$ of debonding is imposed, and $\partial I_{(-\infty,0]}$ its convex analysis subdifferential. The positive coefficients a_k^0 and a_k^1, depending on the parameter k in view of a discussion of different scalings in the adhesive-to-brittle limit $k \to \infty$, are the phenomenological specific energies per area stored and dissipated by disintegrating the adhesive. Finally, $\partial\mathcal{G}_k$ is the (formally written) subdifferential of the gradient term

$$\mathcal{G}_k(z) := \begin{cases} \mathrm{b}_k|\mathrm{D}z|(\Gamma_{\mathrm{C}}) & \text{if } z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}), \\ \infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where $\mathrm{b}_k > 0$ again depends on k, $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ is the space of the special bounded variation functions on Γ_{C}, taking values in $\{0,1\}$, and $|\mathrm{D}z|(\Gamma_{\mathrm{C}})$ denotes the variation on Γ_C of the Radon measure $\mathrm{D}z$. Indeed, we are imposing that z only takes the values 0 and 1, so that it can be identified with the characteristic function of a set $Z \subset \Gamma_C$ with finite perimeter $P(Z;\Gamma_C) = |\mathrm{D}z|(\Gamma_C)$. Thus our adhesive contact model (and the limiting brittle delamination system) accounts for just two states of the bonding between Ω_+ and Ω_-, i.e. the fully effective and the completely ineffective ones. The constraint $z \in \{0,1\}$ makes ours akin to a model for crack propagation (along a prescribed $(d{-}1)$-dimensional interface).

Due to the expected poor time regularity of the delamination variable z, the adhesive contact system (1.3) has to be weakly formulated in a suitable way, reflecting its mixed rate-independent/rate-dependent character. For this, we shall resort to an *energetic*-type solvability concept, generalizing the notion of (global) energetic solution of a purely rate-independent system, cf. reference[42]. First introduced in the article[52], this concept has been recently analyzed from a more abstract viewpoint in the paper[50]. We shall recall this solution concept in a general and abstract setting in the upcoming Definition 2.1; in the specific context of the adhesive contact system, we call a pair (u,z) with suitable temporal and spatial regularity a *semistable energetic solution* of system (1.3) if it fulfills the weak formulation of the momentum balance

$$\begin{aligned}\int_\Omega \varrho \ddot{u}(t) v \,\mathrm{d}x + \int_{\Omega\setminus\Gamma_C} \left(\mathbb{D}e(\dot{u}(t))\colon e(v) + \mathbb{C}e(u(t)) : e(v)\right) \mathrm{d}x \\ + \int_{\Gamma_C} kz [\![u]\!]\, [\![v]\!] \,\mathrm{d}\mathcal{H}^{d-1} = \langle \mathbf{f}(t), v\rangle_{H^1(\Omega;\mathbb{R}^d)}\end{aligned} \tag{1.5}$$

for almost all $t \in (0,T)$ and for every $v \in H^1(\Omega;\mathbb{R}^d)$ with $v = 0$ on $(0,T)\times\Gamma_D$ (with $\mathcal{H}^{d-1}$ the $(d{-}1)$-dimensional Hausdorff measure, and the function $\mathbf{f} : (0,T) \to H^1(\Omega;\mathbb{R}^d)^*$ subsuming the bulk force F and the applied traction f), and the weak formulation of the flow rule (1.3d). The latter is akin to the (global) energetic formulation for rate-independent systems, in that it features

- an *energy-dissipation (in)equality*, involving the stored energy of the adhesive contact system

$$\begin{aligned}\mathcal{E}_k(t,u,z) := \int_{\Omega\setminus\Gamma_C} \tfrac{1}{2}\mathbb{C}e(u) : e(u)\,\mathrm{d}x - \langle \mathbf{f}(t), u\rangle_{H^1(\Omega;\mathbb{R}^d)} \\ + \int_{\Gamma_C} \left(\tfrac{k}{2} z |[\![u]\!]|^2 - a_k^0 z\right) \mathrm{d}\mathcal{H}^{d-1} + \mathcal{G}_k(z)\end{aligned} \tag{1.6}$$

  and the dissipated energy

$$\mathcal{R}_k(\dot{z}) := \begin{cases} \int_{\Gamma_C} a_k^1 |\dot{z}| \,\mathrm{d}\mathcal{H}^{d-1} & \text{if } \dot{z} \le 0 \text{ a.e. in } \Gamma_C, \\ \infty & \text{otherwise,}\end{cases} \tag{1.7}$$

- coupled with the *semistability* condition

$$\mathcal{E}_k(t,u(t),z(t)) \le \mathcal{E}_k(t,u(t),\tilde{z}) + \mathcal{R}_k(\tilde{z}-z(t)) \tag{1.8}$$

  imposed for all $\tilde{z} \in L^1(\Gamma_C)$ and at all $t \in [0,T]$.

In fact, (1.8) reflects the mixed character of the evolution, in that stability is only tested for z, while the rate-dependent variable u is kept fixed as a solution of (1.5).

The first result of this paper, **Theorem 2.1**, states the existence of semistable energetic solutions of (the Cauchy problem for) the adhesive contact system (1.3), in fact satisfying the energy-dissipation *balance* along any interval $[s,t] \subset [0,T]$. We shall derive Thm. 2.1 from a general existence result for damped inertial systems with a mixed rate-dependent/rate-independent character, which was proved in reference[50]. With Thm. 2.1 we will also provide a series of a priori estimates on families of semistable energetic solutions $(u_k, z_k)_k$, uniform with respect to the parameter $k \in \mathbb{N}$.

The adhesive-to-brittle limit: analytical challenges and our results

The asymptotic analysis as $k \to \infty$ for the *purely rate-independent* adhesive contact system, coupling the flow rule (1.3d) (with no regularizing gradient term), with the *static* momentum balance, was carried out in reference[55] by resorting to the evolutionary Γ-convergence results for rate-independent processes from reference[43]. Loosely speaking, the main observation is that the adhesive contact contribution $\int_{\Gamma_C} \frac{k}{2} z |[\![u]\!]|^2 \,\mathrm{d}\mathcal{H}^{d-1}$ to $\mathcal{E}_k$ (1.6) penalizes displacement jumps in points with positive z, and leads as $k \to \infty$ to the brittle constraint $z|[\![u]\!]| = 0$ a.e. in $\Gamma_{\rm C}$, incorporated in the Γ-limit of $(\mathcal{E}_k)_k$, cf. (1.12) below.

The adhesive-to-brittle asymptotics is remarkably more complicated in the case of mixed rate-independent/rate-dependent evolution, where one has to pass to the limit separately in the momentum balance (1.5) featuring the semistable delamination variables, and in the semistability inequality (1.8) featuring the solution of the momentum balance. This problem was tackled in reference[49] for a system also encompassing the heat equation, but without inertia in the momentum balance. Analogous arguments were used in the *purely rate-independent* case in reference[56], to address the adhesive-to-brittle limit combined with time discretization and leading to *local solutions* (cf. [54, Sec. 3]), of the brittle system.

In what follows, we will illustrate these analytical difficulties and hint at our methods, which could in fact be adapted to handle the coupling with the temperature equation, as well. We have however chosen to confine our analysis to the isothermal case, in order to highlight the techniques specifically developed in the present paper to deal with inertia in the momentum balance. The very first problem is due to the

(1) blow-up of the bounds on the adhesive contact term $kz[\![u]\!]$ in the momentum balance (1.5) as $k \to \infty$.

This reflects the fact that, for the limiting brittle system the momentum balance has to be tested with test functions fulfilling the brittle constraint (1.2), which will be satisfied by the limiting displacement u. We will in fact prove that any pair (u, z), arising in the limit as $k \to \infty$ of a sequence of semistable energetic solutions

$(u_k, z_k)_k$ of the adhesive contact system (1.3), $k \in \mathbb{N}$, fulfills

$$\int_\Omega \varrho \ddot{u}(t) v \,\mathrm{d}x + \int_{\Omega\setminus\Gamma_{\mathrm{C}}} (\mathbb{D}e(\dot{u}(t))\colon e(v) + \mathbb{C}e(u(t)) : e(v)) \,\mathrm{d}x = \langle \mathbf{f}(t), v\rangle_{H^1(\Omega;\mathbb{R}^d)} \quad \text{for all } v \in H^1(\Omega;\mathbb{R}^d) \text{ with } v = 0 \text{ on } \Gamma_{\mathrm{D}} \text{ and } [\![v]\!] = 0 \text{ on } \operatorname{supp} z(t) \subset \Gamma_{\mathrm{C}} \tag{1.9}$$

and for almost all $t \in (0,T)$. In order to obtain (1.9), we shall resort to the arguments from reference[49] and provide for every admissible test function v for (1.9) a *recovery sequence* $(v_k)_k$, suitably converging to v, fulfilling the brittle constraint (1.2) already at level $k \in \mathbb{N}$, i.e.

$$[\![v_k]\!] = 0 \qquad \text{a.e. on } \operatorname{supp} z_k(t)\,, \tag{1.10}$$

with $t \in (0,T)$ fixed out of a negligible set. This will allow us to bypass problem (1). The key ingredient in the construction of the sequence $(v_k)_k$, starting from a test function v such that $[\![v]\!] = 0$ on $\operatorname{supp} z(t)$, is a relation between the supports of the approximate, semistable delamination variables z_k, and the support of the semistable limit z. This is provided by the property of *support convergence*

$$\operatorname{supp} z_k(t) \subset \operatorname{supp} z(t) + B_{\rho(k,t)}(0) \quad \text{and} \quad \rho(k,t) \to 0 \text{ as } k \to \infty, \tag{1.11}$$

that was proved in reference[49] via arguments from geometric measure theory. In turn, these arguments heavily rely on the fact that the delamination variables z_k take value in $\{0,1\}$, and on the regularizing perimeter term from (1.4) contributing to the energy functional (1.6) for the adhesive contact system.

In reference[49], addressing the case without inertia, the above arguments were sufficient to pass to the limit in the momentum balance (1.5), tested with the recovery test functions v_k satisfying (1.10). In the present case, we have to face an additional difficulty, clearly related to problem (1), namely the

(2) blow-up as $k \to \infty$ of the estimates (by comparison) on the inertial terms $\ddot{u}_k$ in (1.5).

We will overcome this by a careful refinement of the method from reference[49]. This will lead us to construct a sequence of *recovery spaces* for the space of test functions in the weak momentum balance (1.9) for the brittle system. The crucial point will then be to observe that the terms $(\ddot{u}_k)_k$ are in fact suitably estimated in these spaces, which will allow for compactness arguments and, ultimately, the limit passage in (1.5). The limit passage in the energy-dissipation inequality for the adhesive contact system will essentially follow from lower semicontinuity, while for the semistability condition we will make use of the by now standard *mutual recovery sequence* argument from reference[43].

In this way we will obtain the **main result of our paper**, **Theorem 2.2**, stating the convergence of semistable energetic solutions of the adhesive contact systems to a semistable energetic solution of the brittle one, fulfilling

- the weak momentum balance (1.9),
- the energy-dissipation (in)equality,

- the semistability condition.

The latter two relations feature the dissipation potential $\mathcal{R}_\infty$ arising in the limit of the energies $(\mathcal{R}_k)_k$ from (1.7), and the energy functional

$$\begin{aligned} \mathcal{E}_\infty(t,u,z) := \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}e(u):e(u)\,\mathrm{d}x - \ \langle \mathbf{f}(t),u\rangle_{H^1(\Omega;\mathbb{R}^d)} \\ + \int_{\Gamma_{\mathrm{C}}} \big(J_\infty([\![u]\!],z) - a_\infty^0 z\big)\ \mathrm{d}\mathcal{H}^{d-1} + \mathcal{G}_\infty(z) \end{aligned} \tag{1.12}$$

with $J_\infty([\![u]\!],z)$ the indicator function of the brittle constraint (1.2), i.e. $J_\infty([\![u]\!],z) = 0$ if (1.2) is satisfied and $J_\infty([\![u]\!],z) = \infty$ otherwise, and $\mathcal{G}_\infty$ the Γ-limit as $k \to \infty$ of the perimeter energies $(\mathcal{G}_k)_k$ from (1.4). By adapting some arguments from reference[14], we shall prove that along semistable energetic solutions of the brittle system, the energy-dissipation inequality actually holds as a balance, along any arbitrary interval $[s,t] \subset [0,T]$ for almost all $s \leq t \in (0,T)$, and for $s = 0$.

Our ansatz for $\mathcal{G}_k$ and $\mathcal{R}_k$, cf. (1.4) and (1.7), will allow for different scalings of the parameters a_k^0, a_k^1, and b_k, cf. (2.15). In this way, we can obtain different fracture models in the brittle limit. We will discuss the different options in Sections 2.3 & 2.4.

Plan of the paper. In Section 2 we give our weak solvability notion for damped inertial systems with a mixed rate-independent/rate-dependent character. In particular, in Sec. 2.1 we specify it in the context of the adhesive contact model and then state the existence of semistable energetic solutions for the adhesive system. As for the brittle system, in Sec. 2.2 we give the notion of semistable energetic solutions, while in Sec. 2.3 we present our main existence result, Theorem 2.2, which we compare with other existing results on dynamic fracture in Sec. 2.4. The existence of semistable energetic solutions of the adhesive contact system is proved in Section 3, while the proof of Theorem 2.2 is carried out in Section 4.

2. Setup, solution concepts for the adhesive and brittle problems, and preliminary results

We start by fixing some general notation that will be used throughout the paper: We will denote by $\|\cdot\|_X$ the norm of a Banach space X, and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. If X is a Hilbert space, its inner product shall be denoted by $(\cdot,\cdot)_X$. The symbols (1) $\mathrm{B}([0,T];X)$, (2) $\mathrm{BV}[0,T];X)$, (3) $\mathrm{C}^0_{\mathrm{weak}}([0,T];X)$ shall denote the spaces of functions with values in X that are defined at every $t \in [0,T]$ and are (1) bounded and measurable, (2) with bounded variation, (3) continuous with respect to the weak topology, respectively. Moreover, we shall often denote by the symbols c, $\tilde{c}$, C, $\tilde{C}$ various positive constants, whose meaning may vary from line to line, depending only on known quantities.

Setup & semistable energetic solutions for damped inertial systems. Let us now specify the concept of *abstract damped inertial system*, and the associated notion of *semistable energetic solution*, that will later apply both to the adhesive contact, and to the brittle systems. We draw the following definitions from reference[50], where the semistable energetic solution concept, originally introduced in the work[52] for a class of mixed rate-dependent/rate-independent systems in continuum mechanics, was generalized to an abstract setting. Let us mention that, in reference[50] a fairly broad class of damped inertial systems was tackled, in particular encompassing a dissipation potential $\mathcal{V}$ with general superlinear growth at infinity, and a non-convex (but still with appropriate properties) dependence $u \mapsto \mathcal{E}(t,u,z)$. However, in view of the target adhesive contact and brittle systems, it will be sufficient to confine the discussion to a *quadratic* dissipation potential, and to the case the mapping $u \mapsto \mathcal{E}(t,u,z)$ is *convex*.

In what follows, we will consider an abstract damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ given by:

- two Hilbert spaces

$$\begin{gathered} \mathbf{V} \text{ and } \mathbf{W},\ \mathbf{W} \text{ identified with its dual } \mathbf{W}^*, \text{ such that} \\ \mathbf{V} \Subset \mathbf{W} \text{ compactly and densely,} \end{gathered} \tag{2.1a}$$

  with $\mathbf{V} \subset \mathbf{W} = \mathbf{W}^* \subset \mathbf{V}^*$ continuously and densely, and $\langle w, u \rangle_{\mathbf{V}} = (w,u)_{\mathbf{W}}$ for all $u \in \mathbf{V}$, $w \in \mathbf{W}$;
- a separable Banach space $\mathbf{Z}$;
- a dissipation potential $\mathcal{V} : \mathbf{V} \to [0,\infty)$ given by

$$\begin{gathered} \mathcal{V}(v) = \tfrac{1}{2} a(v,v) \quad \text{ with} \\ a : \mathbf{V} \times \mathbf{V} \to \mathbb{R} \text{ a continuous coercive bilinear form;} \end{gathered} \tag{2.1b}$$

- a dissipation potential $\mathcal{R} : \mathbf{Z} \to [0,\infty]$, with domain $\mathrm{dom}(\mathcal{R})$, lower semicontinuous, convex, positively 1-homogeneous and coercive i.e.,

$$\begin{aligned} &\mathcal{R}(\lambda\zeta) = \lambda\mathcal{R}(\zeta) \quad \text{ for all } \zeta \in \mathbf{Z} \text{ and } \lambda \geq 0, \\ &\exists\, C_R > 0 \ \ \forall\, \zeta \in \mathbf{Z} \qquad \mathcal{R}(\zeta) \geq C_R \|\zeta\|_{\mathbf{Z}}; \end{aligned} \tag{2.1c}$$

- a *kinetic energy* $\mathcal{K} : \mathbf{W} \to [0,\infty)$, $\mathcal{K}(v) := \frac{1}{2}\|v\|_{\mathbf{W}}^2$,
- an *energy functional* $\mathcal{E} : [0,T] \times \mathbf{V} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$, with proper domain $\mathrm{dom}(\mathcal{E}) = [0,T] \times \mathbf{D}_u \times \mathbf{D}_z$, such that

$$\begin{aligned} &t \mapsto \mathcal{E}(t,u,z) \text{ is differentiable} && \text{for all } (u,z) \in \mathbf{D}_u \times \mathbf{D}_z, \\ &(u,z) \mapsto \mathcal{E}(t,u,z) \text{ is lower semicontinuous} && \text{for all } t \in [0,T], \\ &u \mapsto \mathcal{E}(t,u,z) \text{ is convex} && \text{for all } (t,z) \in [0,T] \times \mathbf{D}_z\,. \end{aligned} \tag{2.1d}$$

Hereafter, we shall denote by $\partial_u \mathcal{E} : [0,T] \times \mathbf{V} \times \mathbf{Z} \rightrightarrows \mathbf{V}^*$ the subdifferential of the functional $\mathcal{E}(t,\cdot,z)$ in the sense of convex analysis. Moreover, to account for the coercivity requirements on the map $z \mapsto \mathcal{E}(t,u,z)$, cf. (3.3) ahead for more details, here we also introduce a space $\mathbf{X}$ such that

$$\mathbf{X} \text{ is the dual of a separable Banach space and } \mathbf{X} \Subset \mathbf{Z} \text{ compactly.} \tag{2.1e}$$

We are now in the position to state precisely the semistable energetic solution concept for the damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$, which has been developed in [50, Def. 3.1] based on a time-discrete scheme with alternating (decoupled) minimization w.r.t. the variables u and z.

Definition 2.1 (Semistable energetic solution). We call a pair $(u, z) : [0, T] \to \mathbf{V} \times \mathbf{Z}$ a *semistable energetic solution* of the damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ if

$$u \in W^{1,1}(0, T; \mathbf{V}), \quad \dot{u} \in L^\infty(0, T; \mathbf{W}), \quad \ddot{u} \in L^2(0, T; \mathbf{V}^*), \tag{2.2a}$$

$$z \in \mathrm{B}([0, T]; \mathbf{X}) \cap \mathrm{BV}([0, T]; \mathbf{Z}) \tag{2.2b}$$

fulfill the

- the subdifferential inclusion

$$\varrho\ddot{u}(t) + \partial\mathcal{V}(\dot{u}(t)) + \partial_u\mathcal{E}(t, u(t), z(t)) \ni 0 \qquad \text{in } \mathbf{V}^* \quad \text{ for a.a. } t \in (0, T), \tag{2.3}$$

- the semistability condition

$$\mathcal{E}(t, u(t), z(t)) \le \mathcal{E}(t, u(t), \tilde{z}) + \mathcal{R}(\tilde{z} - z(t)) \text{ for all } \tilde{z} \in \mathbf{Z} \text{ for all } t \in [0, T], \tag{2.4}$$

- the energy-dissipation inequality for all $t \in [0, T]$

$$\begin{aligned} &\tfrac{1}{2}\|\dot{u}(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{u}(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(z, [0, t]) + \mathcal{E}(t, u(t), z(t)) \\ &\le \tfrac{1}{2}\|\dot{u}(0)\|^2_{\mathbf{W}} + \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_t\mathcal{E}(s, u(s), z(s))\,\mathrm{d}s \end{aligned} \tag{2.5}$$

with $\mathrm{Var}_{\mathcal{R}}$ the total variation induced by $\mathcal{R}$, i.e., for a given subinterval $[s, t] \subset [0, T]$,

$$\mathrm{Var}_{\mathcal{R}}(z; [s, t]) := \sup\Big\{ \sum_{j=1}^N \mathcal{R}(z(r_j) - z(r_{j-1})) \; : \; s = r_0 < r_1 < \ldots < r_{N-1} < r_N = t \Big\}.$$

Basic assumptions. Before specifying the above notions in the context of the adhesive contact and brittle systems, let us establish some basic conditions on the domains Ω and Γ_{C}, and on the problem data, in common to the adhesive and brittle models.

Assumptions on the reference domain: We suppose that

$$\Omega \subset \mathbb{R}^d,\ d \ge 2, \text{ is bounded, } \Omega_-,\ \Omega_+,\ \Omega \text{ are Lipschitz domains,} \quad \Omega_+ \cap \Omega_- = \emptyset, \tag{2.6a}$$

$$\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}},\ \Gamma_{\mathrm{D}},\ \Gamma_{\mathrm{N}} \text{ open subsets in } \partial\Omega, \tag{2.6b}$$

$$\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset,\ \overline{\Gamma}_{\mathrm{D}} \cap \overline{\Gamma}_{\mathrm{C}} = \emptyset,\ \mathcal{H}^{d-1}(\Gamma_{\mathrm{D}} \cap \overline{\Omega}_+) > 0,\ \mathcal{H}^{d-1}(\Gamma_{\mathrm{D}} \cap \overline{\Omega}_-) > 0, \tag{2.6c}$$

$$\begin{aligned} &\Gamma_{\mathrm{C}} = \overline{\Omega}_+ \cap \overline{\Omega}_- \subset \mathbb{R}^{d-1} \text{ is a convex ``flat'' surface,} \\ &\qquad \text{i.e. contained in a hyperplane of } \mathbb{R}^d, \\ &\text{such that, in particular, } \mathcal{H}^{d-1}(\Gamma_{\mathrm{C}}) = \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}) > 0, \end{aligned} \tag{2.6d}$$

where $\mathcal{H}^{d-1}$, resp. $\mathcal{L}^{d-1}$, denotes the $(d-1)$-dimensional Hausdorff, resp. Lebesgue measure. The condition that Γ_{C} is contained in a hyperplane has no substantial role for our analysis but to simplify arguments and notation. Instead, the convexity of Γ_{C} is essential for the proof of the adhesive-to-brittle limit passage, cf. the comments following Prop. 4.1. In what follows, we will use the notation

$$H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) := \{v \in H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \,:\, v = 0 \text{ a.e. on } \Gamma_{\mathrm{D}}\}\,.$$

Assumptions on the given data: For the tensors $\mathbb{C}, \mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}$ and the function $\mathbf{f}$ in (1.9), we require that

$$\begin{aligned}
&\mathbb{C}, \mathbb{D} \in \mathbb{R}^{d\times d\times d\times d} \text{ are symmetric and positive definite, i.e.,}\\
&\exists\, C^1_{\mathbb{C}}, C^2_{\mathbb{C}}, C^1_{\mathbb{D}}, C^2_{\mathbb{D}} > 0, \forall\, \eta \in \mathbb{R}^{d\times d}:\\
&C^1_{\mathbb{C}}|\eta|^2 \le \eta : \mathbb{C}\eta \le C^2_{\mathbb{C}}|\eta|^2 \text{ and } C^1_{\mathbb{D}}|\eta|^2 \le \eta : \mathbb{D}\eta \le C^2_{\mathbb{D}}|\eta|^2\,,
\end{aligned} \tag{2.7a}$$

$$\mathbf{f} \in \mathrm{C}^1([0,T];\mathbf{V}^*) \text{ and } \sup_{t\in[0,T]} \big(\|\mathbf{f}(t)\|_{\mathbf{V}^*} + \|\dot{\mathbf{f}}(t)\|_{\mathbf{V}^*}\big) \le C_{\mathbf{f}}\,. \tag{2.7b}$$

Moreover, to keep notation and arguments simple, we prescribe *homogeneous Dirichlet data* on Γ_{D}.

2.1. *Semistable energetic solutions of the adhesive contact systems*

The adhesive contact evolutionary system falls within the class of damped inertial systems, with the following choices of
Function spaces:

$$\mathbf{V} = H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)\,, \tag{2.8a}$$

$$\mathbf{W} = L^2(\Omega;\mathbb{R}^d) \;\text{ endowed with the norm } \|v\|_{\mathbf{W}} := \Big(\int_\Omega \varrho|v|^2\,\mathrm{d}x\Big)^{1/2}, \tag{2.8b}$$

$$\mathbf{Z} = L^1(\Gamma_{\mathrm{C}})\,, \tag{2.8c}$$

$$\mathbf{X} = \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})\,, \tag{2.8d}$$

where the space $\mathbf{X}$ is related to the perimeter regularizing term contributing to the energy functional $\mathcal{E}_k$, cf. (2.12) below. Observe that, due to the positivity and boundedness of the mass density ϱ, the space $\mathbf{W}$ is identified with $L^2(\Omega;\mathbb{R}^d)$.

Dissipation potentials and energy functionals for the adhesive case, $k \in \mathbb{N}$: For each $k \in \mathbb{N}$ the adhesive systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)$ are governed by the functionals corresponding to the kinetic energy $\mathcal{K}$, the viscous dissipation $\mathcal{V}$, the rate-independent dissipation $\mathcal{R}_k$, and the mechanical energy $\mathcal{E}_k$:

$$\mathcal{K}(\dot u) := \tfrac12 \|\dot u\|_{\mathbf{W}}^2 \,, \tag{2.9}$$

$$\mathcal{V} : \mathbf{V} \to [0,\infty)\,,\ \mathcal{V}(\dot u) := \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac12 \mathbb{D} e(\dot u) : e(\dot u)\,\mathrm{d}x\,, \tag{2.10}$$

$$\begin{aligned} &\mathcal{R}_k : \mathbf{Z} \to [0,\infty]\,,\ \mathcal{R}_k(\dot z) := \int_{\Gamma_{\mathrm{C}}} R_k(\dot z)\,\mathrm{d}\mathcal{H}^{d-1}\,, \\ &\qquad\qquad R_k(\dot z) := \begin{cases} a_k^1 |\dot z| & \text{if } \dot z \le 0\,, \\ \infty & \text{otherwise}; \end{cases} \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\mathcal{E}_k : [0,T] \times \mathbf{V} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\},\ \text{ defined for all } t \in [0,T]: \\ &\mathcal{E}_k(t,u,z) := \begin{cases} \widetilde{\mathcal{E}}_k(t,u,z) + \mathcal{J}_k(u,z) & \text{if } (u,z) \in \mathbf{V} \times \mathbf{X}\,, \\ \infty & \text{otherwise} \end{cases} \qquad \text{with} \\ &\widetilde{\mathcal{E}}_k(t,u,z) := \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac12 \mathbb{C} e(u) : e(u)\,\mathrm{d}x - \langle \mathbf{f}(t), u\rangle_{\mathbf{V}} \\ &\qquad\qquad + \int_{\Gamma_{\mathrm{C}}} \big(I_{[0,1]}(z) - a_k^0 z\big)\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}_k P(Z,\Gamma_{\mathrm{C}})\,,\ \text{ and} \end{aligned} \tag{2.12}$$

$$\mathcal{J}_k(u,z) := \int_{\Gamma_{\mathrm{C}}} \tfrac{k}{2} z \left| [\![u]\!] \right|^2 \mathrm{d}\mathcal{H}^{d-1}\,. \tag{2.13}$$

As already mentioned in the Introduction, hereafter Z shall denote a subset of Γ_{C} with finite perimeter $P(Z,\Gamma_{\mathrm{C}})$ in Γ_{C} such that $z = \chi_Z \in \{0,1\}$ is its characteristic function.

Observe that, for every $k \in \mathbb{N}$ the functional $u \mapsto \mathcal{E}_k(t,u,z)$ is Gâteaux-differentiable, in addition to being convex. Therefore, at every $(t,u,z) \in [0,T] \times \mathbf{D}_u \times \mathbf{D}_z$ its subdifferential $\partial_u \mathcal{E}_k(t,u,z)$ reduces to a singleton, whose unique element, still denoted by $\partial_u \mathcal{E}_k$, is given for all $v \in \mathbf{V}$ by

$$\begin{aligned} &\langle \partial_u \mathcal{E}_k(t,u,z), v\rangle_{\mathbf{V}} = \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathbb{C} e(u) : e(v)\,\mathrm{d}x - \langle \mathbf{f}(t), v\rangle_{\mathbf{V}} + \langle \partial_u \mathcal{J}_k(u,z), v\rangle_{\mathbf{V}}, \\ &\text{with} \\ &\langle \partial_u \mathcal{J}_k(u,z), v\rangle_{\mathbf{V}} = \int_{\Gamma_{\mathrm{C}}} kz [\![u]\!]\, [\![v]\!]\,\mathrm{d}\mathcal{H}^{d-1}\,. \end{aligned} \tag{2.14}$$

Therefore, taking into account of the form (2.10) of the dissipation potential $\mathcal{V}$, the subdifferential inclusion (2.3) yields the momentum equation (1.5). With regard to the coefficients $a_k^0, a_k^1, \mathrm{b}_k$ occuring in (2.11)–(2.13), our analysis will cover the two cases (2.15), discussed with more detail in Sec. 2.4:

$$a_k^0 = a^0 = \text{const.}\,,\ a_k^1 = a^1 = \text{const.}\,,\ \mathrm{b}_k = \mathrm{b} = \text{const.}\,, \quad \text{or} \tag{2.15a}$$

$$a_k^0 = \tfrac{a^0}{k}\,,\ a_k^1 = \tfrac{a^1}{k}\,,\ \mathrm{b}_k = \tfrac{\mathrm{b}}{k}\,. \tag{2.15b}$$

Since with the scaling (2.15b) for the coefficients b_k, the constraint $z \in \{0,1\}$ is no longer ensured in the brittle limit $k \to \infty$, the term $I_{[0,1]}$ contributing to $\mathcal{E}_k$ (and to $\mathcal{E}_\infty$, cf. (2.33) below) has the role to enforce, for $k \in \mathbb{N} \cup \{\infty\}$, that z takes values in

$[0, 1]$. This is crucial not only for the physical consistency of the model, but also for technical reasons related to the construction of the recovery sequence for the limit passage as $k \to \infty$ in the semistability condition.

The existence of *semistable energetic solutions* (u_k, z_k) for the evolutionary adhesive contact systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)$ will be deduced from the abstract results of [50] in Section 3 ahead, where we will also derive estimates (2.17) & (2.18) on the functions $(u_k, z_k)_k$. Independently from the bound (2.17b) on the total variation of z_k induced by $\mathcal{R}_k$, we also need to derive estimate (2.17d) for z_k in $\mathrm{BV}([0, T]; L^1(\Gamma_{\mathrm{C}}))$, due to the possible degeneracy of the coefficients a_k^1 when scaled as in (2.15b).

Theorem 2.1 (Existence of semistable energetic solutions for $k \in \mathbb{N}$ fixed, uniform bounds, uniqueness). *Assume* (2.6)–(2.7)*. For each $k \in \mathbb{N}$ the damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)$ admits a semistable energetic solution (u_k, z_k) in the sense of Def. 2.1 starting from initial data $(u_0, u_1, z_0) \in \mathbf{V} \times \mathbf{W} \times \mathbf{Z}$ fulfilling the semistability* (2.4) *at $t = 0$ with $\mathcal{E}_k$ and $\mathcal{R}_k$, cf.* (3.7)*.*

In addition, for every $k \in \mathbb{N}$ the energy-dissipation inequality even holds as an equality along any interval $[s, t] \subset [0, T]$:

$$\begin{aligned} &\tfrac{1}{2}\|\dot{u}_k(t)\|_{\mathbf{W}}^2 + \int_s^t 2\mathcal{V}(\dot{u}_k(\tau))\,\mathrm{d}\tau + \mathrm{Var}_{\mathcal{R}_k}(z_k, [s, t]) + \mathcal{E}_k(t, u_k(t), z_k(t)) \\ &\qquad = \tfrac{1}{2}\|\dot{u}_k(s)\|_{\mathbf{W}}^2 + \mathcal{E}_k(s, u_k(s), z_k(s)) + \int_s^t \partial_t \mathcal{E}_k(\tau, u_k(\tau), z_k(\tau))\,\mathrm{d}\tau\,. \end{aligned} \tag{2.16}$$

Furthermore, for a given $z \in L^\infty(0, T; \mathrm{SBV}(\Gamma_{\mathrm{C}}; \{0, 1\}))$, if (u, z) and $(\tilde{u}, z)$ both satisfy the adhesive momentum balance with the same initial data u_0 and u_1, then $\tilde{u} = u$.

Finally, there exists a constant $C > 0$, only depending on the initial data (u_0, u_1, z_0) and on the given data, such that the functions $(u_k, z_k)_k$ satisfy the following bounds, uniform in $k \in \mathbb{N}$:

$$\sup_{t \in [0,T]} \big(\mathcal{E}_k(t, u_k(t), z_k(t)) + \partial_t \mathcal{E}_k(t, u_k, z_k)\big) \le C\,, \tag{2.17a}$$

$$\int_0^T \mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_k}(z_k, [0, T]) \le C\,, \tag{2.17b}$$

$$\|u_k\|_{H^1(0,T;\mathbf{V})} + \|\dot{u}_k\|_{L^\infty(0,T;\mathbf{W})} \le C\,, \tag{2.17c}$$

$$\|z_k\|_{\mathrm{BV}([0,T];L^1(\Gamma_{\mathrm{C}}))} \le C\,, \tag{2.17d}$$

$$\sup_{t \in [0,T]} \big(P(Z_k(t), \Gamma_{\mathrm{C}}) + \|z_k(t)\|_{L^\infty(\Gamma_{\mathrm{C}})}\big) \le C\,, \tag{2.17e}$$

$$\|\ddot{u}_k + \partial_u \mathcal{J}_k(\cdot, u_k, z_k)\|_{L^2(0,T;\mathbf{V}^*)} \le C\,. \tag{2.17f}$$

Furthermore, (u_k, z_k) satisfy the following k-dependent bounds for a.a. $t \in (0, T)$:

$$\exists\, c, C > 0,\ \forall\, k \in \mathbb{N}: \quad \|\partial_u \mathcal{J}_k(u_k, z_k)\|_{L^2(0,T;\mathbf{V}^*)} \le \sqrt{k}C + c\,, \tag{2.18a}$$

$$\exists\, \tilde{c}, \widetilde{C} > 0,\ \forall\, k \in \mathbb{N}: \quad \|\ddot{u}_k\|_{L^2(0,T;\mathbf{V}^*)} \le \sqrt{k}\widetilde{C} + \tilde{c}\,. \tag{2.18b}$$

2.2. *Semistable energetic solutions for the brittle system*

Let us now specify the functional analytic setting for the brittle system:
Function spaces: In addition to the spaces $\mathbf{V}$, $\mathbf{W}$, $\mathbf{Z}$, $\mathbf{X}$ from (2.8), we will work with the following family of time-dependent spaces, defined for all $t \in [0,T]$:

$$\begin{gathered} \mathbf{V}_z(t) = \{ v \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \, : \, [\![v]\!] = 0 \text{ a.e. on } \operatorname{supp} z(t) \subset \Gamma_{\mathrm{C}} \} \,, \\ \text{for a given } z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})) \cap \mathrm{BV}([0,T];L^1(\Gamma_{\mathrm{C}})) \text{ such that} \\ z(t_2) \le z(t_1) \text{ for all } 0 \le t_1 \le t_2 \le T \,. \end{gathered} \tag{2.19}$$

These will be the spaces for the test functions in the weak formulation of the momentum balance for the brittle system. Observe that, with these spaces we are enforcing a constraint slightly stronger than $z|[\![v]\!]| = 0$, cf. also Remark 4.1 ahead.

As we will see (cf. Proposition 2.1 later on), $\mathbf{V}_z(t)$, endowed with the norm induced by $H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$, is a closed subspace of $\mathbf{V}$. Hence, by the Hahn-Banach theorem every $\xi \in \mathbf{V}_z(t)^*$ can be extended to a functional $\tilde\xi \in \mathbf{V}^*$ such that

$$\langle \xi, v \rangle_{\mathbf{V}_z(t)} = \langle \tilde\xi, v \rangle_{\mathbf{V}} \quad \text{for all } v \in \mathbf{V}_z(t) \quad \text{and} \quad \|\xi\|_{\mathbf{V}_z(t)^*} = \|\tilde\xi\|_{\mathbf{V}^*} \tag{2.20}$$

Moreover, $\mathbf{V}_z(t)$ is continuously and densely embedded in $\mathbf{W}$, cf. (2.8). Hence, $\mathbf{W}$ is continuously and densely embedded in the dual space $\mathbf{V}_z(t)^*$ and for every $t \in [0,T]$ there holds

$$\langle \xi, v \rangle_{\mathbf{V}_z(t)} = (\xi, v)_{\mathbf{W}} \qquad \text{for all } v \in \mathbf{V}_z(t) \text{ and all } \xi \in \mathbf{W}. \tag{2.21}$$

Moreover, due to the monotonicity of the function $z(\cdot,x) : [0,T] \to \{0,1\}$ for a.a. $x \in \Gamma_{\mathrm{C}}$, we have that $\operatorname{supp}(z(t_2))) \subset \operatorname{supp}(z(t_1))$ for every $0 \le t_1 \le t_2 \le T$, and therefore

$$\mathbf{V}_z(t_1) \subset \mathbf{V}_z(t_2) \qquad \text{for all } 0 \le t_1 \le t_2 \le T. \tag{2.22}$$

Accordingly, for every $\xi \in \mathbf{V}_z(t_2)^*$ we can consider its restriction to $\mathbf{V}_z(t_1)$, which gives an element of $\mathbf{V}_z(t_1)^*$ defined by $\langle \xi|_{\mathbf{V}_z(t_1)}, v \rangle_{\mathbf{V}_z(t_1)} = \langle \xi, v \rangle_{\mathbf{V}_z(t_2)}$. The restriction map is continuous and is indeed the adjoint of the embedding $\mathbf{V}_z(t_1) \subset \mathbf{V}_z(t_2)$. Therefore, there holds

$$\mathbf{V}_z(t_2)^* \subset \mathbf{V}_z(t_1)^* \text{ continuously for all } 0 \le t_1 \le t_2 \le T. \tag{2.23}$$

We will also work with the space

$$L^2(0,T;\mathbf{V}_z) := \{ v \in L^2(0,T;\mathbf{V}) \, : \, v(t) \in \mathbf{V}_z(t) \text{ for a.a. } t \in (0,T) \}, \tag{2.24}$$

endowed with the norm $\|\cdot\|_{L^2(0,T;\mathbf{V})}$, and with

$$\begin{aligned} L^2(0,T;\mathbf{V}_z^*) := \{ \xi \in L^2(0,T;\mathbf{V}_z(0)^*) \, : \, & \\ & \xi(t) \in \mathbf{V}_z(t)^* \text{ for a.a. } t \in (0,T), \\ & t \mapsto \|\xi(t)\|_{\mathbf{V}_z(t)^*} \in L^2(0,T) \}, \end{aligned} \tag{2.25a}$$

endowed with the norm

$$\|\xi\|_{L^2(0,T;\mathbf{V}_z^*)} := \sup_{v \in L^2(0,T;\mathbf{V}_z)} \left| \int_0^T \langle \xi(t), v(t) \rangle_{\mathbf{V}_z(t)} \, \mathrm{d}t \right| . \tag{2.25b}$$

Observe that, underlying definition (2.25) is the fact that $\mathbf{V}_z(t)^* \subset \mathbf{V}_z(0)^*$ for all $t \in [0,T]$; we also refer to Prop. 2.1 for more details. Finally, let us also introduce the Sobolev space

$$H^2_{\#}(0,T;\mathbf{V}_z^*) := \{u \in H^1(0,T;\mathbf{V}^*) \, : \ddot{u} \in L^2(0,T;\mathbf{V}_z^*)\}, \tag{2.26}$$

with $\ddot{u}$ the (second-order in time) *weak* distributional derivative, to be understood at almost all $t \in (0,T)$ as the weak limit of the difference quotients $\frac{\dot{u}(t+h)-\dot{u}(t)}{h}$ in $\mathbf{V}_z(t)^*$, namely

$$\langle \ddot{u}(t), v\rangle_{\mathbf{V}_z(t)} = \lim_{h\to 0} \Big\langle \frac{\dot{u}(t+h)-\dot{u}(t)}{h}, v\Big\rangle_{\mathbf{V}_z(t)} \qquad \text{for all } v \in \mathbf{V}_z(t). \tag{2.27}$$

We now collect the basic properties of the previously introduced spaces in Proposition 2.1; its proof can be found in the *Appendix*. The statements are given in terms of a time-dependent set $M(t)$: this applies to the closed set $\operatorname{supp} z(t)$, but also to suitable enlargements of $\operatorname{supp} z(t)$, cf. (4.18) and Proposition 4.4.

Proposition 2.1. *Let $(M(t))_{t\in[0,T]}$ be a family of closed subsets of Γ_{C} and set*

$$\begin{aligned} \mathbf{V}_M(t) &:= \{v \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \, : \, [\![v]\!] = 0 \;\; \mathcal{H}^{d-1}\text{-a.e. on } M(t) \subset \Gamma_{\mathrm{C}}\} \\ &= H^1_{\mathrm{D}}\big((\Omega\backslash\Gamma_{\mathrm{C}}) \cup M(t);\mathbb{R}^d\big)\,. \end{aligned} \tag{2.28}$$

Then, $\mathbf{V}_M(t)$ is a closed subspace of $\mathbf{V}$, and thus it is a reflexive and separable Banach space, and so is its dual $\mathbf{V}_M(t)^*$, which is isometrically isomorphic to the quotient space

$$\mathbf{V}/\mathbf{V}_M(t)^\perp \quad \text{with } \mathbf{V}_M(t)^\perp : = \{\tilde{\xi} \in \mathbf{V}^* \, : \, \langle \tilde{\xi}, v\rangle_{\mathbf{V}} = 0 \text{ for every } v \in M(t)\}$$

the annihilator of $\mathbf{V}_M(t)$. Furthermore, $\mathbf{V}_M(t)$ is dense in $\mathbf{W}$.

Also the space

$$L^2(0,T;\mathbf{V}_M) := \big\{v \in L^2(0,T;\mathbf{V}) : \, v(t) \in \mathbf{V}_M(t) \text{ for a.a. } t \in (0,T)\big\} \tag{2.29}$$

with $\|\cdot\|_{L^2(0,T;\mathbf{V})}$ is reflexive and separable.

Finally, suppose that the sets $(M(t))_{t\in[0,T]}$ are monotonically decreasing, i.e.

$$M(t_2) \subset M(t_1) \qquad \text{for all } 0 \le t_1 \le t_2 \le T. \tag{2.30}$$

Then, $L^2(0,T;\mathbf{V}_M)^*$ endowed with the norm $\|\cdot\|_{L^2(0,T;\mathbf{V}_M)^*} := \sup_{v\in L^2(0,T;\mathbf{V}_M)} |\langle \cdot, v\rangle_{L^2(0,T;\mathbf{V})}|$, is isometrically isomorphic to

$$\begin{aligned} L^2(0,T;\mathbf{V}_M^*) : = \{\xi \in L^2(0,T;\mathbf{V}_M(0)^*) :& \\ &\xi(t) \in \mathbf{V}_M(t)^* \text{ for a.a. } t \in (0,T) \text{ and} \\ &t \mapsto \|\xi(t)\|_{\mathbf{V}_M(t)^*} \in L^2(0,T)\}, \end{aligned} \tag{2.31}$$

with the norm $\|f\|_{L^2(0,T;\mathbf{V}_M^*)} := \sup_{v\in L^2(0,T;\mathbf{V}_M)} \left| \int_0^T \langle f(t), v(t)\rangle_{\mathbf{V}_M(t)} \,\mathrm{d}t \right|$.

Dissipation potentials and energy functionals for the brittle case, $k = \infty$: The brittle system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_\infty, \mathcal{E}_\infty)$ is governed by the kinetic energy $\mathcal{K}$ as in (2.9), the (quadratic) viscous dissipation $\mathcal{V}$ as in (2.10), and by the following rate-independent dissipation and energy functional:

$$
\begin{aligned}
&\mathcal{R}_\infty : \mathbf{Z} \to [0,\infty]\,,\ \mathcal{R}_\infty(\dot z) := \int_{\Gamma_{\mathrm{C}}} R_\infty(\dot z)\,\mathrm{d}\mathcal{H}^{d-1}\,,\\
&\qquad\qquad R_\infty(\dot z) := \begin{cases} a^1_\infty|\dot z| & \text{if } \dot z \le 0,\\ \infty & \text{otherwise,}\end{cases}
\end{aligned}
\tag{2.32}
$$

$\mathcal{E}_\infty : [0,T]\times\mathbf{V}\times\mathbf{Z} \to \mathbb{R}\cup\{\infty\}$ defined for all $t\in[0,T]$:

$$
\mathcal{E}_\infty(t,u,z) := \begin{cases} \widetilde{\mathcal{E}}_\infty(t,u,z) + \mathcal{J}_\infty(u,z) & \text{if } (u,z)\in\mathbf{V}\times\mathbf{X}_\infty,\\ \infty & \text{otherwise,}\end{cases} \qquad \text{where} \tag{2.33}
$$

$$
\begin{aligned}
&\mathcal{J}_\infty(u,z) := \int_{\Gamma_{\mathrm{C}}} J_\infty(\llbracket u\rrbracket, z)\,\mathrm{d}\mathcal{H}^{d-1} \ \text{ with}\\
&J_\infty\left(\llbracket u\rrbracket, z\right) = \begin{cases} 0 & \text{if } \llbracket u\rrbracket = 0 \text{ a.e. on } \operatorname{supp} z,\\ +\infty & \text{otherwise,}\end{cases}
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
\widetilde{\mathcal{E}}_\infty(t,u,z) := &\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac12 \mathbb{C}e(u):e(u) - \langle \mathbf{f}(t), u\rangle_{\mathbf{V}}\\
&+ \int_{\Gamma_{\mathrm{C}}} \left(I_{[0,1]}(z) - a^0_\infty z\right)\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}_\infty P(Z,\Gamma_{\mathrm{C}}).
\end{aligned}
$$

Here, $\mathbb{C}$ and $\mathbf{f}$ are as in (2.7a) and (2.7b), the support of z is defined in a measure-theoretic sense by

$$
\operatorname{supp} z := \bigcap\{A\subset\mathbb{R}^{d-1};\ A \text{ closed },\ \mathcal{H}^{d-1}(Z\backslash A) = 0\}, \tag{2.35}
$$

based on the identification of z with the set Z such that $z = \chi_Z$, and, in correspondence with (2.15), the coefficients and the space $\mathbf{X}_\infty$ satisfy the following

$$
\begin{aligned}
&\mathbf{X}_\infty = \mathbf{X} = \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}),\\
&a^0_\infty = a^0 = \text{const.}\,,\ a^1_\infty = a^1 = \text{const.}\,,\ \mathrm{b}_\infty = \mathrm{b} = \text{const.}\,,\quad \text{or}
\end{aligned}
\tag{2.36a}
$$

$$
\mathbf{X}_\infty = L^\infty(\Gamma_{\mathrm{C}}), \qquad a^0_\infty = a^1_\infty = \mathrm{b}_\infty = 0\,. \tag{2.36b}
$$

Observe that the functional $\mathcal{E}_\infty(t,\cdot,z) : \mathbf{V}\to\mathbb{R}\cup\{\infty\}$ is convex and that its proper domain is $\mathbf{V}_z$. Its subdifferential with respect to u, which appears in the subdifferential inclusion (2.3), takes the form $\partial_u\mathcal{E}_\infty = \partial_u\widetilde{\mathcal{E}}_\infty + \partial_u\mathcal{J}_\infty$ by the sum rule. Now, $\partial_u\widetilde{\mathcal{E}}_\infty$ is the singleton given by the Gâteaux-differential of $u\mapsto\widetilde{\mathcal{E}}_\infty(t,u,z)$, while $\partial_u\mathcal{J}_\infty : \mathbf{V} \rightrightarrows \mathbf{V}^*$ is a multi-valued operator, but we check that for a.a. $t\in(0,T)$:

$$
\forall\,\zeta\in\partial_u\mathcal{J}_\infty(u(t),z(t)),\ \forall\,v\in\mathbf{V}_z(t):\quad \langle\zeta, v\rangle_{\mathbf{V}} = 0\,. \tag{2.37}
$$

This relation can be verified from the definition of $\partial_u\mathcal{J}_\infty(u(t),z(t))$, which reads $\langle\zeta(t), v-u(t)\rangle_{\mathbf{V}} \le 0$ for any $v\in\mathbf{V}_z(t)$. Using the test functions $v = 2u(t)\in\mathbf{V}_z(t)$ and $v = 0\in\mathbf{V}_z(t)$ we first deduce that $\langle\zeta(t), u(t)\rangle_{\mathbf{V}} = 0$. Thus, by testing with v and $-v\in\mathbf{V}_z(t)$ we also find $\langle\zeta(t), v\rangle_{\mathbf{V}} = 0$ for any $v\in\mathbf{V}_z(t)$.

In view of observation (2.37), the notion of semistable energetic solution for the brittle system is not stated with the general subdifferential inclusion (2.3) in $\mathbf{V}^*$, but with its restriction to the domain $\mathbf{V}_z(t) \subset \mathbf{V}$, which in fact increases with $t \in [0,T]$ since z monotonically decreases in time. This restriction results in the momentum balance (2.40) below.

Definition 2.2 (Semistable energetic solution for the brittle system). Let $\varrho \geq 0$. Given $(u_0, u_1, z_0) \in \mathbf{V} \times \mathbf{W} \times \mathbf{Z}$, we call a pair $(u,z) : [0,T] \to \mathbf{V} \times \mathbf{Z}$ a *semistable energetic solution* of the evolutionary brittle system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_\infty, \mathcal{E}_\infty)$ if

$$u \in H^1(0,T;\mathbf{V}), \quad \dot{u} \in L^\infty(0,T;\mathbf{W}), \quad u \in H^2_\#(0,T;\mathbf{V}_z^*), \tag{2.38}$$

z fulfills (2.2b), and the pair (u,z) fulfill the Cauchy conditions

$$u(0) = u_0,\ \dot{u}(0) = u_1,\ z(0) = z_0, \tag{2.39}$$

and the

- weak formulation of the mometum balance in the brittle case

$$u(t) \in \mathbf{V}_z(t) \qquad \text{for every } t \in [0,T], \tag{2.40a}$$

$$\int_\Omega \varrho \ddot{u}(t) v \,\mathrm{d}x + \int_{\Omega\setminus\Gamma_\mathrm{C}} (\mathbb{D}e(\dot{u}(t)) \colon e(v) + \mathbb{C}e(u(t)) : e(v)) \,\mathrm{d}x = \langle \mathbf{f}(t), v\rangle_\mathbf{V} \quad \text{for every } v \in \mathbf{V}_z(t) \quad \text{for a.a. } t \in (0,T); \tag{2.40b}$$

- the semistability condition (2.4) with $\mathcal{R}_\infty$ from (2.32) and $\mathcal{E}_\infty$ from (2.33);
- the energy-dissipation inequality (2.5) with $\mathcal{V}$ from (2.10), $\mathcal{R}_\infty$ (2.32), and $\mathcal{E}_\infty$ (2.33).

2.3. *Main result: Passage from adhesive to brittle*

Following [41], we refer to the adhesive-to-brittle convergence stated in Thm. 2.2 below as an *evolutionary* Γ*-convergence result*. With this limit passage we obtain a semistable energetic solution of the brittle system, with the enhanced regularity property that $\ddot{u} \in H^2(0,T;\mathbf{V}(0)^*)$. Furthermore, the second of (2.43) below will allow us to test the momentum balance for the brittle system (2.40b) by $\dot{u}$, which is the key to the energy-dissipation *identity* (2.46). Like for the adhesive contact systems, we also obtain a uniqueness result for the displacements of the brittle system corresponding to a given semistable z. For this, a pivotal role is played by two separate energy balances for the displacement and for the internal variable i.e. (2.47) & (2.48) below.

Theorem 2.2 (Evolutionary Γ-convergence of the adhesive systems to the brittle limit). *Assume* (2.6)–(2.7). *For each* $k \in \mathbb{N}$ *let* (u_k, z_k) *be a semistable energetic solution of system* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)$. *Assume that* $(u_k(0), \dot{u}_k(0), z_k(0)) = (u_0^k, u_1^k, z_0^k)$, *and that*

$$(u_0^k, u_1^k, z_0^k) \to (u_0, u_1, z_0) \text{ in } \mathbf{V}\times\mathbf{W}\times\mathbf{Z} \text{ and } \mathcal{E}_k(0, u_0^k, z_0^k) \to \mathcal{E}_\infty(0, u_0, z_0), \tag{2.41}$$

such that the pair (u_0, z_0) *satisfies the stability condition* (2.4) *at* $t = 0$*, with* $\mathcal{E}_\infty$ *and* $\mathcal{R}_\infty$ *given by* (2.32)–(2.36)*. Then there exists a (not relabeled) subsequence* $(u_k, z_k)_k$ *and a pair* (u, z) *with the properties:*

(1) ***immediate convergences:*** *the following convergences hold true*

$$\begin{aligned} & u_k \rightharpoonup u \text{ in } H^1(0,T;\mathbf{V}), \quad \text{and} \\ & u_k(t) \rightharpoonup u(t) \text{ in } \mathbf{V} \text{ for all } t \in [0,T]\,, \end{aligned} \tag{2.42a}$$

$$\begin{aligned} & z_k(t) \to z(t) \text{ in } \mathbf{Z}, \\ & z_k(t) \to z(t) \text{ in } L^q(\Gamma_{\mathrm{C}}),\ q \in [1,\infty), \text{ for all } t \in [0,T]\,, \end{aligned} \tag{2.42b}$$

$$\begin{aligned} & z_k \overset{*}{\rightharpoonup} z \text{ in } L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})), \\ & z_k(t) \overset{*}{\rightharpoonup} z(t) \text{ in } \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}) \cap L^\infty(\Gamma_{\mathrm{C}}) \text{ for all } t \in [0,T]\,, \end{aligned} \tag{2.42c}$$

$$\exists\, \lambda \in L^2(0,T;\mathbf{V}^*), \quad \varrho\ddot{u}_k + \partial_u \mathcal{J}_k \rightharpoonup \lambda \text{ in } L^2(0,T;\mathbf{V}^*)\,, \tag{2.42d}$$

(2) ***semistable energetic solution & brittle constraint:*** *the limit pair* (u, z) *is a semistable energetic solution of the brittle system* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_\infty, \mathcal{E}_\infty)$ *in the sense of Def. 2.1. It satisfies the initial condition* (2.39) *and the brittle constraint*

$$\begin{aligned} & [\![u(t)]\!]|_{\mathrm{supp}\, z(t)} = 0\ \mathcal{H}^{d-1}\text{-a.e. in } \Gamma_{\mathrm{C}} \text{ for every } t \in [0,T], \quad \text{and} \\ & [\![\dot{u}(t)]\!]|_{\mathrm{supp}\, z(t)} = 0\ \mathcal{H}^{d-1}\text{-a.e. in } \Gamma_{\mathrm{C}} \text{ for almost all } t \in (0,T)\,. \end{aligned} \tag{2.43}$$

Moreover, for $\lambda \in L^2(0,T;\mathbf{V}^*)$ *obtained in* (2.42d) *we have*

$$\text{for a.a. } t \in (0,T): \quad \lambda(t) = \ddot{u}(t) \text{ in } \mathbf{V}_z(t)^*\,, \tag{2.44}$$

(3) ***regularity of*** u***:*** *in addition to* (2.38)*, the limit* u *fulfills*

$$u \in H^2(0,T;\mathbf{V}_z(0)^*)\,, \tag{2.45}$$

(4) ***weak temporal continuity:*** $\dot{u} \in \mathrm{C}^0_{\mathrm{weak}}([0,T];\mathbf{W})$,

(5) ***energy-dissipation balance:*** *the pair* (u, z) *satisfies the energy-dissipation inequality on* $(0,t)$ *and on* (s,t)*, for every* $t \in (0,T]$ *and almost every* $s \in (0,t)$*, and as an* identity *along the interval* $(s,t) \subset [0,T]$ *for almost all* $s, t \in (0,T)$ *and for* $s = 0$*,*

$$\begin{aligned} & \tfrac{1}{2}\|\dot{u}(t)\|^2_{\mathbf{W}} + \int_s^t 2\mathcal{V}(\dot{u}(\tau))\,\mathrm{d}\tau + \mathrm{Var}_{\mathcal{R}_\infty}(z,[s,t]) + \mathcal{E}_\infty(t,u(t),z(t)) \\ & \quad = \tfrac{1}{2}\|\dot{u}(s)\|^2_{\mathbf{W}} + \mathcal{E}_\infty(s,u(s),z(s)) + \int_s^t \partial_t \mathcal{E}_\infty(\tau,u(\tau),z(\tau))\,\mathrm{d}\tau\,. \end{aligned} \tag{2.46}$$

In addition, u*, resp.* z*, fulfills the following separate balance of the bulk, resp. surface, energy terms along the interval* $(s,t) \subset (0,T)$ *for a.a.* $s < t \in (0,T)$

and for $s = 0$:

$$\begin{aligned}
&\tfrac{1}{2}\|\dot u(t)\|_{\mathbf{W}}^2 + \int_s^t 2\mathcal{V}(\dot u(\tau))\,\mathrm{d}\tau + \int_{\Omega\setminus\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}e(u(t)) : e(u(t))\,\mathrm{d}x \\
&\quad - \langle \mathbf{f}(t), u(t)\rangle_{\mathbf{V}} \\
&= \tfrac{1}{2}\|\dot u(s)\|_{\mathbf{W}}^2 + \int_{\Omega\setminus\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}e(u(s)) : e(u(s))\,\mathrm{d}x - \langle \mathbf{f}(s), u(s)\rangle_{\mathbf{V}} \\
&\quad - \int_s^t \langle \dot{\mathbf{f}}(\tau), u(\tau)\rangle_{\mathbf{V}}\,\mathrm{d}\tau\,,
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
&\mathrm{b}_\infty P(Z(t),\Gamma_{\mathrm{C}}) + \int_{\Gamma_{\mathrm{C}}} a_\infty^0 z(t)\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{Var}_{\mathcal{R}_\infty}(z,[s,t]) \\
&= \mathrm{b}_\infty P(Z(s),\Gamma_{\mathrm{C}}) + \int_{\Gamma_{\mathrm{C}}} a_\infty^0 z(s)\,\mathrm{d}\mathcal{H}^{d-1}\,,
\end{aligned} \tag{2.48}$$

(6) ***enhanced convergences:*** *there hold the enhanced convergences for almost all $t \in (0,T)$:*

$$\dot u_k(t) \to \dot u(t) \ \text{in } \mathbf{W}, \tag{2.49a}$$

$$\int_0^t \mathcal{V}(\dot u_k(s))\,\mathrm{d}s \to \int_0^t \mathcal{V}(\dot u(s))\,\mathrm{d}s, \tag{2.49b}$$

$$\mathrm{Var}_{\mathcal{R}_k}(z_k,[0,t]) \to \mathrm{Var}_{\mathcal{R}_\infty}(z,[0,t]), \tag{2.49c}$$

$$\mathcal{E}_k(t,u_k(t),z_k(t)) \to \mathcal{E}_\infty(t,u(t),z(t)), \tag{2.49d}$$

(7) ***enhanced initial condition:*** *the Cauchy datum u_1 is even attained in the sense of difference quotients*

$$\lim_{h\downarrow 0} \frac{1}{h}\int_0^h \|\dot u(t) - u_1\|_{\mathbf{W}}^2\,\mathrm{d}t = 0\,, \tag{2.50}$$

(8) ***uniqueness of the displacements for given semistable $z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))$:*** *let (u,z) and $(\tilde u,z)$ be semistable energetic solutions of the brittle system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_\infty,\mathcal{E}_\infty)$, satisfying the brittle momentum balance (2.40) with the same initial data u_0 and u_1. Then, $\tilde u = u$.*

The proof of Theorem 2.2 will be carried out in detail in Section 4.

2.4. *Discussion of our result*

We now discuss our result and compare it with other existing results, in particular focusing on [14,16,17].

Momentum balance: Integrating the k-momentum balance (1.5) over $(0,T)$, using test function $\eta v \in L^2(0,T;\mathbf{V})$ with $\eta \in C_0^\infty(0,T)$ and $v \in \mathbf{V}$, convergences (2.42) allow us to pass $k \to \infty$ in (1.5) using weak-strong convergence arguments.

By localization via the fundamental lemma of the Calculus of Variations, we obtain the limit equation

$$\langle\lambda(t),v\rangle_{\mathbf{V}}+\int_{\Omega\setminus\Gamma_{\mathrm{C}}}\big(\mathbb{C}e(u(t))+\mathbb{D}e(\dot u(t))\big):e(v)\,\mathrm{d}x=\langle\mathbf{f}(t),u(t)\rangle_{\mathbf{V}}\text{ for all }v\in\mathbf{V} \quad (2.51)$$

for a.a. $t\in(0,T)$. In general we cannot identify $\lambda(t)=\varrho\ddot u(t)+\zeta(t)$ with $\zeta(t)\in\partial_u\mathcal{J}_\infty(u(t),z(t))$, cf. (2.33). This is mainly due to the fact that $\ddot u_k$ and $\partial_u\mathcal{J}_k(u_k,z_k)$ only as a sum are uniformly bounded by a constant independent of k, cf. (2.17f), whereas their separate bounds blow up with $k\to\infty$, cf. (2.18). Therefore, relation (2.44) establishes a link between the balance (2.51) in $\mathbf{V}^*$ and the brittle momentum balance (2.40) which is restricted to the domain $\mathbf{V}_z(t)$ of $\mathcal{E}_\infty(t,\cdot,z(t))$ that increases with time. Here, in these closed subspaces $\mathbf{V}_z(t)$ of $\mathbf{V}$, we indeed have $\lambda=\varrho\ddot u(t)+\zeta(t)$ in $\mathbf{V}_z(t)^*$ with $\zeta(t)\in\partial_u\mathcal{J}_\infty(u(t),z(t))$, since $\partial_u\mathcal{J}_\infty(u(t),z(t))\subset\mathbf{V}_z(t)^\perp$ by (2.37). Spaces akin to $\mathbf{V}_z(t)$ are also used in [14,16] to formulate the momentum balance.

Energy balance: In the brittle case, we obtain the energy-dissipation balance (2.46) along intervals $(s,t)\subset(0,T)$ with s and t Lebesgue points for $\dot u$, cf. the forthcoming Lemmata 4.11 and 4.12. In particular, this balance splits into the bulk balance (2.47) for the displacements and the surface balance (2.48) for the delamination variable. Such a pure bulk balance, where terms related to crack growth do not show up, is also obtained in [14], and therefore (2.47) is in accord with the results therein. However, in contrast to (2.46), [14,16] observe their energy-dissipation balance to hold along *all* subintervals $[s,t]\subset[0,T]$. This is strongly related to (indeed, implies, cf. the proof of [14, Lemma 3.10]) the fact that [14,16] find solutions such that the map $\dot u:t\mapsto\dot u(t)$ is continuous from time into the space $\mathbf{W}$ (in the dynamic, damped case of [14]), $\mathbf{V}_z(t)^*$ (in the dynamic, undamped case of [16]). In our setting we only manage to prove that $\dot u\in\mathrm{C}^0_{\mathrm{weak}}([0,T];\mathbf{W})$.

Observe that the enhanced energy-dissipation balance and regularity properties proved in [14] do not stem from assuming suitable temporal regularity for the *prescribed* crack evolution. Given a (unique) solution u of the momentum balance in $[0,T]$, their argument to obtain the enhanced energy-dissipation balance is based on choosing an arbitrary time $t_0\in(0,T)$, which is not a Lebesgue point, and to solve the momentum balance on (t_0,T) with the new initial data $u_0=u(t_0)$, $u_1=\dot u(t_0)$. Glueing the solution in $(0,t_0)$ with the new solution on (t_0,T) and exploiting the previously proved uniqueness of the displacement leads to the enhancements. Since our existence result for the brittle system arises from an adhesive contact approximation relying on the *well-preparedness of the initial data* (2.41), we cannot choose an arbitrary time t_0 as a new initial time to solve momentum balance of the brittle system, without implicitly requiring the enhanced convergences (2.49) to hold at the arbitrary time $t_0\in(0,T)$. A further reason why we are not able to reproduce the arguments from [14] leading to the energy-dissipation balance at all times is the presence in our own balance of the surface energy terms due to delamination, which may jump at countably many times. Akin to our (2.46) the energy balance obtained in [16] along all subintervals of $[0,T]$ also features the dissipation due to crack growth. However, it is obtained in a 2D-setting with connected cracks that evolve continuously in time and piecewise even more smoothly.

Crack propagation criterion: Both in the adhesive and in the brittle models, crack propagation is governed by the semistability inequality (2.4). It expresses that the semistable delamination variable $z(t)$, among all possible competitors $\tilde{z} \in \mathbf{Z}$, is a minimizer of the functional $\mathcal{E}(t,u(t),\cdot) + \mathcal{R}(\cdot - z(t))$, with $u(t)$ kept fixed as the solution of the momentum balance (with $z(t)$) in $\mathbf{V}_z(t)^*$ at time t.

In the fully rate-independent setting (i.e., no viscosity and inertia for the displacements), solutions of rate-independent systems for damage and delamination fulfilling the semistability, in place of the *global stability* condition in the standard concept of semistable energetic solutions [46,39], have been termed *local solutions* in, e.g., [54,56]. This higlights that the semistability, as a minimality property, is local in the sense that the displacemements are not modified (but observe that the concept from [54,56] differs from the weaker notion of local solution defined in [40, Def. 4.5, 42, Sec. 1.8]). Let us also mention that, when taking the vanishing-viscosity & inertia limit in the momentum balance, semistable energetic solutions in the sense of Def. 2.1 converge to local solutions (in the sense of [54,56]) of the quasistatic limit system, as shown in the case of damage in [36].

In contrast to the above described approach, models that govern crack propagation by Griffith's fracture criterion, cf. e.g., [47,32,31] in the dynamic setting, are rather based on *global* minimality, and therefore correspond to *(global) energetic* solutions in the fully rate-independent context. The dynamic evolution criterion, rephrased in our notation for easier comparison, is in fact a constrained *global* minimization problem, namely

$$\min_{\tilde{z}(t)\in\mathbf{Z}} \big(\mathcal{E}(t, v_{\tilde{z}}(t), \tilde{z}(t)) + \mathcal{K}(\dot{v}_{\tilde{z}}(t)) + \mathcal{R}(\tilde{z}(t))\big) \tag{2.52}$$

where $v_{\tilde{z}}$ is the *unique* solution of the momentum balance on $[0,t]$, corresponding to the (given) delamination variable $\tilde{z}$; observe that, both in the adhesive and in the brittle cases, we also have uniqueness of the displacements for given $z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))$.

[17] advance a yet different formulation of dynamic fracture for their model of peeling test of a thin film. There, the debonded part of the film is subject to a wave equation. The evolution of the debonded front is governed by a Griffith criterion expressed in terms of the dynamic energy release rate, which includes a kinetic term. Hence, the propagation criterion is rate-dependent. The existence and uniqueness of a solution is proved relying on the 1D-geometry of the model.

Finally, semistable energetic solutions in the sense of Def. 2.1, with semistability as the evolution criterion for the internal variable, are obtained from alternating (decoupled) minimization on the time-discrete level, cf. [50], whereas solutions satisfying the Griffith criterion (2.52), are rather based on simultaneous minimization in the two variables. However, it has been observed that energetic solutions of rate-independent systems, based on global, simultaneous minimization, tend to jump rather early in time. This has motivated the design of alternative solution concepts, resp. crack propagation criteria, cf. e.g., [30,34,56] within crack propagation and delamination. In this spirit also the concept discussed in [16] has to be understood: Therein, the crack propagation criterion is a local-in-time reformulation of Griffith's fracture criterion, inspired by so-called ϵ-stable solutions from [34]. It would be interesting to advance the study of *alternative* solution notions to the brittle system, by combining the adhesive-to-brittle limit with a vanishing-viscosity approximation, cf. [44].

The different scalings (2.15) **and the resulting limit models** (2.32)–(2.36)**:** In the brittle setting it has been noted (cf. [56]) that the semistability condition governing crack propagation, reducing to

$$\begin{aligned}&\int_{\Gamma_{\mathrm{C}}} J_\infty\left(\llbracket u(t)\rrbracket, z(t)\right)\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}_\infty P(Z(t),\Gamma_{\mathrm{C}})\\ &\leq \int_{\Gamma_{\mathrm{C}}} J_\infty\left(\llbracket u\rrbracket, \tilde z\right)\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}_\infty P(\tilde Z,\Gamma_{\mathrm{C}}) + (a_\infty^0 + a_\infty^1)\int_{\Gamma_{\mathrm{C}}}(z(t)-\tilde z)\,\mathrm{d}\mathcal{H}^{d-1}\end{aligned} \tag{2.53}$$

for all $\tilde z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ and for all $t\in[0,T]$, does not feature any term of positive, finite value that depends on the displacements and thus forces z to decrease, as a function of time. In other words, crack growth seems to be rather induced by the perimeter regularization, than by the attempt to reduce the mechanical stresses. However, the solutions of the brittle systems obtained in Theorem 2.2 are selected by approximation with solutions of the adhesive contact models $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_k,\mathcal{E}_k)_k$. Since the semistability condition for finite $k\in\mathbb{N}$, i.e.

$$\begin{aligned}&\int_{\Gamma_{\mathrm{C}}} \tfrac{k}{2} z_k(t)\left|\llbracket u_k(t)\rrbracket\right|^2\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}_k P(Z_k(t),\Gamma_{\mathrm{C}})\\ &\leq \int_{\Gamma_{\mathrm{C}}} \tfrac{k}{2}\tilde z\left|\llbracket u_k(t)\rrbracket\right|^2\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}_k P(\tilde Z,\Gamma_{\mathrm{C}}) + (a_k^0 + a_k^1)\int_{\Gamma_{\mathrm{C}}}(z_k(t)-\tilde z)\,\mathrm{d}\mathcal{H}^{d-1}\end{aligned} \tag{2.54}$$

for all $\tilde z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ and for all $t\in[0,T]$, features the displacement-dependent adhesive contact term, which can drive crack propagation, the solutions of the brittle system obtained by Theorem 2.2 should inherit this information.

On these grounds, in the work[56] (see also [49, Sec. 7]) the alternative scaling from (2.15b) has been proposed. Note, when the coefficients a_k^0, a_k^1, b_k are scaled as in (2.15b), multiplying the semistability inequality (2.54) by k leads to

$$\begin{aligned}&\int_{\Gamma_{\mathrm{C}}} \tfrac{k^2}{2} z_k(t)\left|\llbracket u_k(t)\rrbracket\right|^2\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}P(Z_k(t),\Gamma_{\mathrm{C}})\\ &\leq \int_{\Gamma_{\mathrm{C}}} \tfrac{k^2}{2}\tilde z\left|\llbracket u_k(t)\rrbracket\right|^2\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{b}P(\tilde Z,\Gamma_{\mathrm{C}}) + (a^0 + a^1)\int_{\Gamma_{\mathrm{C}}}(z_k(t)-\tilde z)\,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

for all $\tilde z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ and for all $t\in[0,T]$, which accounts, at least formally, for the magnitude of the stresses. Indeed, from the contact surface boundary condition

(1.3c) we read, taking into account that $z_k(t) \in \{0,1\}$, that

$$\int_{\Gamma_C} \tfrac{k^2}{2} z_k(t) \left| [\![u_k(t)]\!] \right|^2 \, \mathrm{d}\mathcal{H}^{d-1} = \int_{Z_k(t)\cap\{|[\![u_k(t)]\!]|>0\}} |(\mathbb{D}\dot{e}_k(t) + \mathbb{C}e_k(t))|^2 \, \mathrm{d}\mathcal{H}^{d-1},$$

provided that the solutions are sufficiently smooth as to ensure that the stress term on the r.h.s. makes sense. Therefore, under the assumption of convergence and sufficient regularity of the solutions, and taking into account that $\mathcal{H}^{d-1}(\{|z_k(t)[\![u_k(t)]\!]| > 0\}) \to 0$ as $k \to \infty$, one expects the *rescaled* brittle model obtained from (2.15b) to contain a term of the form $\int_{Z(t)\cap\partial\{|[\![u(t)]\!]|>0\}} |(\mathbb{D}\dot{e}(t) + \mathbb{C}e(t))|^2 \, \mathrm{d}\mathcal{H}^{d-1}$. This conveys the information that, also in the brittle limit a decrease of the semistable function z is not only triggered by the perimeter regularization, but by the mechanical stresses as well.

The alternative scaling in (2.15b) also relates our brittle model to the one studied in [14]. However, in contrast to [14], solutions of our limit model carry the additional information that crack growth is rate-independent and that the crack set has finite perimeter, which they inherit from the approximating adhesive contact models, thanks to the bounds (2.17d) & (2.17e).

3. Semistable energetic solutions for adhesive contact (k fixed)

In [49], the existence of semistable energetic solutions of an adhesive contact system with perimeter regularization was proved in the case of a *quasistatic* momentum balance. In [RR11] the fully dynamic case was tackled, but the flow rule for the delamination parameter did not feature the perimeter regularization term we consider here. Hence, in this section we will briefly address the existence of semistable energetic solutions for the adhesive contact systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)$, with $k \in \mathbb{N}$ fixed, by resorting to the abstract existence result for damped inertial systems proved in [50, Thm. 4]. In what follows, for the reader's convenience we shall first revisit the conditions of [50, Thm. 4] on an abstract damped inertial system, and then verify that systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)$ do fulfill them.

Let $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ be a damped inertial system satisfying the basic conditions (2.1): [50, Thm. 4] puts the following additional requirements on the functionals $\mathcal{V} : \mathbf{V} \to [0,\infty)$, $\mathcal{R} : \mathbf{Z} \to [0,\infty]$, and $\mathcal{E} : [0,T] \times \mathbf{V} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$:

Boundedness from below & Weak lower semicontinuity:

$$\exists\, C_0 > 0,\ \forall\, (t,u,z) \subset \mathrm{dom}(\mathcal{E}) :\ \mathcal{E}(t,u,z) \geq C_0\,; \tag{3.1a}$$

$$\text{for all } t \in [0,T] \quad \mathcal{E}(t,\cdot,\cdot) \text{ is weakly sequentially lower semicontinuous on } \mathbf{V} \times \mathbf{Z}\,. \tag{3.1b}$$

Temporal regularity and power control:

$$\left.\begin{array}{l}\forall\,(u,z)\in\mathbf{D}_u\times\mathbf{D}_z,\ \text{the map } t\mapsto\mathcal{E}(t,u,z) \text{ is differentiable}\\ \text{with derivative } \partial_t\mathcal{E}(t,u,z) \text{ s.t.}\\ \exists\,C_1,C_2>0,\forall\,(t,u,z)\in\mathrm{dom}(\mathcal{E}):\ |\partial_t\mathcal{E}(t,u,z)|\le C_1(\mathcal{E}(t,u,z)+C_2)\\ \text{and fulfilling}\\ \text{for all sequences } t_n\to t,\ u_n\to u \text{ in } \mathbf{V},\ z_n\to z \text{ in } \mathbf{Z}\\ \ \text{with } \sup_n\mathcal{E}(t_n,u_n,z_n)\le C\\ \text{that } \limsup_{n\to\infty}\partial_t\mathcal{E}(t_n,u_n,z_n)\le\partial_t\mathcal{E}(t,u,z)\,.\end{array}\right\}\tag{3.2}$$

Coercivity:

$$\left.\begin{array}{l}\text{there exist } \tau_o>0 \text{ such that for all } (t,u_o,z_o)\in[0,T]\times\mathbf{V}\times\mathbf{Z}\\ \text{the map } (u,z)\mapsto\mathcal{E}(t,u,z)+\tau_o\mathcal{V}\big(\frac{u-u_o}{\tau_o}\big)+\mathcal{R}(z-z_o)\\ \ \text{has sublevels bounded in } \mathbf{V}\times\mathbf{X}\,.\end{array}\right\}\tag{3.3}$$

Mutual recovery sequence condition ensuring the closedness of stable sets:

$$\left.\begin{array}{l}\text{Let } (t_n,u_n,z_n)_n\subset\mathrm{dom}(\mathcal{E}) \text{ for all } n\in\mathbb{N} \text{ satisfy semistability (2.4),}\\ \text{let } t_n\to t,(u_n,z_n)\rightharpoonup(u,z) \text{ in } \mathbf{V}\times\mathbf{Z} \text{ with}\\ \sup_n\mathcal{E}(t,u_n,z_n)\le C \text{ for all } t\in[0,T].\\ \text{Then, for every } \tilde z\in\mathbf{Z} \text{ there exists } \tilde z_n\rightharpoonup\tilde z \text{ in } \mathbf{Z} \text{ such that}\\ \limsup_{n\to\infty}\big(\mathcal{E}(t_n,u_n,\tilde z_n)+\mathcal{R}(\tilde z_n-z_n)-\mathcal{E}(t_n,u_n,z_n)\big)\\ \quad\le\mathcal{E}(t,u,\tilde z)+\mathcal{R}(\tilde z-z)-\mathcal{E}(t,u,z)\,.\end{array}\right\}\tag{3.4}$$

[50, Thm. 4] also imposes a condition on the differentials $\partial_u\mathcal{E}$ in the spirit of Minty's trick:

Continuity:

For all sequences $(\mathrm{t}_n)_n,\ \mathrm{t}_n:[0,T]\to[0,T],\ (u_n)_n\subset L^\infty(0,T;\mathbf{V})\cap H^1(0,T;\mathbf{V})$, $(z_n)_n\subset L^\infty(0,T;\mathbf{X})\cap\mathrm{BV}([0,T];\mathbf{Z}),(\partial_u\mathcal{E}(\mathrm{t}_n,u_n,z_n))_n\subset L^2(0,T;\mathbf{V}^*)$ s.t. $\exists\,C>0,\ \forall\,n\in\mathbb{N},\ \forall\,t\in[0,T]:\ \mathcal{E}(t,u_n(t),z_n(t))\le C$ and

$$\left\{\begin{array}{l}\mathrm{t}_n\to\mathrm{t} \text{ pointwise a.e. in } (0,T)\,,\\ u_n\overset{*}{\rightharpoonup}u \text{ in } L^\infty(0,T;\mathbf{V})\cap H^1(0,T;\mathbf{V})\,,\\ z_n\overset{*}{\rightharpoonup}z \text{ in } L^\infty(0,T;\mathbf{X})\,,\ z_n(t)\overset{*}{\rightharpoonup}z(t) \text{ in } \mathbf{X} \text{ for all } t\in[0,T]\,,\\ \partial_u\mathcal{E}(\mathrm{t}_n,u_n,z_n)\rightharpoonup\xi \text{ in } L^2(0,T;\mathbf{V}^*),\\ \limsup_{n\to\infty}\int_0^T\langle\partial_u\mathcal{E}(\mathrm{t}_n,u_n,z_n),u_n\rangle_{\mathbf{V}}\,\mathrm{d}t\le\int_0^T\langle\xi,u\rangle_{\mathbf{V}}\,\mathrm{d}t\,,\end{array}\right\}\tag{3.5}$$

then there holds $\xi(t)=\partial_u\mathcal{E}(\mathrm{t}(t),u(t),z(t))$ for a.a. $t\in(0,T)$.

Finally, to find a bound on the inertial term, a further requirement of [50, Thm. 4] is the following

Subgradient estimate:

There exist constants $C_3,C_4,C_5>0$ and $\sigma\in[1,\infty)$ such that (3.6)

$$\forall\,(t,u,z)\in\mathrm{dom}(\mathcal{E}):\quad\|\partial_u\mathcal{E}(t,u,z)\|_{\mathbf{V}^*}^\sigma\le C_3\mathcal{E}(t,u,z)+C_4\|u\|_{\mathbf{V}}+C_5\,.$$

Theorem 3.1 ([50, Thm. 4]). *Let* $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ *fulfill* (2.1) *&* (3.1)–(3.6).

Then for every $(u_0,u_1,z_0)\in\mathbf{V}\times\mathbf{W}\times\mathbf{Z}$ *fulfilling the semistability* (2.4) *at* $t=0$, *i.e.*

$$\mathcal{E}(0,u_0,z_0)\le\mathcal{E}(0,u_0,\tilde z)+\mathcal{R}(\tilde z-z_0)\qquad\text{for all } \tilde z\in\mathbf{Z} \tag{3.7}$$

there exists a semistable energetic solution (in the sense of Definition 2.1) of $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ *satisfying the Cauchy condition* $(u(0),\dot u(0),z(0))=(u_0,u_1,z_0)$.

The existence Thm. 2.1 for the adhesive contact system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_k,\mathcal{E}_k)$, deduced from Thm. 3.1, also guarantees the validity of the energy-dissipation *balance* (2.16). This would follow from [50, Prop. 3.5, Thm. 3.6], but we prefer to actually prove (2.16) for later reference in the proof of Thm. 2.2.

Proof of Theorem 2.1: We first verify the existence of semistable energetic solutions for system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_k,\mathcal{E}_k)$ by checking the assumptions of Theorem 3.1, i.e., (2.1) & (3.1)–(3.6).

On (2.1)**:** In view of (2.11), (2.10), and (2.7a) conditions (2.1b) and (2.1c) on $\mathcal{R}_k$ and $\mathcal{V}$ are verified. From the definition of $\mathcal{E}_k$ we see that the functionals have the proper domain $\mathrm{dom}(\mathcal{E}_k)=[0,T]\times\mathbf{V}\times\mathbf{X}$.

On (3.1)**:** To check (3.1a), we calculate in view of (2.12), using Korn's and Young's inequality:

$$\begin{aligned}\mathcal{E}_k(t,u,z)\ge{}&\tfrac{C^1_{\mathbb{C}}}{2}\|e(u)\|^2_{L^2}-\|\mathbf{f}(t)\|_{\mathbf{V}^*}\|u\|_{\mathbf{V}}+\int_{\Gamma_{\mathrm{C}}}\tfrac{k}{2}z\left|[\![u]\!]\right|^2\,\mathrm{d}\mathcal{H}^{d-1}\\&\qquad+\mathrm{b}_k(P(Z,\Gamma_{\mathrm{C}})+\|z\|_{L^1(\Gamma_{\mathrm{C}})})-(a^0_k+\mathrm{b}_k)\mathcal{H}^{d-1}(\Gamma_{\mathrm{C}})\\\ge{}&\tfrac{C^1_{\mathbb{C}}C^2_K}{2}\|u\|^2_{\mathbf{V}}-\tfrac{C^1_{\mathbb{C}}C^2_K}{4}\|u\|^2_{\mathbf{V}}-\tfrac{1}{C^1_{\mathbb{C}}C^2_K}\|\mathbf{f}(t)\|^2_{\mathbf{V}^*}\\&+\int_{\Gamma_{\mathrm{C}}}\tfrac{k}{2}z\left|[\![u]\!]\right|^2\,\mathrm{d}\mathcal{H}^{d-1}+\mathrm{b}_k\|z\|_{\mathrm{SBV}(\Gamma_{\mathrm{C}})}-(a^0_k+\mathrm{b}_k)\mathcal{H}^{d-1}(\Gamma_{\mathrm{C}})\\\ge{}&-c_*C^2_{\mathbf{f}}+\int_{\Gamma_{\mathrm{C}}}\tfrac{k}{2}z\left|[\![u]\!]\right|^2\,\mathrm{d}\mathcal{H}^{d-1}-(a^0_k+\mathrm{b}_k)\mathcal{H}^{d-1}(\Gamma_{\mathrm{C}})\,,\end{aligned}\tag{3.8}$$

where we used that $\|\mathbf{f}(t)\|_{\mathbf{V}^*}\le C_{\mathbf{f}}$, as well as that $\|z\|_{L^1(\Gamma_{\mathrm{C}})}\le\mathcal{H}^{d-1}(\Gamma_{\mathrm{C}})$ due to $z\in\{0,1\}$, and set $\frac{1}{C^1_{\mathbb{C}}C^2_K}=c_*$. This proves (3.1a). For $\mathcal{E}_k(t,u,z)\le E$ we then find that

$$\begin{aligned}&\|u\|^2_{\mathbf{V}}\le\tfrac{4}{C^1_{\mathbb{C}}C^2_K}(E+c_*C^2_{\mathbf{f}}+(a^0+\mathrm{b})\mathcal{H}^{d-1}(\Gamma_{\mathrm{C}}))\;\;\text{and}\\&\|z\|_{\mathrm{SBV}(\Gamma_{\mathrm{C}})}\le\mathrm{b}_k^{-1}(E+c_*C^2_{\mathbf{f}}+(a^0+\mathrm{b})\mathcal{H}^{d-1}(\Gamma_{\mathrm{C}}))\,.\end{aligned}\tag{3.9}$$

The weak lower semicontinuity property (3.1b) can be straightforwardly checked.

On (3.2)**:** Observe that $\partial_t\mathcal{E}(t,u,z)=-\langle\dot{\mathbf{f}}(t),u\rangle_{\mathbf{V}}$. In view of the regularity assumption (2.7b) we have $\dot{\mathbf{f}}(t_n)\to\dot{\mathbf{f}}(t)$ in $\mathbf{V}^*$ for $t_n\to t$ in $[0,T]$, which immediately gives the upper semicontinuity property of the powers. In view of (3.9) and Young's inequality we find the following power-control estimate:

$$\begin{aligned}|\partial_t \mathcal{E}_k(t,u,z)| &\leq C_{\mathbf{f}}\|u\|_{\mathbf{V}} \leq \tfrac{1}{2}C_{\mathbf{f}}^2 + \tfrac{1}{2}\|u\|_{\mathbf{V}}^2 \\ &\leq \tfrac{1}{2}C_{\mathbf{f}}^2 + \tfrac{2}{C_{\mathbb{C}}^1 C_K^2}\big(\mathcal{E}_k(t,u,z) + c_* C_{\mathbf{f}}^2 + (a_k^0 + \mathrm{b}_k)\mathcal{H}^{d-1}(\Gamma_{\mathrm{C}})\big)\,.\end{aligned}$$

On (3.3)**:** It directly follows from the coercivity of $\mathcal{V}$ and the coercivity estimate (3.9) for $\mathcal{E}_k$.

On (3.4)**:** We refer to [49, Sec. 5.2] for the construction of the mutual recovery sequence.

On (3.5)**:** Recall that, for all $(t,u,z) \in \mathrm{dom}(\mathcal{E}_k)$ the mapping $u \mapsto \mathcal{E}_k(t,u,z)$ is Gâteaux-differentiable, with Gâteaux-derivative given by (2.14). Due to the quadratic nature of $\mathcal{E}_k(t,\cdot,z)$, it is easy to verify the continuity condition (3.5).

On (3.6)**:** Using (2.7b) and Hölder's inequality to estimate the terms in (2.14) we thus obtain for every $(t,u,z) \in \mathrm{dom}(\mathcal{E}_k)$ with $\mathcal{E}_k(t,u,z) \leq E$ and for all $v \in \mathbf{V}$

$$\begin{aligned}&|\langle \partial_u \mathcal{E}_k(t,u,z), v\rangle_{\mathbf{V}}| \\ &\leq (C_{\mathbb{C}}^2\|e(u)\|_{L^2} + C_{\mathbf{f}})\|v\|_{\mathbf{V}} + k\Big(\int_{\Gamma_{\mathrm{C}}} z\big|[\![u]\!]\big|^2\,\mathrm{d}\mathcal{H}^{d-1}\Big)^{1/2}\Big(\int_{\Gamma_{\mathrm{C}}} \big|[\![v]\!]\big|^2\,\mathrm{d}\mathcal{H}^{d-1}\Big)^{1/2} \\ &\overset{(1)}{\leq} \|v\|_{\mathbf{V}}\Big(C_{\mathbf{f}} + 2\sqrt{k}\max\{\bar{C}, 4C_{\mathbb{C}}^2/C_{\mathbb{C}}^1\}\big(1 + \tfrac{C_{\mathbb{C}}^1}{4}\|e(u)\|_{L^2}^2 + k\int_{\Gamma_{\mathrm{C}}} z\big|[\![u]\!]\big|^2\,\mathrm{d}\mathcal{H}^{d-1}\big)^{1/2}\Big) \\ &\overset{(2)}{\leq} \|v\|_{\mathbf{V}}\Big(C_{\mathbf{f}} + 4\sqrt{k}\max\{\bar{C}, 4C_{\mathbb{C}}^2/C_{\mathbb{C}}^1\}\big(1 + \mathcal{E}_k(t,u,z)\big)\Big)\,.\end{aligned} \tag{3.10}$$

where in (1) $\bar{C}$ is the constant associated with the continuous embedding $\mathbf{V} \subset L^2(\Gamma_{\mathrm{C}};\mathbb{R}^d)$, while (2) is obtained using the the fact that $\mathcal{E}_k(t,u,z)$ bounds both $\frac{C_{\mathbb{C}}^1}{4}\|e(u)\|_{L^2}^2$ and $\int_{\Gamma_{\mathrm{C}}} \frac{k}{2} z\big|[\![u]\!]\big|^2\,\mathrm{d}\mathcal{H}^{d-1}$, cf. (3.8).

Thus, the existence of a semistable energetic solution for system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_k,\mathcal{E}_k)$ is established. We now turn to prove the remaining statements of Theorem 2.1.

Energy equality (2.16)**:** For each $k \in \mathbb{N}$, $\dot{u}_k \in L^2(0,T;\mathbf{V}) \cap H^1(0,T;\mathbf{V}^*)$ is an admissible test function in the k-momentum balance. Thus, applying this test and integrating the k-momentum balance over $[0,t]$ for any $t \in [0,T]$, yields for the term resulting from the viscous damping that

$$\int_0^t \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathbb{D}e(\dot{u}_k(s)) \colon e(\dot{u}_k(s))\,\mathrm{d}s = \int_0^t 2\mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s\,, \tag{3.11}$$

cf. (2.10). For the external loading term we find by integration by parts

$$\int_0^t \langle -\mathbf{f}(s), \dot{u}_k(s)\rangle_{\mathbf{V}}\,\mathrm{d}s = \langle -\mathbf{f}(t), u_k(t)\rangle_{\mathbf{V}} - \langle -\mathbf{f}(0), u_k(0)\rangle_{\mathbf{V}} - \int_0^t \langle -\dot{\mathbf{f}}(s), u_k(s)\rangle_{\mathbf{V}}\,\mathrm{d}s\,.$$

Moreover, since $(\mathbf{V},\mathbf{W},\mathbf{V}^*)$ is a Gelfand triple, the inertial term satisfies the following chain rule:

$$\int_0^t \langle \varrho \ddot{u}_k(s), \dot{u}_k(s)\rangle_{\mathbf{V}}\,\mathrm{d}s = \tfrac{1}{2}\|\dot{u}_k(t)\|^2_{\mathbf{W}} - \tfrac{1}{2}\|\dot{u}_k(0)\|^2_{\mathbf{W}}\,.$$

We also observe that the elastic bulk energy satisfies

$$\begin{aligned}&\int_0^t \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathbb{C}e(u_k(s)) : e(\dot{u}_k(s))\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}e(u_k(t)) : e(u_k(t))\,\mathrm{d}x - \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathbb{C}e(u_k(0)) : e(u_k(0))\,\mathrm{d}x\,.\end{aligned} \tag{3.12}$$

In order to treat the term related surface energy $\int_0^t\int_{\Gamma_{\mathrm{C}}} kz_k[\![u_k]\!][\![\dot{u}_k]\!]\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s$ we proceed as in [53, (4.69)-(4.75)], cf. also [RR11, (8.15)]: Using a well-chosen partition of the time interval $[0,t]$ a Riemann sum argument is applied to the semistability condition to deduce the chain-rule type inequality

$$\begin{aligned}&\int_0^t \int_{\Gamma_{\mathrm{C}}} kz_k(s)\,[\![u_k(s)]\!]\,[\![\dot{u}_k(s)]\!]\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s \\ &\leq \int_{\Gamma_{\mathrm{C}}} \tfrac{k}{2}z_k(t)\big|[\![u_k(t)]\!]\big|^2\,\mathrm{d}\mathcal{H}^{d-1} - \int_{\Gamma_{\mathrm{C}}} \tfrac{k}{2}z_k(0)\big|[\![u_k(0)]\!]\big|^2\,\mathrm{d}\mathcal{H}^{d-1} + \mathrm{Var}_{\mathcal{R}_k}(z_k,[0,t])\,.\end{aligned}$$

In view of the last estimate, putting all the above terms together in the k-momentum balance, results in the energy-dissipation inequality opposite to (2.5)

$$\begin{aligned}&\tfrac{1}{2}\|\dot{u}_k(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_k}(z_k,[0,t]) + \mathcal{E}_k(t,u_k(t),z_k(t)) \\ &\qquad\geq \tfrac{1}{2}\|\dot{u}_k(0)\|^2_{\mathbf{W}} + \mathcal{E}_k(0,u_k(0),z_k(0)) + \int_0^t \partial_t\mathcal{E}_k(s,u_k(s),z_k(s))\,\mathrm{d}s\,.\end{aligned}$$

Thus, in combination with (2.5), we have an equality that holds for all $t\in[0,T]$. Subtracting the energy equality given on $[0,s]$ from the one given on $[0,t]$, where $s<t$, results in (2.16).

Uniqueness of the displacements for given $z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))$: Suppose that the pairs (u,z) and $(\tilde{u},z)$ both satisfy the momentum balance (1.5) and the same Cauchy conditions. Then, $w := u-\tilde{u}$ fulfills (1.5) for $\mathbf{f}=0$, with $w(0)=\dot{w}(0)=0$. We now choose the test function $v=\dot{w}$ and, exploiting the chain rule for each of the terms in (1.5), as well as the positive definiteness (2.7a) of the viscosity and elasticity tensors $\mathbb{D}$ and $\mathbb{C}$ and Korn's inequality, we obtain for a.a. $t\in(0,T)$

$$\begin{aligned}&\tfrac{1}{2}\|\dot{w}(t)\|^2_{\mathbf{W}} + C^1_{\mathbb{D}}C^2_K\int_0^t \|\dot{w}\|^2_{\mathbf{V}}\,\mathrm{d}s + \frac{C^1_{\mathbb{C}}C^2_K}{2}\|w(t)\|^2_{\mathbf{V}} \\ &\leq k\int_0^t\int_{\Gamma_{\mathrm{C}}} z\,\big|[\![w]\!]\big|\,\big|[\![\dot{w}]\!]\big|\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s \\ &\leq \frac{C^1_{\mathbb{D}}C^2_K}{2}\int_0^t\|\dot{w}\|^2_{\mathbf{V}}\,\mathrm{d}s + C\int_0^t\|w\|^2_{\mathbf{V}}\,\mathrm{d}s\,,\end{aligned} \tag{3.13}$$

where the second estimate follows from the arguments previously used for (3.10), and Young's inequality. We now absorb the first term on the r.h.s. into the l.h.s., and use the Gronwall Lemma to deal with the second one. We thus conclude that $w \equiv 0$ on $[0,t]$ for every $t \in (0,T]$, whence the desired uniqueness.

Uniform bounds (2.17)**:** Following the arguments of [40, Prop. 6.3] using a Gronwall estimate and the boundedness of the given data, the power control condition (3.2) yields that

$$\sup_{t\in[0,T]} \left(\tfrac{1}{2}\|\dot{u}_k(t)\|^2_{\mathbf{W}} + \mathcal{E}_k(t,u_k(t),z_k(t)) + \int_0^t \mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_k}(z_k,[0,t])\right) \leq C\,.$$

Again invoking (3.2) thus gives estimates (2.17a) & (2.17b). From this we deduce (2.17c) using (3.9). For (2.17d) we argue that (2.17a) & (2.17b) imply for each $k \in \mathbb{N}$ and all $t \in [0,T]$ that $0 \leq z_k(t) \leq 1$ and that z_k is monotonically decreasing in time. Hence, for every $t \in [0,T]$

$$\|z_k\|_{\mathrm{BV}([0,t];L^1(\Gamma_{\mathrm{C}}))} = \int_0^T \int_{\Gamma_{\mathrm{C}}} (z_k(t) - z_k(0))\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \int_{\Gamma_{\mathrm{C}}} (1 - z_k(0))\,\mathrm{d}x\,\mathrm{d}t \leq C\,.$$

Since the coefficients b_k may tend to zero as $k \to \infty$ (cf. the scaling (2.15b)), we cannot deduce estimate (2.17e) for the perimeter terms $P(Z_k(t),\Gamma_{\mathrm{C}})$ from the energy estimate (2.17a) (using that $\mathcal{E}$ estimates the perimeter, cf. (3.8)). Instead, we will resort to the k-semistability inequality, implying the estimate

$$\mathrm{b}_k P(Z_k(t),\Gamma_{\mathrm{C}}) \leq \mathrm{b}_k P(\widetilde{Z}_k(t),\Gamma_{\mathrm{C}}) + \mathcal{R}_k(\tilde{z}_k - z_k) + a_k^0 \int_{\Gamma_{\mathrm{C}}} (z_k(t) - \tilde{z}_k)\,\mathrm{d}\mathcal{H}^{d-1}$$

for any finite perimeter set $\widetilde{Z}_k \subset Z_k(t)$, in the sense that its characteristic function $\tilde{z}_k$ satisfies $\tilde{z}_k \leq z_k(t)$ $\mathcal{H}^{d-1}$-a.e. in Γ_{C}. In view of the allowed scalings of the coefficients, cf. (2.15), we may therefore cancel out the k-dependence of the coefficients by multiplying by a suitable power of k and thus find

$$\mathrm{b}_k P(Z_k(t),\Gamma_{\mathrm{C}}) \leq \mathrm{b}_k P(\widetilde{Z}_k(t),\Gamma_{\mathrm{C}}) + \mathcal{R}_1(\tilde{z}_k - z_k(t)) + a_k^0 \int_{\Gamma_{\mathrm{C}}} (z_k(t) - \tilde{z}_k)\,\mathrm{d}\mathcal{H}^{d-1}\,.$$

Choosing the particular competitor $\widetilde{Z}_k = \emptyset, \tilde{z}_k = 0$ a.e. in Γ_{C} yields the uniform bound (2.17e). Since the delamination variables z_k take values in $\{0,1\}$, the second bound in (2.17e) is immediate. The uniform bound (2.17f) on $\varrho\ddot{u}_k + \partial_u\mathcal{J}_k(\cdot,u_k,z_k)$ follows by comparison in the k-momentum balance using that

$$\begin{aligned}\left|\,\langle \varrho\ddot{u}_k + \partial_u\mathcal{J}_k(\cdot,u_k,z_k), v\rangle_{\mathbf{V}}\,\right| &= \left|\int_{\Omega\setminus\Gamma_{\mathrm{C}}} (\mathbb{C}e(u_k) + \mathbb{D}e(\dot{u}_k)) : e(v)\,\mathrm{d}x - \langle \mathbf{f}, v\rangle_{\mathbf{V}}\right| \\ &\leq \|v\|_{\mathbf{V}}\left(C_{\mathbb{C}}^2\|u_k\|_{\mathbf{V}} + C_{\mathbb{D}}^2\|\dot{u}_k\|_{\mathbf{V}} + C_{\mathbf{f}}\right), \quad \text{for all } v \in \mathbf{V}\end{aligned}$$

and the right-hand side is uniformly bounded by estimates (2.17c) and (2.7).

k-dependent bounds (2.18)**:** The bound (2.18a) on $\partial_u\mathcal{J}_k(u_k,z_k)$ follows from the calculations developed in (3.10). Then, (2.18b) ensues combining (2.18a) with the previously proved (2.17f). □

4. Passage from adhesive to brittle: Proof of Main Theorem 2.2

We start by proving convergences (2.42), relying on compactness arguments based on the bounds (2.17).

Lemma 4.1 (Statement 1. of Theorem 2.2: convergences (2.42)**).** *There is a subsequence* $(u_k, z_k)_k$ *of the adhesive contact systems* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)_k$ *and a limit* (u, z) *satisfying convergences* (2.42)*.*

Proof. The uniform bound (2.17b) allows us to find a subsequence $(u_k)_k$ and a limit u such that $u_k \rightharpoonup u$ in $H^1(0,T;\mathbf{V})$. Since $H^1(0,T;\mathbf{V}) \Subset \mathrm{C}^0_{\mathrm{weak}}([0,T];\mathbf{V})$ (the latter being the space of weakly continuous functions with values in $\mathbf{V}$) by Aubin-Lions type arguments, cf. [57], we also conclude the second of (2.42a). Hence $u(0) = u_0$ in view of (2.41).

In view of the bound for $(z_k)_k$ in $\mathrm{BV}([0,T];L^1(\Gamma_{\mathrm{C}}))$, and taking into account the continuous embedding of $L^1(\Gamma_{\mathrm{C}})$ into the space $\mathrm{M}(\Gamma_{\mathrm{C}})$ of Radon measures on Γ_{C}, we may apply a suitable version of Helly's selection principle for functions with values in the dual of a separable Banach space, cf. [12, Lemma 7.2], and find a (not relabeled) subsequence $(z_k)_k$ and a limit function z such that $z_k(t) \rightharpoonup z(t)$ in $\mathrm{M}(\Gamma_{\mathrm{C}})$ for every $t \in [0,T]$. Taking into account that the functions z_k are uniformly bounded in $L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))$, a fortiori we conclude the pointwise convergence of $z_k(t)$, for all $t \in [0,T]$, w.r.t. the weak* topology of $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}) \cap L^\infty(\Gamma_{\mathrm{C}})$, i.e. the second of (2.42c). Taking into account that $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}) \Subset L^1(\Gamma_{\mathrm{C}})$, we have that $z_k(t) \to z(t)$ strongly in $\mathbf{Z}$ and, ultimately, by the bound in $L^\infty(\Gamma_{\mathrm{C}})$ we infer strong convergence in $L^q(\Gamma_{\mathrm{C}})$ for all $1 \le q < \infty$. This gives (2.42b). Finally, the Aubin-Lions type compactness result from [51, Cor. 7.9] combined with the Banach Alaouglu Bourbaki theorem also ensures that, up to a further subsequence, there exists $\tilde{z} \in L^\infty((0,T)\times\Gamma_{\mathrm{C}})$ such that $z_k \to \tilde{z}$ weakly* in $L^\infty((0,T)\times\Gamma_{\mathrm{C}})$ and strongly in $L^q((0,T)\times\Gamma_{\mathrm{C}})$ for all $1 \le q < \infty$. Therefore, $\tilde{z} = z$ a.e. in $(0,T)$. This gives the first of (2.42c).

Finally, the existence of an element $\lambda \in L^2(0,T;\mathbf{V}^*)$ and convergence (2.42d) follow from (2.17f). □

4.1. *Limit passage in the semistability condition & fine properties of semistable sets for perimeter-regularized models with unidirectionality*

The limit passage in the semistability condition as $k \to \infty$ results from the construction of a mutual recovery sequence in correspondence with the sequence $(u_k, z_k)_k$, converging to the candidate semistable energetic solution (u, z) of the brittle model as specified in (2.42). Namely, we show that for every $\tilde{z} \in \mathbf{X}$ there exists a sequence $(\tilde{z}_k)_k \subset \mathbf{X}$ such that for every $t \in [0,T]$

$$\begin{aligned}&\limsup_{k\to\infty} \left(\mathcal{E}_k(t,u_k(t),\tilde{z}_k) - \mathcal{E}_k(t,u_k(t),z_k(t)) + \mathcal{R}_k(\tilde{z}_k - z_k(t))\right)\\ &\le \mathcal{E}_\infty(t,u(t),\tilde{z}) - \mathcal{E}_\infty(t,u(t),z(t)) + \mathcal{R}_\infty(\tilde{z} - z(t)),\end{aligned}$$

so that the positivity of the l.h.s. in the above inequality, granted by the semistability inequality for the adhesive system, entails the positivity of the r.h.s., hence the semistability condition in the brittle limit. We refer to [49, Sec. 5.2] for all the details of the construction; regarding scaling (2.15b) we also point to [49, Sec. 7] and to reference[56]. In this way, we conclude

Lemma 4.2 (Limit passage in the semistability condition). *The limit pair* (u, z) *obtained by convergences* (2.42) *satisfies the semistability inequality* (2.4) *for the brittle system* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_\infty, \mathcal{E}_\infty)$.

We now discuss consequences of the above semistability result, which are true for all $k \in \mathbb{N} \cup \{\infty\}$, since they rely on the unidirectionality of the delamination process encoded in the 1-homogeneous dissipation potentials $\mathcal{R}_k$ (2.11), resp. $\mathcal{R}_\infty$ (2.32). In fact, it was already observed in [49, Sec. 6, (6.5)], that the unidirectionality allows us to extend the semistability inequality for $k \in \mathbb{N} \cup \{\infty\}$ (cf. (2.53)–(2.54)), to a more general inequality that compares the perimeter of (semi-)stable sets and their competitors with their volume difference, cf. (4.1) below.

Lemma 4.3 (Consequence of semistability). *Let* $t \in [0, T]$ *be fixed and, for* $k \in \mathbb{N} \cup \{\infty\}$ *let* $z_k(t)$ *be semistable for* $\mathcal{E}_k(t, u_k(t), \cdot)$ *in the sense of* (2.4). *Then the finite-perimeter set* $Z_k(t)$ *with characteristic function* $z_k(t)$ *satisfies the following inequality for all* $\widetilde{Z} \subset Z_k(t)$*:*

$$\mathrm{b}_k P(Z_k(t), \Gamma_{\mathrm{C}}) \leq \mathrm{b}_k P(\widetilde{Z}, \Gamma_{\mathrm{C}}) + (a_k^0 + a_k^1) \mathcal{L}^{d-1}(Z_k \backslash \widetilde{Z}) \,. \tag{4.1}$$

It was deduced in [49, Thm. 6.3] that finite-perimeter sets satisfying (4.1) have an additional regularity property, introduced by Campanato as the Property $\mathfrak{a}$, cf. e.g. [7,8,26,28] and called *lower density estimate* in e.g. [21,1]:

Proposition 4.1 (Lower density estimate for semistable sets). *Keep* $t \in [0, T]$ *fixed and assume that the finite-perimeter set* $Z_k(t) \subset \Gamma_{\mathrm{C}}$ *satisfies* (4.1), *with* $k \in \mathbb{N} \cup \{\infty\}$. *Then, for all* $k \in \mathbb{N} \cup \{\infty\}$, $Z_k(t)$ *fulfills the following* lower density estimate*: there are constants* $R, \mathfrak{a}(\Gamma_{\mathrm{C}}) > 0$ *depending solely on* $\Gamma_{\mathrm{C}} \subset \mathbb{R}^{d-1}$, *on* d, *and on the parameters* $a_k^0, a_k^1, \mathrm{b}_k > 0$, *such that*

$$\begin{aligned} &\forall\, y \in \operatorname{supp} z_k(t) \;\forall\, \rho_\star > 0 : \\ &\mathcal{L}^{d-1}(Z_k(t) \cap B_{\rho_\star}(y)) \geq \begin{cases} \mathfrak{a}(\Gamma_{\mathrm{C}}) \rho_\star^{d-1} & \text{if } \rho_\star < R, \\ \mathfrak{a}(\Gamma_{\mathrm{C}}) R^{d-1} & \text{if } \rho_\star \geq R. \end{cases} \end{aligned} \tag{4.2}$$

Here, $B_{\rho_\star}(y)$ *denotes the open ball of radius* $\rho_\star$ *with center in* y *and the support of the* SBV*-function* $z_k(t)$ *is defined as in* (2.35).

Sets satisfying the lower density estimate (4.2), are sometimes also called $(d{-}1)$-thick, see e.g. [37,19]. The proof of Proposition 4.1 is carried out by contradiction to (4.1). The lower bound $\mathfrak{a}(\Gamma_{\mathrm{C}}) \rho_\star^{d-1}$, which holds uniformly for all radii $\rho_\star$ and at every point of $\operatorname{supp} z_k(t)$, is obtained with the aid of a uniform relative isoperimetric inequality proved in [58, Thm. 3.2]. In turn, for the proof of the latter result it is crucial that Γ_{C} is convex, as required with (2.6d).

Now, let us highlight a simple consequence, yet crucial for our arguments, of Proposition 4.1 in Lemma 4.4 below. It involves the *essential closure* of the semistable sets Z_k, $k \in \mathbb{N} \cup \{\infty\}$. We recall that the essential closure of a measurable set $E \subset \Gamma_{\mathrm{C}}$ is defined as follows (cf. e.g. [48, p. 21]):

$$\mathrm{cl}_* E := \Big\{ x \in \mathbb{R}^{d-1} \, : \, \limsup_{r \to 0} \frac{\mathcal{H}^{d-1}(E \cap B_r(x))}{\mathcal{H}^{d-1}(B_r(x))} > 0 \Big\} . \tag{4.3}$$

The set $\mathrm{cl}_* E$ is not necessarily (topologically) closed. However, the following key property holds (cf. [48, Cor. 1.5.3])

$$\mathcal{H}^{d-1}((E \backslash \mathrm{cl}_* E) \cup (\mathrm{cl}_* E \backslash E)) = 0 \, . \tag{4.4}$$

Lemma 4.4. *Keep $t \in [0,T]$ fixed and, for $k \in \mathbb{N} \cup \{\infty\}$ let $z_k(t)$ be semistable for $\mathcal{E}_k(t, u_k(t), \cdot)$ in the sense of* (2.4)*, with associated finite-perimeter set $Z_k(t)$. Then,*

$$\mathrm{supp}\, z_k(t) \subset \mathrm{cl}_* Z_k(t) \tag{4.5}$$

and, therefore,

$$\mathcal{H}^{d-1}(\mathrm{supp}\, z_k(t) \backslash Z_k(t)) = 0. \tag{4.6}$$

Observe that, since $\mathrm{cl}_* Z_k(t)$ need not be (topologically) closed, (4.5) does not follow from the definition (2.35) of $\mathrm{supp}\, z_k(t)$, which guarantees $\mathrm{supp}\, z_k(t) \subset A$ for every set such that $\mathcal{H}^{d-1}(Z_k(t) \backslash A) = 0$, provided that A is closed. Indeed, (4.5) is due to the lower density estimate (4.2) in the case $\rho_\star \in (0, R)$, yielding

$$\forall\, y \in \mathrm{supp}\, z_k(t) : \quad \lim_{\rho_\star \to 0} \frac{\mathcal{H}^{d-1}(Z_k(t) \cap B_{\rho_\star}(y))}{\mathcal{H}^{d-1}(B_{\rho_\star}(y))} \geq \frac{\mathfrak{a}(\Gamma_{\mathrm{C}})}{\mathcal{H}^{d-1}(B_1(0))} > 0,$$

since $\mathcal{H}^{d-1}(B_{\rho_\star}(y)) / \mathcal{H}^{d-1}(B_1(0)) = \rho_\star^{d-1}$. Then, (4.6) directly ensues from (4.4) combined with (4.5).

The second, key consequence of the lower density estimate (4.2) is a *support convergence* result, proved in [49] and recalled in Prop. 4.2 below, which further strengthens the convergence of the delamination variables z_k for the adhesive contact models. In fact, it states one part for Hausdorff convergence of the supports of the sequence $(z_k)_k$ to the support of the limit z.

Proposition 4.2 (Support convergence [49, Thm. 6.1]). *Let $t \in [0,T]$ be fixed. For all $k \in \mathbb{N} \cup \{\infty\}$ assume that the finite-perimeter sets $Z_k(t) \subset \Gamma_{\mathrm{C}}$ satisfy* (4.1) *and that the associated characteristic functions $z_k(t) \overset{*}{\rightharpoonup} z(t)$ in $\mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$ for some $z \in L^\infty(0,T; \mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\}))$. For all $k \in \mathbb{N}$ set*

$$\rho(k,t) := \inf \big\{ \rho > 0 : \, \mathrm{supp}\, z_k(t) \subset \mathrm{supp}\, z(t) + B_\rho(0) \big\}. \tag{4.7}$$

Then

$$\mathrm{supp}\, z_k(t) \subset \mathrm{supp}\, z(t) + B_{\rho(k,t)}(0) \quad \textit{and} \quad \rho(k,t) \to 0 \ \textit{as} \ k \to \infty. \tag{4.8}$$

In particular, if $\operatorname{supp} z(t) = \emptyset$, *then also* $\operatorname{supp} z_k(t) = \emptyset$ *for all* $k \geq k_0$ *from a particular index* $k_0 \in \mathbb{N}$ *on.*

The counterpart to (4.8), namely $\operatorname{supp} z(t) \subset \operatorname{supp} z_k(t) + B_{\tilde\rho(k,t)}(0)$ with $\tilde\rho(k,t) \to 0$ as $k \to \infty$, can be obtained directly from the pointwise strong $L^1(\Gamma_{\mathrm{C}})$-convergence of the sequence (z_k), cf. (2.42b), see [49, Cor. 6.8] for the proof.

4.2. *Recovery test functions for the momentum balance and proof of the brittle constraint*

The limit passage in the momentum balance for the adhesive system (1.5) as $k \to \infty$ requires, for each test function $v \in \mathbf{V}_z(t)$, with $t \in (0,T)$ fixed, the construction of a *recovery sequence* $(v_k)_k$ fulfilling the following (minimal) convergence properties as $k \to \infty$: $\int_{\Gamma_{\mathrm{C}}} k z_k(t) [\![u_k(t)]\!] [\![v_k]\!] \, \mathrm{d}\mathcal{H}^{d-1} \to 0$ and $v_k \to v$ in $\mathbf{V} = H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$. Based on the knowledge of support convergence from Prop. 4.2 these features can be guaranteed by a construction of recovery sequences for the test functions of (1.5), developed in [45, Prop. 8, Cor. 2]. Since this construction will also be the starting point for proving that the union of the recovery spaces $\mathbf{Y}_n^s$ is dense in $L^2(s,T;\mathbf{V}_z(s)^*)$ for every $s \in (0,T)$, we shall illustrate it in detail in the ensuing Proposition 4.3. We will state the latter result for a fixed $z \in L^\infty(\Gamma_{\mathrm{C}})$ and we will later apply it to $z(t)$, $t \in [0,T]$ fixed, with $z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}}))$ a limiting curve for the sequence $(z_k)_k$ of the adhesive contact delamination variables.

The construction of the recovery test functions (for the adhesive momentum balance) is based on the fact that any function $v \in \mathbf{V}_z = \{v \in \mathbf{V} : [\![v]\!] = 0 \text{ on } \operatorname{supp} z\}$ can be written in terms of its symmetric v_{sym} and its antisymmetric part v_{anti} with respect to the plane $x_1 = 0$. Rewriting any $x \in \Omega$ as $x = (x_1, y)$ for $y = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$, this is

$$\begin{aligned} v(x_1,y) &:= v_{\mathrm{sym}}(x_1,y) + v_{\mathrm{anti}}(x_1,y) \in \mathbf{V}_z, \quad \text{with} \\ v_{\mathrm{sym}}(x_1,y) &:= \tfrac{1}{2}(v(x_1,y) + v(-x_1,y)) \quad \in H^1(\Omega;\mathbb{R}^d) \;\; \text{and} \\ v_{\mathrm{anti}}(x_1,y) &:= \tfrac{1}{2}(v(x_1,y) - v(-x_1,y)) \quad \in H^1((\Omega\backslash\Gamma_{\mathrm{C}}) \cup \operatorname{supp} z;\mathbb{R}^d), \end{aligned} \tag{4.9}$$

where we assume here and in what follows that the domain Ω is oriented in a coordinate system such that the origin is contained in Γ_{C} and the normal n to Γ_{C} points in the x_1-direction, cf. Figure 1 on p. 3. Note that construction (4.9) ensures that $v_{\mathrm{sym}} \in H^1(\Omega;\mathbb{R}^d)$ does not jump across Γ_{C}, whereas $v_{\mathrm{anti}} \in H^1((\Omega\backslash\Gamma_{\mathrm{C}}) \cup \operatorname{supp} z;\mathbb{R}^d)$ satisfies $v_{\mathrm{anti}} = 0$ on $\operatorname{supp} z$, but may jump across $\Gamma_{\mathrm{C}} \backslash \operatorname{supp} z$. The following result gives the definition of the recovery sequence and its convergence properties.

Proposition 4.3 (Recovery sequence for the test functions, [45,49,56]). *Consider* $z \in L^\infty(\Gamma_{\mathrm{C}})$ *and let* $M := \operatorname{supp} z$ *be a* $(d-1)$*-thick subset of* Γ_{C}. *Let*

$$d_M(x) := \min_{\hat x \in M} |x - \hat x| \quad \text{for all } x \in \overline{\Omega}_\pm. \tag{4.10}$$

Let $v \in H^1(\Omega_- \cup M \cup \Omega_+;\mathbb{R}^d)$, *such that* $v = 0$ *on* Γ_{D} *in the trace sense. With*

$\xi_M^\rho(x) := \min\{\frac{1}{\rho}(d_M(x)-\rho)^+, 1\}$ *set*

$$\begin{aligned} r(\rho,M,v) := v_M^\rho(x_1,y) :&= v_{\mathrm{sym}}(x_1,y) + \xi_M^\rho(x_1,y)\, v_{\mathrm{anti}}(x_1,y) \\ &\text{for all } \rho>0 \text{ and } v\in H^1(\Omega_-\cup M\cup\Omega_+;\mathbb{R}^d), \end{aligned} \tag{4.11}$$

with v_{sym} *and* v_{anti} *as in* (4.9)*. Then, the following statements hold:*

(i) $v_M^\rho \to v$ *strongly in* $H^1(\Omega_-\cup\Omega_+;\mathbb{R}^d)$ *as* $\rho\to 0$,
(ii) $v\in H^1(\Omega_-\cup M\cup\Omega_+,\mathbb{R}^d) \Rightarrow v_M^\rho \in H^1(\Omega_-\cup(M{+}B_\rho(0))\cup\Omega_+;\mathbb{R}^d)$.

Later on, we will apply Prop. 4.3 to $z = z(t)$, $t\in[0,T]$ fixed and $z\in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}}))$ a limiting curve for the sequence $(z_k)_k$, with $\rho=\rho(k,t)$ the sequence of radii for which support convergence holds. We thus obtain for every test function $v\in\mathbf{V}_z(t)$ a sequence $(v_k = v^{\rho(k,t)})_k$ which converges to v even strongly in $\mathbf{V} = H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ and which in fact has $[\![v_k]\!]=0$ on $\operatorname{supp} z(t) + B_{\rho(k,t)}(0) \supset \operatorname{supp} z_k(t)$. We refer to 45, Prop. 8, Cor. 2] for a proof.

Remark 4.1. The proof of Prop. 4.3 uses a Hardy's inequality given in [38, p. 190] for closed sets M of arbitrarily low regularity, in L^p-spaces with $p>d$. Only recently, such a Hardy's inequality has been obtained in [19] under much weaker integrability assumptions on the displacements, with only slightly strengthened regularity assumptions on the closed set M. More precisely, the additional regularity imposed on M in [19, Thm. 3.1] for Hardy's inequality to hold, is the lower density estimate (4.2); exactly the fine regularity property deduced in Proposition 4.1 for finite-perimeter sets being semistable in the sense of (4.1). Observe though, that for $p>d$ it is $[\![v]\!]\in \mathrm{C}^0(\overline{\Gamma}_{\mathrm{C}},\mathbb{R}^d)$ for any $v\in W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d)$. Thus, if $z|[\![v]\!]|=0$ a.e. on Γ_{C} for a given function $z\in L^\infty(\Gamma_{\mathrm{C}})$, then in particular $[\![v]\!]\equiv 0$ on $\operatorname{supp} z$. This conclusion is no longer valid for $p\le d$ and therefore the above property is directly incorporated in the definition of $\mathbf{V}_z$ in (2.19). This is essential, because we will exploit the support convergence (4.8) for the construction of the recovery sequence and, for this, the usage of the *closed* set $\operatorname{supp} z$ is important. This fact also motivates the definition of the functional $\mathcal{J}_\infty$ from (2.34) in terms of the constraint $[\![u]\!]=0$ a.e. on $\operatorname{supp} z$, which is stronger than just requiring $z[\![u]\!]=0$.

A first consequence of Prop. 4.3, joint with Lemma 4.4, is the Mosco-convergence of the functionals $\mathcal{J}_k(\cdot,z_k)$ from (2.13) to $\mathcal{J}_\infty(\cdot,z)$, with $(z_k)_k$ converging to z weakly* in $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ and z_k, z satisfying (4.1). In fact, (4.1) guarantees the lower density estimate (4.2), which in turn ensures the support convergence (4.8) and thus Prop. 4.3, at the core of the proof of the lim sup-inequality. Estimate (4.2) also yields the validity of Lemma 4.4, at the basis of the proof of the lim inf-inequality. We will only detail the proof of the latter estimate, which in turn will allow us to deduce the brittle constraint.

Lemma 4.5. *Let* $(z_k)_k\subset \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ *fulfill* $z_k \overset{*}{\rightharpoonup} z$ *in* $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$*, and suppose that* z_k *for every* $k\in\mathbb{N}$*, and* z *(and the associated finite-perimeter sets* Z_k*,* Z*) satisfy* (4.1)*. Then, the functionals* $\mathcal{J}_k(\cdot,z_k):\mathbf{V}\to\mathbb{R}$ *Mosco-converge to the* *functional* $\mathcal{J}_\infty(\cdot,z) : \mathbf{V} \to \mathbb{R}\cup\{\infty\}$ *w.r.t. the topology of* $H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$*, i.e., there holds*

– $\liminf$**-*inequality:*** *for every* $u \in H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ *and* $(u_k)_k \subset H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ *there holds*

$$u_k \rightharpoonup u \quad \text{weakly in } H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \ \Rightarrow\ \liminf_{k\to\infty} \mathcal{J}_k(u_k,z_k) \geq \mathcal{J}_\infty(u,z); \tag{4.12a}$$

– $\limsup$-***inequality:*** *for every* $v \in H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ *there is a sequence* $(v_k)_k \subset H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ *such that*

$$v_k \to v \quad \text{strongly in } H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \text{ and } \limsup_{k\to\infty} \mathcal{J}_k(v_k,z_k) \leq \mathcal{J}_\infty(v,z). \tag{4.12b}$$

Proof. In order to prove (4.12a), we may confine the discussion to the case $\liminf_{k\to\infty}\mathcal{J}_k(u_k,z_k) < \infty$. Therefore, up to a subsequence we have $\sup_{k\in\mathbb{N}}\mathcal{J}_k(u_k,z_k) \leq C$ and there holds

$$\int_{\Gamma_{\mathrm{C}}} \tfrac{1}{2}z\left|[\![u]\!]\right|^2 \,\mathrm{d}\mathcal{H}^{d-1} \leq \liminf_{k\to\infty}\int_{\Gamma_{\mathrm{C}}} \tfrac{1}{2}z_k\left|[\![u_k]\!]\right|^2 \,\mathrm{d}\mathcal{H}^{d-1} \leq \lim_{k\to\infty}\frac{C}{k} = 0,$$

where the first inequality follows from combining the strong convergence $z_k \to z$ in $L^1(\Gamma_{\mathrm{C}})$ (due to $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}) \Subset L^1(\Gamma_{\mathrm{C}})$), with the weak convergence $[\![u_k]\!] \rightharpoonup [\![u]\!]$ in $L^2(\Gamma_{\mathrm{C}};\mathbb{R}^d)$ (due to $u_k \rightharpoonup u$ in $H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ via trace theorems and Sobolev embeddings). Hence

$$z[\![u]\!] = 0 \text{ a.e. on } \Gamma_{\mathrm{C}} \ \Leftrightarrow\ [\![u]\!] = 0 \text{ a.e. on } Z \ \Leftrightarrow\ [\![u]\!] = 0 \text{ a.e. on } \operatorname{supp} z \tag{4.13}$$

the last implication due to the fact that $\operatorname{supp} z$ and Z coincide, up to a $\mathcal{H}^{d-1}$-negligible set, thanks to (4.4) and (4.6). Then, $\mathcal{J}_\infty(u,z) = 0 \leq \liminf_{k\to\infty}\mathcal{J}_k(u_k,z_k)$. The proof of the $\limsup$-inequality follows from adapting that for [49, Prop. 5.4]. □

We can now conclude the brittle constraint for the limit pair (u,z) from the lower Γ-limit (4.12a). For the the pair $(\dot u, z)$, the brittle constraint will be obtained arguing on difference quotients.

Lemma 4.6 (Brittle constraint (2.43)**).** *The pair* (u,z) *obtained by convergences* (2.42) *satisfies* (2.43)*.*

Proof. Let $t \in [0,T]$ be fixed. We may apply Lemma 4.5 to the sequence $(z_k(t))_k$ and to $z(t)$, which satisfy the semistability condition for the energies $\mathcal{E}_k(t,u_k(t),\cdot)$ and $\mathcal{E}_\infty(t,u(t),\cdot)$, respectively, and thus inequality (4.1). Therefore, for every $t \in [0,T]$ the energies $\mathcal{J}_k(\cdot,z_k(t))$ Mosco-converge to $\mathcal{J}_\infty(\cdot,z(t))$ in $\mathbf{V}$. Since $u_k(t) \rightharpoonup u(t)$ in $\mathbf{V}$ by (2.42a), the $\liminf$-inequality (4.12a) ensures that $\mathcal{J}_\infty(u(t),z(t)) \leq \liminf_{k\to\infty}\mathcal{J}_k(u_k(t),z_k(t)) \leq C$, where we have used that $\mathcal{J}_k(u_k(t),z_k(t)) \leq \mathcal{E}_k(t,u_k(t),z_k(t)) + \tilde C \leq C$ for constants C, $\tilde C$ uniform w.r.t. $t \in [0,T]$ and $k \in \mathbb{N}$ (cf. (2.17a)). Hence $\mathcal{J}_\infty(u(t),z(t)) = 0$ for every $t \in [0,T]$, whence the brittle constraint (2.43) for u.

To deduce the brittle constraint for the time-derivative $\dot{u}$ given that $\big|[\![u(t)]\!]\big|^2\big|_{\mathrm{supp}\,z(t)} = 0$, we argue with the aid of difference quotients. It follows from the definition of the Bochner space $W^{1,p}(0,T;\mathbf{B})$, with $\mathbf{B}$ a reflexive Banach space, cf. e.g. [6, Def. A.1, p. 140], that for every $v \in W^{1,p}(0,T;\mathbf{B})$ there holds $\frac{v(t)-v(t-h)}{h} \to \dot{v}(t)$ strongly in $\mathbf{B}$ at every Lebesgue point t of $\dot{v}$. Namely,

$$\begin{aligned} &\lim_{0<h\to 0} \left\| \tfrac{v(t)-v(t-h)}{h} - \dot{v}(t) \right\|_{\mathbf{B}} \\ &= \lim_{0<h\to 0} \left\| \tfrac{1}{h} \int_{t-h}^{t} \dot{v}(s)\mathrm{d}s - \dot{v}(t) \right\|_{\mathbf{B}} = 0 \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{4.14}$$

For $v = [\![u]\!] \in H^1(0,T;L^2(\Gamma_{\mathrm{C}}))$ such that $v(t,x) = 0$ for a.a. $x \in \mathrm{supp}\,z(t)$, for a.a. $t \in (0,T)$ we also have that $v(t-h,x) = 0$ for a.a. $x \in \mathrm{supp}\,z(t)$, since $\mathrm{supp}\,z(t) \subset \mathrm{supp}\,z(t-h)$ (due to the fact that $z(t) \le z(t-h)$ a.e. in Γ_{C} by unidirectionality). Thus, in view of (4.14), denoting with $\mathcal{X}_{\mathrm{supp}\,z(t)}$ the characteristic function of $\mathrm{supp}\,z(t)$, we obtain for a.a. $t \in (0,T)$ that

$$\begin{aligned} \int_{\Gamma_{\mathrm{C}}} \mathcal{X}_{\mathrm{supp}\,z(t)} |\dot{v}(t)|^2 \,\mathrm{d}\mathcal{H}^{d-1} &= \int_{\Gamma_{\mathrm{C}}} \mathcal{X}_{\mathrm{supp}\,z(t)} \left| \tfrac{v(t)-v(t-h)}{h} - \dot{v}(t) \right|^2 \mathrm{d}\mathcal{H}^{d-1} \\ &\le \int_{\Gamma_{\mathrm{C}}} \left| \tfrac{v(t)-v(t-h)}{h} - \dot{v}(t) \right|^2 \mathrm{d}\mathcal{H}^{d-1} \to 0\,. \end{aligned}$$

Since the integrand on the left-hand side of the above inequality is positive, we conclude that, indeed, $\dot{v}(t) = [\![\dot{u}(t)]\!] = 0$ a.e. on $\mathrm{supp}\,z(t)$ for a.a. $t \in (0,T)$. □

4.3. *Recovery spaces for the momentum balance and their properties*

Recall the weak formulation of the momentum balance for the adhesive contact systems

$$\begin{aligned} \langle \varrho \ddot{u}_k(t), v \rangle_{\mathbf{V}} &+ \int_{\Omega\backslash\Gamma_{\mathrm{C}}} (\mathbb{D}e(\dot{u}_k(t)) \colon e(v) + \mathbb{C}e(u_k(t)) : e(v)) \,\mathrm{d}x \\ &+ \int_{\Gamma_{\mathrm{C}}} k z_k(t) \left[\!\left[u_k(t) \right]\!\right] \left[\!\left[v \right]\!\right] \mathrm{d}\mathcal{H}^{d-1} = \langle \mathbf{f}(t), v \rangle_{\mathbf{V}} \end{aligned} \tag{4.15}$$

for every $v \in \mathbf{V}$ and for a.a. $t \in (0,T)$. In the limit passage as $k \to \infty$, one has to face two problems:

(1) the blow-up of the bounds on the adhesive contact term $kz_k[\![u_k]\!]$ tested against $v \in \mathbf{V}$ as $k \to \infty$;
(2) the consequent blow-up of the bounds (by comparison) on the inertial terms $(\ddot{u}_k)_k$.

In Section 4.2 we have illustrated the construction of the recovery sequence for the test functions of the brittle momentum balance. In [49], such a construction allowed us to overcome problem (1) in the *quasistatic* (viscous) setting, where inertial terms in the momentum balance were neglected.

In what follows, we will exploit a *refinement of this method* in order to tackle problem (2), by costructing a sequence of *recovery spaces* for the space $\mathbf{V}_z(t)$ (cf. (2.19)) of the test functions for the momentum balance in the brittle limit. The definition of these recovery spaces and the proof of their properties relies on the support convergence $\operatorname{supp} z_k(t) \subset \operatorname{supp} z(t) + B_{\rho(k,t)}(0)$ for every $t \in [0,T]$, with $\rho(k,t) \to 0$ as $k \to \infty$, of the semistable solutions z_k for the adhesive systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_k, \mathcal{E}_k)$ (cf. Prop. 4.2) This convergence is intended along the very same sequence of indices k such that convergences (2.42b) hold. In particular, the extracted sequence $(\rho(k,\cdot))_k$ of radii is independent of $t \in [0,T]$. Moreover, due to the temporally monotonically decreasing nature of (semistable) z_k we also have

$$\forall\, k \in \mathbb{N} \cup \{\infty\}, \forall\, t > s \in [0,T] : \ \operatorname{supp} z_k(t) \subset \operatorname{supp} z_k(s) \,. \tag{4.16}$$

Note that there is in general no monotonicity relation between $\rho(k,t)$ and $\rho(k,s)$, because $\operatorname{supp} z$ and $\operatorname{supp} z_k$ need not decrease with the same speed. We now choose a *nonincreasing* sequence $(\varepsilon_n)_n$ with $\varepsilon_n \downarrow 0$. Then, thanks to (4.16) for any $k \in \mathbb{N} \cup \{\infty\}$, we also have for every $s \in [0,T)$ and $t \in [s,T]$:

$$\begin{aligned} &\text{If } \operatorname{supp} z_k(s) \subset \operatorname{supp} z(s) + \overline{B_{\varepsilon_n}(0)}, \\ &\quad \text{then also } \operatorname{supp} z_k(t) \subset \operatorname{supp} z(s) + \overline{B_{\varepsilon_n}(0)}\,. \end{aligned} \tag{4.17}$$

This relation will be of great use later on, when deducing sufficient compactness results for the adhesive inertial terms on the intervals $[s,T]$. That is why, for the above chosen sequence $(\varepsilon_n)_n$ with $\varepsilon_n \downarrow 0$, we now introduce the following *recovery spaces* for all $n \in \mathbb{N}$ and for $t \in [0,T]$ and $s \in [0,T)$:

$$\begin{aligned} \mathbf{V}_z(\varepsilon_n,t) &:= \{ v \in H^1\big((\Omega \backslash \Gamma_{\mathrm{C}}) \cup (\operatorname{supp} z(t) + \overline{B_{\varepsilon_n}(0)}); \mathbb{R}^d\big) \ : \ v = 0 \text{ on } \Gamma_{\mathrm{D}} \} \\ &= \{ v \in \mathbf{V} \ : \ [\![v]\!] = 0 \text{ a.e. on } \Gamma_{\mathrm{C}} \cap (\operatorname{supp} z(t) + \overline{B_{\varepsilon_n}(0)}) \}, \\ \mathbf{Y}_n^s &:= L^2(s,T; \mathbf{V}_z(\varepsilon_n,s)), \\ \widetilde{\mathbf{Y}}_n^s &:= \Big\{ (v_1,v_2,v_3) \in (\mathbf{Y}_n^s)^* \times (\mathbf{Y}_n^s)^* \times (\mathbf{Y}_n^s)^* \ : \ \text{ for } i = 1,2 : \\ &\qquad \textstyle\int_{s+h}^T \big\langle v_i(t) - v_i(t-h) - \int_{t-h}^t v_{i+1}(s)\,\mathrm{d}s, \phi(t) \big\rangle_{\mathbf{V}} \,\mathrm{d}t = 0 \\ &\qquad\qquad \text{for all } \phi \in \mathbf{Y}_n^s,\ h \in (0,T-s) \} \Big\}\,. \end{aligned} \tag{4.18}$$

Observe that the definition of the spaces $\widetilde{\mathbf{Y}}_n^s$ encompasses the information that v_{i+1} is the time-derivative of the function v_i in $(\mathbf{Y}_n^s)^*$, for $i = 1,2$. Indeed, choosing test functions $\phi = \eta\varphi$ with $\eta \in \mathrm{C}_0^\infty(s,T)$ such that $\operatorname{supp}\eta \subset (s+h,T)$, and $\varphi \in \mathbf{V}_z(\varepsilon_n,s)$, the fundamental lemma of the Calculus of Variations yields that $\big\langle v_i(t) - v_i(t-h) - \int_{t-h}^t v_{i+1}(s)\,\mathrm{d}s, \varphi \big\rangle_{\mathbf{V}} = 0$ for a.a. $t \in (s+h,T)$ and for $h \in (0,T-s)$. Hence, $v_i(t) = v_i(t-h) - \int_{t-h}^t v_{i+1}(s)\,\mathrm{d}s$ in $\mathbf{V}_z(\varepsilon_n,s)^*$, which corresponds to the time-derivative in Bochner-spaces, cf. [6, p. 140, Def. A.1]. We now state further properties of the above defined recovery spaces.

Proposition 4.4. *For all $s,t \in [0,T]$, let the spaces $\mathbf{V}_z(t), \mathbf{V}_z(\varepsilon_n,t), \mathbf{Y}_n^s$, and $\widetilde{\mathbf{Y}}_n^s$ be as in* (2.19) *and* (4.18). *Then,* *(1) for every $n \in \mathbb{N}$ and every $t \in [0,T]$ the space $\mathbf{V}_z(\varepsilon_n, t)$ is a closed subspace of $\mathbf{V}_z(t)$ endowed with the norm $\|\cdot\|_{\mathbf{V}}$. For every $n \in \mathbb{N}$ and all $s \le t$ the space $\mathbf{V}_z(\varepsilon_n, s)$ is a closed subspace of $\mathbf{V}_z(\varepsilon_n, t)$. Moreover, since the sequence $(\varepsilon_n)_n$ is monotonically decreasing, there holds*

$$\mathbf{V}_z(\varepsilon_n, t) \subset \mathbf{V}_z(\varepsilon_{n+1}, t) \quad \text{for every } t \in [0,T] \text{ and } n \in \mathbb{N}; \tag{4.19}$$

(2) for every $s \in (0,T]$, the space $\mathbf{Y}_n^s$ is a closed subspace of $L^2(s,T,\mathbf{V}_z(s))$. Hence, $\mathbf{Y}_n^s$ endowed with the norm $\big(\int_0^T \|\cdot\|_{\mathbf{V}}^2\,\mathrm{d}t\big)^{1/2}$ is a reflexive Banach space, and so is $(\mathbf{Y}_n^s)^* \cong L^2(s,T;\mathbf{V}_z(\varepsilon_n,s)^*)$;

(3) for every $s \in [0,T]$, the union $\cup_{n\in\mathbb{N}}\mathbf{Y}_n^s$ is dense in $L^2(s,T;\mathbf{V}_z(s))$;

(4) for every $s \in [0,T]$ and every $n \in \mathbb{N}$, the space $\widetilde{\mathbf{Y}}_n^s$ endowed with the norm $\|\cdot\|_{(\mathbf{Y}_n^s\cdot)^*} \times \|\cdot\|_{(\mathbf{Y}_n^s\cdot))^*} \times \|\cdot\|_{(\mathbf{Y}_n^s\cdot)^*}$ is a reflexive Banach space.

Proof. On Item 1.: For each $s < t \in [0,T]$ it holds $\operatorname{supp} z(t) \subset \operatorname{supp} z(s) \subset \Gamma_{\mathrm{C}}$ and, thus, $\mathbf{V}(\varepsilon_n, s) \subset \mathbf{V}(\varepsilon_n, t) \subset \mathbf{V}$. Similarly $\operatorname{supp} z(t) \subset \overline{\operatorname{supp} z(t) + B_{\varepsilon_n}(0)}$ and, hence, $\mathbf{V}(\varepsilon_n, t) \subset \mathbf{V}_z(t)$. It is also a standard matter to verify that $\mathbf{V}_z(\varepsilon_n, t) \subset \mathbf{V}_z(\varepsilon_{n+1}, t)$, since $B_{\varepsilon_{n+1}}(0) \subset B_{\varepsilon_n}(0)$ for $(\varepsilon_n)_n$ decreasing. The closedness then follows by Prop. 2.1 for each of the spaces.

On Item 2.: It can be straightforwardly verified that $\mathbf{Y}_n^s$ is a subspace of $L^2(s,T;\mathbf{V}_z(s))$; its closedness follows from the very same argument as in the proof of Prop. 2.1. The representation formula for $(\mathbf{Y}_n^s)^*$ is a standard fact in the theory of Bochner spaces, cf., e.g., [18].

On Item 3.: In order to verify that $\cup_{n\in\mathbb{N}}\mathbf{Y}_n^s$ is dense in $L^2(s,T;\mathbf{V}_z(s))$, we fix a function $v \in L^2(s,T;\mathbf{V}_z(s))$ and prove the existence of a sequence $(v_n)_n \subset \cup_{n\in\mathbb{N}}\mathbf{Y}_n^s$ satisfying $v_n \to v$ in $L^2(s,T;\mathbf{V})$. For this, we pick from the equivalence class v a selection $\bar v$ that is defined for every $t \in [s,T]$, for instance

$$\bar v(t) := \begin{cases} \lim_{h\to 0} \frac{1}{h}\int_{t-h}^t v(r)\,\mathrm{d}r & \text{if } t \text{ is a Lebesgue point of } v, \\ 0 & \text{otherwise.} \end{cases} \tag{4.20}$$

Using this representative $\bar v$ and the recovery operator r from (4.11), for every $n \in \mathbb{N}$ we set

$$v_n(t) := r(\varepsilon_n, \operatorname{supp} z(s), \bar v(t)) \qquad \text{for all } t \in [0,T]. \tag{4.21}$$

By construction of the recovery operator, cf. (4.11), we have for $\bar v(t) \in \mathbf{V}_z(s)$ that $v_n(t) \in \mathbf{V}(\varepsilon_n, s)$ for all $t \in [s,T]$. Moreover, $\|v_n(t) - \bar v(t)\|_{H^1(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d)} \to 0$ as $n \to \infty$ and, in addition, there holds:

$$\begin{aligned} \|v_n(t) - \bar v(t)\|_{H^1(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d)} &\le \|\bar v_{\mathrm{anti}}(t)(\xi_{\varepsilon_n}^{\operatorname{supp} z(s)} - 1)\|_{H^1(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d)} \\ &\le \|\bar v_{\mathrm{anti}}(t)\|_{H^1(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d)}\,. \end{aligned}$$

The dominated convergence theorem then implies that $v_n \to \bar v$ in $L^2(s,T;H^1(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d))$, and establishes **3.**.

On Item 4.: We now show that $\widetilde{\mathbf{Y}}_n^s$ endowed with the norm $\|\cdot\|_{(\mathbf{Y}_n^s)^*} \times \|\cdot\|_{(\mathbf{Y}_n^s)^*} \times \|\cdot\|_{(\mathbf{Y}_n^s)^*}$ is a reflexive Banach space. For this, we argue that $\widetilde{\mathbf{Y}}_n^s$ is a closed subspace of the reflexive Banach space $(\mathbf{Y}_n^s)^* \times (\mathbf{Y}_n^s)^* \times (\mathbf{Y}_n^s)^*$: Consider a sequence $(v_1^k, v_2^k, v_3^k)_k \subset \widetilde{\mathbf{Y}}_n^s$ such that $(v_1^k, v_2^k, v_3^k) \to (v_1, v_2, v_3)$ in $(\mathbf{Y}_n^s)^* \times (\mathbf{Y}_n^s)^* \times (\mathbf{Y}_n^s)^*$, which means for each $i \in \{1, 2, 3\}$ that

$$\sup_{\phi \in \mathbf{Y}_n^s, \|\phi\|_{\mathbf{V}}=1} \left| \int_s^T \langle v_i^k(t) - v_i(t), \phi(t) \rangle_{\mathbf{V}} \, \mathrm{d}t \right| \to 0 \qquad \text{as } k \to \infty. \tag{4.22}$$

This allows us to pass to the limit in the terms $\int_{s+h}^T \langle v_i^k(t), \phi(t) \rangle_{\mathbf{V}} \, \mathrm{d}t$ and $\int_{s+h}^T \langle v_i^k(t - h), \phi(t) \rangle_{\mathbf{V}} \, \mathrm{d}t$ for $i = 1, 2$ and all $\phi \in \mathbf{Y}_n^s$. Moreover, for the integral term involving v_{i+1} we observe that for all $\phi \in \mathbf{Y}_n^s$, for almost all $t \in (s, T)$, we have that

$$\int_{t-h}^t \langle v_{i+1}^k(\tau) - v_{i+1}(\tau), \phi(t) \rangle_{\mathbf{V}} \, \mathrm{d}\tau \to 0 \, . \tag{4.23}$$

Indeed, for $\phi \in \mathbf{Y}_n^s$, we have by definition $\phi(t) \in \mathbf{V}(\varepsilon_n, s)$ for a.a. $t \in (s, T)$. For all $\tau \in [s, T]$ we may thus set $\varphi(\tau) := \phi(t)$ and understand $\varphi \in \widetilde{\mathbf{Y}}_n^s$ as a function constant in time. Using φ as a particular choice in (4.22), we conclude (4.23). Since also

$$\Big| \int_{t-h}^t \langle v_{i+1}^k(\tau) - v_{i+1}(\tau), \phi(t) \rangle_{\mathbf{V}} \, \mathrm{d}\tau \Big| \leq \sup_{\phi \in \mathbf{Y}_n^s} \int_s^T \langle v_{i+1}^k(\tau) - v_{i+1}(\tau), \phi(t) \rangle_{\mathbf{V}} \, \mathrm{d}\tau \leq C \, ,$$

the dominated convergence theorem implies that

$$\int_{s+h}^T \int_{t-h}^t \langle v_{i+1}^k(\tau) - v_{i+1}(\tau), \phi(t) \rangle_{\mathbf{V}} \, \mathrm{d}\tau \, \mathrm{d}t \to 0 \, .$$

From this we ultimately conclude that $(v_1, v_2, v_3) \in \widetilde{\mathbf{Y}}_n^s$. □

4.4. *Compactness for the inertial terms & limit in the momentum balance*

With the first result of this section, Lemma 4.7, we pass from adhesive to brittle in the momentum balance. In fact, this limit passage will go hand in hand with establishing sufficient compactness for the inertial terms. These arguments rely on the recovery spaces $\mathbf{Y}_n^s$ and $\widetilde{\mathbf{Y}}_n^s$ introduced in (4.18), which are just small enough to prevent the blow-up of the functional derivatives of the adhesive contact term, but still large enough to carry the information on the support of the limit delamination variable. That is why, compactness and limit passage cannot be separated.

Lemma 4.7 (Compactness for the inertial terms & limit passage in the momentum balance). *The following statements hold true:*

1. ***Compactness & brittle momentum balance in*** $\mathbf{V}_z(s)^*$ ***for every*** $s \in [0, T)$ ***fixed:***
   For every $s \in [0, T)$ *there exists a (not relabeled) subsequence of* $(u_k)_k$,

possibly depending on s*, and a function* $\mu^s \in L^2(s,T;\mathbf{V}_z(s)^*)$*, such that*

$$(u_k,\dot u_k,\ddot u_k) \rightharpoonup (u,\dot u,\mu^s) \text{ in } \widetilde{\mathbf{Y}}_n^s \text{ as } k\to\infty \quad \text{for every } n\in\mathbb{N}, \text{ and} \tag{4.24a}$$

$$\begin{aligned}\langle \int_{t-h}^t \mu_s(\tau)\,\mathrm{d}\tau,\mathsf{v}\rangle_{\mathbf{V}} = \langle \dot u(t)-\dot u(t-h),\mathsf{v}\rangle_{\mathbf{V}} = 0 \quad \text{for all } \mathsf{v}\in\mathbf{V}_z(s), \\ \text{for a.a. } t\in(s+h,T), \text{ and for all } h\in(0,T-s),\end{aligned} \tag{4.24b}$$

whence

$$\begin{aligned}\langle \mu_s(t),\mathsf{v}\rangle_{\mathbf{V}_z(s)} = \lim_{h\downarrow 0} \langle \frac{\dot u(t)-\dot u(t-h)}{h},\mathsf{v}\rangle_{\mathbf{V}_z(s)} \\ \text{for all } \mathsf{v}\in\mathbf{V}_z(s) \quad \text{for a.a. } t\in(s,T).\end{aligned} \tag{4.24c}$$

Furthermore, the momentum balance holds with test functions in $\mathbf{V}_z(s)$*, i.e., for a.a.* $t\in(s,T)$*:*

$$\begin{aligned}\langle \varrho\mu_s(t),\mathsf{v}\rangle_{\mathbf{V}} + \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathbb{C}e(u(t))+\mathbb{D}e(\dot u(t))\big) : e(\mathsf{v})\,\mathrm{d}x = \langle \mathbf{f}(t),\mathsf{v}\rangle_{\mathbf{V}} \\ \text{for every } \mathsf{v}\in\mathbf{V}_z(s)\,.\end{aligned} \tag{4.24d}$$

2. ***Compactness independent of*** $s\in[0,T)$***:***
Let $D\subset(0,T]$ *be a dense and countable subset. There exist a (not relabeled) subsequence of* $(u_k,\dot u_k,\ddot u_k)_k$ *and a function*

$$\mu\in L^2(0,T;\mathbf{V}_z(0)^*)\cap\bigcap_{s\in D} L^2(s,T;\mathbf{V}_z(s)^*) \quad \text{such that} \tag{4.25a}$$

$$\begin{aligned}(u_k,\dot u_k,\ddot u_k) \rightharpoonup (u,\dot u,\mu) \text{ in } \widetilde{\mathbf{Y}}_n^s \\ \text{for all } s\in D\cup\{0\} \text{ and every } n\in\mathbb{N}\,, \text{ and s.t.}\end{aligned} \tag{4.25b}$$

$$(u,\dot u,\mu) \text{ fulfill (4.24b)–(4.24d) for all } s\in D\cup\{0\}\,. \tag{4.25c}$$

3. ***Brittle momentum balance in*** $\mathbf{V}_z(t)^*$ ***for a.a.*** $t\in(0,T)$***:***
The function μ *satisfies for almost all* $t\in(0,T)$

$$\mu(t)\in\mathbf{V}_z(t)^* \quad \text{and} \tag{4.26a}$$

$$\tfrac{\dot u(t+h)-\dot u(t)}{h} \rightharpoonup \mu(t) \text{ in } \mathbf{V}_z(t)^* \quad \text{as } h\downarrow 0, \tag{4.26b}$$

hence $\mu(t)=\ddot u(t)$ *in the sense of* (2.27)*. Moreover, the momentum balance* (2.40b) *holds, and for a.a.* $t\in(h,T)$ *for every* $h\in(0,T)$ *it is*

$$\langle \dot u(t)-\dot u(t-h),v\rangle_{\mathbf{V}} = \langle \int_{t-h}^t \ddot u(\tau)\,\mathrm{d}\tau,\mathsf{v}\rangle_{\mathbf{V}} \quad \text{for all } v\in\mathbf{V}_z(t)\,. \tag{4.26c}$$

Moreover, $\lambda\in L^2(0,T;\mathbf{V}^*)$ *obtained in* (2.42d)*, cf. also* (2.51)*, satisfies relation* (2.44)*, i.e.,*

$$\lambda(t)=\ddot u(t) \quad \text{in } \mathbf{V}_z(t)^* \text{ for a.a. } t\in(0,T)\,. \tag{4.27}$$

An inspection of the proof of Lemma 4.7 will reveal that, in fact, (4.24b) & (4.26c) also hold with the forward differences $\dot u(t+h)-\dot u(t)$.

Before carrying out the details, we briefly summarize the **main ideas of the proof**:

On Item 1. Compactness & limit balance in $\mathbf{V}_z(s)^*$ for every $s \in [0,T)$ fixed: Using the recovery spaces $\mathbf{Y}_n^s$ and $\widetilde{\mathbf{Y}}_n^s$, for arbitrary fixed $s \in [0,T)$ (which is the starting point of the time-intervals (s,T) taken into account in $\mathbf{Y}_n^s$ and $\widetilde{\mathbf{Y}}_n^s$), and $n \in \mathbb{N}$, we extract a convergent (s-dependent) subsequence $(u_k, \dot{u}_k, \ddot{u}_k)_k \subset \widetilde{\mathbf{Y}}_n^s$ and a limit triple $(u, \dot{u}, \mu_n^s) \in \widetilde{\mathbf{Y}}_n^s$. By the definition (4.18) of $\widetilde{\mathbf{Y}}_n^s$ we are entitled to say that $\mu_n^s = \ddot{u}$ in $(\mathbf{Y}_n^s)^*$. This allows us to pass to the limit in the momentum balance integrated over (s,T) and to obtain a limit balance in $(\mathbf{Y}_n^s)^*$. By a diagonal sequence argument over $n \in \mathbb{N}$ we can extract a subsequence converging for all $n \in \mathbb{N}$ to find (4.24a) & (4.24b). Due to the density result Prop. 4.4, Item 3., we can pass $n \to \infty$ to find (4.24c) and the limit momentum balance to hold for a.a. $t \in (s,T)$ with test functions $v \in \mathbf{V}_z(s)$, i.e., (4.24d).

On Item 2. Compactness independent of $s \in [0,T)$: The subsequences and their limit obtained by a diagonal procedure over $n \in \mathbb{N}$ in Item 1, depend on $s \in [0,T)$. By arguing on the countable dense set $D \subset [0,T]$ we can essentially repeat the demonstration of Item 1 with a further diagonal procedure over the elements of $D \cup \{0\}$ to conclude statements (4.25).

On Item 3. Brittle momentum balance in $\mathbf{V}_z(t)^*$ for a.a. $t \in (0,T)$: In order to show that $\mu(t) \in \mathbf{V}_z(t)^*$ and to extend the brittle momentum balance to hold in $\mathbf{V}_z(t)^*$ we adapt the arguments of [14, Lemma 2.2]. The basis for this is a further density result [14, Lemma 2.3], which in our setting guarantees that

$$\begin{aligned}
&\textit{for the monotonically increasing sequence of}\\
&\textit{closed linear subspaces } (\mathbf{V}_z(t))_{t\in[0,T]} \textit{ of the}\\
&\textit{separable Hilbert space } \mathbf{V} \textit{ there exists}\\
&\textit{an at most countable set } S \subset [0,T] \textit{ such that:}\\
&\mathbf{V}_z(t) = \overline{\cup_{s<t}\mathbf{V}_z(s)} \quad \text{for all } t \in [0,T]\backslash S\,.
\end{aligned} \tag{4.28}$$

In this way, we can approximate a test function $\phi \in \mathbf{V}_z(t)$ for any $t \in (0,T)$ out of a set of zero Lebesgue measure by a sequence $(\phi_m)_m \in \cup_m \mathbf{V}_z(s_m)$ with $s_m \nearrow t$ and $(s_m)_m \subset D$. Hence, (4.25) holds along $(s_m)_m$ and by approximation we may ultimately infer statements (4.26). Finally, relation (4.27) ensues by direct comparison of the brittle momentum balance (2.40) with (2.51).

Proof of Lemma 4.7: The proof will be carried out according to the items outlined above.

On Item 1. Compactness & brittle momentum balance in $\mathbf{V}_z(s)^*$ for every $s \in [0,T]$ fixed: Observe that, by the very definition (4.18) of $\mathbf{Y}_n^s$, for every $v \in \mathbf{Y}_n^s$ and almost all $t \in (s,T)$ there holds $[\![v(t)]\!] = 0$ on $\mathrm{supp}(z(s)) + \overline{B_{\varepsilon_n}(0)}$, and thus on $\mathrm{supp}(z(t)) + \overline{B_{\varepsilon_n}(0)}$ since $\mathrm{supp}(z(t)) \subset \mathrm{supp}(z(s))$ (cf. (2.22)). Moreover, in dependence of $s \in [0,T]$ and $n \in \mathbb{N}$ we find, thanks to support convergence (4.8) an index $k(s,n)$, such that for all $k \geq k(s,n)$ it is $\operatorname{supp} z_k(s) \subset \operatorname{supp}(z(s)) + B_{\varepsilon_n}(0)$ and thus, by (4.17), also $\operatorname{supp} z_k(t) \subset \operatorname{supp}(z(s)) + B_{\varepsilon_n}(0)$. All in all, we conclude that

$$\begin{aligned} &\forall\, s \in [0,T), \forall\, n \in \mathbb{N}\; \exists\, k(s,n)\; \forall\, k \geq k(s,n): \\ &\langle \partial_u \mathcal{J}_k(z_k(t), u_k(t)), v(t) \rangle_{\mathbf{V}} = 0 \quad \text{for all } v \in \mathbf{Y}_n^s, \text{ for a.a. } t \in (s,T). \end{aligned} \tag{4.29}$$

Therefore, by comparison in the k-momentum balance, we can deduce the following uniform bounds for the inertial terms, which are independent of $k \in \mathbb{N}$:

$$\exists\, C > 0\; \forall\, s \in [0,T) \forall\, n \in \mathbb{N}\; \exists\, k(s,n)\; \forall\, k \geq k(s,n): \quad \|\ddot{u}_k\|_{(\mathbf{Y}_n^s)^*} \leq C\,. \tag{4.30}$$

In addition, from the uniform bound (2.17c), i.e. $\|u_k\|_{H^1(0,T;\mathbf{V})} \leq C$, we also deduce that $\|u_k\|_{(\mathbf{Y}_n^s)^*} \leq C$ as well as $\|\dot{u}_k\|_{(\mathbf{Y}_n^s)^*} \leq C$, since $L^2(s,T;\mathbf{V}) \subset (\mathbf{Y}_n^s)^*$ continuously. In particular, we observe for all $n \in \mathbb{N}$ and all $v \in \mathbf{Y}_n^s$ that

$$\begin{aligned} &\int_h^T \left| \left\langle \tfrac{u_k(t)-u_k(t-h)}{h} - \dot{u}_k(t), v(t) \right\rangle_{\mathbf{V}} \right| \mathrm{d}t \\ &\leq \int_h^T \left\| \tfrac{u_k(t)-u_k(t-h)}{h} - \dot{u}_k(t) \right\|_{\mathbf{V}^*} \|v(t)\|_{\mathbf{V}}\, \mathrm{d}t \\ &\leq C \int_h^T \left\| \tfrac{u_k(t)-u_k(t-h)}{h} - \dot{u}_k(t) \right\|_{\mathbf{V}} \|v(t)\|_{\mathbf{V}}\, \mathrm{d}t \\ &\leq \|v\|_{L^2(0,T;\mathbf{V})} \left(\int_h^T \left\| \tfrac{u_k(t)-u_k(t-h)}{h} - \dot{u}_k(t) \right\|_{\mathbf{V}}^2 \mathrm{d}t \right)^{1/2} \to 0 \end{aligned}$$

as $h \to 0$ since $u_k \in H^1(0,T;\mathbf{V})$. Hence, $\dot{u}_k$ indeed is the partial time derivative of u_k also in the space $(\mathbf{Y}_n^s)^*$. In the same way, taking into account that $u_k \in H^2(0,T;\mathbf{V}^*)$ and that $L^2(s,T;\mathbf{V}^*) \subset (\mathbf{Y}_n^s)^*$ continuously, we argue that $\ddot{u}_k$ is the partial time derivative of $\dot{u}_k$ in $(\mathbf{Y}_n^s)^*$. Due to these observations we deduce the following uniform bounds

$$\begin{aligned} &\exists\, C > 0\; \forall\, s \in [0,T]\; \forall\, n \in \mathbb{N}\; \exists\, k(s,n)\; \forall\, k \geq k(s,n): \\ &\|u_k\|_{H^1(0,T;\mathbf{V})} + \|u_k\|_{\widetilde{\mathbf{Y}}_n^s} \leq C\,, \end{aligned} \tag{4.31}$$

where $\widetilde{\mathbf{Y}}_n^s$ is defined in (4.18).

Again, keep $s \in [0,T]$ fixed. By (4.31) and by the reflexivity of the spaces granted by Prop. 4.4, for every $n \in \mathbb{N}$ we can extract a convergent subsequence. To be more precise, here we extract this subsequence by a diagonal procedure: Starting with $n = 1$, from the corresponding bound (4.31), we find a (not relabeled, s-dependent) subsequence such that

$$u_k \rightharpoonup u^1 \text{ in } H^1(0,T;\mathbf{V}),\; (u_k, \dot{u}_k, \ddot{u}_k) \rightharpoonup (u^1, \dot{u}^1, \mu_s^1) \text{ in } \widetilde{\mathbf{Y}}_1^s\,. \tag{4.32}$$

Observe that, in view of convergence (2.42a), taking into account the continuous embedding $L^2(s,T;\mathbf{V}) \subset (\mathbf{Y}_n^s)^*$, we can identify the limits u^1 and $\dot{u}^1$, i.e. we have $u^1 = u|_{(s,T)}$ and $\dot{u}^1 = \dot{u}|_{(s,T)}$. Now, for $n = 2$ the above subsequence satisfies the corresponding bound (4.31), so that we can extract a further (not relabeled, s-dependent) subsequence satisfying

$$(u_k, \dot{u}_k, \ddot{u}_k) \rightharpoonup (u, \dot{u}, \mu_s^2) \text{ in } \widetilde{\mathbf{Y}}_2^s .$$

Due to the monotonicity property $\mathbf{V}_z(\varepsilon_1, s) \subset \mathbf{V}_z(\varepsilon_2, s)$ for $\varepsilon_2 < \varepsilon_1$, it holds $L^2(s,T;\mathbf{V}_z(\varepsilon_1,s)) \subset L^2(s,T;\mathbf{V}_z(\varepsilon_2,s))$. Hence we find that the restriction of the element $\mu_s^2 \in L^2(s,T;\mathbf{V}_z(\varepsilon_2,s))^*$ to $\mathbf{Y}_1^s = L^2(s,T;\mathbf{V}_z(\varepsilon_1,s))$ coincides with μ_s^1. Proceeding this way, we obtain a (not relabeled, s-dependent) sequence $(u_k)_k$ and a sequence of limits $(u, \dot{u}, \mu_s^n)_n$ such that, for every $n \in \mathbb{N}$:

$$\begin{aligned} u_k \rightharpoonup u \text{ in } H^1(0,T;\mathbf{V}),\ (u_k, \dot{u}_k, \ddot{u}_k) \rightharpoonup (u, \dot{u}, \mu_s^n) \text{ in } \widetilde{\mathbf{Y}}_n^s \text{ as } k \to \infty \text{ and} \\ \mu_s^n|_{\mathbf{Y}_{n-1}^s} = \mu_s^{n-1} \quad \text{for every } n \in \mathbb{N}\,. \end{aligned} \tag{4.33}$$

For each $n \in \mathbb{N}$, due to the weak convergence of the sequence in $\widetilde{\mathbf{Y}}_n^s$, we also have that

$$\begin{aligned} \int_{s+h}^T \langle \dot{u}(t) - \dot{u}(t-h) - \int_{t-h}^t \mu_s^n(\tau)\,\mathrm{d}\tau, \phi(t) \rangle_{\mathbf{V}}\,\mathrm{d}t = 0 \\ \text{for all } \phi \in \mathbf{Y}_n^s \text{ for all } h \in (0, T-s)\,. \end{aligned} \tag{4.34}$$

Let us now extend the functions $(\mu_s^n)_n$ to an element $\mu_s \in L^2(s,T;\mathbf{V}_z(s)^*)$ by setting

$$\langle \mu_s, \phi \rangle_{L^2(s,T;\mathbf{V}_z(s))} := \begin{cases} \langle \mu_s^n, \phi \rangle_{\mathbf{Y}_n^s} & \text{if } \phi \in \mathbf{Y}_n^s \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \phi \in L^2(s,T;\mathbf{V}_z(s)) \setminus \cup_{n\in\mathbb{N}} \mathbf{Y}_n^s. \end{cases} \tag{4.35}$$

Observe that μ_s is well-defined. Indeed, suppose that $\phi \in \mathbf{Y}_{n_1}^s \cap \mathbf{Y}_{n_2}^s$ for some $n_1 < n_2 \in \mathbb{N}$. By the monotonicity property (4.19), there holds $\mathbf{Y}_{n_1}^s \subset \mathbf{Y}_{n_2}^s$ and, thanks to (4.33), $\langle \mu_s^{n_2}, \phi \rangle_{\mathbf{Y}_{n_2}^s} = \langle \mu_s^{n_1}, \phi \rangle_{\mathbf{Y}_{n_1}^s}$. Observe that $\|\mu_s\|_{L^2(s,T;\mathbf{V}_z(s)^*)} \leq \sup_{n\in\mathbb{N}} \|\mu_s^n\|_{(\mathbf{Y}_n^s)^*} \leq C$, and (4.24a) follows from (4.33).

Using the density of $\cup_n \mathbf{Y}_n^s$ in $L^2(s,T;\mathbf{V}_z(s))$, we now show (4.24b). For this, let $\phi \in L^2(s,T;\mathbf{V}_z(s))$. Using the construction of the recovery sequence (4.21) we find $\phi_n(\tau) = r(\varepsilon_n, \mathrm{supp}(z(s)), \phi) \in \mathbf{Y}_n^s$ with the property $\phi_n \to \phi$ strongly in $L^2(s,T;\mathbf{V}_z(s))$. Using weak-strong convergence arguments and the dominated convergence theorem we deduce that

$$\int_{s+h}^T \langle \int_{t-h}^t \mu_s^n(\tau)\mathrm{d}\tau, \phi_n(t) \rangle_{\mathbf{V}}\mathrm{d}t \to \int_{s+h}^T \langle \int_{t-h}^t \mu_s(\tau)\mathrm{d}\tau, \phi(t) \rangle_{\mathbf{V}}\mathrm{d}t\,,$$

essentially arguing along the lines of the proof of Item 4 in Prop. 4.4. Hence, from (4.34), we infer

$$\begin{aligned} \int_{s+h}^T \langle \dot{u}(t) - \dot{u}(t-h) - \int_{t-h}^t \mu_s(\tau)\,\mathrm{d}\tau, \phi(t) \rangle_{\mathbf{V}}\,\mathrm{d}t = 0 \\ \text{for all } \phi \in L^2(s,T;\mathbf{V}_z(s)) \text{ and for all } h \in (0, T-s)\,. \end{aligned} \tag{4.36}$$

Choosing $\phi \in L^2(s,T;\mathbf{V}_z(s))$ such that $\phi(t,x) = \eta(t)\mathsf{v}(x)$ with $\eta \in C_0^\infty(s,T)$ and $\mathsf{v} \in \mathbf{V}_z(s)$ the fundamental lemma of the Calculus of Variations yields (4.24b). As $h \to 0$, this yields (4.24c), cf. (2.27).

In order to prove (4.24d), we shall pass to the limit as $k \to \infty$ in the k-momentum balance, with test functions $v \in \mathbf{Y}_n^s$, for $n \in \mathbb{N}$ *fixed*. To this aim, we test (4.15) by $v \in \mathbf{Y}_n^s$ and integrate over (s,T). Convergences (4.24a) then allow us to pass to the limit $k \to \infty$ for $n \in \mathbb{N}$ fixed:

$$\int_s^T \Big(\langle \varrho \ddot{u}_k, v \rangle_{\mathbf{V}} + \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathbb{C}e(u_k) + \mathbb{D}e(\dot{u}_k) \big) : e(v) \,\mathrm{d}x \Big) \,\mathrm{d}t = \int_s^T \langle \mathbf{f}, v \rangle_{\mathbf{V}} \,\mathrm{d}t$$
$$\downarrow$$
$$\int_s^T \Big(\langle \varrho \mu_s, v \rangle_{\mathbf{V}} + \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathbb{C}e(u) + \mathbb{D}e(\dot{u}) \big) : e(v) \,\mathrm{d}x \Big) \,\mathrm{d}t = \int_s^T \langle \mathbf{f}, v \rangle_{\mathbf{V}} \,\mathrm{d}t \quad (4.37)$$

for all $v \in \mathbf{Y}_n^s$.

We now obtain the (integrated) brittle momentum balance, first with test functions in $L^2(s,T;\mathbf{V}_z(s))$. Indeed, let $v \in L^2(s,T;\mathbf{V}_z(s))$ be fixed, and let $(v_n)_n \subset \mathbf{Y}_n^s$ be the corresponding recovery sequence given in (4.21). Taking into account that $v_n \to v$ in $L^2(0,T;H^1(\Omega\backslash\Gamma_{\mathrm{C}},\mathbb{R}^d))$ as $n \to \infty$, we pass to the limit with n in (4.37) and finally obtain

$$\int_s^T \Big(\langle \varrho \mu_s, v \rangle_{\mathbf{V}} + \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathbb{C}e(u) + \mathbb{D}e(\dot{u}) \big) : e(v) \,\mathrm{d}x \Big) \,\mathrm{d}t = \int_s^T \langle \mathbf{f}, v \rangle_{\mathbf{V}} \,\mathrm{d}t$$

for every $v \in L^2(s,T;\mathbf{V}_z(s))$. Again, choosing test functions v of the form $v(t,x) = \eta(t)\mathsf{v}(x)$ with $\eta \in C_0^\infty(s,T)$ and $\mathsf{v} \in \mathbf{V}_z(s)$ we obtain (4.24d). This concludes the proof of Item 1.

On Item 2. Compactness independent of $s \in [0,T]$: First of all, we apply the previously proven Item 1 for $s=0$ and find a not relabeled subsequence $(u_k,\dot{u}_k,\ddot{u}_k)_k$, and $\mu := \mu_0 \in L^2(0,T;\mathbf{V}_z(0)^*)$ such that (4.24a)–(4.24d) hold for μ and $s=0$. We now apply it for $\bar{s}$ in the countable dense set D, and find a further subsequence, and $\mu_{\bar{s}} \in L^2(\bar{s},T;\mathbf{V}_z(\bar{s})^*)$, fulfilling (4.24a)–(4.24d). Observe that, since $\bar{s} \geq 0$, there holds $\mathbf{V}_z(0) \subset \mathbf{V}_z(\bar{s})$. Therefore, from (4.24d) at $s=0$ and at $s=\bar{s}$ we read

$$\langle \varrho \mu_{\bar{s}}(t), \mathsf{v} \rangle_{\mathbf{V}_z(\bar{s})} = \langle \mathbf{f}(t), \mathsf{v} \rangle_{\mathbf{V}} - \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathbb{C}e(u(t)) + \mathbb{D}e(\dot{u}(t)) \big) : e(\mathsf{v}) \,\mathrm{d}x = \langle \mu_0(t), \mathsf{v} \rangle_{\mathbf{V}_z(0)}$$
$$\text{for every } \mathsf{v} \in \mathbf{V}_z(0) \quad \text{for a.a. } t \in (\bar{s},T),$$

whence $\mu_{\bar{s}}(t) = \mu_0(t) = \mu(t)$ in $\mathbf{V}_z(0)^*$ for a.a. $t \in (\bar{s},T)$. Item 2 follows from a diagonal procedure.

On Item 3. Brittle momentum balance in $\mathbf{V}_z(t)$ for a.a. $t \in (0,T)$: From Item 1 it follows that for every $s \in D \cup \{0\}$ there exists a set N_s with zero Lebesgue measure, such that for all $t \in [s,T] \setminus N_s$ there holds

$$\begin{gathered} \mu(t) \in \mathbf{V}_z(s)^*, \ \tfrac{\dot{u}(t+h)-\dot{u}(t)}{h} \rightharpoonup \mu(t) \text{ in } \mathbf{V}_z(s)^* \text{ as } h \downarrow 0, \\ \mu(t) \text{ fulfills (4.24d) in } \mathbf{V}_z(s)^*. \end{gathered} \qquad (4.38)$$

In order to show (4.26a), we shall now adapt an argument from the proof of [14, Lemma 2.2]. Indeed, set $N := \cup_{s \in D \cup \{0\}} N_s$, with N_s the negligible set out of which

(4.38) holds. Then, N is also negligible and for every $t \in [0,T] \setminus N$ properties (4.38) hold at every $s \in D \cup \{0\}$ with $s < t$. Now, to the monotonically increasing family of closed sets $(\mathbf{V}_z(t))_{t\in[0,T]}$ we apply (4.28). Hence, let us fix $t \in [0,T] \setminus (S \cup N)$, and let us pick an increasing sequence $(s_m)_m \subset D$ with $s_m \uparrow t$. Due to (4.28), for every $\phi \in \mathbf{V}_z(t)$ there exists a sequence $(\phi_m)_m$, with $\phi_m \in \mathbf{V}_z(s_m)$ for every $m \in \mathbb{N}$, such that $\phi_m \to \phi$ in $\mathbf{V}_z(t)$. Observe that, in particular, $\mu(t)$ fulfills (4.24d) with the test functions ϕ_m for all $m \in \mathbb{N}$. Therefore,

$$
\begin{aligned}
&\exists \lim_{m\to\infty} \langle \varrho\mu(t), \phi_m \rangle_{\mathbf{V}_z(s_m)} \\
&= \lim_{m\to\infty} \Big(\langle \mathbf{f}(t), \phi_m \rangle_{\mathbf{V}} - \int_{\Omega\setminus\Gamma_{\mathrm{C}}} \big(\mathbb{C}e(u(t)) + \mathbb{D}e(\dot{u}(t)) \big) : e(\phi_m) \,\mathrm{d}x \Big) \\
&= \langle \mathbf{f}(t), \phi \rangle_{\mathbf{V}} - \int_{\Omega\setminus\Gamma_{\mathrm{C}}} \big(\mathbb{C}e(u(t)) + \mathbb{D}e(\dot{u}(t)) \big) : e(\phi) \,\mathrm{d}x \,.
\end{aligned}
\tag{4.39}
$$

Since $\phi \in \mathbf{V}_z(t)$ is arbitrary, the right-hand side of (4.39) defines an element in $\mathbf{V}_z(t)^*$, which, for the time being, we denote by $\tilde{\mu}(t)$. Observe that, in fact

$$
\begin{aligned}
&\langle \varrho\tilde{\mu}(t), \phi \rangle_{\mathbf{V}_z(t)} = \lim_{m\to\infty} \langle \varrho\ddot{u}(t), \phi_m \rangle_{\mathbf{V}_z(s_m)} \\
&\text{for } \textit{every } (\phi_m)_m \subset \cup_{s<t} \mathbf{V}_z(s) \text{ with } \phi_m \to \phi \text{ in } \mathbf{V}_z(t),
\end{aligned}
\tag{4.40}
$$

whence

$$
\langle \varrho\tilde{\mu}(t), \phi \rangle_{\mathbf{V}_z(t)} = \langle \varrho\mu(t), \phi \rangle_{\mathbf{V}_z(s)} \qquad \text{for all } \phi \in \mathbf{V}_z(s) \text{ and all } s < t,
$$

choosing the constant sequence $\phi_m \equiv \phi$ in (4.40). Repeating the very same argument as in the proof of [14, Lemma 2.2], we may in fact check that, for $t \in [0,T] \setminus (S \cup N)$ fixed, (4.26b) holds, which ultimately entitles us to denote $\mu(t)$ by $\ddot{u}(t)$, cf. (2.27). Clearly, (4.39) yields that $\ddot{u}(t)$ satisfies the brittle momentum balance, with test functions in $\mathbf{V}_z(t)$, for all $t \in [0,T] \setminus (S \cup N)$. This gives (2.40b). Furthermore, again using (4.40) we may extend (4.24b) to test functions v in $\mathbf{V}_z(t)$, namely we conclude (4.26c) for every $h \in (0,T)$.

A comparison argument in (2.40b), taking into account that $u \in H^1(0,T;\mathbf{V})$ and that $\mathbf{f} \in L^2(0,T;\mathbf{V}^*)$, shows that the map $t \mapsto \sup_{v\in\mathbf{V}_z(t)} |\langle \ddot{u}(t), v \rangle_{\mathbf{V}_z(t)}| \doteq \|\mu(t)\|_{\mathbf{V}_z(t)^*}$ is in $L^2(0,T)$, whence $\ddot{u} \in L^2(0,T;\mathbf{V}_z^*)$. Thus, $u \in H^2_{\#}(0,T;\mathbf{V}_z^*)$, cf. (2.26).

To find relation (4.27) at time $t \in (0,T)$ out of a negligible set, we test both (2.40) and (2.51) with arbitrary $v \in \mathbf{V}_z(t)$. Subtracting the two equations from each other yields $\langle \varrho\ddot{u}(t) - \lambda(t), v \rangle_{\mathbf{V}_z(t)} = 0$. This concludes the proof of Item 3. □

By exploiting the validity of the brittle momentum balance (2.40) and relation (4.26b) we now deduce the **3. additional regularity** (2.45) of the limit $\ddot{u}$.

Lemma 4.8 (Regularity (2.45) **of the limit** $\ddot{u}$**).** *There holds*

$$
\tfrac{\dot{u}(\cdot+h)-\dot{u}(\cdot)}{h} \to \ddot{u} \text{ strongly in } L^2(0,T;\mathbf{V}_z(0)^*),
\tag{4.41}
$$

therefore $u \in H^2(0,T;\mathbf{V}_z(0)^*)$.

Proof. In order to show (4.41), we will check that for every sequence $(h_n)_n$ with $h_n \to 0$

$$\left(\tfrac{\dot u(\cdot+h_n)-\dot u(\cdot)}{h_n}\right)_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } L^2(0,T;\mathbf{V}_z(0)^*). \tag{4.42}$$

To this aim, we observe that for almost all $t\in(0,T)$ and all $v \in \mathbf{V}_z(0)$

$$\begin{aligned}
&\langle \tfrac{\dot u(\cdot+h_n)-\dot u(\cdot)}{h_n}, v\rangle_{\mathbf{V}_z(0)} \\
&\overset{(1)}{=} \tfrac{1}{h_n}\langle \int_t^{t+h_n} \ddot u(\tau)\,\mathrm{d}\tau, v\rangle_{\mathbf{V}_z(0)} \\
&\overset{(2)}{=} \tfrac{1}{\varrho h_n}\Big(\langle \int_t^{t+h_n} \mathbf{f}(\tau), v\rangle_{\mathbf{V}} - \int_t^{t+h_n}\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathbb{C}e(u(\tau)) : e(v)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad - \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathbb{D}e(u(t+h_n)) : e(v)\,\mathrm{d}x + \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathbb{D}e(u(t)) : e(v)\,\mathrm{d}x\Big),
\end{aligned}$$

where (1) follows from (4.26c) with test functions v in $\mathbf{V}_z(0)\subset\mathbf{V}_z(t)$, and (2) ensues from the momentum balance (2.40b). Taking into account that $\mathbf{f}\in \mathrm{C}^1([0,T];\mathbf{V}^*)$ and that $u\in H^1(0,T;\mathbf{V})$, from the above identity we conclude (4.42). Therefore, there exists $w\in L^2(0,T;\mathbf{V}_z(0)^*)$ such that $\frac{\dot u(\cdot+h_n)-\dot u(\cdot)}{h_n}\to w$ in $L^2(0,T;\mathbf{V}_z(0)^*)$, whence, up to a subsequence, $\frac{\dot u(t+h_n)-\dot u(t)}{h_n}\to w(t)$ in $\mathbf{V}_z(0)^*$ for almost all $t\in(0,T)$. Taking into account (4.26b), we ultimately conclude that $w(t)=\ddot u(t)$ in $\mathbf{V}_z(0)^*$, and (4.41) ensues. □

Since regularity (2.45), i.e., $u\in H^2(0,T;\mathbf{V}_z(0)^*)$, holds true and since the spaces $\mathbf{V}_z(0)\subset\mathbf{W}\subset\mathbf{V}_z(0)^*$ form a Gelfand triple, we in fact have that $\dot u\in L^\infty(0,T;\mathbf{W})\cap \mathrm{C}^0([0,T];\mathbf{V}_z(0)^*)$. Therefore, we are now in the position to deduce the **4. weak temporal continuity of** $\dot u$ as a corollary of (2.45).

Corollary 4.1 (Weak continuity of $\dot u$, Theorem 2.2, Item 4). *We have*

$$\dot u(t)\in\mathbf{W} \text{ and } \|\dot u(t)\|_{\mathbf{W}}\le C \quad \text{for every } t\in[0,T], \tag{4.43}$$

$$t\mapsto \dot u(t) \text{ is weakly continuous from } [0,T] \text{ to } \mathbf{W}. \tag{4.44}$$

Proof. Given $t\in[0,T]$, using that $\dot u\in L^\infty(0,T;\mathbf{W})$ we find that there exists a sequence $(t_n)_n\subset[0,T]$ with $t_n\to t$, such that $\|\dot u(t_n)\|_{\mathbf{W}}\le C$. Since $\dot u(t_n)\to\dot u(t)$ in $\mathbf{V}_z(0)^*$ we conclude from the continuous embedding $\mathbf{W}\hookrightarrow\mathbf{V}_z(0)^*$ that also $\dot u(t)\in\mathbf{W}$ with $\|\dot u(t)\|_{\mathbf{W}}\le C$ and $\dot u(t_n)\rightharpoonup\dot u(t)$ in $\mathbf{W}$. This proves (4.43). The same argument with arbitrary $t,(t_n)_n\subset[0,T]$ such that $t_n\to t$ yields (4.44). □

Observe that, by (2.41) and (2.42a), the limit displacement u satisfies the initial condition $u(0)=u_0$ in $\mathbf{V}$. It remains to verify that $\dot u(0)=u_1$ in $\mathbf{W}$. For this, we will prove the pointwise-in-time weak $\mathbf{W}$-convergence of $\dot u_k(t)$ to $\dot u(t)$, cf. (4.45) below. In fact, in view of convergences (2.42), (4.45) is also the missing piece for passing to the limit in the energy balance as an inequality, via lower semicontinuity arguments, in Sec. 4.5.

Lemma 4.9 (Pointwise-in-time weak L^2-convergence & initial condition $\dot u(0) = u_1$). *Along the same sequence as in Lemma 4.7, Item 1, it holds*

$$\dot u_k(t) \rightharpoonup \dot u(t) \quad \text{in } \mathbf{W} \quad \text{for every } t \in [0,T], \tag{4.45}$$

therefore $\dot u(0) = u_1$ *thanks to* (2.41).

Proof. It follows from (4.25b), for $s = 0$, and from the previously obtained estimates, that the sequence $(\dot u_k)_k$ is bounded in $L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{W}) \cap H^1(0,T;\mathbf{V}_z(\varepsilon_n,0)^*)$ for every $n \in \mathbb{N}$. Since for each n the space $\mathbf{V}_z(\varepsilon_n,0)$ is densely and compactly embedded in $\mathbf{W}$, we have that $\mathbf{W} \subset \mathbf{V}_z(\varepsilon_n,0)^*$ densely and compactly. By a Aubin-Lions compactness argument (cf. e.g. [57, Cor. 5, p. 86]), we conclude

$$\dot u_k \to \dot u \quad \text{in } L^p(0,T;\mathbf{W}) \cap \mathrm{C}^0([0,T];\mathbf{V}_z(\varepsilon_n,0)^*) \tag{4.46}$$

for some fixed $n \in \mathbb{N}$. Ultimately, we infer convergence (4.45): indeed, for every $t \in [0,T]$, every subsequence of the sequence $(\dot u_k(t))_k$, bounded in $\mathbf{W}$, admits a further subsequence weakly converging in $\mathbf{W}$ to some limit v_t. In view of (4.46), we have that $v_t = \dot u(t)$: since the limit does not depend on the extracted subsequence, convergence (4.45) holds. □

4.5. *Limit passage in the energy balance & Proof of Thm. 2.2, Items 5.-8.*

We deduce the energy balance (2.46) for the brittle limit system. First, in Lemma 4.10, we will obtain the inequality $\leq$, *at all* $t \in [0,T]$, by suitable lower semicontinuity arguments, cf. (4.47) below.

Lemma 4.10 (Upper energy-dissipation estimate via lower semicontinuity). *We have*

$$\begin{aligned} \tfrac{1}{2}\|\dot u(t)\|_{\mathbf{W}}^2 + \int_0^t 2\mathcal{V}(\dot u(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_\infty}(z,[0,t]) + \mathcal{E}_\infty(t,u(t),z(t)) \\ \leq \tfrac{1}{2}\|\dot u(0)\|_{\mathbf{W}}^2 + \mathcal{E}_\infty(0,u(0),z(0)) + \int_0^t \partial_t\mathcal{E}_\infty(s,u(s),z(s))\,\mathrm{d}s \end{aligned} \tag{4.47}$$

for all $t \in [0,T]$.

Proof. Inequality (4.47) follows by passing to the limit as $k \to \infty$ in the energy-dissipation inequality for the adhesive system. On the left-hand side, we exploit convergences (2.42), which give $\int_0^t \mathcal{V}(\dot u(s))\,\mathrm{d}s \leq \liminf_{k\to\infty} \int_0^t \mathcal{V}(\dot u_k(s))\,\mathrm{d}s$ and $\mathrm{Var}_{\mathcal{R}_\infty}(z,[0,t]) \leq \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{R}_k}(z_k,[0,t])$. Furthermore, the pointwise convergences for u and z in (2.42) give $\mathcal{E}_\infty(t,u(t),z(t)) \leq \liminf_{k\to\infty} \mathcal{E}_k(t,u_k(t),z_k(t))$ via (4.12a), and the limit passage in the term $\frac{1}{2}\|\dot u_k(t)\|_{\mathbf{W}}^2$ is guaranteed by (4.45). On the right-hand side, we use the convergence for the initial data (2.41) and again (2.42), which allows us to pass to the limit in $\int_0^t \partial_t\mathcal{E}_k(s,u_k(s),z_k(s))\,\mathrm{d}s$. □

The energy-dissipation inequality opposite to (4.47) will be proved in Lemma 4.12 ahead. For this we will have to test the brittle momentum balance (2.40) by $\dot{u}$. Note that this is admissible since also $\dot{u}$ satisfies the brittle constraint (2.43). Also observe that the quadratic bulk term and the external loading term fulfill a chain rule. The missing piece is thus a chain-rule inequality involving the kinetic term $\varrho\ddot{u}$ and the Gelfand triple $(\mathbf{V}_z(t), \mathbf{W}, \mathbf{V}_z(t)^*)$, established now in Lemma 4.11 (cf. (4.48) and (4.49) ahead). For the proof of Lemma 4.11, Item 1, we adapt the arguments from [14, Lemma 3.5]. Lemma 4.11, Item 2, can then be concluded following the lines of [14, Lemma 3.6], exploiting the weak continuity of $\dot{u}$ proved in Lemma 4.1.

Lemma 4.11 (Chain rule for the inertial term). *Let $u \in H^2(0,T;\mathbf{V}_z(0)^*)$ comply with the regularity properties* (2.38) *&* (4.44)*, and with the brittle momentum balance for given $z \in \mathrm{BV}(0,T;L^1(\Gamma_{\mathrm{C}})) \cap \mathrm{B}([0,T];\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))$, semistable as in* (2.4) *for all $t \in [0,T]$. Then:*

(1) for all $s,t \in (0,T]$ such that s and t are Lebesgue points for $\|\dot{u}(\cdot)\|^2_{\mathbf{W}}$ there holds:

$$\tfrac{1}{2}\|\dot{u}(t)\|^2_{\mathbf{W}} - \tfrac{1}{2}\|\dot{u}(s)\|^2_{\mathbf{W}} = \int_s^t \langle \varrho\ddot{u}(\tau), \dot{u}(\tau)\rangle_{\mathbf{V}_z(\tau)}\,\mathrm{d}\tau\,, \tag{4.48}$$

(2) $\dot{u}$ fulfills the integral chain-rule inequality

$$\tfrac{1}{2}\|\dot{u}(t)\|^2_{\mathbf{W}} - \tfrac{1}{2}\|u_1\|^2_{\mathbf{W}} \geq \int_0^t \langle \varrho\ddot{u}(\tau), \dot{u}(\tau)\rangle_{\mathbf{V}_z(\tau)}\,\mathrm{d}\tau \tag{4.49}$$

holds true for every Lebesgue point $t \in (0,T]$ of $\|\dot{u}(\cdot)\|^2_{\mathbf{W}}$.

Proof. We prove the two statements of the lemma separately.

On Item 1.: In order to prove (4.48) we adapt the argument from the proof of [14, Lemma 3.5]. A straightforward calculation shows that

$$\tfrac{1}{h}\left(\|\dot{u}(t)\|^2_{\mathbf{W}} - \|\dot{u}(t-h)\|^2_{\mathbf{W}}\right) = \langle \varrho\tfrac{\dot{u}(t)-\dot{u}(t-h)}{h}, \dot{u}(t)+\dot{u}(t-h)\rangle_{\mathbf{V}_z(t)} \tag{4.50}$$

for all $h \in (0,T)$ and for almost all $t \in (h,T)$. Integrating (4.50) on (s,t) yields for every $h \in (0,s)$

$$\begin{aligned} &\tfrac{1}{2h}\int_{t-h}^t \|\dot{u}(\tau)\|^2_{\mathbf{W}}\,\mathrm{d}\tau - \tfrac{1}{2h}\int_{s-h}^s \|\dot{u}(\tau)\|^2_{\mathbf{W}}\,\mathrm{d}\tau \\ &= \tfrac{1}{2}\int_s^t \langle \varrho\tfrac{\dot{u}(\tau)-\dot{u}(\tau-h)}{h}, \dot{u}(\tau)+\dot{u}(\tau-h)\rangle_{\mathbf{V}_z(\tau)}\,\mathrm{d}\tau\,. \end{aligned} \tag{4.51}$$

Now, since $\dot{u} \in L^2(0,T;\mathbf{V})$, there holds

$$\dot{u}(\tau)+\dot{u}(\tau-h) \to 2\dot{u}(\tau) \quad \text{in } \mathbf{V} \quad \text{for a.a. } \tau \in (0,T) \tag{4.52}$$

along a sequence $h \downarrow 0$. Combining this with the weak convergence $\frac{\dot{u}(\tau)-\dot{u}(\tau-h)}{h} \rightharpoonup \ddot{u}(\tau)$ in $\mathbf{V}_z(\tau)^*$ for almost all $\tau \in (0,T)$, we infer the pointwise convergence

$$a_h(\tau) := \langle \varrho\tfrac{\dot{u}(\tau)-\dot{u}(\tau-h)}{h}, \dot{u}(\tau)+\dot{u}(\tau-h)\rangle_{\mathbf{V}_z(\tau)} \to 2\,\langle \varrho\ddot{u}(\tau), \dot{u}(\tau)\rangle_{\mathbf{V}_z(\tau)} \tag{4.53}$$

for a.a. $\tau \in (0,T)$. In order to pass to the limit in the integral on the right-hand side of (4.51), we shall apply a variant of the dominated convergence theorem, cf. e.g. [20, Thm. 5.3, p. 261]. For this, we further observe that

$$\begin{aligned}|a_h(\tau)| &\leq \|\varrho\tfrac{\dot u(\tau)-\dot u(\tau-h)}{h}\|_{\mathbf{V}_z(\tau)^*}\Big(\|\dot u(\tau)\|_{\mathbf{V}_z(\tau)}+\|\dot u(\tau-h)\|_{\mathbf{V}_z(\tau)}\Big)\\ &= \|\varrho\tfrac{\dot u(\tau)-\dot u(\tau-h)}{h}\|_{\mathbf{V}_z(\tau)^*}\Big(\|\dot u(\tau)\|_{\mathbf{V}}+\|\dot u(\tau-h)\|_{\mathbf{V}}\Big) =: M_h(\tau)\,.\end{aligned}$$

We now introduce the short-hands $l_h(\tau) := \|\varrho\frac{\dot u(\tau)-\dot u(\tau-h)}{h}\|_{\mathbf{V}_z(\tau)^*}$ and $m_h(\tau) := \big(\|\dot u(\tau)\|_{\mathbf{V}}+\|\dot u(\tau-h)\|_{\mathbf{V}}\big)$, so that $M_h(\tau)=l_h(\tau)m_h(\tau)$. Since $\dot u\in L^2(0,T;\mathbf{V})$ we have

$$m_h \to 2\|\dot u\|_{\mathbf{V}} \quad \text{in } L^2(0,T)\,, \tag{4.54}$$

while, thanks to (4.26c), we find for l_h that

$$\begin{aligned}&l_h(\tau)=\|\tfrac1h\int_{\tau-h}^{\tau}g(s)\,\mathrm{d}s\|_{\mathbf{V}_z(\tau)^*}=\|\tfrac1h\int_{\tau-h}^{\tau}g(s)\,\mathrm{d}s\|_{\mathbf{V}^*} \text{ with } g:(0,T)\to\mathbf{V}^* \text{ given by}\\ &\langle g(s),v\rangle_{\mathbf{V}}=\langle \mathbf{f}(s),v\rangle_{\mathbf{V}}-\int_{\Omega\backslash\Gamma_{\mathrm{C}}}(\mathbb{D}e(\dot u(s))\colon e(v)+\mathbb{C}e(u(s))\colon e(v))\,\mathrm{d}x\,.\end{aligned}$$

Now, since $g\in L^2(0,T;\mathbf{V}^*)$, the sequence of functions $\tau\mapsto\frac1h\int_{\tau-h}^{\tau}g(s)\,\mathrm{d}s$ converges to g in $L^2(0,T;\mathbf{V}^*)$. Therefore, $(l_h)_h$ converges to $\|g(\cdot)\|_{\mathbf{V}^*}$ in $L^2(0,T)$. Together with (4.54) we infer that $(M_h)_h\subset L^1(0,T)$ and $M_h\to 2\|g(\cdot)\|_{\mathbf{V}^*}\|\dot u(\cdot)\|_{\mathbf{V}}$ in $L^1(0,T)$. Now, the dominated convergence theorem, c.f. e.g. [20, Thm. 5.3, p. 261], allows us to conclude that $\int_s^t a_h(\tau)\,\mathrm{d}\tau\to\int_s^t 2\langle\varrho\ddot u(\tau),\dot u(\tau)\rangle_{\mathbf{V}_z(\tau)}\,\mathrm{d}\tau$, and thus the convergence of the right-hand side of (4.51). Additionally, the integrals on the left-hand side of (4.51) converge to the left-hand side of (4.48) since s and t are Lebesgue points for $\|\dot u(\cdot)\|^2_{\mathbf{W}}$. Thus, we conclude (4.48).

On Item 2.: The proof can be adapted from [14, Lemma 3.6]. More precisely, we choose a sequence of Lebesgue points $(t_j)_{j\in\mathbb{N}}$ of $\|\dot u(\cdot)\|^2_{\mathbf{W}}$ such that $t_j\searrow 0$. By Lemma 4.1, $\dot u:[0,T]\to\mathbf{W}$ is weakly continuous, i.e., we have $\dot u(t_j)\rightharpoonup\dot u(0)$ in $\mathbf{W}$. Moreover, for all Lebesgue points t_j and t the chain rule (4.48) holds true. Hence, we find:

$$\begin{aligned}\int_0^t\langle\varrho\ddot u(\tau),\dot u(\tau)\rangle_{\mathbf{V}_z(\tau)}\,\mathrm{d}\tau &= \liminf_{j\to\infty}\int_{t_j}^t\langle\varrho\ddot u(\tau),\dot u(\tau)\rangle_{\mathbf{V}_z(\tau)}\,\mathrm{d}\tau\\ &\leq \limsup_{j\to\infty}\tfrac12\big(\|\dot u(t)\|^2_{\mathbf{W}}-\|\dot u(t_j)\|^2_{\mathbf{W}}\big)\\ &\leq \tfrac12\|\dot u(t)\|^2_{\mathbf{W}}-\liminf_{j\to\infty}\tfrac12\|\dot u(t_j)\|^2_{\mathbf{W}}\\ &\leq \tfrac12\|\dot u(t)\|^2_{\mathbf{W}}-\tfrac12\|\dot u(0)\|^2_{\mathbf{W}}=\tfrac12\|\dot u(t)\|^2_{\mathbf{W}}-\tfrac12\|u_1\|^2_{\mathbf{W}}\,.\end{aligned}$$

Above, the last inequality follows from the lower semicontinuity of $\|\cdot\|^2_{\mathbf{W}}$ with respect to weak convergence in $L^2(\Omega)$ and the last equality is due to the initial condition $\dot u(0)=u_1$ verified in Lemma 4.9. □

By the chain rule (4.49) we are now in the position to prove the energy-dissipation inequality opposite to (4.47), i.e. (4.55) below. For this we will test the brittle momentum balance (2.40) by $\dot{u}$, apply chain rules separately to each of the energy terms, and combine the obtained relation with the brittle semistability condition, arguing as in the proof of the *adhesive* energy-dissipation balance (2.16).

Lemma 4.12 (Lower energy-dissipation estimate, balance (2.46)**).** *The limit pair* (u,z) *obtained by convergences* (2.42) *satisfies the following lower energy-dissipation estimate*

$$\begin{aligned} &\tfrac12\|\dot u(t)\|_{\mathbf{W}}^2+\int_s^t 2\mathcal{V}(\dot u(\tau))\,\mathrm{d}\tau+\mathrm{Var}_{\mathcal{R}_\infty}(z,[s,t])+\mathcal{E}_\infty(t,u(t),z(t))\\ &\qquad\geq \tfrac12\|\dot u(s)\|_{\mathbf{W}}^2+\mathcal{E}_\infty(s,u(s),z(s))+\int_s^t\partial_t\mathcal{E}_\infty(\tau,u(\tau),z(\tau))\,\mathrm{d}\tau \\ &\qquad\qquad \text{for all } s,t\in[0,T]\backslash\mathsf{L} \text{ with } s<t \text{ and for } s=0, \end{aligned} \tag{4.55}$$

where L *denotes the set of Lebesgue points of* $\|\dot u(\cdot)\|_{\mathbf{W}}^2$*. Hence, the energy-dissipation balance* (2.46) *holds true as well as the bulk energy-dissipation balance* (2.47) *and the surface energy-dissipation balance* (2.48)*.*

Proof. We test the momentum balance (2.40b) of the brittle limit system by $\dot{u}$, which is admissible according to (2.43). We argue as in the proof of the k-energy balance (2.16), i.e. using integration by parts on the loading term, and exploiting the analogues of (3.11) and (3.12) (since $u\in H^1(0,T;\mathbf{V})$), for the viscous and the bulk energy terms, respectively. For the inertial term, we use (4.48) and (4.49). Thus

$$\begin{aligned} &\tfrac12\|\dot u(t)\|_{\mathbf{W}}^2-\tfrac12\|\dot u(s)\|_{\mathbf{W}}^2\\ &+\langle-\mathbf{f}(t),u(t)\rangle_{\mathbf{V}}-\langle-\mathbf{f}(s),u(s)\rangle_{\mathbf{V}}-\int_s^t\langle-\dot{\mathbf{f}}(\tau),u(\tau)\rangle_{\mathbf{V}}\,\mathrm{d}\tau\\ &+\int_{\Omega\backslash\Gamma_{\mathrm{C}}}\tfrac12\big(\mathbb{C}e(u(t)):e(u(t))-\mathbb{C}e(u(s)):e(u(s))\big)\,\mathrm{d}x\\ &+\int_s^t\mathcal{V}(e(\dot u(\tau)))\,\mathrm{d}\tau\geq 0 \end{aligned} \tag{4.56}$$

for almost all $s,t\in(0,T)$ with $s<t$ and for $s=0$. The semistability inequality for the brittle limit at time s, tested with $\tilde z=z(t)$ for arbitrary $t\in[0,T]$ reduces to

$$\begin{aligned} &\mathcal{R}_\infty(z(t)-z(s))+\mathrm{b}_\infty P(Z(t),\Gamma_{\mathrm{C}})\\ &-a_\infty^0\int_{\Gamma_{\mathrm{C}}}z(t)\,\mathrm{d}\mathcal{H}^{d-1}-\mathrm{b}_\infty P(Z(s),\Gamma_{\mathrm{C}})+a_\infty^0\int_{\Gamma_{\mathrm{C}}}z(s)\,\mathrm{d}\mathcal{H}^{d-1}\geq 0\,. \end{aligned} \tag{4.57}$$

Summing up (4.56) and (4.57) results in (4.55) valid in Lebesgue points s,t of $\|u\|_{\mathbf{W}}^2$. Then, finally, the energy-dissipation balance (2.46) follows from combining (4.47) with (4.55).

Observe that (2.46) rewrites as $A + B = 0$, where A and B stand for the left-hand sides of inequalities (4.56) & (4.57), which in turn state $A \geq 0$ and $B \geq 0$. Therefore, as a by-product we have that $A = 0 = B$. In particular, from $A = 0$ it is immediate to conclude that the chain-rule inequality (4.49) also holds as an *equality* at the Lebesgue points of $\|\dot{u}(\cdot)\|^2_{\mathbf{W}}$. □

Exploiting energy-dissipation balance (2.46) for the brittle system, convergence (2.41) of the initial data, and convergences (2.42), we can now deduce the enhanced convergences (2.49).

Lemma 4.13 (Enhanced convergences (2.49)**).** *Convergences* (2.49) *hold true. Thus* (u, z) *fulfill the upper energy-dissipation estimate* (4.47) *on the interval* (s, t)*, for every* $t \in (0, T]$ *and a.a.* $s \in (0, t)$.

Proof. The previously proved convergences as well as (2.41) yield

$$
\begin{aligned}
&\tfrac{1}{2}\|\dot{u}(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{u}(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_\infty}(z,[0,t]) + \mathcal{E}_\infty(t,u(t),z(t)) \\
&\leq \liminf_{k\to\infty} \tfrac{1}{2}\|\dot{u}_k(t)\|^2_{\mathbf{W}} + \liminf_{k\to\infty} \int_0^t 2\mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s + \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{R}_k}(z_k,[0,t]) \\
&\quad + \liminf_{k\to\infty} \mathcal{E}_k(t,u_k(t),z_k(t)) \\
&\leq \limsup_{k\to\infty} \left(\tfrac{1}{2}\|\dot{u}_k(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_k}(z_k,[0,t]) + \mathcal{E}_k(t,u_k(t),z_k(t)) \right) \\
&= \lim_{k\to\infty} \tfrac{1}{2}\|u_1^k\|^2_{\mathbf{W}} + \lim_{k\to\infty} \mathcal{E}_k(0,u_0,z_0) + \lim_{k\to\infty} \int_0^t \partial_t\mathcal{E}_k(s,u_k(s),z_k(s))\,\mathrm{d}s \\
&= \tfrac{1}{2}\|u_1\|^2_{\mathbf{W}} + \mathcal{E}_\infty(0,u(0),z(0)) + \int_0^t \partial_t\mathcal{E}_\infty(s,u(s),z(s))\,\mathrm{d}s \\
&= \tfrac{1}{2}\|\dot{u}(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{u}(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_\infty}(z,[0,t]) + \mathcal{E}_\infty(t,u(t),z(t))
\end{aligned}
$$

where the last equality follows from the energy equality (2.46), at almost all $t \in (0, T)$. Hence, all inequalities turn out to hold as equalities, and convergences (2.49) ensue from a standard argument.

To obtain (4.47) for a.e. $s \in (0, t)$, as in the proof of Lemma 4.10 we pass to the limit as $k \to \infty$ in the upper energy-dissipation inequality for the adhesive system: we can now do so on the interval (s, t) for every $t \in (0, T]$ and for $s \in (0, t)$, out of a negligible set, such that convergences (2.49a) & (2.49d) hold for s. This concludes the proof. □

Next, we deduce the enhanced validity of the initial condition stated in Theorem 2.2, Item 7.

Lemma 4.14 (Enhanced initial condition (2.50)**).** *Let* $u \in \mathrm{C}^0_{\mathrm{weak}}([0, T]; \mathbf{W}) \cap H^2(0, T; \mathbf{V}_z(0)^*)$ *enjoy the regularity properties* (2.38)*, with the brittle momentum* *balance for given* $z \in \mathrm{B}(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})) \cap \mathrm{BV}(0,T;L^1(\Gamma_{\mathrm{C}}))$, *semistable as in* (2.4) *for all* $t \in [0,T]$, *and with the bulk energy balance* (2.47) *for the brittle system. Then,* (2.50) *holds true, and in particular, along any sequence of Lebesgue points* $(t_j)_j$ *of* $\|\dot{u}(\cdot)\|_{\mathbf{W}}$ *with* $t_j \to 0$ *it holds* $\dot{u}(t_j) \to u_1$ *strongly in* $\mathbf{W}$.

Proof. We adapt the arguments of [14, p. 10]: Thanks to (4.44) we have $\dot{u}(t_j) \rightharpoonup u_1$ in $\mathbf{W}$. Thus, in order to verify that $\dot{u}(t_j) \to u_1$ strongly in $\mathbf{W}$, it is sufficient to show that $\limsup_{j\to\infty} \|\dot{u}(t_j)\|^2_{\mathbf{W}} \le \|u_1\|^2_{\mathbf{W}}$. From the bulk energy-dissipation balance (2.47) we deduce

$$\begin{aligned}\tfrac{1}{2}\|\dot{u}(t_j)\|^2_{\mathbf{W}} \le \tfrac{1}{2}\|u_1\|^2_{\mathbf{W}} + \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}e(u_0) : e(u_0) - \tfrac{1}{2}\mathbb{C}e(u(t_j)) : e(u(t_j))\,\mathrm{d}x \\ +\langle \mathbf{f}(t_j), u(t_j)\rangle_{\mathbf{V}} - \langle \mathbf{f}(0), u_0\rangle_{\mathbf{V}} - \int_0^{t_j} \langle \dot{\mathbf{f}}(\tau), u(\tau)\rangle_{\mathbf{V}}\,\mathrm{d}\tau \\ \to \tfrac{1}{2}\|u_1\|^2_{\mathbf{W}} \quad \text{as } t_j \to 0\,.\end{aligned}$$

Here, the convergence of the terms on the right-hand side is due to the regularity property $u \in H^1(0,T;\mathbf{V})$, which ensures that $u(t_j) \to u_0$ strongly in $H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$, and to assumption (2.7) on $\mathbf{f}$. Hence, the desired lim sup inequality, and thus the enhanced initial condition (2.50), ensue. □

Thanks to the above proved enhanced validity of the initial condition we are now in the position to conclude the uniqueness result stated in Theorem 2.2, Item 8.

Lemma 4.15 (Uniqueness of the displacements for a given $z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))$). *The uniqueness of the displacements holds true in the sense of Theorem 2.2, Item 8.*

Proof. Suppose that (u,z) and $(\tilde{u},z)$ both are semistable energetic solutions of the evolutionary brittle system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_\infty,\mathcal{E}_\infty)$ and that they both satisfy the brittle momentum balance (2.40) with the same initial data u_0 and u_1. Then, $w := u - \tilde{u}$ fulfills (2.40) for almost all $t \in (0,T)$, with $\mathbf{f} = 0$ and $w(0) = \dot{w}(0) = 0$. For $t \in (0,T)$ fixed we test (2.40) $v(t) = \dot{w}(t)$, which is admissible since it satisfies $[\![\dot{w}(t)]\!] = 0$ a.e. on $\operatorname{supp} z(t)$. To treat the quadratic bulk terms and the external loading term resulting from this test we have suitable chain rules at our disposal, cf. also (4.56). It remains to verify a chain rule for the inertial term $\langle \ddot{w}(t), \dot{w}(t)\rangle_{\mathbf{W}}$.

For this, we use the information that both u and v satisfy the bulk energy-dissipation balance (2.47), and hence, the enhanced initial condition in the sense of the above Lemma 4.14. Thus, picking a sequence $(t_j)_j$, which are Lebesgue points for both functions $\|\dot{u}(\cdot)\|_{\mathbf{W}}$ and $\|\dot{v}(\cdot)\|_{\mathbf{W}}$, and which satisfies $t_j \to 0$ as $j \to \infty$, we conclude by Lemma 4.14 that also $\dot{w}(t_j) \to (u_1 - v_1) = 0$ strongly in $\mathbf{W}$. Moreover, observe that the chain rule equality (4.48) holds true also for w in all Lebesgue points s, t of $\|\dot{w}(\cdot)\|^2_{\mathbf{W}}$, since w solves the momentum balance. Thus, choosing $s = t_j$

in (4.48) and letting $j \to \infty$, the previously deduced strong convergence $\dot{w}(t_j) \to (u_1 - v_1) = 0$ in $\mathbf{W}$ now yields the chain rule with initial datum for w, namely $\frac{1}{2}\|\dot{w}(t)\|^2_{\mathbf{W}} - \frac{1}{2}\|\dot{w}(0)\|^2_{\mathbf{W}} = \int_0^t \langle \varrho \ddot{w}(\tau), \dot{w}(\tau)\rangle_{\mathbf{V}_z(\tau)} \,\mathrm{d}\tau$. Hence, by exploiting the chain rule for each of the terms in (2.3), we readily obtain for almost all $t \in (0,T)$

$$\tfrac{1}{2}\|\dot{w}(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{w}(s))\,\mathrm{d}s + \int_{\Omega\setminus\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}(e(w(t)) : e(w(t))\,\mathrm{d}x = 0\,.$$

This implies that each of the positive terms on the left-hand side has to be zero separately, which shows that $w \equiv 0$ and $\dot{w} \equiv 0$ a.e. in $[0,T]$. Then, $w \equiv 0$ *everywhere* in $[0,T]$. □

Appendix A. Appendix: Proof of Proposition 2.1

Let $(v_n)_n \subset \mathbf{V}_M(t)$ with $v_n \to v$ in $\mathbf{V}$. Taking into account that the jump operator $[\![\cdot]\!] : \mathbf{V} \to H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^d)$ is continuous by the trace theorem, one immediately checks that $[\![v]\!] = 0$ a.e. on the *closed* set $M(t)$. Thus $\mathbf{V}_M(t)$ is a closed, linear subspace of $\mathbf{V}$, whence its reflexivity and separability. Then, its dual $\mathbf{V}_M(t)^*$ is also reflexive and separable. Since $\mathbf{V}_M(t)$ is a closed subspace of $\mathbf{V}$, one of the corollaries of the Hahn-Banach theorem applies, yielding $\mathbf{V}_M(t)^*$ is isometrically isomorphic to the quotient space $\mathbf{V}^*/\mathbf{V}_M(t)^\perp$ through the operator $L : \mathbf{V}_M(t)^* \to \mathbf{V}^*/\mathbf{V}_M(t)^\perp$ which maps an element ξ of $\mathbf{V}_M(t)^*$ to $\xi + \mathbf{V}_M(t)^\perp$, where we denote by the same symbol the extension of ξ to $\mathbf{V}$.

The density of $\mathbf{V}_M(t)$ in the space $\mathbf{W}$ can be concluded from the fact that $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d) \subset \mathbf{V}_M(t)$ and $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$ is dense in $\mathbf{W}$ given that Ω is a Lipschitz domain.

Observe that $L^2(0,T;\mathbf{V}_M)$ is a closed subspace of $L^2(0,T;\mathbf{V})$: Given a sequence $(v_n)_n \subset L^2(0,T;\mathbf{V}_M)$ with $v_n \to v$ in $L^2(0,T;\mathbf{V})$, there holds (for a not relabeled subsequence) $v_n(t) \to v(t)$ in $\mathbf{V}$, whence $v(t) \in \mathbf{V}_M(t)$ for a.a. $t \in (0,T)$. Thus, $L^2(0,T;\mathbf{V}_M)$ inherits the reflexivity and separability of $L^2(0,T;\mathbf{V})$. Clearly, also its dual $L^2(0,T;\mathbf{V}_M)^*$ is reflexive and separable.

Finally, in order to verify the equivalence $L^2(0,T;\mathbf{V}_M)^* = L^2(0,T;\mathbf{V}_M^*)$ stated along with (2.31), we first observe that $L^2(0,T;\mathbf{V})^* \cong L^2(0,T;\mathbf{V}^*)$. We will now show that the annihilator of $L^2(0,T;\mathbf{V}_M)$, namely

$$\begin{aligned} &L^2(0,T;\mathbf{V}_M)^\perp \\ &= \Big\{\xi \in L^2(0,T;\mathbf{V})^* : \ \textstyle\int_0^T \langle \xi(t), v(t)\rangle_{\mathbf{V}}\ \mathrm{d}t = 0 \ \text{ for all } v \in L^2(0,T;\mathbf{V}_M)\Big\}, \end{aligned}$$

is isometrically isomorphic to

$$L^2(0,T;\mathbf{V}_M^\perp) := \big\{\xi \in L^2(0,T;\mathbf{V}^*) : \xi(t) \in \mathbf{V}_M(t)^\perp \text{ for a.a. } t \in (0,T)\big\}.$$

To this aim, we observe that, due to the monotonicity property (2.30), there holds

$$\mathbf{V}_M(t_1) \subset \mathbf{V}_M(t_2) \quad \text{and} \quad \mathbf{V}_M(t_2)^\perp \subset \mathbf{V}_M(t_1)^\perp \quad \text{for all } 0 \le t_1 \le t_2 \le T. \tag{A.1}$$

Then, we have that

$$\begin{aligned} &\xi \in L^2(0,T;\mathbf{V}_M)^\perp \\ &\ \Leftrightarrow\ \Big(\xi(t) \in L^2(0,T;\mathbf{V}^*)\ \&\ \forall\, \mathsf{v}\in\mathbf{V}_M(t):\ \langle \xi(t),\mathsf{v}\rangle_{\mathbf{V}} = 0 \text{ for a.a. } t\in(0,T)\Big). \end{aligned} \tag{A.2}$$

The right-to-left implication is obvious. As for the converse one, for all $\xi \in L^2(0,T;\mathbf{V}_M)^\perp$ there holds

$$\frac{1}{h}\int_t^{t+h} \langle \xi(s),\mathsf{v}\rangle_{\mathbf{V}}\,\mathrm{d}s = 0 \text{ for every } h>0,\ \mathsf{v}\in\mathbf{V}_M(t) \text{ and } t\in[0,T-h]. \tag{A.3}$$

This follows from choosing $v(t) := \frac{1}{h}\chi_{(t,t+h)}\mathsf{v}$, which satisfies $v \in L^2(0,T;\mathbf{V}_M)$ thanks to (A.1), in the identity $\int_0^T \langle \xi(t),v(t)\rangle_{\mathbf{V}}\,\mathrm{d}t = 0$ fulfilled by $\xi \in L^2(0,T;\mathbf{V}_M)^\perp$. Then, letting $h\downarrow 0$ in (A.3) yields $\xi(t)\in\mathbf{V}_M(t)^\perp$. Thus, the left-to-right implication holds true. Taking into account the representation of $L^2(0,T;\mathbf{V}_M)^\perp$ and the Hahn-Banach theorem we find that

$$\begin{aligned} L^2(0,T;\mathbf{V}_M)^* &\cong L^2(0,T;\mathbf{V})^*/L^2(0,T;\mathbf{V}_M)^\perp \\ &\cong L^2(0,T;\mathbf{V}^*)/L^2(0,T;\mathbf{V}_M^\perp) \cong L^2(0,T;\mathbf{V}_M^*)\,, \end{aligned}$$

which concludes the proof. ■

Acknowledgment

R.R. has been partially supported by a MIUR-PRIN'10-11 grant for the project "Calculus of Variations". M.T. has been partially supported by the Deutsche Forschungsgemeinschaft in the Priority Program 1748 "Reliable simulation techniques in solid mechanics. Development of non- standard discretization methods, mechanical and mathematical analysis" within the project "Finite element approximation of functions of bounded variation and application to models of damage, fracture, and plasticity". R.R. and M.T. are also grateful for the support from the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This research was carried out during several visits of R.R. at the Weierstrass Institute, supported by the European Research Council through the ERC Advanced Grant *Analysis of Multiscale Systems Driven by Functionals (267802)*, and of M.T. at the University of Brescia; the kind hospitality at both institutions is gratefully acknowledged.

References

1. L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems* (Oxford University Press, 2005).
2. L. Ambrosio and V.M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.* **43** (1990) 999–1036.

3. J.-F. Babadjian and V. Millot, Unilateral gradient flow of the Ambrosio-Tortorelli functional by minimizing movements, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31** (2014) 779–822.
4. B. Bourdin, G.A. Francfort, and J.-J. Marigo, *The variational approach to fracture* (Springer, New York, 2008). Reprinted from J. Elasticity **9** (2008).
5. B. Bourdin, C.J. Larsen, and C.L. Richardson, A time-discrete model for dynamic fracture based on crack regularization, *Int. J. Fracture* **168** (2011) 133–143.
6. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, (North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973).
7. S. Campanato, Proprietà di hölderianità di alcune classi di funzioni, *Ann. Scuola Norm. Sup. Pisa (3)* **17** (1963) 175–188.
8. S. Campanato, Proprietà di una famiglia di spazi funzionali, *Ann. Scuola Norm. Sup. Pisa (3)* **18** (1964) 137–160.
9. A. Chambolle, A density result in two-dimensional linearized elasticity, and applications, *Arch. Ration. Mech. Anal.* **167** (2003) 211–233.
10. G. Dal Maso and R. Toader, A model for the quasi-static growth of brittle fractures: existence and approximation results, *Arch. Ration. Mech. Anal.*, **162** (2002) 101–135.
11. G. Dal Maso, G. Francfort, and R. Toader, Quasistatic crack growth in nonlinear elasticity, *Arch. Rat. Mech. Anal.* **176** (2005) 165–225.
12. G. Dal Maso, A. DeSimone, and M.G. Mora, Quasistatic evolution problems for linearly elastic-perfectly plastic materials, *Arch. Rational Mech. Anal.* **180** (2006) 237–291.
13. G. Dal Maso and G. Lazzaroni, Quasistatic crack growth in finite elasticity with non-interpenetration, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010) 257–290.
14. G. Dal Maso and C.J. Larsen, Existence for wave equations on domains with arbitrary growing cracks, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **22** (2011) 387–408.
15. G. Dal Maso and I. Lucardesi, The wave equation on domains with cracks growing on a prescribed path: existence, uniqueness, and continuous dependence on the data, SISSA Preprint, Trieste (2015). *Appl. Math. Res. Express*, to appear.
16. G. Dal Maso, C.J. Larsen, and R. Toader, Existence for constrained dynamic griffith fracture with a weak maximal dissipation condition, *J. Mech. Phys. Solids* **95** (2016) 697–707.
17. G. Dal Maso, G. Lazzaroni, and L. Nardini, Existence and uniqueness of dynamic evolutions for a peeling test in dimension one, *J. Differential Equations* **261** (2016) 4897–4923.
18. J. Diestel and J. J. Uhl, Jr, *Vector measures*, (American Mathematical Society, Providence, R.I., 1977).
19. M. Egert, R. Haller-Dintelmann, and J. Rehberg, Hardy's inequality for functions vanishing on a part of the boundary, *Potential Anal.* **43** (2015) 49–78.
20. J. Elstrodt, *Maß- und Integrationstheorie*, (Springer-Lehrbuch. [Springer Textbook]. Springer-Verlag, Berlin, fourth edition, 2005).
21. I. Fonseca and G. Francfort, Relaxation in BV versus quasiconvexification in $W^{1,p}$; a model for the interaction between fracture and damage, *Calc. Var. Partial Differential Equations* **3** (1995) 407–446.
22. G.A. Francfort and C.J. Larsen, Existence and convergence for quasi-static evolution in brittle fracture, *Comm. Pure Appl. Math.* **56** (2003) 1465–1500.
23. G. A. Francfort and J.-J. Marigo, Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids* **46** (1998) 1319–1342.

24. M. Frémond and B. Nedjar, Damage, gradient of damage and principle of virtual power, *Int. J. Solids Struct.* **33** (1996) 1083–1103.
25. M. Frémond, *Non-Smooth Thermomechanics*, (Springer-Verlag Berlin Heidelberg, 2002).
26. M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, (Priceton University Press, Princeton, NJ, 1983).
27. A. Giacomini, Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fracture, *Calc. Var. Partial Differential Equations* **22** (2005) 129–172.
28. J.A. Griepentrog, Linear elliptic boundary value problems with non-smooth data: Campanato spaces of functionals, *Mathematische Nachrichten* **243** (2002) 19–42.
29. B. Halphen and Q.S. Nguyen, Sur les matériaux standards généralisés, *J. Mécanique* **14** (1975) 39–63.
30. D. Knees, A. Mielke, and C. Zanini, On the inviscid limit of a model for crack propagation, *Math. Models Methods Appl. Sci.* **18** (2008) 1529–1569.
31. A. Lalegname and A.-M. Sändig, Wave-crack interaction in finite elastic bodies, *Int. J. Fract.* **172** (2011) 131–149.
32. A. Lalegname, A.-M. Sändig, and G. Sewell, Analytical snd numerical treatment of a dynamic crack model, *Int. J. Fract.* **152** (2008) 97–125.
33. C.J. Larsen, C. Ortner, and E. Süli, Existence of solutions to a regularized model of dynamic fracture, *Math. Models Methods Appl. Sci.* **20** (2010) 1021–1048.
34. C.J. Larsen, Epsilon-stable quasi-static brittle fracture evolution, *Comm. Pure Appl. Math.* **63** (2010) 630–654.
35. G. Lazzaroni, R. Bargellini, P.-E. Dumouchel, and J.-J. Marigo, On the role of kinetic energy during unstable propagation in a heterogeneous peeling test. *Int. J. Fract.* **175** (2012) 127–150.
36. G. Lazzaroni, R. Rossi, M. Thomas, and R. Toader, Rate-independent damage in thermo-viscoelastic materials with inertia, WIAS Preprint 2025 (2014).
37. J. Lehrbäck, Pointwise Hardy inequalities and uniformly fat sets, *Proc. Amer. Math. Soc.* **136** (2008) 2193–2200.
38. J.L. Lewis, Uniformly fat sets, *Trans. Amer. Math. Soc.* **308** (1988) 177–196.
39. A. Mielke, Evolution in rate-independent systems (Ch.6), In C.M. Dafermos and E. Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, pages 461–559 (Elsevier B.V., Amsterdam, 2005).
40. A. Mielke, Differential, energetic and metric formulations for rate-independent processes. In *Nonlinear PDE's and applications*, volume 2028 of *Lecture Notes in Mathematics*, pages xiv+224 (Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2011).
41. A. Mielke, On evolutionary Γ-convergence for gradient systems. In *Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity*, eds. A. Muntean, J. Rademacher, A. Zagaris; proceedings of Summer School in Twente University (June 2012), Lecture Notes in Appl. Math. Mech., pages 187-249 (Springer, 2016).
42. A. Mielke and T. Roubíček, *Rate-independent Systems. Theory and Application*, (Springer, New York, 2015).
43. A. Mielke, T. Roubíček, and U. Stefanelli, Γ-limits and relaxations for rate-independent evolutionary problems, *Calc. Var. Partial Differential Equations* **31** (2008) 387–416.
44. A. Mielke, R. Rossi, and G. Savaré, BV solutions and viscosity approximations of rate-independent systems, *ESAIM Control Optim. Calc. Var.* **18** (2012) 36–80.
45. A. Mielke, T. Roubíček, and M. Thomas, From damage to delamination in nonlinearly elastic materials at small strains, *J. Elasticity* **109** (2012) 235–273.

46. A. Mielke and F. Theil, On rate–independent hysteresis models, *NoDEA Nonlinear Differential Equations Appl.* **11** (2004) 151–189.
47. S. Nicaise and A.-M. Sändig, Dynamic crack propagation in a 2d elastic body: The out-of-plane case, *J. Math. Anal. Appl* **329** (2007) 1–30.
48. W.F. Pfeffer, *Derivation and Integration*, (Cambridge University Press, Cambridge, 2001).
RR11. R. Rossi and T. Roubíček, Thermodynamics and analysis of rate-independent adhesive contact at small strains, *Nonlinear Anal.* **74** (2011) 3159–3190.
49. R. Rossi and M. Thomas, From an adhesive to a brittle delamination model in thermo-visco-elasticity, *ESAIM Control Optim. Calc. Var.* **21** (2015) 1–59.
50. R. Rossi and M. Thomas, Coupling rate-independent and rate-dependent processes: Existence results, *SIAM J. Math. Anal.*, 2017. In print.
51. T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, (Birkhäuser Verlag, Basel, 2005).
52. T. Roubíček, Rate-independent processes in viscous solids at small strains, *Math. Methods Appl. Sci.* **32** (2009) 825–862.
53. T. Roubíček, Thermodynamics of rate-independent processes in viscous solids at small strains, *SIAM J. Math. Anal.* **42** (2010) 256–297.
54. T. Roubíček, Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity, *SIAM J. Math. Anal.* **45** (2013) 101–126.
55. T. Roubíček, L. Scardia, and C. Zanini, Quasistatic delamination problem, *Continuum Mech. Thermodynam.* **21** (2009) 223–235.
56. T. Roubíček, M. Thomas, and C.G. Panagiotopoulos, Stress-driven local-solution approach to quasistatic brittle delamination, *Nonlinear Anal. Real World Appl.* **22** (2015) 645–663.
57. J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **46** (1987) 65–96.
58. M. Thomas, Uniform Poincaré-Sobolev and isoperimetric inequalities for classes of domains, *Discrete Contin. Dyn. Syst.* **35** (2015) 2741–2761.