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Global existence for a contact problem with adhesion

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SUMMARY

In this paper, we analyze a contact problem with irreversible adhesion between a viscoelastic body and a rigid support. On the basis of Frémond's theory, we detail the derivation of the model and of the resulting partial differential equation system. Hence, we prove the existence of global in time solutions (to a suitable variational formulation) of the related Cauchy problem by means of an approximation procedure, combined with monotonicity and compactness tools, and with a prolongation argument. In fact the approximate problem (for which we prove a local well-posedness result) models a contact phenomenon in which the occurrence of repulsive dynamics is allowed for. We also show local uniqueness of the solutions, and a continuous dependence result under some additional assumptions. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: contact; adhesion; irreversibility; existence; uniqueness

1. INTRODUCTION

In this paper, we aim to investigate the phenomenon of contact with adhesion between a viscoelastic body and a rigid support. To this aim, we refer to the modeling approach proposed by Frémond, see [1, Chapter 14]. Such an approach stems from this key remark: although the basic unilateral contact theory does not allow for any resistance to tension, in the so-called adhesion phenomenon resistance to tension is due to microscopic bonds between the surfaces of the solids which are in contact. Thus, in order to provide an efficient and predictive theory of the contact with adhesion, the description of the contact surface has to include a description of the state of the bonds between the two solids. What is more, one has to take into account that the breach of the contact results from microscopic motions counteracting such bonds.

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As a matter of fact, Frémond's model has been obtained combining the damage theory in continuum mechanics (cf. e.g. [1, Chapter 12; 2, 3]), with the theory for unilateral contact (see [1, Chapter 14]). Moreover, since the phenomenon is investigated on a macroscopic level (in view of engineering applications), the model features a macroscopic phase parameter χ, characterizing the level of damage of the micro-bonds. The evolution of χ is then described in terms of the phase field theory.

Now, let us introduce the model and derive a corresponding initial and boundary value problem. We investigate the mechanical behavior of a viscoelastic body, located in a smooth bounded domain $\Omega\subset\mathbb{R}^3$, and lying on a rigid support on a part Γ_c of its boundary Γ, in a time interval $(0,T)$.

Derivation of the model: The state variables of the model, describing the mechanical equilibrium of the system, are the symmetric linearized strain tensor $\varepsilon(\mathbf{u})$, defined in $\Omega\times(0,T)$ by

$$\varepsilon_{ij}(\mathbf{u})=\tfrac{1}{2}(\partial_{x_i}u_j+\partial_{x_j}u_i)$$

where $i,j=1,2,3$ and $\mathbf{u}=(u_1,u_2,u_3)$ is the vector of small displacements, and the damage parameter χ of the bonds acting on the contact surface Γ_c. In fact, χ is defined on $\Gamma_c\times(0,T)$ and denotes the fraction of active or unbroken bonds between the solids. In particular, when χ satisfies the constraint $\chi\in[0,1]$, the value $\chi=0$ corresponds to the case of completely broken bonds (i.e. when there is no adhesion), $\chi=1$ to unbroken bonds (active contact), and $\chi\in(0,1)$ to an intermediate situation. Then, we also introduce as a state variable the surface gradient $\nabla_{\mathrm{s}}\chi$ (∇_{s} stands for the gradient suitably defined on Γ_c), describing the local interactions on Γ_c between the points of the surface. For the sake of simplicity, we do not take into account any thermal effects. Hence, the dissipative variables responsible for the evolution of the adhesive contact are the time derivatives $\varepsilon(\mathbf{u}_t)$ and χ_t. In addition, we let $\Gamma=\overline{\Gamma_1}\cup\overline{\Gamma_2}\cup\overline{\Gamma_c}$, where Γ_1, Γ_2, and Γ_c are open subsets in the relative topology of Γ, each of them with a smooth boundary and disjoint one from another; further, we assume that the contact surface Γ_c and the region Γ_1 have strictly positive measure. We let $\mathbf{u}=\mathbf{0}$ on $\Gamma_1\times(0,T)$, while a known traction is applied on $\Gamma_2\times(0,T)$.

As far as the equations of the model are concerned, we follow the idea by Frémond of taking into account in the principle of virtual power the power of the microscopic forces, which create and break bonds. In particular, assuming that the only macroscopic variable related to the microscopic velocities is χ_t, we let the power of interior (microscopic) forces depend on χ_t. Thus, for all times $t\in[0,T]$ the virtual power of the internal forces, written for any virtual macroscopic velocity $\mathbf{v}$ with $\mathbf{v}=\mathbf{0}$ on Γ_1 and for any virtual microscopic velocity γ, reads

$$\mathscr{P}_{\mathrm{int}}(\mathbf{v},\gamma)=-\int_\Omega \sigma:\varepsilon(\mathbf{v})\,\mathrm{d}\Omega-\int_{\Gamma_c}\mathbf{R}\cdot\mathbf{v}\,\mathrm{d}\Gamma-\int_{\Gamma_c}(B\gamma+\mathbf{H}\cdot\nabla_{\mathrm{s}}\gamma)\,\mathrm{d}\Gamma \quad (1)$$

where σ is the stress tensor, $\mathbf{R}$ is the interaction force between the body and the support, while B and $\mathbf{H}$ are interior microscopic forces acting on the contact surface Γ_c.

Next, the virtual power of the exterior forces is

$$\mathscr{P}_{\mathrm{est}}(\mathbf{v},\gamma)=\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\Omega+\int_{\Gamma_2}\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\Gamma+\int_{\Gamma_c}A\gamma\,\mathrm{d}\Gamma+\int_{\partial\Gamma_c}a\gamma\,\mathrm{d}(\partial\Gamma) \quad (2)$$

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where **f** and **g** are, respectively, the volume and the surface forces applied to the body, while A and a are, resp., the surface and the line sources of microscopic work which can break the bonds. For the latter, one may think of chemical and optical actions (indeed, it is known that light damages the glue). Hence, neglecting any acceleration effect, the principle of virtual power (see [4]) leads to the following partial differential equation (PDE) system

$$-\operatorname{div}\sigma = \mathbf{f} \quad \text{in } \Omega\times(0,T) \tag{3}$$

$$\mathbf{u}=\mathbf{0} \quad \text{in } \Gamma_1\times(0,T), \qquad \sigma\mathbf{n}=\mathbf{g} \quad \text{in } \Gamma_2\times(0,T) \tag{4}$$

$$\sigma\mathbf{n}+\mathbf{R}=\mathbf{0} \quad \text{in } \Gamma_c\times(0,T) \tag{5}$$

$$B-\operatorname{div}_{\mathrm{s}}\mathbf{H}=A \quad \text{in } \Gamma_c\times(0,T), \qquad \mathbf{H}\cdot\mathbf{n}_{\mathrm{s}}=a \quad \text{in } \partial\Gamma_c\times(0,T) \tag{6}$$

Here, $\operatorname{div}_{\mathrm{s}}$ stands for the divergence operator on Γ_c, $\mathbf{n}_{\mathrm{s}}$ is the outward unit normal vector to the boundary $\partial\Gamma_c$, and $\mathbf{n}$ the outward unit normal vector to the boundary Γ.

In order to introduce the constitutive laws for the mechanical quantities involved in (3)–(6), we need to specify the free energy and the pseudo-potential of dissipation (cf. [1,5]). The volume part Ψ_{v} of the free energy is written, as usual in elasticity, as

$$\Psi_{\mathrm{v}}(\varepsilon(\mathbf{u}))=\tfrac{1}{2}\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}) \tag{7}$$

K being the rigidity matrix, while the contact surface free energy Ψ_{s} is chosen as follows

$$\Psi_{\mathrm{s}}(\mathbf{u},\chi,\nabla_{\mathrm{s}}\chi)=\frac{k}{2}\chi|\mathbf{u}|^2+I_{(-\infty,0]}(\mathbf{u}\cdot\mathbf{n})+w_{\mathrm{s}}(1-\chi)+\frac{k_{\mathrm{s}}}{2}|\nabla_{\mathrm{s}}\chi|^2+\psi(\chi) \tag{8}$$

where $\psi(\chi)$ is a function accounting for the phase transition (between the damaged and the undamaged state), and possibly including some physical constraint on the phase variable. For instance, in order to render $\chi\in[0,1]$ one can choose $\psi=I_{[0,1]}$, i.e. the indicator function of the interval $[0,1]$, defined by $I_{[0,1]}(y)=0$ if $y\in[0,1]$ and $I_{[0,1]}(y)=+\infty$ otherwise. Analogously, $I_{(-\infty,0]}$ being the indicator function of the interval $(-\infty,0]$, the term $I_{(-\infty,0]}(\mathbf{u}\cdot\mathbf{n})$ in (8) yields the impenetrability condition on the contact surface, since it implies that $\mathbf{u}\cdot\mathbf{n}\leqslant 0$. Let us point out that, here and in the sequel, we shall simply denote by $\mathbf{u}$ the trace of the vector $\mathbf{u}$ on the boundary Γ. The constants $k, w_{\mathrm{s}}, k_{\mathrm{s}}$ are positive.

Moreover, we prescribe that the evolution of the adhesion is dissipative and irreversible (i.e. the bonds cannot repair themselves once they are broken): irreversibility indeed corresponds to requiring that $\chi_t\leqslant 0$. This is encoded in the pseudo-potential of dissipation, which describes the dissipation of the system in terms of the time derivatives χ_t and $\varepsilon(\mathbf{u}_t)$ and is written for a volume part Φ_{v} and a contact one Φ_{s}, defined on Γ_c. More precisely,

$$\Phi_{\mathrm{v}}:=\frac{1}{2}\varepsilon(\mathbf{u}_t)K_{\mathrm{v}}\varepsilon(\mathbf{u}_t), \quad \Phi_{\mathrm{s}}:=\frac{c_{\mathrm{s}}}{2}|\chi_t|^2+I_{(-\infty,0]}(\chi_t) \tag{9}$$

where $c_{\mathrm{s}}>0$, K_{v} is a symmetric positive viscosity matrix and the term $I_{(-\infty,0]}(\chi_t)$ forces χ_t to assume non-positive values.

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The constitutive laws are specified for the dissipative and the non-dissipative contributions, and they are recovered from the pseudo-potential and the free energy, respectively, by

$$\sigma=\sigma^d+\sigma^{nd}=\frac{\partial\Phi_{\mathrm{v}}}{\partial\varepsilon(\mathbf{u}_t)}+\frac{\partial\Psi_{\mathrm{v}}}{\partial\varepsilon(\mathbf{u})} \tag{10}$$

$$B=B^d+B^{nd}=\frac{\partial\Phi_{\mathrm{s}}}{\partial\chi_t}+\frac{\partial\Psi_{\mathrm{s}}}{\partial\chi} \tag{11}$$

$$\mathbf{H}=\mathbf{H}^{nd}=\frac{\partial\Psi_{\mathrm{s}}}{\partial\nabla_{\mathrm{s}}\chi} \tag{12}$$

$$\mathbf{R}=\mathbf{R}^{nd}=\frac{\partial\Psi_{\mathrm{s}}}{\partial\mathbf{u}} \tag{13}$$

Finally, we substitute (10)–(13), combined with (7)–(9), into system (3)–(6). In the sequel, for the sake of simplicity, we assume that no work involving microscopic motions is provided to the system, so that we let $A=a=0$ in (6). Thus, we obtain the following PDE system

$$-\mathrm{div}(K\varepsilon(\mathbf{u})+K_{\mathrm{v}}\varepsilon(\mathbf{u}_t))=\mathbf{f}\quad\text{in }\Omega\times(0,T) \tag{14}$$

$$\mathbf{u}=\mathbf{0}\quad\text{in }\Gamma_1\times(0,T),\qquad(K\varepsilon(\mathbf{u})+K_{\mathrm{v}}\varepsilon(\mathbf{u}_t))\mathbf{n}=\mathbf{g}\quad\text{in }\Gamma_2\times(0,T) \tag{15}$$

$$(K\varepsilon(\mathbf{u})+K_{\mathrm{v}}\varepsilon(\mathbf{u}_t))\mathbf{n}+k\chi\mathbf{u}+\partial I_{(-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\ni\mathbf{0}\quad\text{in }\Gamma_c\times(0,T) \tag{16}$$

$$c_{\mathrm{s}}\chi_t-k_{\mathrm{s}}\Delta_{\mathrm{s}}\chi+\partial I_{(-\infty,0]}(\chi_t)+\beta(\chi)\ni w_{\mathrm{s}}-\frac{k}{2}|\mathbf{u}|^2\quad\text{in }\Gamma_c\times(0,T) \tag{17}$$

$$\partial_{\mathbf{n}_{\mathrm{s}}}\chi=0\quad\text{in }\partial\Gamma_c\times(0,T) \tag{18}$$

Here, Δ_{s} is the Laplacian on Γ_c, while $\beta=\partial\psi$ and $\partial I_{(-\infty,0]}$ denote the subdifferentials of ψ and $I_{(-\infty,0]}$.

Let us now spend a few words on the condition expressed by (16) and derived from (8). We first consider the case in which $\chi\in[0,1]$ (enforced by taking $\psi=I_{[0,1]}$ in (8)). If $\chi=0$, i.e. when there is no adhesion between the body and the support, actually (16) complies with the so-called Signorini conditions. Indeed, if $\mathbf{u}\cdot\mathbf{n}=0$ on Γ_c, then $\sigma\mathbf{n}=(K\varepsilon(\mathbf{u})+K_{\mathrm{v}}\varepsilon(\mathbf{u}_t))\mathbf{n}=-\lambda\mathbf{n}$ for some $\lambda\geqslant 0$, while in the case in which $\mathbf{u}\cdot\mathbf{n}<0$, then $\sigma\mathbf{n}=\mathbf{0}$. Conversely, when $0<\chi\leqslant 1$, the glue is active and prevents the separation when a tension is applied (note that, if $\chi>0$, a reaction on the boundary appears to counteract separation). Instead, if χ is allowed to assume negative values, from (16) we read that repulsive dynamics are possible, i.e. the body may tend to separate from the support. Next, concerning relation (17), we remark that the evolution of the parameter χ is mainly ruled by the sign of the right-hand side of (17). The positive constant w_{s} therein represents the cohesion of the adhesion, while the quadratic term plays the role of a source of microscopic damage, due to the macroscopic movements.

Related literature: Before proceeding, let us review some results in the literature on contact models with adhesion, based on Frémond's papers [6, 7]. First of all, we recall the contributions [8–10], focusing on adhesive contact between a viscoelastic body and a deformable foundation. In this case, contact properties are described by a normal compliance law, which allows for the

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interpenetration of surface asperities. Let us point out that, from the analytical point of view, such a condition may be regarded as a regularization of the Signorini conditions. In particular, a quasistatic model is analyzed in [8], where the evolution of the bonding field is described by a first-order ordinary differential equation (ODE). Existence, uniqueness, and continuous dependence results for weak solutions are proved. Dynamic processes of adhesive contact are instead considered in [9, 10], where the rate of the bonding field is assumed to be reversible and irreversible, respectively. Hence, well-posedness results and some numerical simulations are obtained.

Unilateral contact is instead considered in the papers [11, 12]. The latter also couples friction with adhesion and unilateral contact. Assuming that the friction coefficient is sufficiently small, an existence result for an incremental formulation of the problem is given and some numerical schemes are proposed. In [11], the adhesive properties are described assuming that the displacement on the contact surface is different from zero only in the case when all the bonds are broken (i.e. $\chi=0$). Moreover, the evolution of the bonding field is governed by an integro-differential equation. An existence theorem and some numerical computations are shown.

Our results: In the present paper, we couple unilateral contact with irreversible adhesion and we obtain, as main result, a *global in time* existence theorem for the Cauchy problem related to a generalized version of system (14)–(18). Since the constraints on the internal variables are recovered by subdifferentials of indicator functions, the major analytical difficulties connected with this problem are due to the presence of three multivalued nonlinearities in (16)–(17), which yield, in particular, the doubly nonlinear character of (17). Another mathematical peculiarity of our system is the unusual coupling between equations (14) and (17), which are in fact posed on manifolds of different dimension.

These problems shall be carefully handled in our argument for existence, which consists of the following steps. First, system (14)–(18) is approximated by regularization of the maximal monotone nonlinearities in (16)–(17). A local in time existence result for the approximate system is proved by a Schauder fixed point argument. The local character of the solution is related to the form of the free energy (8). Indeed, in the case in which χ is allowed to be negative, the contact surface free energy is not convex w.r.t. $\mathbf{u}$ and we cannot deduce a control for $\mathbf{u}$ at equilibrium, in contrast with the case $\chi>0$, in which the minimum of $\Psi_s(\cdot,\chi,\nabla_s\chi)$ is attained at $\mathbf{u}=\mathbf{0}$. Hence, passing to the limit in the regularization parameter, we obtain the existence of a *local solution* to the Cauchy problem for (14)–(18), see Theorem 1. This obstruction to global existence is removed if χ is forced to take non-negative values: in that case, it is possible to perform on the aforementioned local solution a refined prolongation procedure, whose technicalities are essentially motivated by the fact that some of the *a priori* estimates needed for the extension can be rigourously justified only in the approximate framework. We refer to Section 5 for further explanations and for a detailed outline of our technique. In this way, we prove our global existence result, Theorem 3 later on.

As far as the uniqueness issue is concerned, we shall prove a continuous dependence result (Theorem 2), yielding uniqueness of the solutions, for the simplified system obtained by replacing the possibly multivalued nonlinearity β in (17) with a Lipschitz continuous function. Furthermore, we shall give a second uniqueness result for a special class of solutions, corresponding to a suitable choice of the initial data and of the involved nonlinearities (cf. Propositions 2.3 and 2.4). In the general case, uniqueness is not to be expected, essentially due to the doubly nonlinear character of (17). On the contrary, dealing with the simplified physical situation of reversible phenomena of adhesion (i.e. without the term $\partial I_{(-\infty,0]}(\chi_t)$ in (17)), Reference [13] shows the well posedness of the initial-value problem for (14)–(18) and investigates the long-time behavior of the solutions.

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Plan of the paper: In Section 2, we introduce the variational formulation of (14)–(18) and state our main results; Section 3 is devoted to the proof of the continuous dependence Theorem 2. The proof of Theorem 1 is taken up in Section 4, while Theorem 3 is proved in Section 5. Finally, in Section 6 we prove a uniqueness result for a special class of solutions (Proposition 2.3), leading to a local in time uniqueness result for the original problem (under a suitable choice of the potentials and the initial datum χ_0). Throughout the paper we also give some comments and remarks.

2. MATHEMATICAL FORMULATION AND MAIN RESULTS

In this section, we rigorously state the variational formulation of (14)–(18) combined with the initial conditions

$$\mathbf{u}(\cdot,0)=\mathbf{u}_0 \text{ in } \Omega, \quad \chi(\cdot,0)=\chi_0 \text{ in } \Gamma_c \tag{19}$$

Hereafter, for the sake of simplicity, we assume that Ω is a bounded smooth set of $\mathbb{R}^3$, such that Γ_c is a smooth bounded domain of $\mathbb{R}^2$ (one may think of a flat surface).

Before proceeding, we also introduce some useful notation. Given a Banach space X, we denote by ${}_{X'}\langle\cdot,\cdot\rangle_X$ the duality pairing between X' and X itself, while by the same symbol $\|\cdot\|_X$ we indicate both the norm in a Banach space X and in any power of X. We introduce the Hilbert triplet $V \hookrightarrow H \hookrightarrow V'$, with

$$H:=L^2(\Omega), \quad V:=H^1(\Omega)$$

As usual, H is identified with its dual space. Then, we set

$$W:=\{\mathbf{v}\in V^3 : \mathbf{v}=\mathbf{0} \quad \text{a.e. on } \Gamma_1\}$$

endowed with the natural norm induced by V^3.

As usual in elasticity theory, we assume the material to be homogeneous and isotropic, so that the rigidity matrix K in relations (7), (14)–(16) may be represented by

$$K\varepsilon(\mathbf{u})=\lambda \operatorname{tr}\varepsilon(\mathbf{u})\mathbf{1}+2\mu\varepsilon(\mathbf{u})$$

where $\lambda,\mu>0$ are the so-called Lamé constants and $\mathbf{1}$ is the identity matrix. Hence, to introduce the variational formulation for system (14)–(18), let us consider the continuous bilinear symmetric forms $a,b: W\times W\to\mathbb{R}$ defined by

$$a(\mathbf{u},\mathbf{v}):=\lambda\int_\Omega \operatorname{div}(\mathbf{u})\operatorname{div}(\mathbf{v})+2\mu\sum_{i,j=1}^{3}\int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \forall\, \mathbf{u},\mathbf{w}\in W \tag{20}$$

$$b(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{3}\int_\Omega b_{ij}\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \quad \forall\, \mathbf{u},\mathbf{w}\in W \tag{21}$$

where (b_{ij}) is the viscosity matrix K_{v}, cf. (9), (14)–(16). In the sequel, just for the sake of simplicity but without loss of generality, we let

$$b_{ij}=1, \quad i,j=1,2,3, \text{ so that } b(\mathbf{u},\mathbf{u})=\|\varepsilon(\mathbf{u})\|_H^2 \tag{22}$$

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Since Γ_1 has positive measure, by Korn's inequality we deduce that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are W-elliptic, i.e. there exist $C_a, C_b > 0$ such that for all $\mathbf{u} \in W$

$$a(\mathbf{u},\mathbf{u}) \geqslant C_a \|\mathbf{u}\|_W^2 \tag{23}$$

$$b(\mathbf{u},\mathbf{u}) \geqslant C_b \|\mathbf{u}\|_W^2 \tag{24}$$

Now, we introduce the set

$$\mathscr{X}_- := \{\mathbf{v} \in (H^{1/2}(\Gamma_c))^3 : \mathbf{v}\cdot\mathbf{n} \leqslant 0 \quad \text{a.e. in } \Gamma_c\}$$

$\mathbf{n}$ being the outward unit normal vector on Γ_c, and we let $I_{\mathscr{X}_-}$ be the indicator function of $\mathscr{X}_-$. In the duality between $(H^{1/2}(\Gamma_c))^3$ and $(H^{-1/2}(\Gamma_c))^3$. For simplicity, hereafter we shall use the notation $\int_{\Gamma_c}$ ($\int_{\Gamma_2}$, resp.) for the duality pairing ${}_{(H^{-1/2}(\Gamma_c))^3}\langle\cdot,\cdot\rangle_{(H^{1/2}(\Gamma_c))^3}$ (for ${}_{(H^{-1/2}(\Gamma_2))^3}\langle\cdot,\cdot\rangle_{(H^{1/2}(\Gamma_2))^3}$, resp.), we define the operator $\partial I_{\mathscr{X}_-} : (H^{1/2}(\Gamma_c))^3 \rightarrow 2^{(H^{-1/2}(\Gamma_c))^3}$ by

$$\begin{gathered}\boldsymbol{\eta} \in (H^{-1/2}(\Gamma_c))^3 \text{ belongs to } \partial I_{\mathscr{X}_-}(\mathbf{y}) \text{ if and only if} \\ \mathbf{y} \in \mathscr{X}_-, \quad \int_{\Gamma_c} \boldsymbol{\eta}\cdot(\mathbf{v}-\mathbf{y}) \leqslant 0 \quad \forall\ \mathbf{v} \in \mathscr{X}_-\end{gathered} \tag{25}$$

Let us point out that the subdifferential introduced by (25) cannot be interpreted as a force a.e. defined. However, we can consider it as a constraint on the sign of $\mathbf{u}\cdot\mathbf{n}$: in this way, we render the physical constraint of impenetration between the body and its support.

Actually, our results apply to a more general maximal monotone graph (in the space $(H^{1/2}(\Gamma_c))^3 \times (H^{-1/2}(\Gamma_c))^3$), in relation (16) (see (41) and (42)). More precisely, we introduce a functional

$$\begin{gathered}\varphi : (H^{1/2}(\Gamma_c))^3 \rightarrow [0,+\infty] \text{ proper, convex and lower semicontinuous} \\ \text{with } \varphi(\mathbf{0}) = 0 = \min\varphi, \text{ and} \\ \text{we set } \alpha := \partial\varphi : (H^{1/2}(\Gamma_c))^3 \rightarrow 2^{(H^{-1/2}(\Gamma_c))^3}\end{gathered} \tag{26}$$

Analogously, in relation (17) we will be able to handle a more general maximal monotone graph than the subdifferential $\partial I_{(-\infty,0]}$ (see (43) and (44)). To this aim, we consider

$$\rho : \mathbb{R} \rightarrow 2^{\mathbb{R}} \text{ a maximal monotone operator such that } 0 \in \rho(0) \tag{27}$$

As for the function ψ in (8), we assume that

$$\psi : \mathbb{R} \rightarrow [0,+\infty] \text{ proper, convex and lower semicontinuous with } \psi(0) = 0 = \min\psi \tag{28}$$

It is well known that $\beta := \partial\psi : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a maximal monotone operator such that $0 \in \beta(0)$. For every $r \in \mathrm{dom}(\beta)$, we shall denote by $\beta^0(r)$ the element of minimal norm in $\beta(r)$.

Remark 2.1

Let us point out that we do not suppose that the domain of β is bounded or a subset of $[0,+\infty)$. This corresponds to the fact that our model includes situations in which repulsive dynamics may occur between the body and the support, leading to $\chi < 0$ (and forcing, in this case, separation due to (16)).

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Next, concerning the Cauchy conditions (19), we suppose that

$$\mathbf{u}_0 \in W \quad \text{and} \quad \mathbf{u}_0 \in \mathrm{dom}(\varphi)$$

$$\chi_0 \in H^2(\Gamma_c), \quad \partial_{\mathbf{n}_s}\chi_0 = 0 \quad \text{a.e. in } \partial\Gamma_c, \quad \psi(\chi_0) \in L^1(\Gamma_c) \tag{29}$$

and

$$\beta^0(\chi_0) \in L^2(\Gamma_c) \tag{30}$$

As far as the body force **f** and the surface traction **g** are concerned, we prescribe that

$$\mathbf{f} \in L^2(0,T;H^3) \tag{31}$$

$$\mathbf{g} \in L^2(0,T;(H^{-1/2}(\Gamma_2))^3) \tag{32}$$

and we define $\mathbf{F}:(0,T)\to W'$ by

$$_{W'}\langle \mathbf{F}(t),\mathbf{v}\rangle_W := \int_\Omega \mathbf{f}(t)\cdot\mathbf{v} + \int_{\Gamma_2} \mathbf{g}(t)\cdot\mathbf{v} \quad \forall \mathbf{v}\in W \ \text{ for a.e. } t\in(0,T) \tag{33}$$

Of course, thanks to (31)–(32), $\mathbf{F}\in L^2(0,T;W')$.

Remark 2.2
Assumption (32) on **g** could be slightly refined by taking into account that, in the variational formulation of (14) (see (41)), **g** is in fact tested with functions that are traces on Γ_2 of elements of $H^1(\Omega)^3$ having null trace on Γ_1. Such functions describe the space

$$(H^{1/2}_{00,\Gamma_1}(\Gamma_2))^3 = \{\mathbf{w}\in(H^{1/2}(\Gamma_2))^3 : \exists\ \tilde{\mathbf{w}}\in(H^{1/2}(\Gamma))^3 \text{ with } \tilde{\mathbf{w}}=\mathbf{w} \text{ in } \Gamma_2,\ \tilde{\mathbf{w}}=\mathbf{0} \text{ in } \Gamma_1\} \tag{34}$$

where the latter identities are of course meant in the sense of traces and the notation $(H^{1/2}_{00,\Gamma_1}(\Gamma_2))^3$ is in accordance with the notation introduced in [14, Chapter 11]. We may endow $(H^{1/2}_{00,\Gamma_1}(\Gamma_2))^3$ with the norm

$$\|\mathbf{w}\| := \inf\{\|\tilde{\mathbf{w}}\|_{H^{1/2}(\Gamma)} : \tilde{\mathbf{w}}\in(H^{1/2}(\Gamma))^3 \text{ with } \tilde{\mathbf{w}}=\mathbf{w} \text{ in } \Gamma_2,\ \tilde{\mathbf{w}}=\mathbf{0} \text{ in } \Gamma_1\} \tag{35}$$

thus obtaining a Banach (in fact a Hilbert) space, continuously and densely embedded in $(H^{1/2}(\Gamma_2))^3$, so that $(H^{-1/2}(\Gamma_2))^3 \subset \left(H^{1/2}_{00,\Gamma_1}(\Gamma_2)^3\right)'$ with a continuous (and dense) embedding. Thus, we might replace (32) with the slightly finer requirement that

$$\mathbf{g}\in L^2\left(0,T;(H^{1/2}_{00,\Gamma_1}(\Gamma_2)^3)'\right)$$

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Analogously, in agreement with definition (34), we might consider the space $(H^{1/2}_{00,\Gamma_1}(\Gamma_c))^3$ and deal with a maximal monotone operator

$$\alpha\colon \left(H^{1/2}_{00,\Gamma_1}(\Gamma_c)\right)^3 \to 2^{(H^{1/2}_{00,\Gamma_1}(\Gamma_c)^3)'}$$

in (41) and (42). However, for the sake of simplicity, we prefer to focus on the framework prescribed by the previous assumptions, since the results we obtain are in any case meaningful both for analysis and applications.

Finally, without loss of generality, we set the coefficients k_{s}, k and c_{s} in the system (14)–(18) equal to 1, while we do not normalize the constant w_{s} in (17).

We may now specify the variational problem we are dealing with.

Problem (P)
Find $\mathsf{t}\in(0,T]$ and a quintuple $(\mathbf{u},\chi,\boldsymbol{\eta},\xi,\zeta)$ such that

$$\mathbf{u}\in H^1(0,\mathsf{t};W), \tag{36}$$

$$\chi\in W^{1,\infty}(0,\mathsf{t};L^2(\Gamma_c))\cap H^1(0,\mathsf{t};H^1(\Gamma_c))\cap L^\infty(0,\mathsf{t};H^2(\Gamma_c)) \tag{37}$$

$$\boldsymbol{\eta}\in L^2(0,\mathsf{t};(H^{-1/2}(\Gamma_c))^3) \tag{38}$$

$$\xi\in L^\infty(0,\mathsf{t};L^2(\Gamma_c)) \tag{39}$$

$$\zeta\in L^\infty(0,\mathsf{t};L^2(\Gamma_c)) \tag{40}$$

fulfilling (19) and

$$b(\mathbf{u}_t,\mathbf{v})+a(\mathbf{u},\mathbf{v})+\int_{\Gamma_c}(\chi\mathbf{u}+\boldsymbol{\eta})\cdot\mathbf{v}={}_{W'}\langle\mathbf{F},\mathbf{v}\rangle_W \quad \forall\,\mathbf{v}\in W \quad \text{a.e. in } (0,\mathsf{t}) \tag{41}$$

$$\boldsymbol{\eta}\in\alpha(\mathbf{u}) \quad \text{a.e. in } (0,\mathsf{t}) \tag{42}$$

$$\chi_t-\Delta\chi+\xi+\zeta=w_{\mathrm{s}}-\tfrac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_c\times(0,\mathsf{t}) \tag{43}$$

$$\xi\in\rho(\chi_t) \quad \text{a.e. in } \Gamma_c\times(0,\mathsf{t}) \tag{44}$$

$$\zeta\in\beta(\chi) \quad \text{a.e. in } \Gamma_c\times(0,\mathsf{t}) \tag{45}$$

$$\partial_{\mathbf{n}_{\mathrm{s}}}\chi=0 \quad \text{a.e. in } \partial\Gamma_c\times(0,\mathsf{t}) \tag{46}$$

Note that (37) yields the regularity $\chi\in C^0_w([0,\mathsf{t}];H^2(\Gamma_c))$ (the space of weakly continuous functions on $[0,\mathsf{t}]$ valued in $H^2(\Gamma_c)$), which is in agreement with (30).

We now list the main results of the paper. Our first existence theorem for Problem (P) is only local in time. Nonetheless, as we mentioned in the Introduction, this is expected from the physical point of view, as the model under consideration is physically feasible in the framework of small deformations and the fact that χ may assume negative values yields the displacement $\mathbf{u}$ to be very large at equilibrium (in this way, the separation between the body and its support is favored). On the contrary, forcing $\chi\geqslant 0$ we can control its evolution on the whole time interval

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$(0,T)$ (see Theorem 3), as in this case the boundary contribution on Γ_c in (41) prevents $\mathbf{u}$ from taking too large values on the boundary (cf. also (8)). We point out the most interesting situation for applications is clearly the one described by Theorem 3.

Theorem 1
Under assumptions (26)–(32), there exist a final time $\widehat{T}\in(0,T]$ and a quintuple $(\mathbf{u},\chi,\boldsymbol{\eta},\xi,\zeta)$ solving Problem (P) on the time interval $(0,\widehat{T})$.

The proof of this result will be carried out in Section 4 by means of a fixed point technique, combined with regularization tools.

As we have already mentioned, we have just been able to give partial answers to the uniqueness issue for Problem (P). First, assuming $\beta:\mathbb{R}\to\mathbb{R}$ to be (single-valued and) Lipschitz continuous, we establish the following continuous dependence result, from which uniqueness follows. In fact, in this case (43) loses its doubly nonlinear character, which is the main obstruction for uniqueness.

Theorem 2
In addition to assumptions (26)–(32), suppose that

$$\beta:\mathbb{R}\to\mathbb{R} \text{ is Lipschitz continuous}$$

Let $(\mathbf{u}_0^1,\chi_0^1,\mathbf{f}_1,\mathbf{g}_1)$ and $(\mathbf{u}_0^2,\chi_0^2,\mathbf{f}_2,\mathbf{g}_2)$ be two sets of data for Problem (P), and, accordingly, let $(\mathbf{u}_1,\chi_1,\boldsymbol{\eta}_1,\xi_1,\zeta_1)$ and $(\mathbf{u}_2,\chi_2,\boldsymbol{\eta}_2,\xi_2,\zeta_2)$ be two quintuples solving Problem (P) on some time interval $(0,\mathsf{t})$. Set

$$m:=\max_{i=1,2}\{\|\mathbf{u}_0^i\|_W^2+\varphi(\mathbf{u}_0^i)^{1/2}+\|\chi_0^i\|_{H^1(\Gamma_c)}+\|\psi(\chi_0^i)\|_{L^1(\Gamma_c)}^{1/2}+\|\mathbf{F}_i\|_{L^2(0,T;W')}\}$$

where $\mathbf{F}_i$, $i=1,2$, are defined as in (33). Then, there exists a positive constant M, only depending on m and on C_a, C_b, w_{s}, T, Ω, and Γ_c, such that

$$\begin{aligned}&\|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty(0,\mathsf{t};W)}+\|\chi_1-\chi_2\|_{H^1(0,\mathsf{t};L^2(\Gamma_c))\cap L^\infty(0,\mathsf{t};H^1(\Gamma_c))}\\&\quad\leqslant M(\|\mathbf{u}_0^1-\mathbf{u}_0^2\|_W+\|\chi_0^1-\chi_0^2\|_{H^1(\Gamma_c)}+\|\mathbf{F}_1-\mathbf{F}_2\|_{L^2(0,T;W')})\end{aligned}\tag{47}$$

In particular, the solution to Problem (P) is unique.

Theorem 3
In addition to assumptions (26)–(32), suppose that

$$\mathrm{dom}(\beta)\subseteq[0,+\infty)\tag{48}$$

Then, there exists a quintuple $(\mathbf{u},\chi,\boldsymbol{\eta},\xi,\zeta)$ solving Problem (P) on the whole interval $(0,T)$.

The proof of this result will be carried out in Section 5 and relies on the local well posedness provided by Theorems 1 and 2, combined with a careful and non-standard prolongation procedure.

In the particular case of $\rho=\partial I_{(-\infty,0]}$ and $\beta=\partial I_{[0,1]}$, which is interesting from the physical point of view, it is possible to obtain uniqueness in a special class of solutions (see (50)). Hence, requiring in addition that the initial datum satisfies $\chi_0\in(0,1)$, we obtain local uniqueness of the solution to Problem (P). These results are stated by the following propositions, whose proofs are shown in Section 6.

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Proposition 2.3
Assume (26), (29)–(32), and suppose further that

$$\rho=\partial I_{(-\infty,0]}, \quad \beta=\partial I_{[0,1]} \tag{49}$$

Let $0<T_0\leqslant T$, and let $(\mathbf{u}_1,\chi_1,\boldsymbol{\eta}_1,\xi_1,\zeta_1)$, $(\mathbf{u}_2,\chi_2,\boldsymbol{\eta}_2,\xi_2,\zeta_2)$ be two quintuples of solutions of Problem (P) on the interval $(0,T_0)$ such that

$$0<\chi_i(x,t)<1 \quad \forall (x,t)\in\Gamma_c\times(0,T_0), \ \ i=1,2 \tag{50}$$

Then, for a.e. $t\in(0,T_0)$ we have $\zeta_1(t)\equiv\zeta_2(t)\equiv 0$ and

$$\mathbf{u}_1(t)\equiv\mathbf{u}_2(t), \quad \chi_1(t)\equiv\chi_2(t), \quad \boldsymbol{\eta}_1(t)\equiv\boldsymbol{\eta}_2(t), \quad \xi_1(t)\equiv\xi_2(t) \tag{51}$$

In fact, the following result guarantees that, under the assumptions of Proposition 2.3, if the initial datum χ_0 stays away from 0 and 1, then any solution of Problem (P) complies with the property (50) locally in time (and uniqueness follows).

Proposition 2.4
Assume (26), (29)–(32), and (49). Suppose that

$$\exists\delta\in(0,1) \quad \text{s.t.} \quad \delta\leqslant\chi_0(x)<1 \quad \forall x\in\Omega \tag{52}$$

Then, for any corresponding solution $(\mathbf{u},\chi,\boldsymbol{\eta},\xi,\zeta)$ of Problem (P) there exists $T_0\in(0,T]$ such that

$$\frac{\delta}{2}\leqslant\chi(x,t)<1 \quad \forall (x,t)\in\Gamma_c\times[0,T_0] \tag{53}$$

Before proceeding, we collect here some properties that shall be useful in the sequel. We recall that, by trace theorems and by Sobolev's embedding result, there exists a positive constant c such that

$$\|\mathbf{v}\|_{L^4(\Gamma_c)}+\|\mathbf{v}\|_{H^{1/2}(\Gamma_2)}\leqslant c\|\mathbf{v}\|_W \quad \forall\mathbf{v}\in W \tag{54}$$

Moreover, we shall use the elementary inequality

$$ab\leqslant(\delta/2)a^2+(2\delta)^{-1}b^2 \quad \forall a,b\in\mathbb{R}, \ \ \delta>0 \tag{55}$$

Finally, we warn that, in the following proofs, for the sake of simplicity we shall employ the same symbols c, C for different constants, even in the same formula.

3. PROOF OF THEOREM 2

Proof of Theorem 2
Let $(\mathbf{u}_1,\chi_1,\boldsymbol{\eta}_1,\xi_1,\zeta_1)$ and $(\mathbf{u}_2,\chi_2,\boldsymbol{\eta}_2,\xi_2,\zeta_2)$ be two solutions to Problem (P), respectively, corresponding to the data $(\mathbf{u}_0^1,\chi_0^1,\mathbf{f}_1,\mathbf{g}_1)$ and $(\mathbf{u}_0^2,\chi_0^2,\mathbf{f}_2,\mathbf{g}_2)$. Being β Lipschitz continuous, we have $\zeta_1=\beta(\chi_1)$ and $\zeta_2=\beta(\chi_2)$. We set

$$\begin{aligned}&\widetilde{\mathbf{u}}_0:=\mathbf{u}_0^1-\mathbf{u}_0^2, \quad \widetilde{\chi}_0:=\chi_0^1-\chi_0^2, \quad \widetilde{\mathbf{F}}:=\mathbf{F}_1-\mathbf{F}_2\\ &\widetilde{\mathbf{u}}:=\mathbf{u}_1-\mathbf{u}_2, \quad \widetilde{\chi}:=\chi_1-\chi_2, \quad \widetilde{\boldsymbol{\eta}}:=\boldsymbol{\eta}_1-\boldsymbol{\eta}_2, \quad \widetilde{\xi}:=\xi_1-\xi_2\end{aligned} \tag{56}$$

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Then, we subtract (41) written for $(\mathbf{u}_2, \chi_2, \boldsymbol{\eta}_2)$ from (41) written for $(\mathbf{u}_1, \chi_1, \boldsymbol{\eta}_1)$, we test it by $\widetilde{\mathbf{u}}$ and we integrate from 0 to t, with $0<t<\mathsf{t}$. Thanks to (23) and (24), we find

$$\frac{C_b}{2}\|\widetilde{\mathbf{u}}(t)\|_W^2 + C_a\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} + \int_0^t\int_{\Gamma_c}\widetilde{\boldsymbol{\eta}}\cdot\widetilde{\mathbf{u}}$$
$$\leqslant -\int_0^t\int_{\Gamma_c}\widetilde{\chi}\mathbf{u}_2\cdot\widetilde{\mathbf{u}} - \int_0^t\int_{\Gamma_c}\chi_1|\widetilde{\mathbf{u}}|^2 + C\|\widetilde{\mathbf{u}}_0\|_W^2 + \int_0^t {}_{W'}\langle\widetilde{\mathbf{F}},\widetilde{\mathbf{u}}\rangle_W \tag{57}$$

The integral on the left-hand side of (57) is non-negative, thanks to the monotonicity of α. Moreover, we estimate the right-hand side of (57) owing to the Hölder inequality, (54), (55), and assumptions (29)–(30) on the initial data. We obtain

$$\int_0^t\int_{\Gamma_c}\widetilde{\chi}\mathbf{u}_2\cdot\widetilde{\mathbf{u}} \leqslant \int_0^t \|\widetilde{\chi}\|_{L^2(\Gamma_c)}\|\mathbf{u}_2\|_{L^4(\Gamma_c)}\|\widetilde{\mathbf{u}}\|_{L^4(\Gamma_c)}$$
$$\leqslant c\|\widetilde{\chi}\|^2_{L^2(0,t;L^2(\Gamma_c))} + c\int_0^t\|\mathbf{u}_2\|_W^2\|\widetilde{\mathbf{u}}\|_W^2$$
$$\leqslant c\int_0^t\|\widetilde{\chi}_t\|^2_{L^2(0,s;L^2(\Gamma_c))}\,\mathrm{d}s + c\|\mathbf{u}_2\|^2_{L^\infty(0,t;W)}\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} \tag{58}$$

$$\int_0^t\int_{\Gamma_c}\chi_1|\widetilde{\mathbf{u}}|^2 \leqslant \int_0^t\|\chi_1\|_{L^2(\Gamma_c)}\|\widetilde{\mathbf{u}}\|^2_{L^4(\Gamma_c)}$$
$$\leqslant c\int_0^t\|\chi_1\|_{L^2(\Gamma_c)}\|\widetilde{\mathbf{u}}\|_W^2 \leqslant c\|\chi_1\|_{L^\infty(0,t;L^2(\Gamma_c))}\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} \tag{59}$$

$$\int_0^t {}_{W'}\langle\widetilde{\mathbf{F}},\widetilde{\mathbf{u}}\rangle_W \leqslant \frac{1}{2}\|\widetilde{\mathbf{F}}\|^2_{L^2(0,t;W')} + \frac{1}{2}\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} \tag{60}$$

Next, we take the difference between (43) written for $(\mathbf{u}_1, \chi_1, \xi_1, \zeta_1)$ and for $(\mathbf{u}_2, \chi_2, \xi_2, \zeta_2)$, we multiply it by $\widetilde{\chi}_t$ and we integrate over $\Gamma_c\times(0,t)$. Thanks to the monotonicity of ρ, the term $\int_0^t\int_{\Gamma_c}\widetilde{\xi}\widetilde{\chi}_t$ is non-negative, and we find

$$\|\widetilde{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + \frac{1}{2}\|\nabla\widetilde{\chi}(t)\|^2_{L^2(\Gamma_c)} + \int_0^t\int_{\Gamma_c}(\beta(\chi_1)-\beta(\chi_2))\widetilde{\chi}_t$$
$$\leqslant -\frac{1}{2}\int_0^t\int_{\Gamma_c}(\mathbf{u}_1+\mathbf{u}_2)\widetilde{\mathbf{u}}\widetilde{\chi}_t + \frac{1}{2}\|\nabla\widetilde{\chi}_0\|^2_{L^2(\Gamma_c)} \tag{61}$$

Being β Lipschitz continuous, we have

$$\int_0^t\int_{\Gamma_c}|(\beta(\chi_1)-\beta(\chi_2))\widetilde{\chi}_t| \leqslant c\int_0^t\|\widetilde{\chi}\|_{L^2(\Gamma_c)}\|\widetilde{\chi}_t\|_{L^2(\Gamma_c)}$$
$$\leqslant \frac{1}{4}\|\widetilde{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + c\|\widetilde{\chi}\|^2_{L^2(0,t;L^2(\Gamma_c))}$$
$$\leqslant \frac{1}{4}\|\widetilde{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + c\int_0^t\|\widetilde{\chi}_t\|^2_{L^2(0,s;L^2(\Gamma_c))}\,\mathrm{d}s \tag{62}$$

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Moreover,

$$\int_0^t \int_{\Gamma_c} |(\mathbf{u}_1+\mathbf{u}_2)\widetilde{\mathbf{u}}\widetilde{\chi}_t| \leqslant \int_0^t \|\mathbf{u}_1+\mathbf{u}_2\|_{L^4(\Gamma_c)}\|\widetilde{\chi}_t\|_{L^2(\Gamma_c)}\|\widetilde{\mathbf{u}}\|_{L^4(\Gamma_c)}$$

$$\leqslant \frac{1}{4}\|\widetilde{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + c\|\mathbf{u}_1+\mathbf{u}_2\|^2_{L^\infty(0,t;W)}\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} \tag{63}$$

Finally, we add (57) and (61). On account of (58)–(60), (62)–(63), and of Lemma 5.2 proved later on (which yields a bound on $\|\mathbf{u}_1\|_{L^\infty(0,t;W)}$, $\|\mathbf{u}_2\|_{L^\infty(0,t;W)}$, and $\|\chi_1\|_{L^\infty(0,t;L^2(\Gamma_c))}$), we find

$$\frac{C_b}{2}\|\widetilde{\mathbf{u}}(t)\|^2_W + C_a\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} + \frac{1}{2}\|\widetilde{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + \frac{1}{2}\|\nabla\widetilde{\chi}(t)\|^2_{L^2(\Gamma_c)}$$

$$\leqslant C\|\widetilde{\mathbf{u}}_0\|^2_W + \frac{1}{2}\|\widetilde{\chi}_0\|^2_{H^1(\Gamma_c)} + c\|\widetilde{\mathbf{F}}\|^2_{L^2(0,T;W')} + c\int_0^t \|\widetilde{\chi}_t\|^2_{L^2(0,s;L^2(\Gamma_c))}\,\mathrm{d}s + c\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)}$$

Hence, we apply the Gronwall Lemma (see, e.g. [15]), and conclude the continuous dependence estimate (47).

In particular, when $\widetilde{\mathbf{u}}_0 = \widetilde{\mathbf{F}} = \mathbf{0}$ and $\widetilde{\chi}_0 = 0$, we of course obtain $\mathbf{u}_1 = \mathbf{u}_2$ a.e. in $\Omega\times(0,\mathrm{t})$ and $\chi_1 = \chi_2$ in $\Gamma_c\times(0,\mathrm{t})$. A comparison in (41) (resp. in (43)), gives also $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$ in $(H^{-1/2}(\Gamma_c))^3$ a.e. in $(0,\mathrm{t})$ (resp. $\xi_1 = \xi_2$ a.e. in $\Gamma_c\times(0,\mathrm{t})$). Thus, the uniqueness statement of Theorem 2 follows. □

4. PROOF OF THEOREM 1

Let us briefly sketch our proof of Theorem 1: as a first step, in Section 4.1 we approximate Problem (P) by regularizing the maximal monotone operators α, ρ, and β. In this way, we obtain a family of problems (P_ε), for each of which we provide a (unique) local in time solution, cf. Section 4.2. Thus, we end up with a sequence of approximate solutions, which we show to converge to a local solution of Problem (P) in Section 4.3.

4.1. Approximating problems

For any $\varepsilon > 0$, we consider the respective Yosida regularizations (cf. [16, Chapter II; 17, Chapter II.1.2])

$$\rho_\varepsilon, \beta_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad \alpha_\varepsilon : (H^{1/2}(\Gamma_c))^3 \to (H^{-1/2}(\Gamma_c))^3$$

It follows from the theory of maximal monotone operators in Hilbert spaces (cf. [16, Proposition II.2.6]) that for any $\varepsilon > 0$ ρ_ε and β_ε are monotone and Lipschitz continuous on $\mathbb{R}$, with Lipschitz constant $1/\varepsilon$. We also recall that

$$|\beta_\varepsilon(r)| \leqslant |\beta^0(r)| \quad \forall r \in \mathrm{dom}(\beta) \tag{64}$$

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In the same way, [17, Proposition II.1.1] ensures that the operator α_ε is single valued, monotone, bounded and demicontinuous from $(H^{1/2}(\Gamma_c))^3$ to $(H^{-1/2}(\Gamma_c))^3$, i.e. $\alpha_\varepsilon(x_n) \rightharpoonup \alpha_\varepsilon(x)$ in $(H^{-1/2}(\Gamma_c))^3$ if $x_n \to x$ in $(H^{1/2}(\Gamma_c))^3$. For any $\varepsilon>0$, we also denote by φ_ε the Moreau–Yosida approximation of φ. For later use, we collect here some properties of φ_ε (see, e.g. [17, pp. 42, 57]). Indeed, for all $\varepsilon>0$, φ_ε is Fréchet differentiable on $(H^{1/2}(\Gamma_c))^3$ and

$$D\varphi_\varepsilon = \alpha_\varepsilon, \qquad 0 \leqslant \varphi_\varepsilon(\mathbf{u}) \leqslant \varphi(\mathbf{u}) \quad \forall \mathbf{u} \in \mathrm{dom}(\varphi) \tag{65}$$

Analogously, we consider the Moreau–Yosida approximation ψ_ε of the functional ψ, which fulfils

$$D\psi_\varepsilon = \beta_\varepsilon, \quad 0 \leqslant \psi_\varepsilon(x) \leqslant \psi(x) \quad \forall x \in \mathrm{dom}(\psi) \tag{66}$$

We may now introduce the following regularized version of Problem (P). For simplicity, the subscript ε will be temporarily omitted in denoting the solution.

Problem (P_ε)
Find $\mathsf{t} \in (0, T]$ and a pair $(\mathbf{u}, \chi)$ such that

$$\mathbf{u} \in H^1(0, \mathsf{t}; W) \tag{67}$$

$$\chi \in W^{1,\infty}(0, \mathsf{t}; L^2(\Gamma_c)) \cap H^1(0, \mathsf{t}; H^1(\Gamma_c)) \cap L^\infty(0, \mathsf{t}; H^2(\Gamma_c)) \tag{68}$$

fulfilling (19) and

$$b(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} (\chi \mathbf{u} + \alpha_\varepsilon(\mathbf{u})) \cdot \mathbf{v} = {}_{W'}\langle \mathbf{F}, \mathbf{v} \rangle_W \quad \forall \mathbf{v} \in W \quad \text{a.e. in } (0, \mathsf{t}) \tag{69}$$

$$\chi_t - \Delta\chi + \rho_\varepsilon(\chi_t) + \beta_\varepsilon(\chi) = w_s - \tfrac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_c \times (0, \mathsf{t}) \tag{70}$$

$$\partial_{\mathbf{n}_s} \chi = 0 \quad \text{a.e. in } \partial\Gamma_c \times (0, \mathsf{t}) \tag{71}$$

Proposition 4.1
Assume (26)–(32). Then, there exists a final time $\widehat{T} \in (0, T]$ such that for any $\varepsilon > 0$ there exists a unique pair $(\mathbf{u}, \chi)$ solving problem (P_ε) on the time interval $(0, \widehat{T})$.

The proof of Proposition 4.1 will be carried out by applying the Schauder theorem to a suitable operator $\mathscr{T}$ which we construct below.

Construction of the Schauder operator: For $R > 0$ and $2 < p < 4$, let us define the set $Y(\widehat{T}, R)$ by

$$Y(\widehat{T}, R) = \{ v \in L^2(0, \widehat{T}; L^p(\Gamma_c)) : \|v\|_{L^2(0,\widehat{T}; L^p(\Gamma_c))} \leqslant R \} \tag{72}$$

where $\widehat{T} \in (0, T]$ will be determined later in such a way that $\mathscr{T} : Y(\widehat{T}, R) \to Y(\widehat{T}, R)$ is a compact and continuous operator.

We consider the following auxiliary problems, whose well posedness is guaranteed by Propositions 4.2 and 4.5 later on.

Let $\widehat{\chi} \in Y(\widehat{T}, R)$ be fixed and let $\overline{\mathbf{u}} := \mathscr{T}_1(\widehat{\chi})$ be the unique solution to the following problem.

Problem $(P_{\varepsilon 1})$
Given $\widehat{\chi}\in Y(\widehat{T},R)$, find $\overline{\mathbf{u}}$ such that

$$\overline{\mathbf{u}} \in H^1(0,\widehat{T};W) \tag{73}$$

$$b(\overline{\mathbf{u}}_t,\mathbf{v})+a(\overline{\mathbf{u}},\mathbf{v})+\int_{\Gamma_c}(\widehat{\chi}\,\overline{\mathbf{u}}+\alpha_\varepsilon(\overline{\mathbf{u}}))\cdot\mathbf{v} = {}_{W'}\langle \mathbf{F},\mathbf{v}\rangle_W \quad \forall \mathbf{v}\in W \text{ a.e. in } (0,\widehat{T}) \tag{74}$$

$$\overline{\mathbf{u}}(\cdot,0)=\mathbf{u}_0 \quad \text{a.e. in } \Omega \tag{75}$$

Now, given such a function $\overline{\mathbf{u}}$, consider its restriction on Γ_c in the sense of traces (which belongs to $H^1(0,\widehat{T};(L^4(\Gamma_c))^3)$, cf. (54)) and let $\overline{\chi}:=\mathscr{T}_2(\overline{\mathbf{u}})$ be the unique solution of the following problem.

Problem $(P_{\varepsilon 2})$
Given $\overline{\mathbf{u}}\in H^1(0,\widehat{T};(L^4(\Gamma_c))^3)$, find $\overline{\chi}$ such that

$$\overline{\chi}\in W^{1,\infty}(0,\widehat{T};L^2(\Gamma_c))\cap H^1(0,\widehat{T};H^1(\Gamma_c))\cap L^\infty(0,\widehat{T};H^2(\Gamma_c)) \tag{76}$$

$$\overline{\chi}_t-\Delta\overline{\chi}+\rho_\varepsilon(\overline{\chi}_t)+\beta_\varepsilon(\overline{\chi}) = w_s-\tfrac{1}{2}|\overline{\mathbf{u}}|^2 \quad \text{a.e. in } \Gamma_c\times(0,\widehat{T}) \tag{77}$$

$$\overline{\chi}(\cdot,0)=\chi_0 \quad \text{a.e. in } \Gamma_c \tag{78}$$

$$\partial_{\mathbf{n}_s}\overline{\chi}=0 \quad \text{a.e. in } \partial\Gamma_c\times(0,\widehat{T}) \tag{79}$$

Finally, we define the operator $\mathscr{T}$ as the composition $\mathscr{T}_2\circ\mathscr{T}_1$. In what follows we show that, at least for small times, the map $\mathscr{T}$ complies with the conditions of the Schauder theorem.

Well posedness for Problem $(P_{\varepsilon 1})$
Hereafter, we shall sometimes use the notation ${}_{W'}\langle\cdot,\cdot\rangle_W$ for the duality pairing ${}_{(H^{-1/2}(\Gamma_c))^3}\langle\cdot,\cdot\rangle_{(H^{1/2}(\Gamma_c))^3}$.

Proposition 4.2
Under assumptions (26), (29), (31) and (32), for any $2<p<4$ and any $\widehat{\chi}\in L^2(0,\widehat{T};L^p(\Gamma_c))$ there exists a unique solution $\mathbf{u}$ to Problem $(P_{\varepsilon 1})$.

Since we are indeed going to obtain a *global* existence result for Problem $(P_{\varepsilon 1})$, to simplify notation we shall directly work on the time interval $(0,T)$ instead of $(0,\widehat{T})$ within the proof of Proposition 4.2.

We are going to prove the existence of solutions to the Cauchy problem (74)–(75) by passing to the limit in a suitable time discretization scheme. Indeed, let us fix a time step $\tau=T/N>0$, $N\in\mathbb{N}$, inducing a partition $t_0=0<t_1<\cdots<t_n<\cdots<t_{N-1}<t_N=T$ of the interval $(0,T)$, with $t_n:=n\tau$, $n\leqslant N$. Accordingly, we define

$$\chi_\tau^n(x):=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\widehat{\chi}(x,t)\,\mathrm{d}t \quad \text{for a.e. } x\in\Gamma_c,\ \ n=1,\ldots,N \tag{80}$$

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Since $\chi_\tau^n \in L^p(\Gamma_c)$ for all $n=1,\dots,N$, for any $\mathbf{w}\in(L^{q_p}(\Gamma_c))^3$, with $1/p+1/q_p+\frac{1}{4}=1$, we have that $\chi_\tau^n\mathbf{w}$ is in $(H^{-1/2}(\Gamma_c))^3$, and

$$\|\chi_\tau^n\mathbf{w}\|_{H^{-1/2}(\Gamma_c)}\leqslant\begin{cases}\|\chi_\tau^n\|_{L^p(\Gamma_c)}\|\mathbf{w}\|_{L^{q_p}(\Gamma_c)}\\ \|\chi_\tau^n\|_{L^2(\Gamma_c)}\|\mathbf{w}\|_{L^4(\Gamma_c)}\end{cases}\tag{81}$$

Furthermore, for all $n=1,\dots,N$ we define $\mathbf{F}_\tau^n\in W'$ by

$$_{W'}\langle\mathbf{F}_\tau^n,\mathbf{v}\rangle_W:=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}{}_{W'}\langle\mathbf{F}(t),\mathbf{v}\rangle_W\,\mathrm{d}t\quad\forall\mathbf{v}\in W\tag{82}$$

We shall construct a sequence of approximate solutions by means of the following finite difference scheme:

$$\begin{aligned}&\text{given }\mathbf{u}_\tau^0:=\mathbf{u}_0,\ \text{find }\mathbf{u}_\tau^1,\dots,\mathbf{u}_\tau^N\in W,\ \text{such that }\forall n=1,\dots,N\\ &b\left(\frac{\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1}}{\tau},\mathbf{v}\right)+a(\mathbf{u}_\tau^n,\mathbf{v})+\int_{\Gamma_c}\chi_\tau^n\mathbf{u}_\tau^{n-1}\cdot\mathbf{v}+\int_{\Gamma_c}\alpha_\varepsilon(\mathbf{u}_\tau^n)\cdot\mathbf{v}\\ &\quad={}_{W'}\langle\mathbf{F}_\tau^n,\mathbf{v}\rangle_W\quad\forall\mathbf{v}\in W\end{aligned}\tag{83}$$

It is easy to see that for any fixed $\tau>0$ (83) admits a solution $\{\mathbf{u}_\tau^n\}_{n=1}^N$: indeed, the above variational equation can be rephrased in the operator form as

$$B\mathbf{u}_\tau^n+\tau A\mathbf{u}_\tau^n+\tau H_\varepsilon\mathbf{u}_\tau^n=\tau J_\tau^n(\mathbf{u}_\tau^{n-1})+B\mathbf{u}_\tau^{n-1}\tag{84}$$

where $A,\ B:W\to W'$ are the operators associated with the W-elliptic forms a and b, while $H_\varepsilon, J_\tau^n:W\to W'$ are defined by

$$_{W'}\langle H_\varepsilon(\mathbf{u}),\mathbf{v}\rangle_W:=\int_{\Gamma_c}\alpha_\varepsilon(\mathbf{u})\cdot\mathbf{v}\quad\forall\mathbf{u},\mathbf{v}\in W$$

$$_{W'}\langle J_\tau^n(\mathbf{u}),\mathbf{v}\rangle_W:=-\int_{\Gamma_c}\chi_\tau^n\mathbf{u}\cdot\mathbf{v}+{}_{W'}\langle\mathbf{F}_\tau^n,\mathbf{v}\rangle_W\quad\forall\mathbf{u},\mathbf{v}\in W$$

Now, owing to [17, Corollary II.1.3], the operator $B+\tau(A+H_\varepsilon)$ is maximal monotone and coercive (by (23)–(24)). Hence, the range of $B+\tau(A+H_\varepsilon)$ is W', and for all $n=1,\dots,N$, (84) admits a solution $\mathbf{u}_\tau^n$, unique by coercivity.

Let $\overline{U}_\tau$ and $\underline{U}_\tau$ be, respectively, the left-continuous and the right-continuous piecewise-constant interpolant of the values $\{\mathbf{u}_\tau^n\}_{n=1}^N$, fulfilling

$$\overline{U}_\tau(t)=\mathbf{u}_\tau^n\ \ \forall t\in(t_{n-1},t_n],\quad \underline{U}_\tau(t)=\mathbf{u}_\tau^{n-1}\ \ \forall t\in[t_{n-1},t_n),\quad n=1,\dots,N\tag{85}$$

We also introduce the piecewise linear interpolant U_τ of $\{\mathbf{u}_\tau^n\}_{n=1}^N$, defined by

$$U_\tau(t):=\frac{t-t_{n-1}}{\tau}\mathbf{u}_\tau^n+\frac{t_n-t}{\tau}\mathbf{u}_\tau^{n-1}\quad\forall t\in[t_{n-1},t_n),\ \ n=1,\dots,N\tag{86}$$

and, finally, we consider the left-continuous piecewise constant interpolants $\overline{\chi}_\tau$ and $\overline{\mathbf{F}}_\tau$ of $\{\chi_\tau^n\}_{n=1}^N$ and of $\{\mathbf{F}_\tau^n\}_{n=1}^N$, respectively. Owing to (31), (32) and $\widehat{\chi}\in L^2(0,T;L^p(\Gamma_c))$, there exists a constant $C>0$ such that for all $\tau>0$ we have the estimates

$$\|\overline{\chi}_\tau\|_{L^2(0,T;L^p(\Gamma_c))}+\|\overline{\mathbf{F}}_\tau\|_{L^2(0,T;W')}\leqslant C \tag{87}$$

Furthermore, we have

$$\begin{aligned}\overline{\chi}_\tau &\to \widehat{\chi} \quad \text{in } L^2(0,T;L^p(\Gamma_c)) \text{ as } \tau\downarrow 0\\ \overline{\mathbf{F}}_\tau &\to \mathbf{F} \quad \text{in } L^2(0,T;W') \text{ as } \tau\downarrow 0\end{aligned} \tag{88}$$

We may thus reformulate the discrete equation (83) as

$$\begin{aligned}b({U_\tau}'(t),\mathbf{v})+a(\overline{U}_\tau(t),\mathbf{v})+\int_{\Gamma_c}\overline{\chi}_\tau(t)\underline{U}_\tau(t)\cdot\mathbf{v}+\int_{\Gamma_c}\alpha_\varepsilon(\overline{U}_\tau(t))\cdot\mathbf{v}\\ ={}_{W'}\langle\overline{\mathbf{F}}_\tau(t),\mathbf{v}\rangle_W \quad \forall\mathbf{v}\in W \ \text{ for a.e. } t\in(0,T)\end{aligned} \tag{89}$$

We shall need the following *discrete* version of the Gronwall Lemma (see, e.g. [18, Proposition 2.2.1]).

Lemma 4.3
Let $\psi,\alpha_0,\alpha_1,\ldots,\alpha_n,\ x_0,x_1,\ldots,x_n$ be given non-negative numbers such that

$$x_0\leqslant\psi,\quad x_i\leqslant\psi+\sum_{j=0}^{i-1}\alpha_j x_j \quad \forall 1\leqslant i\leqslant n$$

Then, we have

$$x_i\leqslant\psi\exp\left(\sum_{j=0}^{i-1}\alpha_j\right)\quad\forall 1\leqslant i\leqslant n$$

Lemma 4.4 (*A priori estimates on the approximate solutions*)
Assume (26), (29), (31) and (32). Then, there exists a positive constant C such that for all $\tau>0$

$$\|\overline{U}_\tau\|_{L^\infty(0,T;W)}+\|\underline{U}_\tau\|_{L^\infty(0,T;W)}+\|U_\tau\|_{L^\infty(0,T;W)}\leqslant C \tag{90}$$

$$\|{U_\tau}'\|_{L^2(0,T;W)}\leqslant C \tag{91}$$

$$\|\alpha_\varepsilon(\overline{U}_\tau)\|_{L^2(0,T;W')}\leqslant C \tag{92}$$

Proof
Hereafter, we shall use the short-hand notation

$$\Delta_\tau\mathbf{u}_\tau^n:=\frac{\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1}}{\tau}$$

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We choose the test function $\mathbf{v}:=\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1}$ in (83). Taking into account (24) and the relation $\alpha_\varepsilon=D\varphi_\varepsilon$ (cf. (65)), so that

$$\int_{\Gamma_c}\alpha_\varepsilon(\mathbf{u}_\tau^n)\cdot(\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1})\geqslant\varphi_\varepsilon(\mathbf{u}_\tau^n)-\varphi_\varepsilon(\mathbf{u}_\tau^{n-1})$$

we end up with

$$\begin{aligned}
&C_b\tau\|\Delta_\tau\mathbf{u}_\tau^n\|_W^2+a(\mathbf{u}_\tau^n,\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1})+\varphi_\varepsilon(\mathbf{u}_\tau^n)\\
&\quad\leqslant\varphi_\varepsilon(\mathbf{u}_\tau^{n-1})+\tau\left|\int_{\Gamma_c}\chi_\tau^n\mathbf{u}_\tau^{n-1}\cdot\Delta_\tau\mathbf{u}_\tau^n\right|+|{}_{W'}\langle\mathbf{F}_\tau^n,\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1}\rangle_W|\\
&\quad\leqslant\varphi_\varepsilon(\mathbf{u}_\tau^{n-1})+\tau\|\chi_\tau^n\|_{L^2(\Gamma_c)}\|\mathbf{u}_\tau^{n-1}\|_{L^4(\Gamma_c)}\|\Delta_\tau\mathbf{u}_\tau^n\|_{L^4(\Gamma_c)}+\tau\|\mathbf{F}_\tau^n\|_{W'}\|\Delta_\tau\mathbf{u}_\tau^n\|_W\\
&\quad\leqslant\varphi_\varepsilon(\mathbf{u}_\tau^{n-1})+2\sigma\tau\|\Delta_\tau\mathbf{u}_\tau^n\|_W^2+\frac{c\tau}{4\sigma}\|\chi_\tau^n\|_{L^2(\Gamma_c)}^2\|\mathbf{u}_\tau^{n-1}\|_W^2+\frac{\tau}{4\sigma}\|\mathbf{F}_\tau^n\|_{W'}^2
\end{aligned}\tag{93}$$

for some $0<\sigma<C_b/4$, the latter estimate following from (55) and (54). Summing up (93) for $n=1,\dots,k,\ k\leqslant N$, using the elementary identity

$$a(\mathbf{v},\mathbf{v}-\mathbf{w})=\tfrac{1}{2}a(\mathbf{v},\mathbf{v})-\tfrac{1}{2}a(\mathbf{w},\mathbf{w})+\tfrac{1}{2}a(\mathbf{v}-\mathbf{w},\mathbf{v}-\mathbf{w})\quad\forall\mathbf{v},\mathbf{w}\in W\tag{94}$$

the coercivity estimate (23), as well as the second of (65), we obtain

$$\begin{aligned}
\frac{C_b}{2}\sum_{n=1}^k\tau\|\Delta_\tau\mathbf{u}_\tau^n\|_W^2+\frac{1}{2}C_a\|\mathbf{u}_\tau^k\|_W^2+\varphi_\varepsilon(\mathbf{u}_\tau^k)&\leqslant\frac{1}{2}a(\mathbf{u}_0,\mathbf{u}_0)+\varphi(\mathbf{u}_0)+\frac{c\tau}{4\sigma}\|\chi_\tau^1\|_{L^2(\Gamma_c)}^2\|\mathbf{u}_0\|_W^2\\
&\quad+\frac{c\tau}{4\sigma}\sum_{n=1}^k\|\chi_\tau^{n+1}\|_{L^2(\Gamma_c)}^2\|\mathbf{u}_\tau^n\|_W^2+\frac{\tau}{4\sigma}\sum_{n=1}^k\|\mathbf{F}_\tau^n\|_{W'}^2
\end{aligned}$$

(with the convention $\chi_\tau^{N+1}=0$). Hence, recalling the bounds (87), using the positivity of φ_ε and applying Lemma 4.3, we easily find that there exists a constant C such that

$$\|\mathbf{u}_\tau^k\|_W^2+\sum_{n=1}^k\tau\|\Delta_\tau\mathbf{u}_\tau^n\|_W^2\leqslant C$$

whence (90) and (91).

In the end, (92) follows by comparison in (89), combining (87), (90)–(91), and

$$\|\overline{\chi}_\tau\underline{U}_\tau\|_{L^2(0,T;H^{-1/2}(\Gamma_c))}\leqslant\|\overline{\chi}_\tau\|_{L^2(0,T;L^2(\Gamma_c))}\|\underline{U}_\tau\|_{L^\infty(0,T;L^4(\Gamma_c))}\leqslant C$$

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which ensues from (81), (87), (90) and (54). For later convenience, let us also note that, in view of (83) and of the identity (94) for the form $b(\cdot,\cdot)$, we have for all $n=1,\dots,N$

$$\int_{t_{n-1}}^{t_n}\int_{\Gamma_c}\alpha_\varepsilon(\mathbf{u}_\tau^n)\cdot\mathbf{u}_\tau^n=\tau\int_{\Gamma_c}\alpha_\varepsilon(\mathbf{u}_\tau^n)\cdot\mathbf{u}_\tau^n=-\tau a(\mathbf{u}_\tau^n,\mathbf{u}_\tau^n)-\tau\int_{\Gamma_c}\chi_\tau^n\mathbf{u}_\tau^{n-1}\cdot\mathbf{u}_\tau^n+\tau_{W'}\langle\mathbf{F}_\tau^n,\mathbf{u}_\tau^n\rangle_W-\frac{1}{2}b(\mathbf{u}_\tau^n,\mathbf{u}_\tau^n)$$
$$+\frac{1}{2}b(\mathbf{u}_\tau^{n-1},\mathbf{u}_\tau^{n-1})-\frac{1}{2}b(\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1},\mathbf{u}_\tau^n-\mathbf{u}_\tau^{n-1})$$
$$\leqslant-\int_{t_{n-1}}^{t_n}a(\mathbf{u}_\tau^n,\mathbf{u}_\tau^n)-\int_{t_{n-1}}^{t_n}\int_{\Gamma_c}\chi_\tau^n\mathbf{u}_\tau^{n-1}\cdot\mathbf{u}_\tau^n$$
$$+\int_{t_{n-1}}^{t_n}{}_{W'}\langle\mathbf{F}_\tau^n,\mathbf{u}_\tau^n\rangle_W-\frac{1}{2}b(\mathbf{u}_\tau^n,\mathbf{u}_\tau^n)+\frac{1}{2}b(\mathbf{u}_\tau^{n-1},\mathbf{u}_\tau^{n-1})\qquad(95)$$

□

Proof of Proposition 4.2
Passage to the limit and existence: Preliminarily, note that (91) yields

$$\|\overline{U}_\tau-U_\tau\|_{L^\infty(0,T;W)}\leqslant C\tau^{1/2}\qquad(96)$$

and an analogous estimate holds for $\underline{U}_\tau$.

Standard weak compactness results ensure that, for any vanishing sequence $k\mapsto\tau_k\downarrow 0$ of time steps, there exist a subsequence (still labeled τ_k), and a limit function $\mathbf{u}\in H^1(0,T;W)$ such that as $k\to+\infty$

$$U_{\tau_k}\rightharpoonup\mathbf{u}\quad\text{weakly in } H^1(0,T;W)\qquad(97)$$

By (90)–(91), (96) and [19, Corollary 4], we also conclude

$$\begin{gathered}\overline{U}_{\tau_k},\underline{U}_{\tau_k},U_{\tau_k}\rightharpoonup^*\mathbf{u}\quad\text{in } L^\infty(0,T;W)\\ \overline{U}_{\tau_k}(t),\underline{U}_{\tau_k}(t),U_{\tau_k}(t)\rightharpoonup\mathbf{u}(t)\quad\text{in } W\ \ \forall t\in[0,T]\end{gathered}\qquad(98)$$

the latter convergence being due to a version of the Ascoli-Arzelà theorem in the framework of the weak topology of W. Furthermore, $\overline{U}_{\tau_k},\underline{U}_{\tau_k},U_{\tau_k}\to\mathbf{u}$ in $L^2(0,T;(L^q(\Gamma_c))^3)$ for all $q<4$, due to the compact embedding $W\subset(L^q(\Gamma_c))^3$. In particular, we have

$$\underline{U}_{\tau_k}\to\mathbf{u}\quad\text{in } L^2(0,T;(L^{q_p}(\Gamma_c))^3)\ \text{ with } q_p \text{ s.t. } \frac{1}{p}+\frac{1}{q_p}+\frac{1}{4}=1\qquad(99)$$

Then, combining the estimate (cf. (81))

$$\|\overline{\chi}_{\tau_k}\underline{U}_{\tau_k}-\widehat{\chi}\mathbf{u}\|_{L^1(0,T;H^{-1/2}(\Gamma_c))}\leqslant\|\overline{\chi}_{\tau_k}-\widehat{\chi}\|_{L^2(0,T;L^2(\Gamma_c))}\|\underline{U}_{\tau_k}\|_{L^2(0,T;L^4(\Gamma_c))}$$
$$+\|\widehat{\chi}\|_{L^2(0,T;L^p(\Gamma_c))}\|\underline{U}_{\tau_k}-\mathbf{u}\|_{L^2(0,T;L^{q_p}(\Gamma_c))}$$

with the first of (88) and with (99), we end up with

$$\overline{\chi}_{\tau_k}\underline{U}_{\tau_k}\to\widehat{\chi}\mathbf{u}\quad\text{in } L^1(0,T;(H^{-1/2}(\Gamma_c))^3)\qquad(100)$$

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Finally, thanks to (92) there exists $\overline{\alpha}\in L^2(0,T;W')$ such that, up to a further extraction,

$$\alpha_\varepsilon(\overline{U}_{\tau_k})\rightharpoonup\overline{\alpha} \quad \text{in } L^2(0,T;W') \tag{101}$$

In view of (88), (97), (98), (100), and (101), we deduce $\mathbf{u}(\cdot,0)=\mathbf{u}_0$ and pass to the limit in (89). Hence, $\mathbf{u}$ fulfils

$$b(\mathbf{u}_t,\mathbf{v})+a(\mathbf{u},\mathbf{v})+\int_{\Gamma_c}(\widehat{\chi}\mathbf{u}+\overline{\alpha})\cdot\mathbf{v}={}_{W'}\langle\mathbf{F},\mathbf{v}\rangle_W \quad \forall\mathbf{v}\in W \quad \text{a.e. in } (0,T) \tag{102}$$

Furthermore, we note that for all $t\in[0,T]$

$$\begin{aligned}
&\limsup_{k\to\infty}\int_0^t\int_{\Gamma_c}\alpha_\varepsilon(\overline{U}_{\tau_k})\cdot\overline{U}_{\tau_k}\\
&\leqslant\limsup_{k\to\infty}\left(-\int_0^t a(\overline{U}_{\tau_k},\overline{U}_{\tau_k})-\frac{1}{2}b(\overline{U}_{\tau_k}(t),\overline{U}_{\tau_k}(t))\right)\\
&\quad+\frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0)+\lim_{k\to\infty}\left(-\int_0^t\int_{\Gamma_c}\overline{\chi}_{\tau_k}\underline{U}_{\tau_k}\cdot\overline{U}_{\tau_k}+\int_0^t{}_{W'}\langle\overline{\mathbf{F}}_{\tau_k},\overline{U}_{\tau_k}\rangle_W\right)\\
&\leqslant-\int_0^t a(\mathbf{u},\mathbf{u})-\frac{1}{2}b(\mathbf{u}(t),\mathbf{u}(t))+\frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0)-\int_0^t\int_{\Gamma_c}\widehat{\chi}\mathbf{u}\cdot\mathbf{u}+\int_0^t{}_{W'}\langle\mathbf{F},\mathbf{u}\rangle_W\\
&=\int_0^t\int_{\Gamma_c}\overline{\alpha}\cdot\mathbf{u}
\end{aligned}$$

Indeed, the first inequality easily follows from (95), the second one from the convergences (88), (97), (98), (100), and (101) (also arguing by lower semicontinuity), while the third passage is a consequence of (102). Owing to the monotonicity of α_ε (see [17, Lemma II.1.3]), we may conclude that $\overline{\alpha}=\alpha_\varepsilon(\mathbf{u})$, and hence that $\mathbf{u}$ solves Problem $(P_{\varepsilon 1})$. Finally, the uniqueness statement in Proposition 4.2 can be deduced arguing as in the proof of Theorem 2 (see (57)–(60)). □

Well posedness for Problem $(P_{\varepsilon 2})$.

Proposition 4.5
Assume (27)–(30). Then, for any $\overline{\mathbf{u}}\in H^1(0,\widehat{T};(L^4(\Gamma_c))^3)$ there exists a unique solution of Problem $(P_{\varepsilon 2})$.

Proof of Proposition 4.5
Since $\overline{\mathbf{u}}\in H^1(0,\widehat{T};(L^4(\Gamma_c))^3)$, the right-hand side of (77) is assigned in $H^1(0,\widehat{T};(L^2(\Gamma_c))^3)$ and a solution of (77)–(79) enjoying the regularity specified by (76) can be found by exploiting a time discretization procedure, following the steps of [20, Section 3]. Finally, since β_ε is Lipschitz continuous, the uniqueness result can be deduced in the same way as in the proof of Theorem 2 (see (61)–(63)). □

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4.2. Proof of Proposition 4.1

In order to prove Proposition 4.1, we will show that there exists $\widehat{T}>0$ such that the operator $\mathscr{T}$ complies with the following properties:

$$\mathscr{T} \text{ maps } Y(\widehat{T}, R) \text{ into itself}$$
$$\mathscr{T} \text{ is compact}$$
$$\mathscr{T} \text{ is continuous}$$

To prove the first claim, we need to obtain suitable *a priori* bounds on $\overline{\mathbf{u}}$ and $\overline{\chi}$. We test (74) by $\overline{\mathbf{u}}_t$ and we integrate from 0 to t, with $0<t<\widehat{T}$. Owing to (23), (24), the Hölder inequality, (54), (55), and (65), we have

$$\begin{aligned}
&C_b\|\overline{\mathbf{u}}_t\|^2_{L^2(0,t;W)}+\frac{C_a}{2}\|\overline{\mathbf{u}}(t)\|^2_W+\varphi_\varepsilon(\overline{\mathbf{u}}(t))\\
&\leqslant\frac{c}{2}\|\mathbf{u}_0\|^2_W+\varphi_\varepsilon(\mathbf{u}_0)+\int_0^t\|\widehat{\chi}\|_{L^2(\Gamma_c)}\|\overline{\mathbf{u}}\|_{L^4(\Gamma_c)}\|\overline{\mathbf{u}}_t\|_{L^4(\Gamma_c)}+\|\mathbf{F}\|_{L^2(0,T;W')}\|\overline{\mathbf{u}}_t\|_{L^2(0,t;W)}\\
&\leqslant\frac{c}{2}\|\mathbf{u}_0\|^2_W+\varphi(\mathbf{u}_0)+\frac{C_b}{2}\|\overline{\mathbf{u}}_t\|^2_{L^2(0,t;W)}+c\int_0^t\|\widehat{\chi}\|^2_{L^2(\Gamma_c)}\|\overline{\mathbf{u}}\|^2_W+c\|\mathbf{F}\|^2_{L^2(0,T;W')}
\end{aligned}\tag{103}$$

Recalling that, by the definition of $Y(\widehat{T},R)$, $\|\widehat{\chi}\|^2_{L^2(\Gamma_c)}$ is bounded in $L^1(0,\widehat{T})$, we apply the Gronwall Lemma to (103), and we deduce that there exists a positive constant c_1 such that

$$\|\overline{\mathbf{u}}\|_{H^1(0,\widehat{T};W)}\leqslant c_1\tag{104}$$

Next, we multiply (77) by $\overline{\chi}_t$ and we integrate over $\Gamma_c\times(0,t)$. Taking the monotonicity of ρ_ε into account, applying the Hölder inequality, and again using (54), (55), (66), we conclude

$$\begin{aligned}
&\|\overline{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))}+\frac{1}{2}\|\nabla\overline{\chi}(t)\|^2_{L^2(\Gamma_c)}+\int_{\Gamma_c}\psi_\varepsilon(\overline{\chi}(t))\\
&\leqslant\frac{1}{2}\|\nabla\chi_0\|^2_{L^2(\Gamma_c)}+\int_{\Gamma_c}\psi_\varepsilon(\chi_0)+\frac{1}{2}\int_0^t\|\overline{\chi}_t\|_{L^2(\Gamma_c)}\|\overline{\mathbf{u}}\|^2_{L^4(\Gamma_c)}+\frac{1}{2}\|\overline{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))}+c\\
&\leqslant\frac{1}{2}\|\nabla\chi_0\|^2_{L^2(\Gamma_c)}+\int_{\Gamma_c}\psi(\chi_0)+\frac{3}{4}\|\overline{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))}+c\|\overline{\mathbf{u}}\|^4_{L^4(0,t;W)}+c
\end{aligned}\tag{105}$$

Also thanks to (104), we deduce that there exists a positive constant c_2 such that

$$\|\overline{\chi}\|_{H^1(0,\widehat{T};L^2(\Gamma_c))\cap L^\infty(0,\widehat{T};H^1(\Gamma_c))}\leqslant c_2\tag{106}$$

Now, our aim is to find $\widehat{T}>0$ such that the operator $\mathscr{T}:Y(\widehat{T},R)\to Y(\widehat{T},R)$ turns out to be well defined. Exploiting (106) and using some standard Sobolev's embeddings, we find

$$\|\overline{\chi}\|_{L^2(0,\widehat{T};L^p(\Gamma_c))}\leqslant(\widehat{T})^{1/2}\|\overline{\chi}\|_{L^\infty(0,\widehat{T};L^p(\Gamma_c))}\leqslant c_3(\widehat{T})^{1/2}\tag{107}$$

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for some positive constant c_3. Hence, we fix $\widehat{T}$ such that

$$c_3(\widehat{T})^{1/2} \leqslant R \tag{108}$$

and conclude that $\overline{\chi}$ belongs to $Y(\widehat{T}, R)$.

Next, we observe that the same argument leading to (106) ensures that $\mathscr{T}$ is a compact operator.

Hence, to achieve the proof of the Schauder theorem, it remains to show that $\mathscr{T}$ is continuous with respect to the natural topology induced in $Y(\widehat{T}, R)$ by $L^2(0, \widehat{T}; L^p(\Gamma_c))$. To this aim, we consider a sequence $\widehat{\chi}_n$ in $Y(\widehat{T}, R)$ such that

$$\widehat{\chi}_n \to \widehat{\chi} \text{ strongly in } Y(\widehat{T}, R) \quad \text{as } n \to +\infty \tag{109}$$

Then, we set $\overline{\mathbf{u}}_n := \mathscr{T}_1(\widehat{\chi}_n)$, $\overline{\mathbf{u}} := \mathscr{T}_1(\widehat{\chi})$, and $\widetilde{\mathbf{u}}_n := \overline{\mathbf{u}}_n - \overline{\mathbf{u}}$, we take the difference between the corresponding equations (74), test it by $\widetilde{\mathbf{u}}_n$, and integrate on $(0, t)$. Thanks to (23) and (24), we find

$$\begin{aligned} &\frac{C_b}{2}\|\widetilde{\mathbf{u}}_n(t)\|_W^2 + C_a\|\widetilde{\mathbf{u}}_n\|_{L^2(0,t;W)}^2 + \int_0^t \int_{\Gamma_c} (\alpha_\varepsilon(\overline{\mathbf{u}}_n) - \alpha_\varepsilon(\overline{\mathbf{u}}))(\overline{\mathbf{u}}_n - \overline{\mathbf{u}}) \\ &\quad \leqslant \int_0^t \int_{\Gamma_c} (\widehat{\chi} - \widehat{\chi}_n)\overline{\mathbf{u}} \cdot \widetilde{\mathbf{u}}_n - \int_0^t \int_{\Gamma_c} \widehat{\chi}_n |\widetilde{\mathbf{u}}_n|^2 \end{aligned} \tag{110}$$

The integral on the left-hand side of (110) is non-negative, thanks to the monotonicity of α_ε. Moreover, we estimate the right-hand side of (110) arguing as in (58)–(59) and we obtain

$$\begin{aligned} &\int_0^t \int_{\Gamma_c} (\widehat{\chi} - \widehat{\chi}_n)\overline{\mathbf{u}} \cdot \widetilde{\mathbf{u}}_n - \int_0^t \int_{\Gamma_c} \widehat{\chi}_n |\widetilde{\mathbf{u}}_n|^2 \\ &\quad \leqslant c\|\widehat{\chi}_n - \widehat{\chi}\|_{L^2(0,t;L^2(\Gamma_c))}^2 + c\int_0^t (\|\overline{\mathbf{u}}\|_W^2 + \|\widehat{\chi}_n\|_{L^2(\Gamma_c)})\|\widetilde{\mathbf{u}}_n\|_W^2 \end{aligned} \tag{111}$$

for some positive constant c independent of n. Finally, we apply the Gronwall Lemma to (110) on account of estimate (111): the convergence specified by (109) allows us to deduce

$$\|\overline{\mathbf{u}}_n - \overline{\mathbf{u}}\|_{L^\infty(0,\widehat{T};W)} \to 0 \quad \text{as } n \to +\infty \tag{112}$$

Now, we set $\overline{\chi}_n := \mathscr{T}_2(\overline{\mathbf{u}}_n) = \mathscr{T}(\widehat{\chi}_n)$, $\overline{\chi} := \mathscr{T}_2(\overline{\mathbf{u}}) = \mathscr{T}(\widehat{\chi})$, and $\widetilde{\chi}_n := \overline{\chi}_n - \overline{\chi}$ and we consider the difference between the corresponding equations (77), multiply it by $\widetilde{\chi}_{n,t}$, and integrate over $\Gamma_c \times (0, t)$. Taking the monotonicity of ρ_ε into account, we find

$$\begin{aligned} &\|\widetilde{\chi}_{n,t}\|_{L^2(0,t;L^2(\Gamma_c))}^2 + \frac{1}{2}\|\nabla\widetilde{\chi}_n(t)\|_{L^2(\Gamma_c)}^2 + \int_0^t \int_{\Gamma_c} (\beta_\varepsilon(\overline{\chi}_n) - \beta_\varepsilon(\overline{\chi}))\widetilde{\chi}_{n,t} \\ &\quad \leqslant -\frac{1}{2}\int_0^t \int_{\Gamma_c} (\overline{\mathbf{u}}_n + \overline{\mathbf{u}})\widetilde{\mathbf{u}}_n \widetilde{\chi}_{n,t} \end{aligned} \tag{113}$$

Owing also to the Lipschitz continuity of β_ε (with Lipschitz constant $1/\varepsilon$), we have

$$\begin{aligned}\int_0^t \int_{\Gamma_c} (\beta_\varepsilon(\overline{\chi}_n)-\beta_\varepsilon(\overline{\chi}))\widetilde{\chi}_{n,t} &\leqslant \frac{1}{\varepsilon}\int_0^t \|\widetilde{\chi}_n\|_{L^2(\Gamma_c)}\|\widetilde{\chi}_{n,t}\|_{L^2(\Gamma_c)}\\ &\leqslant \frac{1}{4}\|\widetilde{\chi}_{n,t}\|^2_{L^2(0,t;L^2(\Gamma_c))}+c(\varepsilon)\|\widetilde{\chi}_n\|^2_{L^2(0,t;L^2(\Gamma_c))}\\ &\leqslant \frac{1}{4}\|\widetilde{\chi}_{n,t}\|^2_{L^2(0,t;L^2(\Gamma_c))}+c(\varepsilon)\int_0^t \|\widetilde{\chi}_{n,t}\|^2_{L^2(0,s;L^2(\Gamma_c))}\,\mathrm{d}s \end{aligned} \tag{114}$$

where $c(\varepsilon)$ is a positive constant (independent of n) due to (55). Moreover (cf. (63)), we have

$$\int_0^t \int_{\Gamma_c} (\overline{\mathbf{u}}_n+\overline{\mathbf{u}})\,\widetilde{\mathbf{u}}_n\,\widetilde{\chi}_{n,t} \leqslant \frac{1}{4}\|\widetilde{\chi}_{n,t}\|^2_{L^2(0,t;L^2(\Gamma_c))}+c\|\overline{\mathbf{u}}_n+\overline{\mathbf{u}}\|^2_{L^\infty(0,t;W)}\,\|\widetilde{\mathbf{u}}_n\|^2_{L^2(0,t;W)} \tag{115}$$

for some positive constant c independent of n. Finally, we apply the Gronwall Lemma to (113) on account of (114)–(115): the convergence specified by (112) allows us to deduce that

$$\|\overline{\chi}_n-\overline{\chi}\|_{H^1(0,\widehat{T};L^2(\Gamma_c))\cap L^\infty(0,\widehat{T};H^1(\Gamma_c))}\to 0 \tag{116}$$

as $n\to+\infty$. Hence,

$$\|\overline{\chi}_n-\overline{\chi}\|_{L^2(0,\widehat{T};L^p(\Gamma_c))}\to 0 \tag{117}$$

which concludes the proof of the continuity of the operator $\mathscr{T}$.

Thus, we have proved that $\mathscr{T}$ has a fixed point in $Y(\widehat{T},R)$, i.e. there exists a local in time solution of the Problem (P_ε), defined on the interval $(0,\widehat{T})$.

Finally, the uniqueness result in Proposition 4.1 follows by applying Theorem 2 (recall that the nonlinearity β_ε is Lipschitz continuous).

Remark 4.6

Let us stress that all the constants in estimates (103)–(108) are independent of ε. Thus, the final time $\widehat{T}$ of the solution provided by the fixed point procedure does not depend on ε. This will be crucial in the sequel.

A closer examination of the proof of Proposition 4.1 shows that $\widehat{T}$ does not depend on the specific initial conditions $\mathbf{u}_0$, χ_0 either, but only on quantities related to them. In other words, for any 'ball' of initial data

$$\mathscr{I}(r):=\{(\overline{\mathbf{u}}_0,\overline{\chi}_0)\in W\times H^2(\Gamma_c):\ \|\overline{\mathbf{u}}_0\|_W+\|\overline{\chi}_0\|_{H^2(\Gamma_c)}+\varphi(\overline{\mathbf{u}}_0)+\|\psi(\overline{\chi}_0)\|_{L^1(\Gamma_c)}\leqslant r\}$$

there exists a final time $\widehat{T}_r>0$, only depending on r and on C_a, C_b, w_s, Ω, Γ_c, and $\|\mathbf{F}\|_{L^2(0,T;W')}$, such that, for any $(\overline{\mathbf{u}}_0,\overline{\chi}_0)\in\mathscr{I}(r)$ and for any $\varepsilon>0$, Problem P_ε supplemented with the initial data $(\overline{\mathbf{u}}_0,\overline{\chi}_0)$ admits a (unique) solution on the interval $(0,\widehat{T}_r)$.

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4.3. Local existence for Problem (P)

Hereafter, we shall denote by $(\mathbf{u}_\varepsilon, \chi_\varepsilon)$ the (unique) solution of Problem (P_ε), whose existence is guaranteed on the interval $(0,\widehat{T})$ ($\widehat{T}$ being as in (108)) by Proposition 4.1. In this section, we perform an asymptotic analysis as $\varepsilon\downarrow 0$ of the sequence $\{(\mathbf{u}_\varepsilon, \chi_\varepsilon)\}_\varepsilon$ on the existence interval $(0,\widehat{T})$. In fact, such an analysis is doable in view of Remark 4.6. We thus obtain the following result.

Proposition 4.7
Under the assumptions of Theorem 1, there exist a vanishing sequence $\{\varepsilon_k\}_k$ and a quintuple $(\widehat{\mathbf{u}},\widehat{\chi},\widehat{\boldsymbol{\eta}},\widehat{\xi},\widehat{\zeta})$, with

$$\widehat{\mathbf{u}}\in H^1(0,\widehat{T};W),\quad \widehat{\chi}\in W^{1,\infty}(0,\widehat{T};L^2(\Gamma_c))\cap H^1(0,\widehat{T};H^1(\Gamma_c))\cap L^\infty(0,\widehat{T};H^2(\Gamma_c))$$
$$\widehat{\boldsymbol{\eta}}\in L^2(0,\widehat{T};(H^{-1/2}(\Gamma_c))^3),\quad \widehat{\xi}\in L^\infty(0,\widehat{T};L^2(\Gamma_c)),\quad \widehat{\zeta}\in L^\infty(0,\widehat{T};L^2(\Gamma_c))$$

such that the following convergences hold as $k\to\infty$:

$$\mathbf{u}_{\varepsilon_k}\rightharpoonup\widehat{\mathbf{u}}\quad\text{in } H^1(0,\widehat{T};W) \tag{118}$$
$$\mathbf{u}_{\varepsilon_k}\to\widehat{\mathbf{u}}\quad\text{in } C^0([0,\widehat{T}];(H^{1-s}(\Omega))^3)\quad \forall s>0 \tag{119}$$
$$\mathbf{u}_{\varepsilon_k}\to\widehat{\mathbf{u}}\quad\text{in } C^0([0,\widehat{T}];(L^p(\Gamma_c))^3)\quad \forall 1\leqslant p<4 \tag{120}$$
$$\chi_{\varepsilon_k}\overset{*}{\rightharpoonup}\widehat{\chi}\quad\text{in } W^{1,\infty}(0,\widehat{T};L^2(\Gamma_c))\cap H^1(0,\widehat{T};H^1(\Gamma_c))\cap L^\infty(0,\widehat{T};H^2(\Gamma_c)) \tag{121}$$
$$\chi_{\varepsilon_k}\to\widehat{\chi}\quad\text{in } C^0([0,\widehat{T}];H^{2-s}(\Gamma_c))\quad \forall s>0 \tag{122}$$
$$\alpha_{\varepsilon_k}(\mathbf{u}_{\varepsilon_k})\rightharpoonup\widehat{\boldsymbol{\eta}}\quad\text{in } L^2(0,\widehat{T};(H^{-1/2}(\Gamma_c))^3) \tag{123}$$
$$\beta_{\varepsilon_k}(\chi_{\varepsilon_k})\overset{*}{\rightharpoonup}\widehat{\zeta}\quad\text{in } L^\infty(0,\widehat{T};L^2(\Gamma_c)) \tag{124}$$
$$\rho_{\varepsilon_k}(\chi_{\varepsilon_k t})\overset{*}{\rightharpoonup}\widehat{\xi}\quad\text{in } L^\infty(0,\widehat{T};L^2(\Gamma_c)) \tag{125}$$

Besides, $(\widehat{\mathbf{u}},\widehat{\chi},\widehat{\boldsymbol{\eta}},\widehat{\xi},\widehat{\zeta})$ is a solution of Problem (P) on $(0,\widehat{T})$.

In this way, we have in fact proved the existence of a local solution for Problem (P).

Proof
As a first step, we shall obtain some *a priori* estimates for (suitable norms of) the sequence $\{(\mathbf{u}_\varepsilon, \chi_\varepsilon)\}_\varepsilon$ on the interval $(0,\widehat{T})$. Indeed, it follows from the previous estimates (104) and (106) (cf. also Remark 4.6) that there exists a positive constant c, independent of ε, such that

$$\|\mathbf{u}_\varepsilon\|_{H^1(0,\widehat{T};W)}\leqslant c \tag{126}$$

and

$$\|\chi_\varepsilon\|_{H^1(0,\widehat{T};L^2(\Gamma_c))\cap L^\infty(0,\widehat{T};H^1(\Gamma_c))}\leqslant c \tag{127}$$

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Moreover, by comparison in (69), on account of (126)–(127), we obtain

$$\|\alpha_\varepsilon(\mathbf{u}_\varepsilon)\|_{L^2(0,\widehat{T};H^{-1/2}(\Gamma_c))}\leqslant c \tag{128}$$

Next, we establish a further regularity estimate on χ_ε. We multiply (70) by $(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))_t$ and we integrate over $\Gamma_c\times(0,t)$. We have

$$\begin{aligned}&\frac{1}{2}\|-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\|^2_{L^2(\Gamma_c)}+\|\nabla\chi_{\varepsilon t}\|^2_{L^2(0,t;L^2(\Gamma_c))}+\int_0^t\int_{\Gamma_c}\beta'_\varepsilon(\chi_\varepsilon)(\chi_{\varepsilon t})^2\\&\quad+\int_0^t\int_{\Gamma_c}\rho'_\varepsilon(\chi_{\varepsilon t})|\nabla\chi_{\varepsilon t}|^2+\int_0^t\int_{\Gamma_c}\beta'_\varepsilon(\chi_\varepsilon)\rho_\varepsilon(\chi_{\varepsilon t})\chi_{\varepsilon t}\\&\quad\leqslant\frac{1}{2}\|-\Delta\chi_0+\beta_\varepsilon(\chi_0)\|^2_{L^2(\Gamma_c)}+\int_0^t\int_{\Gamma_c}\left(w_s-\frac{1}{2}|\mathbf{u}_\varepsilon|^2\right)(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))_t\end{aligned} \tag{129}$$

We note that the integrals on the left-hand side of (129) are non-negative, thanks to the monotonicity of ρ_ε and of β_ε, as well as (27). Moreover, an integration by parts leads to

$$\begin{aligned}&\int_0^t\int_{\Gamma_c}\left(w_s-\frac{1}{2}|\mathbf{u}_\varepsilon|^2\right)(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))_t\\&\quad=\int_0^t\int_{\Gamma_c}(\mathbf{u}_\varepsilon\cdot\mathbf{u}_{\varepsilon t})(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))\\&\qquad+\int_{\Gamma_c}\left(w_s-\frac{1}{2}|\mathbf{u}_\varepsilon|^2(t)\right)(-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t)))-\int_{\Gamma_c}\left(w_s-\frac{1}{2}|\mathbf{u}_0|^2\right)(-\Delta\chi_0+\beta_\varepsilon(\chi_0))\\&\quad\leqslant c\int_0^t\|\mathbf{u}_\varepsilon\|_W\|\mathbf{u}_{\varepsilon t}\|_W\|\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)\|_{L^2(\Gamma_c)}+\frac{1}{4}\|-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\|^2_{L^2(\Gamma_c)}+c\|\mathbf{u}_\varepsilon(t)\|^4_W+c\\&\quad\leqslant c\|\mathbf{u}_\varepsilon\|_{L^\infty(0,t;W)}\int_0^t\|\mathbf{u}_{\varepsilon t}\|_W\|\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)\|_{L^2(\Gamma_c)}\\&\qquad+\frac{1}{4}\|-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\|^2_{L^2(\Gamma_c)}+c\|\mathbf{u}_\varepsilon(t)\|^4_W+c\end{aligned} \tag{130}$$

also thanks to (64). Finally, we apply the Gronwall Lemma to (129) taking into account (130) and (126). Thus, we obtain

$$\|-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,\widehat{T};L^2(\Gamma_c))}\leqslant c$$

On the other hand, by the monotonicity of β_ε we have for a.e. $t\in(0,\widehat{T})$

$$\|(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))(t)\|^2_{L^2(\Gamma_c)}\geqslant\|-\Delta\chi_\varepsilon(t)\|^2_{L^2(\Gamma_c)}+\|\beta_\varepsilon(\chi_\varepsilon)(t)\|^2_{L^2(\Gamma_c)}$$

Hence, (127) and standard elliptic regularity results yield that

$$\|\chi_\varepsilon\|_{H^1(0,\widehat{T};H^1(\Gamma_c))\cap L^\infty(0,\widehat{T};H^2(\Gamma_c))}\leqslant c \tag{131}$$

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and, besides,

$$\|\beta_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,\widehat{T};L^2(\Gamma_c))}\leqslant c \tag{132}$$

for some positive constant c independent of ε. Finally, on account of (131), (132), and (126) a comparison in (70) entails

$$\|\rho_\varepsilon(\chi_{\varepsilon t})\|_{L^\infty(0,\widehat{T};L^2(\Gamma_c))}+\|\chi_{\varepsilon t}\|_{L^\infty(0,\widehat{T};L^2(\Gamma_c))}\leqslant c \tag{133}$$

Thanks to the above estimates, by well-known weak and weak* compactness results we deduce that there exists a subsequence $\{(\mathbf{u}_{\varepsilon_k},\chi_{\varepsilon_k})\}_k$ along which convergences (118), (121), and (123)–(125) hold as $k\to\infty$. Moreover, using a classical compactness argument (see [21]) and the generalized Ascoli theorem (see [19], Corollary 4), we also obtain (119), (120) and (122), entailing that the pair $(\widehat{\mathbf{u}},\widehat{\chi})$ fulfils initial condition (19).

It is easy to check that the functions $(\widehat{\mathbf{u}},\widehat{\chi},\widehat{\boldsymbol{\eta}},\widehat{\xi},\widehat{\zeta})$ satisfy (41) and (43). Moreover, (122) combined with (124) allows to conclude that $\widehat{\zeta}\in\beta(\widehat{\chi})$. It remains to prove that $\widehat{\boldsymbol{\eta}}\in\alpha(\widehat{\mathbf{u}})$ and $\widehat{\xi}\in\rho(\widehat{\chi}_t)$. To this aim, we use a semicontinuity-comparison tool, and we prove that

$$\limsup_{\varepsilon_k\downarrow 0}\int_0^t\int_{\Gamma_c}\alpha_{\varepsilon_k}(\mathbf{u}_{\varepsilon_k})\cdot\mathbf{u}_{\varepsilon_k}\leqslant\int_0^t\int_{\Gamma_c}\widehat{\boldsymbol{\eta}}\cdot\widehat{\mathbf{u}} \tag{134}$$

Indeed, let us test (69) by $\mathbf{u}_{\varepsilon_k}$ and we integrate from 0 to t. We have

$$\begin{aligned}\int_0^t\int_{\Gamma_c}\alpha_{\varepsilon_k}(\mathbf{u}_{\varepsilon_k})\cdot\mathbf{u}_\varepsilon &= -\frac{1}{2}b(\mathbf{u}_{\varepsilon_k}(t),\mathbf{u}_{\varepsilon_k}(t))+\frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0)\\ &\quad -\int_0^t a(\mathbf{u}_{\varepsilon_k},\mathbf{u}_{\varepsilon_k})-\int_0^t\int_{\Gamma_c}\chi_{\varepsilon_k}|\mathbf{u}_{\varepsilon_k}|^2+\int_0^t {}_{W'}\langle\mathbf{F},\mathbf{u}_{\varepsilon_k}\rangle_W\end{aligned} \tag{135}$$

We take the lim sup as $\varepsilon_k\downarrow 0$ of both sides of (135). Thanks to the lower semicontinuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ and to the convergences specified by (120)–(122), we obtain

$$\begin{aligned}\limsup_{\varepsilon_k\downarrow 0}\int_0^t\int_{\Gamma_c}\alpha_{\varepsilon_k}(\mathbf{u}_{\varepsilon_k})\cdot\mathbf{u}_{\varepsilon_k} &\leqslant -\frac{1}{2}b(\widehat{\mathbf{u}}(t),\widehat{\mathbf{u}}(t))+\frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0)\\ &\quad -\int_0^t a(\widehat{\mathbf{u}},\widehat{\mathbf{u}})-\int_0^t\int_{\Gamma_c}\widehat{\chi}|\widehat{\mathbf{u}}|^2+\int_0^t {}_{W'}\langle\mathbf{F},\widehat{\mathbf{u}}\rangle_W\end{aligned} \tag{136}$$

Comparing the right-hand side of (136) with the limit relation (41) tested by $\widehat{\mathbf{u}}$, we readily obtain (134). In view of [17, Proposition II.1.1], (134) gives (42).

An analogous procedure allows us to prove (44). As before, we will show that

$$\limsup_{\varepsilon_k\downarrow 0}\int_0^t\int_{\Gamma_c}\rho_{\varepsilon_k}(\chi_{\varepsilon_k t})\chi_{\varepsilon_k t}\leqslant\int_0^t\int_{\Gamma_c}\widehat{\xi}\widehat{\chi}_t \tag{137}$$

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To this aim, we multiply (70) by $\chi_{\varepsilon_k t}$ and we integrate over $\Gamma_c\times(0,t)$. We have

$$\int_0^t\int_{\Gamma_c}\rho_{\varepsilon_k}(\chi_{\varepsilon_k t})\chi_{\varepsilon_k t} = -\|\chi_{\varepsilon_k t}\|^2_{L^2(0,t;L^2(\Gamma_c))}-\frac{1}{2}\|\nabla\chi_{\varepsilon_k}(t)\|^2_{L^2(\Gamma_c)}+\frac{1}{2}\|\nabla\chi_0\|^2_{L^2(\Gamma_c)}$$
$$-\int_{\Gamma_c}\psi_{\varepsilon_k}(\chi_{\varepsilon_k}(t))+\int_{\Gamma_c}\psi_{\varepsilon_k}(\chi_0)+\int_0^t\int_{\Gamma_c}\left(w_s-\frac{1}{2}|\mathbf{u}_{\varepsilon_k}|^2\right)\chi_{\varepsilon_k t} \quad (138)$$

Here, we remark that, since ψ_{ε_k} Mosco converges to ψ (see, e.g. [22]),

$$\liminf_{\varepsilon_k\downarrow 0}\int_{\Gamma_c}\psi_{\varepsilon_k}(\chi_{\varepsilon_k}(t))\geqslant\int_{\Gamma_c}\psi(\widehat{\chi}(t)) \quad (139)$$

We take the lim sup as $\varepsilon_k\downarrow 0$ of both sides of (138). Thanks to (121), (120), (139), (66) and owing to the lower semicontinuity properties of L^p-norms, we have

$$\limsup_{\varepsilon_k\downarrow 0}\int_0^t\int_{\Gamma_c}\rho_{\varepsilon_k}(\chi_{\varepsilon_k t})\chi_{\varepsilon_k t}\leqslant -\|\widehat{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))}-\frac{1}{2}\|\nabla\widehat{\chi}(t)\|^2_{L^2(\Gamma_c)}+\frac{1}{2}\|\nabla\chi_0\|^2_{L^2(\Gamma_c)}$$
$$-\int_{\Gamma_c}\psi(\widehat{\chi}(t))+\int_{\Gamma_c}\psi(\chi_0)+\int_0^t\int_{\Gamma_c}\left(w_s-\frac{1}{2}|\widehat{\mathbf{u}}|^2\right)\widehat{\chi}_t \quad (140)$$

Again, comparing the right-hand side of (140) with the limit relation (43) tested by $\widehat{\chi}_t$, we obtain (137), whence (44). □

5. GLOBAL EXISTENCE FOR PROBLEM (P)

Strategy of the proof of Theorem 3: We shall prove Theorem 3 by showing that the local solution $(\widehat{\mathbf{u}},\widehat{\chi},\widehat{\boldsymbol{\eta}},\widehat{\xi},\widehat{\zeta})$ to Problem (P), on the interval $(0,\widehat{T})$, found in Theorem 1, extends to a global solution, under the additional assumption (48) on β.

This requirement indeed plays a crucial role in our proof, which is based on a careful refinement of a standard extension technique. The latter procedure usually involves performing suitable *a priori* estimates on solutions to Problem (P) on some interval $(0,\mathsf{t})$, $\mathsf{t}\in(0,T]$, and then showing that such estimates in fact do not depend on the final time t. In this way, one considers some maximal extension of the local solution and shows by a classical argument that it is in fact defined on the whole interval $(0,T)$.

In the present case, we can directly obtain *global estimates* only for the solution norms $\|\widehat{\mathbf{u}}\|_{H^1(0,\widehat{T};W)}$, $\|\widehat{\chi}\|_{L^\infty(0,\widehat{T};H^1(\Gamma_c))\cap H^1(0,\widehat{T};L^2(\Gamma_c))}$, (and $\|\widehat{\boldsymbol{\eta}}\|_{L^2(0,\widehat{T};H^{-1/2}(\Gamma_c))}$ by comparison), see Lemma 5.2. Let us stress that this can be done only by substantially exploiting condition (48) (see Remark 5.3). However, Lemma 5.2 alone does not allow us to conclude that $(\widehat{\mathbf{u}},\widehat{\chi},\widehat{\boldsymbol{\eta}},\widehat{\xi},\widehat{\zeta})$ extends to a global solution, since we would need further estimates for $(\widehat{\xi},\widehat{\zeta})$ in the spaces (39)–(40). Such estimates (and the enhanced regularity ones for $\widehat{\chi}$) can be rigorously proved only in the framework of the approximate system (69)–(71) (cf. the proof of Proposition 4.7). On the other hand, as we mentioned in the Introduction no *global in time* existence result seems at hand for the regularized initial boundary-value Problem P_ε (nor in general for Problem (P) without (48)), due to the fact that the solution component χ of the approximate system is not necessarily positive.

Thus, we have developed a careful extension technique (below outlined), which tries to balance the above-mentioned features of the problem. In fact, the key point is that we extend (the $\mathbf{u}$ and χ

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components of) the local solution $(\widehat{\mathbf{u}}, \widehat{\chi}, \widehat{\boldsymbol{\eta}}, \widehat{\xi}, \widehat{\zeta})$ (see Remark 5.5), along with its 'approximability' properties (cf. Definition 5.1). As we shall see, the latter are going to play a crucial role in rigourously carrying out the extension procedure.

Scheme of the proof: Since we are going to highlight the role of the $\mathbf{u}$ and χ components of the solutions of Problem (P), for the sake of convenience hereafter we shall call a *solution* to Problem (P)

$$\begin{gathered}\text{any pair } (\mathbf{u}, \chi), \text{with the regularity (36)–(37)}\\ \text{for which there exists a triplet } (\boldsymbol{\eta}, \xi, \zeta) \text{ complying with (38)–(46)}\end{gathered} \tag{141}$$

Now, we recall that we have denoted by $\{(\mathbf{u}_{\varepsilon_k}, \chi_{\varepsilon_k})\}_k$ the sequence of the (unique by Theorem 2) solutions to Problem (P_{ε_k}) on $(0, \widehat{T})$, which we have shown to converge to the local solution $(\widehat{\mathbf{u}}, \widehat{\chi})$ as $\varepsilon_k \downarrow 0$ in Proposition 4.7. Now, for a fixed $\mu \in (0, T]$ and $k \in \mathbb{N}$ we shall denote by $(\mathbf{u}^{\mu}_{\varepsilon_k}, \chi^{\mu}_{\varepsilon_k})$ the unique solution to (P_{ε_k}) on the interval $(0, \mu)$: clearly, for $\mu \geqslant \widehat{T}$, $(\mathbf{u}^{\mu}_{\varepsilon_k}, \chi^{\mu}_{\varepsilon_k})$, if it exists, is a proper extension of $(\mathbf{u}_{\varepsilon_k}, \chi_{\varepsilon_k})$. We are now in the position of defining the solution notion, based on 'approximability' properties, which we are going to extend to the whole interval $(0, T)$.

Definition 5.1
Let $\mu \in (0, T]$; we say that a pair

$$(\mathbf{u}_\mu, \chi_\mu) \in H^1(0, \mu; W) \times (W^{1,\infty}(0, \mu; L^2(\Gamma_c)) \cap H^1(0, \mu; H^1(\Gamma_c)) \cap L^\infty(0, \mu; H^2(\Gamma_c)))$$

is an *approximable solution* of Problem (P) on $(0, \mu)$ if

1. $(\mathbf{u}_\mu, \chi_\mu)$ solves Problem (P) in the sense of (141) on $(0, \mu)$;
2. there exists a subsequence $\{\varepsilon^{\mu}_n\}_n \subset \{\varepsilon_k\}_k$ such that the related sequence $\{(\mathbf{u}^{\mu}_{\varepsilon_n}, \chi^{\mu}_{\varepsilon_n})\}_n$ of solutions to Problem (P_{ε_n}) on $(0, \mu)$ fulfils

$$\|\mathbf{u}^{\mu}_{\varepsilon_n} - \mathbf{u}_\mu\|_{C^0([0,\mu];H)} + \|\chi^{\mu}_{\varepsilon_n} - \chi_\mu\|_{C^0([0,\mu];H^1(\Gamma_c))} \to 0 \quad \text{as } n \to \infty \tag{142}$$

Since for all $\mu > 0$ $\{\varepsilon^{\mu}_n\}_n \subset \{\varepsilon_k\}_k$, combining (142) with convergences (120) and (122) we conclude that, for all $\mu \in (\widehat{T}, T]$, $(\mathbf{u}_\mu, \chi_\mu)$ is a proper extension of $(\widehat{\mathbf{u}}, \widehat{\chi})$, i.e.

$$\mathbf{u}_\mu(t) = \widehat{\mathbf{u}}(t) \text{ in } H, \quad \chi_\mu(t) = \widehat{\chi}(t) \text{ in } H^1(\Gamma_c) \qquad \forall t \in [0, \widehat{T}] \tag{143}$$

Let us introduce the set

$$\mathcal{T} := \{\mu \in (0, T] : \exists \text{ an } \textit{approximable solution } (\mathbf{u}_\mu, \chi_\mu) \text{ of } (P) \text{ on } (0, \mu)\} \tag{144}$$

Of course, $\mathcal{T} \neq \emptyset$, as $\widehat{T} \in \mathcal{T}$. Thus, letting $T^* := \sup \mathcal{T}$, we have that $0 < T^* \leqslant T$. Indeed, we are going to show that

$$T^* = T \tag{145}$$

yielding, in view of (143), the desired extension of $(\widehat{\mathbf{u}}, \widehat{\chi})$ to a global solution on $(0, T)$. The proof of (145) shall be developed in some technical steps, which we enlist for the sake of clarity:

1. first, we obtain some *a priori* estimates on the solutions $(\mathbf{u}_\mu, \chi_\mu)$ *independent* of μ, see Lemma 5.2;

2. hence, we deduce that there exists a pair

$$(\mathbf{u}^*, \chi^*) \in H^1(0, T^*; W) \times (L^\infty(0, T^*; H^1(\Gamma_c)) \cap H^1(0, T^*; L^2(\Gamma_c))) \tag{146}$$

extending $(\widehat{\mathbf{u}}, \widehat{\chi})$ on $(0, T^*)$, see (154);
3. in Lemma 5.4 we show that $(\mathbf{u}^*, \chi^*)$ is in fact a solution (in the sense of (141)) of Problem (P) on $(0, T^*)$;
4. finally, in Lemma 5.6 we conclude the proof of (145) by a contradiction argument.

Proof of Step 1

Lemma 5.2 (*Global estimates*)
Under the assumptions of Theorem 3, there exists a positive constant S, only depending on C_a, C_b, w_s, T, Ω, and Γ_c, such that for any $\mu > 0$ and for any solution $(\mathbf{u}, \chi)$ of Problem (P) on the interval $(0, \mu)$ the following estimates hold:

$$\begin{aligned} &\|\mathbf{u}\|_{H^1(0,\mu;W)} + \|\chi\|_{L^\infty(0,\mu;H^1(\Gamma_c)) \cap H^1(0,\mu;L^2(\Gamma_c))} \\ &\quad \leqslant S(1 + \|\mathbf{u}_0\|_W + \varphi(\mathbf{u}_0)^{1/2} + \|\chi_0\|_{H^1(\Gamma_c)} + \|\psi(\chi_0)\|^{1/2}_{L^1(\Gamma_c)} + \|\mathbf{F}\|_{L^2(0,T;W')}) \end{aligned} \tag{147}$$

Proof
We test (41) by $\mathbf{u}_t$ and we integrate from 0 to t, with $0 < t < \mu$. Owing to (23) and (24), we have

$$\begin{aligned} &\frac{C_b}{2} \|\mathbf{u}_t\|^2_{L^2(0,t;W)} + \frac{C_a}{2} \|\mathbf{u}(t)\|^2_W + \varphi(\mathbf{u}(t)) + \int_0^t \int_{\Gamma_c} \chi \mathbf{u} \cdot \mathbf{u}_t \\ &\quad \leqslant \frac{c}{2} \|\mathbf{u}_0\|^2_W + \varphi(\mathbf{u}_0) + \frac{1}{2C_b} \int_0^T \|\mathbf{F}\|^2_{W'} \end{aligned} \tag{148}$$

where we have used the chain rule for φ, see [23, Lemma 4.1], and the Hölder inequality. Let us denote by I the last integral on the left-hand side of (148). An integration by parts in time leads to

$$I = \frac{1}{2} \int_0^t \int_{\Gamma_c} \chi(|\mathbf{u}|^2)_t = -\frac{1}{2} \int_0^t \int_{\Gamma_c} \chi_t |\mathbf{u}|^2 + \frac{1}{2} \int_{\Gamma_c} \chi(t) |\mathbf{u}(t)|^2 - \frac{1}{2} \int_{\Gamma_c} \chi_0 |\mathbf{u}_0|^2 \tag{149}$$

Recalling that $\chi \geqslant 0$ a.e. in $\Gamma_c \times (0, t)$ thanks to (48), we obtain

$$I \geqslant -\frac{1}{2} \int_0^t \int_{\Gamma_c} \chi_t |\mathbf{u}|^2 - \frac{1}{2} \int_{\Gamma_c} \chi_0 |\mathbf{u}_0|^2 \tag{150}$$

Next, we multiply (43) by χ_t and we integrate over $\Gamma_c \times (0, t)$. Taking the monotonicity of ρ into account, applying the Hölder inequality, and (55), we have

$$\begin{aligned} &\|\chi_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + \frac{1}{2} \|\nabla \chi(t)\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c} \psi(\chi(t)) \\ &\quad \leqslant \frac{1}{2} \|\nabla \chi_0\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c} \psi(\chi_0) + \int_0^t \int_{\Gamma_c} w_s \chi_t - \frac{1}{2} \int_0^t \int_{\Gamma_c} \chi_t |\mathbf{u}|^2 \\ &\quad \leqslant \frac{1}{2} \|\nabla \chi_0\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c} \psi(\chi_0) + \frac{1}{2} \|\chi_t\|^2_{L^2(0,t;L^2(\Gamma_c))} - \frac{1}{2} \int_0^t \int_{\Gamma_c} \chi_t |\mathbf{u}|^2 + c \end{aligned} \tag{151}$$

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again by the chain rule [16, Lemma 3.3, p.73] for ψ. Finally, we add (148) and (151), taking into account (150). Noting that two terms cancel out, we easily conclude (147). □

Remark 5.3
Let us stress that the positivity constraint on the solution component χ, due to (48), has played a crucial role in the proof of the above estimates: in particular, to deduce (150) from (149).

Proof of Step 2
Let us now select a family $\{(\mathbf{u}_\mu, \chi_\mu)\}_{\mu\in\mathscr{T}}$ and note that, by the definition of approximable solution, for all $\mu\in\mathscr{T}$ the pair $(\mathbf{u}_\mu, \chi_\mu)\in C^0([0,\mu]; W)\times C^0([0,\mu]; H^1(\Gamma_c))$. Hence, we may consider its extension by continuity on $[0, T^*]$ for all $t\in[0, T^*]$

$$\tilde{\mathbf{u}}_\mu(t) := \begin{cases} \mathbf{u}_\mu(t) & \text{if } t\in[0,\mu], \\ \mathbf{u}_\mu(\mu) & \text{if } t\in(\mu, T^*], \end{cases} \qquad \tilde{\chi}_\mu(t) := \begin{cases} \chi_\mu(t) & \text{if } t\in[0,\mu] \\ \chi_\mu(\mu) & \text{if } t\in(\mu, T^*] \end{cases}$$

It follows from (147) that there exists some constant $S_0>0$ such that

$$\|\tilde{\mathbf{u}}_\mu\|_{H^1(0,T^*;W)} + \|\tilde{\chi}_\mu\|_{L^\infty(0,T^*;H^1(\Gamma_c))\cap H^1(0,T^*;L^2(\Gamma_c))} \leqslant S_0 \quad \forall \mu\in\mathscr{T} \tag{152}$$

Let us now fix a sequence $\{\mu_m\}_m$ with $\mu_m\uparrow T^*$ as $m\to\infty$. Possibly extracting a subsequence (which we do not relabel), we conclude that there exists a pair $(\mathbf{u}^*, \chi^*)$ with the regularity (146) such that, by the aforementioned compactness results of [19], $\{(\tilde{\mathbf{u}}_{\mu_m}, \tilde{\chi}_{\mu_m})\}$ converges as $m\to\infty$ to $(\mathbf{u}^*, \chi^*)$ w.r.t. to suitable topologies. Here, we just point out for later convenience that

$$\|\tilde{\mathbf{u}}_{\mu_m} - \mathbf{u}^*\|_{C^0([0,T^*];H)} + \|\tilde{\chi}_{\mu_m} - \chi^*\|_{C^0([0,T^*];L^2(\Gamma_c))} \to 0 \quad \text{as } m\to\infty \tag{153}$$

Then, in view of (143), we deduce that

$$\mathbf{u}^*(t) = \widehat{\mathbf{u}}(t) \;\text{ in } H, \quad \chi^*(t) = \widehat{\chi}(t) \;\text{ in } H^1(\Gamma_c), \;\; \forall t\in[0,\widehat{T}] \tag{154}$$

Proof of Step 3

Lemma 5.4
Under the assumptions of Theorem 3, there exist $\boldsymbol{\eta}^*\in L^2(0,T^*;(H^{-1/2}(\Gamma_c))^3)$, $\xi^*\in L^\infty(0,T^*; L^2(\Gamma_c))$, and $\zeta^*\in L^\infty(0,T^*;L^2(\Gamma_c))$ such that the quintuple $(\mathbf{u}^*, \chi^*, \boldsymbol{\eta}^*, \xi^*, \zeta^*)$ is a solution of Problem (P) on the interval $(0, T^*)$.

Remark 5.5
In order to prove the existence of the triplet $(\boldsymbol{\eta}^*, \xi^*, \zeta^*)$, we shall develop a careful approximation procedure, which in fact relies on the very notion of *approximable solution*. As it will be clear from the proof, the triplet $(\boldsymbol{\eta}^*, \xi^*, \zeta^*)$ is not in general related to the triplet $(\widehat{\boldsymbol{\eta}}, \widehat{\xi}, \widehat{\zeta})$ originally associated with the pair $(\widehat{\mathbf{u}}, \widehat{\chi})$.

Proof. A diagonalization argument:
Let $\{\mu_m\}_m$ be the sequence converging to T^* along which convergences (153) hold. It follows from the definition of *approximable solution* that for every $m\in\mathbb{N}$ there exists a sequence $\{\varepsilon_n^{\mu_m}\}_n\subset\{\varepsilon_k\}_k$,

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with $\varepsilon_n^{\mu_m} \downarrow 0$ as $n \to \infty$, and, accordingly, the sequence of solutions $\{(\mathbf{u}_{\varepsilon_n^{\mu_m}}^{\mu_m}, \chi_{\varepsilon_n^{\mu_m}}^{\mu_m})\}_n$ of Problem $(P_{\varepsilon_n^{\mu_m}})$ fulfilling

$$\|\mathbf{u}_{\varepsilon_n^{\mu_m}}^{\mu_m} - \mathbf{u}_{\mu_m}\|_{C^0([0,\mu_m];H)} + \|\chi_{\varepsilon_n^{\mu_m}}^{\mu_m} - \chi_{\mu_m}\|_{C^0([0,\mu_m];H^1(\Gamma_c))} \to 0 \quad \text{as } n \to \infty$$

Now, by an elementary diagonalization argument we construct a sequence $\varepsilon_m \downarrow 0$ (still fulfilling $\{\varepsilon_m\}_m \subset \{\varepsilon_k\}_k$), and, correspondingly, a sequence $\{(\mathbf{u}_{\varepsilon_m}^{\mu_m}, \chi_{\varepsilon_m}^{\mu_m})\}_m$ such that

$$\begin{aligned} &(\mathbf{u}_{\varepsilon_m}^{\mu_m}, \chi_{\varepsilon_m}^{\mu_m}) \text{ solves Problem } (P_{\varepsilon_m}) \text{ on } (0,\mu_m) \quad \forall m \in \mathbb{N} \\ &\|\mathbf{u}_{\varepsilon_m}^{\mu_m} - \mathbf{u}_{\mu_m}\|_{C^0([0,\mu_m];H)} + \|\chi_{\varepsilon_m}^{\mu_m} - \chi_{\mu_m}\|_{C^0([0,\mu_m];H^1(\Gamma_c))} \leqslant \frac{1}{m} \quad \forall m \in \mathbb{N} \end{aligned} \tag{155}$$

Combining this with (153), we find that, up to a subsequence, there exists $m_* \in \mathbb{N}$ such that

$$\|\mathbf{u}_{\varepsilon_m}^{\mu_m} - \mathbf{u}^*\|_{C^0([0,\mu_m];H)} + \|\chi_{\varepsilon_m}^{\mu_m} - \chi^*\|_{C^0([0,\mu_m];L^2(\Gamma_c))} \leqslant \frac{2}{m} \quad \forall m \geqslant m_* \tag{156}$$

Hereafter, to simplify we shall denote by $\{\mathbf{u}_m\}_m$ and $\{\chi_m\}_m$ the sequences $\{\mathbf{u}_{\varepsilon_m}^{\mu_m}\}_m$ and $\{\chi_{\varepsilon_m}^{\mu_m}\}_m$, respectively. Correspondingly, for all $m \in \mathbb{N}$ we set

$$\boldsymbol{\eta}_m := \alpha_{\varepsilon_m}(\mathbf{u}_m), \quad \xi_m := \rho_{\varepsilon_m}(\chi_{mt}), \quad \zeta_m := \beta_{\varepsilon_m}(\chi_m)$$

A priori estimates: In view of (152) and (155), we obtain the following key estimates on the sequences $\{\mathbf{u}_m\}$ and $\{\chi_m\}$:

$$\|\mathbf{u}_m\|_{C^0([0,\mu_m];H)} + \|\chi_m\|_{L^\infty(0,\mu_m;H^1(\Gamma_c))} \leqslant S_0 + 1 \quad \forall m \in \mathbb{N} \tag{157}$$

(for simplicity, we have neglected some embedding constants). Indeed, relying on (157) we manage to obtain further estimates on the sequence $\{(\mathbf{u}_m, \chi_m, \boldsymbol{\eta}_m, \xi_m, \zeta_m)\}_m$. To see this, it suffices to repeat estimates (126), (127) on the level of the approximate Problem (P_{ε_m}) (let us recall that (126)–(127) were indeed mutuated from the Schauder estimates in the proof of Proposition 4.1). Thus, there exists a constant S_1 such that

$$\|\mathbf{u}_m\|_{H^1(0,\mu_m;W)} + \|\chi_m\|_{L^\infty(0,\mu_m;H^1(\Gamma_c)) \cap H^1(0,\mu_m;L^2(\Gamma_c))} \leqslant S_1 \quad \forall m \in \mathbb{N} \tag{158}$$

$$\|\boldsymbol{\eta}_m\|_{L^2(0,\mu_m;H^{-1/2}(\Gamma_c))} \leqslant S_1 \quad \forall m \in \mathbb{N} \tag{159}$$

the latter estimate again following the comparison in (69). Next, mimicking the computations (129)–(130), we derive the enhanced estimates (cf. with (131)–(133))

$$\begin{aligned} &\|\chi_m\|_{W^{1,\infty}(0,\mu_m;L^2(\Gamma_c)) \cap H^1(0,\mu_m;H^1(\Gamma_c)) \cap L^\infty(0,\mu_m;H^2(\Gamma_c))} + \|\xi_m\|_{L^\infty(0,\mu_m;L^2(\Gamma_c))} \\ &\quad + \|\zeta_m\|_{L^\infty(0,\mu_m;L^2(\Gamma_c))} \leqslant S_2 \ \ \forall m \in \mathbb{N} \end{aligned} \tag{160}$$

again for a positive constant S_2 independent of $m \in \mathbb{N}$. As a by-product of the above estimates (cf. (103) and (105)), we have

$$\|\mathbf{u}_m(\mu_m)\|_W + \|\chi_m(\mu_m)\|_{H^2(\Gamma_c)} + \|\psi_{\varepsilon_m}(\chi_m(\mu_m))\|_{L^1(\Gamma_c)} + \varphi_{\varepsilon_m}(\mathbf{u}_m(\mu_m)) \leqslant S_3 \tag{161}$$

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Let us stress that the constants S_1, S_2, and S_3 depend on C_a, C_b, w_s, T, Ω, Γ_c, $\|\mathbf{u}_0\|_W$, $\|\chi_0\|_{H^2(\Gamma_c)}$, $\varphi(\mathbf{u}_0)$, $\|\psi(\chi_0)\|_{L^1(\Gamma_c)}$, $\|\mathbf{F}\|_{L^2(0,T;W')}$, and are *independent* of m.

Passage to the limit: To fix ideas, let us consider the sequence $T_k \to T^*$ defined by $T_k := T^*(2^k-1)/2^k$ for all $k \geqslant 1$. Of course, for $k=1$ there exists $\bar{m}_1 \in \mathbb{N}$ such that $[0, T_1] \subset [0, \mu_m]$ for all $m \geqslant \bar{m}_1$. Taking into account estimates (158) and (160) for the sequences $\{\mathbf{u}_m\}$, $\{\chi_m\}$ on the interval $(0, T_1)$, first of all we improve convergences (156). Indeed, arguing as in the proof of Proposition 4.7, we obtain

$$\begin{cases} \mathbf{u}_m \rightharpoonup \mathbf{u}^* & \text{in } H^1(0,T_1;W) \\ \mathbf{u}_m \to \mathbf{u}^* & \text{in } C^0([0,T_1];(H^{1-s}(\Omega))^3) \quad \forall s>0 \\ \mathbf{u}_m \to \mathbf{u}^* & \text{in } C^0([0,T_1];(L^p(\Gamma_c))^3) \quad \forall 1\leqslant p<4 \\ \chi_m \overset{*}{\rightharpoonup} \chi^* & \text{in } W^{1,\infty}(0,T_1;L^2(\Gamma_c)) \cap H^1(0,T_1;H^1(\Gamma_c)) \cap L^\infty(0,T_1;H^2(\Gamma_c)) \\ \chi_m \to \chi^* & \text{in } C^0([0,T_1];H^{2-s}(\Gamma_c)) \quad \forall s>0 \end{cases} \tag{162}$$

as $m \to \infty$. Furthermore, estimates (159)–(160) for the sequences $\{\boldsymbol{\eta}_m\}$, $\{\xi_m\}$, and $\{\zeta_m\}$ yield that there exist an increasing subsequence $\{m_j^1\}_j$ (with $m_1^1 \geqslant \bar{m}_1$) and $\boldsymbol{\eta}_1 \in L^2(0,T_1;(H^{-1/2}(\Gamma_c))^3)$, $\xi_1 \in L^\infty(0,T_1;L^2(\Gamma_c))$, and $\zeta_1 \in L^\infty(0,T_1;L^2(\Gamma_c))$ such that the following convergences hold as $j \to \infty$:

$$\begin{cases} \boldsymbol{\eta}_{m_j^1} \rightharpoonup \boldsymbol{\eta}_1 & \text{in } L^2(0,T_1;(H^{-1/2}(\Gamma_c))^3) \\ \xi_{m_j^1} \overset{*}{\rightharpoonup} \xi_1 & \text{in } L^\infty(0,T_1;L^2(\Gamma_c)) \\ \zeta_{m_j^1} \overset{*}{\rightharpoonup} \zeta_1 & \text{in } L^\infty(0,T_1;L^2(\Gamma_c)) \end{cases} \tag{163}$$

In particular, from (159) and (160) we deduce

$$\|\boldsymbol{\eta}_1\|_{L^2(0,T_1;H^{-1/2}(\Gamma_c))} + \|\xi_1\|_{L^\infty(0,T_1;L^2(\Gamma_c))} + \|\zeta_1\|_{L^\infty(0,T_1;L^2(\Gamma_c))} \leqslant S_1 + S_2$$

Arguing exactly in the same way as in the proof of Proposition 4.7, we conclude that

$$\boldsymbol{\eta}_1 \in \alpha(\mathbf{u}^*) \text{ a.e. in } (0,T_1) \quad \text{and} \quad \xi_1 \in \rho(\chi_t^*), \zeta_1 \in \beta(\chi^*) \text{ a.e. in } \Gamma_c \times (0,T_1) \tag{164}$$

Further,

$$\text{the quintuple } (\mathbf{u}^*, \chi^*, \boldsymbol{\eta}_1, \xi_1, \zeta_1) \text{ solves Problem (P) on } (0,T_1) \tag{165}$$

Now, let us fix $k=2$: focusing on the sequence $\{m_j^1\}_j$, we again find there exists an index $j_2 \in \mathbb{N}$ such that $[0,T_2] \subset [0, \mu_{m_j^1}]$ for all $j \geqslant j_2$. Then, arguing exactly in the same way as in the previous lines, we conclude the enhanced convergences (162) for the sequences $\{\mathbf{u}_m\}$, $\{\chi_m\}$ *on the interval* $(0, T_2)$. Furthermore, we end up with an increasing subsequence $\{m_j^2\}_j$ of $\{m_j^1\}_j$ (with of course $m_1^2 \geqslant m_{j_2}^1$) and with a triplet $(\boldsymbol{\eta}_2, \xi_2, \zeta_2) \in L^2(0,T_2;(H^{-1/2}(\Gamma_c))^3) \times L^\infty(0,T_2;L^2(\Gamma_c)) \times L^\infty(0,T_2;L^2(\Gamma_c))$ for which convergences (163) hold, as well as the estimates

$$\|\boldsymbol{\eta}_2\|_{L^2(0,T_2;H^{-1/2}(\Gamma_c))} + \|\xi_2\|_{L^\infty(0,T_2;L^2(\Gamma_c))} + \|\zeta_2\|_{L^\infty(0,T_2;L^2(\Gamma_c))} \leqslant S_1 + S_2$$

Likewise, (164) and (165) hold for the quintuple $(\mathbf{u}^*, \chi^*, \boldsymbol{\eta}_2, \xi_2, \zeta_2)$ on $(0, T_2)$. Eventually, note that, since we have selected 'nested' subsequences, we have

$$\boldsymbol{\eta}_1 \equiv \boldsymbol{\eta}_2, \quad \xi_1 \equiv \xi_2 \quad \text{and} \quad \zeta_1 \equiv \zeta_2 \quad \text{on } (0, T_1) \tag{166}$$

Therefore, we have extended the solution $(\mathbf{u}^*, \chi^*, \boldsymbol{\eta}_1, \xi_1, \zeta_1)$ on the interval $(0, T_2)$. Iterating this construction for any $k \in \mathbb{N}$, we end up with a triplet $(\boldsymbol{\eta}^*, \xi^*, \zeta^*)$ such that

$$\begin{cases} \boldsymbol{\eta}^* \equiv \boldsymbol{\eta}_k \quad \text{on } (0, T_k) \ \ \forall k \in N \\ \text{whence } \|\boldsymbol{\eta}^*\|_{L^2(0,T^*;H^{-1/2}(\Gamma_c))} \leqslant S_1 \end{cases}$$

and analogously for ξ^* and ζ^*. It is immediate to see that the quintuple $(\mathbf{u}^*, \chi^*, \boldsymbol{\eta}^*, \xi^*, \zeta^*)$ solves Problem (P) on $(0, T^*)$. □

Proof of Step 4

Lemma 5.6
Under the assumptions of Theorem 3, $(\mathbf{u}^*, \chi^*)$ is an *approximable solution* of Problem (P) on $(0, T^*)$, and we have $T^* = T$.

Proof
Let us suppose that $T^* < T$. In the following lines, we shall in fact extend the pair $(\mathbf{u}^*, \chi^*)$ to an *approximable solution* $(\tilde{\mathbf{u}}, \tilde{\chi})$ on $(0, T^* + \delta/2)$ for some $\delta > 0$, and thus obtain a contradiction.

Let us consider the sequence $\{(\mathbf{u}_m, \chi_m)\}$ of solutions to P_{ε_m} on $[0, \mu_m]$ constructed in the first part of the proof of Lemma 5.4. We now extend it by continuity to some larger interval. In fact, estimate (161) and Remark 4.6 guarantee that there exists $\delta > 0$, *independent* of m, such that for every $m \in \mathbb{N}$ problem P_{ε_m}, supplemented with the initial data $\mathbf{u}_0^m := \mathbf{u}_m(\mu_m)$ and $\chi_0^m := \chi_m(\mu_m)$, admits a unique solution $(\overline{\mathbf{u}}_m, \overline{\chi}_m)$ on the interval $[\mu_m, \mu_m + \delta] \subset [0, T]$. Furthermore, arguing as in the proof of Proposition 4.7 we see that there exists a positive constant S_4, *independent* of $m \in \mathbb{N}$, such that

$$\begin{aligned} &\|\overline{\mathbf{u}}_m\|_{H^1(\mu_m,\mu_m+\delta;W)} \\ &\quad + \|\overline{\chi}_m\|_{W^{1,\infty}(\mu_m,\mu_m+\delta;L^2(\Gamma_c)) \cap H^1(\mu_m,\mu_m+\delta;H^1(\Gamma_c)) \cap L^\infty(\mu_m,\mu_m+\delta;H^2(\Gamma_c))} \\ &\quad + \|\alpha_{\varepsilon_m}(\overline{\mathbf{u}}_m)\|_{L^2(\mu_m,\mu_m+\delta;H^{-1/2}(\Gamma_c))} + \|\rho_{\varepsilon_m}(\overline{\chi}_{mt})\|_{L^\infty(\mu_m,\mu_m+\delta;L^2(\Gamma_c))} \\ &\quad + \|\beta_{\varepsilon_m}(\overline{\chi}_m)\|_{L^\infty(\mu_m,\mu_m+\delta;L^2(\Gamma_c))} \leqslant S_4 \quad \forall m \in \mathbb{N} \end{aligned} \tag{167}$$

Now, we consider the final time $T^* + \delta/2 < T$. As $\mu_m \to T^*$, we have that $T^* + \delta/2 \leqslant \mu_m + \delta$ for all $m \geqslant m_0$, for some $m_0 > 0$. After these preparations, for every $m \geqslant m_0$ we extend the solution $(\mathbf{u}_m, \chi_m)$ on the interval $[0, T^* + \delta/2]$ by setting

$$\tilde{\mathbf{u}}_m(t) := \begin{cases} \mathbf{u}_m(t), & t \in [0, \mu_m], \\ \overline{\mathbf{u}}_m(t), & t \in \left[\mu_m, T^* + \dfrac{\delta}{2}\right], \end{cases} \qquad \tilde{\chi}_m(t) := \begin{cases} \chi_m(t), & t \in [0, \mu_m] \\ \overline{\chi}_m(t), & t \in \left[\mu_m, T^* + \dfrac{\delta}{2}\right] \end{cases} \tag{168}$$

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Thanks to the *a priori* estimates (158)–(160) and to (167), we may apply standard compactness results to the sequence $\{(\tilde{\mathbf{u}}_m, \tilde{\chi}_m)\}$ in the same way as in the proof of Proposition 4.7. Thus, there exists a quintuple $(\tilde{\mathbf{u}}, \tilde{\chi}, \tilde{\boldsymbol{\eta}}, \tilde{\xi}, \tilde{\zeta})$, with

$$\tilde{\mathbf{u}} \in H^1\left(0, T^* + \frac{\delta}{2}; W\right), \quad \tilde{\boldsymbol{\eta}} \in L^2\left(0, T^* + \frac{\delta}{2}; (H^{-1/2}(\Gamma_c))^3\right)$$

$$\tilde{\chi} \in W^{1,\infty}\left(0, T^* + \frac{\delta}{2}; L^2(\Gamma_c)\right) \cap H^1\left(0, T^* + \frac{\delta}{2}; H^1(\Gamma_c)\right) \cap L^\infty\left(0, T^* + \frac{\delta}{2}; H^2(\Gamma_c)\right)$$
$$\subset C^0_w\left(\left[0, T^* + \frac{\delta}{2}\right]; H^2(\Gamma_c)\right)$$

$$\tilde{\xi} \in L^\infty\left(0, T^* + \frac{\delta}{2}; L^2(\Gamma_c)\right), \quad \tilde{\zeta} \in L^\infty\left(0, T^* + \frac{\delta}{2}; L^2(\Gamma_c)\right)$$

such that, up to a subsequence, convergences (118)–(125) hold on $(0, T^* + \delta/2)$ for $\{(\tilde{\mathbf{u}}_m, \tilde{\chi}_m, \alpha_{\varepsilon_m}(\tilde{\mathbf{u}}_m), \rho_{\varepsilon_m}(\tilde{\chi}_{mt}), \beta_{\varepsilon_m}(\tilde{\chi}_m))\}$ and $(\tilde{\mathbf{u}}, \tilde{\chi}, \tilde{\boldsymbol{\eta}}, \tilde{\xi}, \tilde{\zeta})$. In particular,

$$\|\tilde{\chi}_m - \tilde{\chi}\|_{C^0([0,T^*+\delta/2];H^1(\Gamma_c))} + \|\tilde{\mathbf{u}}_m - \tilde{\mathbf{u}}\|_{C^0([0,T^*+\delta/2];H)} \to 0 \quad \text{as } m \to \infty \tag{169}$$

As a consequence, we conclude by the usual arguments that the quintuple $(\tilde{\mathbf{u}}, \tilde{\chi}, \tilde{\boldsymbol{\eta}}, \tilde{\xi}, \tilde{\zeta})$ solves Problem (P) on the interval $(0, T^* + \delta/2)$.

We claim that

$$\tilde{\mathbf{u}}(t) \equiv \mathbf{u}^*(t) \text{ in } W, \quad \tilde{\chi}(t) \equiv \chi^*(t) \text{ in } H^2(\Gamma_c) \ \ \forall t \in [0, T^*] \tag{170}$$

Indeed, due to the definition (168) of $\tilde{\mathbf{u}}_m$, to (156) and to (169), we check that $\tilde{\mathbf{u}}$ and $\mathbf{u}^*$ coincide on $[0, T^*)$, hence on $[0, T^*]$ by continuity. We argue for $\tilde{\chi}$ and χ^* in the same way. Then, (169) in particular yields that $(\mathbf{u}^*, \chi^*)$ is an *approximable solution* on $(0, T^*)$. Further, $(\tilde{\mathbf{u}}, \tilde{\chi})$ is itself an *approximable solution* of on Problem (P) on the interval $(0, T^* + \delta/2)$. Thus,

$$T^* + \frac{\delta}{2} \in \mathscr{T}$$

which is a contradiction. □

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Proof of Proposition 2.3

As a consequence of (49) and of (50), the solution quintuples $(\mathbf{u}_i, \chi_i, \boldsymbol{\eta}_i, \xi_i, \zeta_i)$, $i = 1, 2$, turn out to solve on $(0, T_0)$ the system given by (41), (42), (44), (46), and

$$\chi_{i,t} - \Delta\chi_i + \xi_i = w_s - \tfrac{1}{2}|\mathbf{u}_i|^2, \quad \zeta_i \equiv 0 \ \text{ a.e. in } \Gamma_c \times (0, T_0), \ \ i = 1, 2 \tag{171}$$

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Now, let us set $\widetilde{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2$, and analogously introduce the functions $\widetilde{\chi}, \widetilde{\boldsymbol{\eta}}, \widetilde{\xi}$ see (56): of course, in this case $\widetilde{\mathbf{F}} = \widetilde{\mathbf{u}}_0 = \mathbf{0}$ and $\widetilde{\chi}_0 = 0$). Arguing as in the proof of Theorem 2, we deduce that the quadruple $(\widetilde{\mathbf{u}}, \widetilde{\chi}, \widetilde{\boldsymbol{\eta}}, \widetilde{\xi})$ fulfils

$$\frac{C_b}{2}\|\widetilde{\mathbf{u}}(t)\|_W^2 + C_a\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} + \frac{1}{2}\|\widetilde{\chi}_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + \frac{1}{2}\|\nabla\widetilde{\chi}(t)\|^2_{L^2(\Gamma_c)}$$
$$\leqslant c\int_0^t \|\widetilde{\chi}_t\|^2_{L^2(0,s;L^2(\Gamma_c))}\,\mathrm{d}s + c\|\widetilde{\mathbf{u}}\|^2_{L^2(0,t;W)} \quad \forall t \in (0, T_0)$$

A straightforward application of the Gronwall Lemma and a comparison in (41) and (171) leads to (51). □

Proof of Proposition 2.4

First of all, let us note that the solution $(\mathbf{u}, \chi, \boldsymbol{\eta}, \xi, \zeta)$ trivially fulfils

$$\|\chi(t) - \chi_0\|_{H^1(\Gamma_c)} \leqslant \int_0^t \|\chi_t(s)\|_{H^1(\Gamma_c)}\,\mathrm{d}s \leqslant Ct^{1/2} \quad \forall t \in [0, T] \tag{172}$$

Then, following the notation of [14, Chapter I.9], let us consider the interpolation space $H^{5/3}(\Gamma_c) = [H^2(\Gamma_c), H^1(\Gamma_c)]_{1/3}$, which is continuously embedded in $L^\infty(\Gamma_c)$. Thus, we have for all $t \in [0, T]$

$$\|\chi(t) - \chi_0\|_{L^\infty(\Gamma_c)} \leqslant c\|\chi(t) - \chi_0\|_{H^{5/3}(\Gamma_c)} \leqslant c\|\chi(t) - \chi_0\|^{2/3}_{H^2(\Gamma_c)}\|\chi(t) - \chi_0\|^{1/3}_{H^1(\Gamma_c)}$$
$$\leqslant c(C + \|\chi_0\|_{H^2(\Gamma_c)})^{2/3} C^{1/3} t^{1/6}$$

where we have combined the interpolation inequality for the norm of $H^{5/3}(\Gamma_c)$ (cf. [14, Theorem I.9.6]) with the estimate (172). Thus, choosing

$$0 < T_0 \leqslant \left(\frac{\delta}{2c(C + \|\chi_0\|_{H^2(\Gamma_c)})^{2/3} C^{1/3}}\right)^6$$

we deduce that

$$\|\chi - \chi_0\|_{L^\infty(\Gamma_c \times (0, T_0))} \leqslant \frac{\delta}{2} \tag{173}$$

On the other hand, due to the constraint $\chi_t \in \mathrm{dom}(\rho) = (-\infty, 0]$ a.e. in $\Gamma_c \times (0, T)$, we have

$$\chi(x, t) \leqslant \chi_0(x) < 1 \quad \forall (x, t) \in \Gamma_c \times [0, T] \tag{174}$$

Collecting (52), (173) and (174), (53) follows. □

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