Well-posedness and Long-time Behaviour for a Model of Contact with Adhesion

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ABSTRACT. This paper addresses the analysis of a model, proposed by M. Frémond, for the phenomenon of contact with reversible adhesion between a viscoelastic body and a rigid support. First of all, we prove existence and uniqueness of global-in-time solutions of (the initial-boundary value problem for) the related PDE system by means of a fixed point technique. Hence, we investigate the long-time behaviour of such solutions and obtain some results on the structure of the associated ω-limit set.

1. INTRODUCTION

This paper is concerned with the analytical treatment of a nonlinear PDE system describing contact with adhesion between a viscoelastic body and a rigid support. The model has been proposed by M. Frémond (see [15], [16], [17]), and mainly combines the damage and contact theories making use of the phase field approach (cf. also [5], [24]). In the phenomenon of contact with adhesion, the resistance to the tension on the contact surface is due to microbonds between the surface of the body and its support. Thus, in the model by Frémond the description of the contact surface includes the description of the state of such bonds, which can break (or get damaged) owing to microscopic motions. Actually, a phase parameter $\chi \in [0,1]$ is introduced to represent the state of damage of the bonds: in particular, the value $\chi = 0$ corresponds to completely damaged bonds, $\chi = 1$ to the undamaged case, and $\chi \in (0,1)$ to an intermediate situation.

The analytical version of this model has recently been addressed in [4], where a global existence result on a finite time interval $(0,T)$ has been proved in the case of an irreversible damage process for the adhesion. In the present paper, we instead deal with the case in which the bonds responsible for the adhesion on the contact

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surface, once damaged, can repair themselves, i.e. the phenomenon of damage is reversible (this behaviour may be for instance observed in materials such as polymers).

We are considering, at first, an isothermal phenomenon. Thus, the system is written in terms of two unknowns: the macroscopic deformation $\varepsilon(\mathbf{u})$ of the viscoelastic body (i.e. the linearized symmetric strain tensor, $\mathbf{u}$ being the vector of small displacements) and the damage parameter χ. The equations are recovered from the principle of virtual power, in which there are included the effects of the contact micro-forces and of the micro-movements breaking the bonds. Thus, the resulting system is given by a balance equation for macroscopic movements, coupled with an equilibrium equation describing the evolution of damage on the contact surface. These two equations are supplemented with suitable initial and boundary conditions; in particular, concerning the interaction between the body and the support we prescribe an impenetrability condition. Since in [4] the derivation of the model is fully detailed, here we just sketch the main ideas and introduce the resulting analytical formulation.

The viscoelastic body is assumed to be located in a smooth bounded domain $\Omega \subset \mathbb{R}^3$; let its boundary be $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_c}$, where Γ_1, Γ_2, and Γ_c are open subsets in the relative topology of Γ with a smooth boundary and disjoint one from another; morever, we assume that the contact surface Γ_c and the region Γ_1 have strictly positive measure. Hence, the system is written on the whole time interval $(0, +\infty)$ as follows:

$$-\operatorname{div}\,(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t)) = \mathbf{f} \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_1 \times (0, +\infty), \qquad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t))\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_2 \times (0, +\infty), \tag{1.2}$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\mathbf{u}_t))\mathbf{n} + k\chi\mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \ni \mathbf{0} \quad \text{in } \Gamma_c \times (0, +\infty), \tag{1.3}$$

$$c_s\chi_t - k_s\Delta_s\chi + \partial I_{[0,1]}(\chi) \ni w_s(\chi) - \frac{k}{2}|\mathbf{u}|^2 + A \quad \text{in } \Gamma_c \times (0, +\infty), \tag{1.4}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{in } \partial\Gamma_c \times (0, +\infty). \tag{1.5}$$

Note that in (1.2)-(1.4) we have omitted the trace symbol for $\mathbf{u}$ and $\mathbf{u}_t$. Hence, Δ_s is the laplacian on Γ_c (see (1.4)), while $\mathbf{n}$ ($\mathbf{n}_s$, respectively) is the outward unit normal vector to Γ (to $\partial\Gamma_c$, resp.). Further, K is the rigidity matrix, K_v the viscosity matrix, $\mathbf{f}$ an applied volume force, $\mathbf{g}$ a traction applied on a part of the boundary from the exterior, and A an exterior source of damage acting on Γ_c. Note that (1.5) implies that no external forces act on the boundary of Γ_c. The function $w_s(\chi)$ in (1.4) represents the cohesion of the adhesion on the contact surface and, in general, it may depend on the state of the bonds of the adhesion, i.e. on the damage parameter χ. Conversely, the quadratic nonlinearity $(k/2)|\mathbf{u}|^2$ plays the role of a damage source. As far as the choice of w_s is concerned, we recall that in [4] w_s was simply taken to be a positive constant. Here, we allow for a more general condition, i.e. we include the possibility that the cohesion may change according to the fact that the adhesion is completely active or partially/completely damaged. For instance, one might think of the case in which $w_s(\chi) \to 0$ if $\chi \to 0$, for it is fairly reasonable that it should be less difficult to damage bonds which have been already partially damaged. As for the boundary condition (1.3), the subdifferential operator $\partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n})$ entails the impenetrability condition, since it implies $\mathbf{u}\cdot\mathbf{n} \leq 0$. As a matter of fact, this operator is defined by $\partial I_{]-\infty,0]}(\mathbf{u}\cdot\mathbf{n}) = 0$ if $\mathbf{u}\cdot\mathbf{n} < 0$ and $\partial I_{]-\infty,0]}(0) = [0,+\infty[$. Analogously, the subdifferential $\partial I_{[0,1]}$ in (1.4) represents a physical constraint on χ, which is forced to take values in the interval $[0,1]$, where $\partial I_{[0,1]}(\chi) = 0$ if $\chi \in]0,1[$, $\partial I_{[0,1]}(0) =]-\infty,0]$, and $\partial I_{[0,1]}(1) = [0,+\infty[$. However, our results apply to more general graphs than the subdifferentials $\partial I_{]-\infty,0]}$ and $\partial I_{[0,1]}$. Finally, the constants k, c_s, k_s are strictly positive: for the sake of simplicity, in the sequel they shall be set equal to 1.

Before moving on, let us make some comments on the connection between [4] and the present paper. As we have already mentioned, in [4] the model has been introduced in the case when the cohesion of the adhesion does not depend on the level of damage of the bonds, i.e. the function w_s has been taken constant. In addition, we have assumed that no external forces can break the bonds. In this sense, the present equation (1.4) covers more general situations. However, in [4] we have tackled an irreversible phenomenon, namely we have required that $\chi_t \leq 0$. This constraint is ensured by the presence of a second subdifferential operator acting on χ_t in the differential inclusion for the dynamics of χ, which therefore displays a doubly nonlinear character. Mainly for this reason, in [4] uniqueness of the solution has been proved only for an approximate system, obtained by regularization of the graph acting on χ (which is in fact the subdifferential $\partial I_{[0,1]}$ in (1.4)).

Instead, in the present paper we can establish existence *and uniqueness* of a solution $(\mathbf{u},\chi)$ to the Cauchy problem (1.1–1.5) on any interval $(0,T)$. In particular, uniqueness follows from a result of continuous dependence on the data. On the basis of this well-posedness result, we further investigate the long-time behaviour of the solution as the time t goes to $+\infty$. More precisely, we prove that the ω-limit associated with our PDE system, i.e. the set of the cluster points as time goes to infinity of the solution trajectories, is non empty and connected. Moreover, we show that the elements of the ω-limit solve the stationary problem associated with (1.1–1.5). Now, note that the presence of a non-constant cohesion w_s in (1.4) may correspond to an antimonotone contribution in the evolution of χ, coming from a non-convex free-energy. Thus, in the general situation we cannot prove that the stationary problem associated with our evolution system admits a unique solution, and we are unable to deduce directly that the *whole* solution trajectories converge as $t \to +\infty$ to the same limit point in the ω-limit. However, imposing suitable conditions on the cohesion function w_s and on the long-time behaviour of the external source A of damage, we can show that the stationary problem admits a unique solution that we find explicitly, concluding the convergence to the equilibrium of the whole solution trajectories.

In the last decade there has been a rich literature on the study of the ω-limit of solutions to systems of phase field type (cf., e.g., among the others [11]). Such systems feature the presence of free-energy functionals which are non convex w.r.t. the phase parameter so that the associated stationary problems do not have in general a unique solution, so that, again, it is not possible to deduce directly that the solution trajectories converge to an equilibrium. Nonetheless, there has recently been a flourishing development of results on the convergence of the solution trajectories of some phase separation and phase field systems (see [26], [13], [20], and the references therein), based on the so-called Łojasiewicz technique. The latter can be however applied only to analytical potentials for the phase parameter, while the most physically meaningful potential in our model is in fact the (non-smooth) indicator function of $[0,1]$. Lately, in the paper [22], convergence results have been obtained for a phase relaxation system with a non-smooth potential by means of refined inequalities, partly based on a careful application of the maximum principle. Such techniques seem, however, out of reach in the present situation. Therefore, the convergence of the solution trajectories of (1.1)–(1.5) to their equilibrium remains open in the general case, as well as the the existence of the universal attractor for the related semigroup of solutions.

Finally, let us hint at the main analytical difficulties connected with our problem. First of all, let us point out that both the equations of our system include multivalued operators (the first equation indeed through the boundary condition (1.2)). Moreover, while (1.1) is set in the domain Ω, (1.4) describes the evolution of the unknown X on a part of the boundary of Ω, so that (1.1) and (1.4) are related in the sense of trace theorems for Sobolev functions. Thus, in the following analysis we shall apply *ad hoc* tools to handle the resulting PDE system.

Here is the plan of the paper. In the next section, we introduce the variational formulation of the problem we are dealing with and make precise the assumptions we need on the data. Hence, we state our main results concerning the existence, uniqueness, and continuous dependence on the data of the solutions, as well as their long-time behaviour. Section 3 is devoted to the proof of the existence and uniqueness result through a Banach fixed point argument and contractive estimates. Finally, in Section 4 we investigate the ω-limit set associated with the trajectories of the solutions.

2. MAIN RESULTS

2.1. Variational formulation and statement of the assumptions

Notation. Throughout the paper, given a Banach space X, we shall denote by ${}_{X'}\langle\cdot,\cdot\rangle_X$ the duality pairing between X' and X itself, and by $\|\cdot\|_X$ both the norm in X and in any power of X. We shall also use the notation $C^0_{\mathrm{w}}([0,T];X)$ for the space of weakly continuous X-valued functions on $[0,T]$.

Furthermore, we shall suppose that Ω is a bounded smooth set of $\mathbb{R}^3$, such that Γ_c is a smooth bounded domain of $\mathbb{R}^2$ (one may think of a flat surface). We shall consider the spaces

$$H := (L^2(\Omega))^3, \quad V := (H^1(\Omega))^3$$

(inducing a Hilbert triplet $V \hookrightarrow H \equiv H' \hookrightarrow V'$), as well as

$$W := \{\mathbf{v} \in V : \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\},$$

endowed with the norm induced by V. For simplicity, we shall hereafter use the notation $\int_{\Gamma_c}$ ($\int_{\Gamma_2}$, resp.), for the duality pairing ${}_{(H^{-1/2}(\Gamma_c))^3}\langle \cdot , \cdot \rangle_{(H^{1/2}(\Gamma_c))^3}$ between $(H^{-1/2}(\Gamma_c))^3$ and $(H^{1/2}(\Gamma_c))^3$ (between $(H^{-1/2}(\Gamma_2))^3$ and $(H^{1/2}(\Gamma_2))^3$, resp.). Finally, given a subset $\mathcal{O} \subset \mathbb{R}^N$, $N = 1, 2, 3$, we shall denote by $|\mathcal{O}|$ its Lebesgue measure.

In order to give a variational formulation to the boundary-value problem (1.1)–(1.5), supplemented with the initial conditions

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \text{ in } \Omega, \qquad \chi(\cdot, 0) = \chi_0 \text{ in } \Gamma_c, \tag{2.1}$$

we let the material under investigation be homogeneous and isotropic, as usual in elasticity theory, so that the rigidity matrix K in (1.1)–(1.3) may be represented as

$$K\varepsilon(\mathbf{u}) = \lambda \mathrm{tr}\, \varepsilon(\mathbf{u})\mathbf{1} + 2\mu\varepsilon(\mathbf{u}),$$

where $\lambda, \mu > 0$ are the so-called Lamé constants and $\mathbf{1}$ is the identity matrix. Also, for the sake of simplicity but without loss of generality, let us set in (1.1)–(1.3)

$$K_v\varepsilon(\mathbf{v}) = \varepsilon(\mathbf{v}).$$

Therefore, (1.1) may be formulated by means of the following continuous bilinear symmetric forms $a, b : W \times W \to \mathbb{R}$, defined by

$$a(\mathbf{u}, \mathbf{v}) := \lambda \int_\Omega \mathrm{div}(\mathbf{u})\, \mathrm{div}(\mathbf{v}) + 2\mu \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \qquad \forall \mathbf{u}, \mathbf{w} \in W,$$

$$b(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \qquad \forall \mathbf{u}, \mathbf{w} \in W.$$

Hence,

$$\exists\, M > 0 : \; |a(\mathbf{u}, \mathbf{v})| + |b(\mathbf{u}, \mathbf{v})| \leq M \|\mathbf{u}\|_W \|\mathbf{v}\|_W \quad \forall \mathbf{u}, \mathbf{v} \in W, \tag{2.2}$$

and

$$b(\mathbf{u}, \mathbf{u}) = \|\varepsilon(\mathbf{u})\|_H^2 .$$

Since Γ_1 has positive measure, by Korn's inequality we deduce that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are W-elliptic, i.e., there exist $C_a, C_b > 0$ such that

$$a(\mathbf{u},\mathbf{u}) \geq C_a \|\mathbf{u}\|_W^2 \qquad \forall\, \mathbf{u} \in W, \tag{2.3}$$

$$b(\mathbf{u},\mathbf{u}) \geq C_b \|\mathbf{u}\|_W^2 \qquad \forall\, \mathbf{u} \in W. \tag{2.4}$$

As it was already pointed out in [4], the boundary condition (1.3) might be rendered by means of the following operator: we consider the set

$$\mathcal{X}_- := \{\mathbf{v} \in (H^{1/2}(\Gamma_c))^3 : \mathbf{v}\cdot\mathbf{n} \leq 0 \text{ a.e. in } \Gamma_c\},$$

and we denote by $I_{\mathcal{X}_-}$ the indicator function of $\mathcal{X}_-$, and by $\partial I_{\mathcal{X}_-} : (H^{1/2}(\Gamma_c))^3 \to 2^{(H^{-1/2}(\Gamma_c))^3}$ its subdifferential, defined by the formula

$$\begin{aligned}&\boldsymbol{\eta} \in (H^{-1/2}(\Gamma_c))^3 \text{ belongs to } \partial I_{\mathcal{X}_-}(\mathbf{y}) \text{ if and only}\\ &\text{if } \mathbf{y} \in \mathcal{X}_-, \int_{\Gamma_c} \boldsymbol{\eta}\cdot(\mathbf{v}-\mathbf{y}) \leq 0 \quad \forall\, \mathbf{v} \in \mathcal{X}_-.\end{aligned}$$

Actually, following the outline of [4] we shall deal with a more general maximal monotone graph in $(H^{1/2}(\Gamma_c))^3 \times (H^{-1/2}(\Gamma_c))^3$. Indeed, let us consider a functional

$$\begin{aligned}&\varphi : (H^{1/2}(\Gamma_c))^3 \to [0,+\infty] \text{ proper, convex and}\\ &\text{lower semicontinuous, with } \varphi(\mathbf{0}) = 0 = \min \varphi,\\ &\text{and let } \alpha := \partial\varphi : (H^{1/2}(\Gamma_c))^3 \to 2^{(H^{-1/2}(\Gamma_c))^3}.\end{aligned} \tag{2.5}$$

Let us also introduce a maximal monotone operator

$$\beta : \mathbb{R} \to 2^{\mathbb{R}}, \quad \text{such that } \operatorname{dom}(\beta) \subseteq [0,+\infty[\text{ and } 0 \in \beta(0). \tag{2.6}$$

In the sequel, for any $r \in \operatorname{dom}(\beta)$ we shall denote by $\beta^0(r)$ the element of minimal norm in $\beta(r)$. Of course, the operator β will generalize the graph $\partial I_{[0,1]}$ in equation (1.4). It is well known that there exists a functional $\psi : \mathbb{R} \to [0,+\infty]$, proper, convex, and lower semicontinuous, such that

$$\psi(0) = 0 = \min \psi, \text{ and } \beta = \partial\psi. \tag{2.7}$$

As for the body force $\mathbf{f}$ and the surface traction $\mathbf{g}$, we require

$$\mathbf{f} \in L^2(0,T;H), \tag{2.8}$$

$$\mathbf{g} \in L^2(0,T;(H^{-1/2}(\Gamma_2))^3), \tag{2.9}$$

so that, defining $\mathbf{F} : (0,T) \to W$ by

$${}_{W'}\langle \mathbf{F}(t),\mathbf{v}\rangle_W := \int_\Omega \mathbf{f}(t)\cdot\mathbf{v} + \int_{\Gamma_2} \mathbf{g}(t)\cdot\mathbf{v} \quad \forall\, \mathbf{v} \in W \quad \text{for a.e. } t \in (0,T), \tag{2.10}$$

we have $\mathbf{F} \in L^2(0,T;W')$. Furthermore, we assume that

(2.11) $\quad w_s : \mathbb{R} \to [0,+\infty)$ is a Lipschitz continuous function (with Lipschitz constant $L_s > 0$),

(2.12) $\quad A \in L^2(0,T;L^2(\Gamma_c))\,.$

Let $\widehat{w_s} \in C^1(\mathbb{R})$ be the primitive of w_s such that $\widehat{w_s}(0) = 0$. Owing to (2.11), there exist positive constants K_L and C_L such that

$$|\widehat{w_s}(x)| \le K_L x^2 + C_L \quad \forall\, x \in \mathbb{R}. \tag{2.13}$$

Finally, we shall consider initial data satisfying

$$\mathbf{u}_0 \in W, \quad \mathbf{u}_{0|_{\Gamma_c}} \in \mathrm{dom}(\varphi)\,, \tag{2.14}$$

$$\chi_0 \in H^1(\Gamma_c),\ \psi(\chi_0) \in L^1(\Gamma_c)\,. \tag{2.15}$$

In this framework, setting the coefficients k_s, k and c_s in (1.3)–(1.4) equal to 1, the variational formulation of the initial boundary-value problem (1.1–1.5, 2.1) reads as follows:

Problem 2.1. Find $(\mathbf{u},\chi,\boldsymbol{\eta},\xi)$ such that

$$\mathbf{u} \in H^1(0,T;W), \tag{2.16}$$

$$\chi \in L^2(0,T;H^2(\Gamma_c)) \cap C^0(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c))\,, \tag{2.17}$$

$$\boldsymbol{\eta} \in L^2(0,T;(H^{-1/2}(\Gamma_c))^3)\,, \tag{2.18}$$

$$\xi \in L^2(0,T;L^2(\Gamma_c))\,, \tag{2.19}$$

(2.1) holds and

$$b(\mathbf{u}_t,\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_c} (\chi\mathbf{u} + \boldsymbol{\eta})\cdot\mathbf{v} = {}_{W'}\langle \mathbf{F},\mathbf{v}\rangle_W \quad \forall\, \mathbf{v} \in W \text{ almost everywhere in } (0,T)\,, \tag{2.20}$$

$$\boldsymbol{\eta} \in \alpha(\mathbf{u}) \quad \text{almost everywhere in } (0,T), \tag{2.21}$$

$$\chi_t - \Delta\chi + \xi = w_s(\chi) - \frac{1}{2}|\mathbf{u}|^2 + A \quad \text{almost everywhere in } \Gamma_c \times (0,T), \tag{2.22}$$

$$\xi \in \beta(\chi) \quad \text{almost everywhere in } \Gamma_c \times (0,T), \tag{2.23}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{almost everywhere in } \partial\Gamma_c \times (0,T)\,. \tag{2.24}$$

Note that, from now, to simplify notation we use the symbol $-\Delta$ for the Laplace operator $-\Delta_s$ on Γ_c (cf. (2.22)).

2.2. Well-posedness and continuous dependence results

Theorem 1. *Assume* (2.5)–(2.6) *and* (2.8)–(2.9), (2.11)–(2.12), *and* (2.14)–(2.15). *Then, Problem 2.1 admits a unique solution* $(\mathbf{u}, \chi, \boldsymbol{\eta}, \xi)$.

Moreover, under the additional assumptions

$$\chi_0 \in H^2(\Gamma_c),\ \partial_{\mathbf{n}_s}\chi_0 = 0 \text{ in } \partial\Gamma_c,\ \beta^0(\chi_0) \in L^2(\Gamma_c), \tag{2.25}$$

$$A \in W^{1,1}(0,T;L^2(\Gamma_c)), \tag{2.26}$$

we have the further regularity

$$\chi \in L^\infty(0,T;H^2(\Gamma_c)) \cap H^1(0,T;H^1(\Gamma_c)) \cap W^{1,\infty}(0,T;L^2(\Gamma_c)) \subset C^0_w([0,T];H^2(\Gamma_c)), \tag{2.27}$$

$$\xi \in L^\infty(0,T;L^2(\Gamma_c)). \tag{2.28}$$

Indeed, the uniqueness statement in Theorem 1 is a consequence of the following continuous dependence result:

Proposition 2.2.
Under the assumptions (2.5)–(2.6) *and* (2.11), *let* $(\mathbf{u}_0^1, \chi_0^1, \mathbf{f}_1, \mathbf{g}_1, A_1)$ *and* $(\mathbf{u}_0^2, \chi_0^2, \mathbf{f}_2, \mathbf{g}_2, A_2)$ *be two pairs of data for Problem 2.1, complying with* (2.14)–(2.15), (2.8)–(2.9), *and* (2.12). *Accordingly, let* $(\mathbf{u}_1, \chi_1, \boldsymbol{\eta}_1, \xi_1)$ *and* $(\mathbf{u}_2, \chi_2, \boldsymbol{\eta}_2, \xi_2)$ *be the corresponding solutions of Problem 2.1. Set*

$$m := \max_{i=1,2} \Big\{ \|\mathbf{u}_0^i\|_W + \|\chi_0^i\|_{L^2(\Gamma_c)} + \|\mathbf{f}_i\|_{L^2(0,T;H)} + \|\mathbf{g}_i\|_{L^2(0,T;H^{-1/2}(\Gamma_2))} + \|A_i\|_{L^2(0,T;L^2(\Gamma_c))} \Big\}. \tag{2.29}$$

Then, there exists a positive constant L, only depending on m *and on* M, C_a, C_b, L_s, T, $|\Omega|$ *and* $|\Gamma_c|$ *such that the following estimate holds:*

$$\begin{aligned} &\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(0,T;W)} + \|\chi_1 - \chi_2\|_{L^2(0,T;H^1(\Gamma_c)) \cap L^\infty(0,T;L^2(\Gamma_c))} \\ &\quad \le L \Big(\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_W + \|\chi_0^1 - \chi_0^2\|_{L^2(\Gamma_c)} + \|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(0,T;H)} \\ &\qquad + \|\mathbf{g}_1 - \mathbf{g}_2\|_{L^2(0,T;H^{-1/2}(\Gamma_2))} + \|A_1 - A_2\|_{L^2(0,T;L^2(\Gamma_c))} \Big). \end{aligned} \tag{2.30}$$

2.3. Long-time behaviour. Let us now turn to examining the long-time behaviour of the solutions to Problem 2.1, which can be in fact extended to $(0,+\infty)$ because our well-posedness result Theorem 1 holds for any $T > 0$, once we have assumed that (2.8)-(2.9) and (2.12) hold for any $T > 0$. More precisely, given a pair of initial data $(\mathbf{u}_0, \chi_0)$, we investigate the cluster points as $t \to +\infty$ of the associated solution trajectory $(\mathbf{u}(t),\chi(t))_{t\geq 0}$, in the topology of $H\times L^2(\Gamma_c)$. To this aim, we define the ω-limit set $\omega(\mathbf{u}_0,\chi_0)$ of $(\mathbf{u}(t),\chi(t))_{t\geq 0}$ as

$$\begin{aligned}(2.31)\quad \omega(\mathbf{u}_0,\chi_0) := \Big\{(\mathbf{u}_\infty,\chi_\infty)\in W\times H^1(\Gamma_c)\ :\\ \exists\,\{t_n\}\subset[0,+\infty),\ \ t_n\nearrow+\infty \ \text{ as } n\uparrow\infty,\\ \text{with }\ (\mathbf{u}(t_n),\chi(t_n))\to(\mathbf{u}_\infty,\chi_\infty)\ \text{ in } H\times L^2(\Gamma_c)\Big\}.\end{aligned}$$

The ensuing Theorem 2 states that any element of $\omega(\mathbf{u}_0,\chi_0)$ solves the stationary problem associated with system (2.20)–(2.24). Indeed, we consider the following additional assumptions:

$$(2.32)\quad \begin{aligned}&\mathbf{f}\in L^\infty(0,+\infty;H) \quad \text{and} \quad \mathbf{f}_t\in L^1(0,+\infty;H),\\ &\mathbf{g}\in L^\infty(0,+\infty;(H^{-1/2}(\Gamma_c))^3) \quad \text{and} \quad \mathbf{g}_t\in L^1(0,+\infty;(H^{-1/2}(\Gamma_c))^3);\end{aligned}$$

$$(2.33)\quad A\in L^\infty(0,+\infty;L^2(\Gamma_c)) \quad \text{and} \quad A_t\in L^1(0,+\infty;L^2(\Gamma_c)).$$

Due to (2.32), the function $\mathbf{F}$ defined by (2.10) is in $L^\infty(0,+\infty;W')$, with $\mathbf{F}_t\in L^1(0,+\infty;W')$. In the case when the domain dom β is unbounded, we shall also suppose that

$$(2.34)\qquad \lim_{x\nearrow+\infty}\frac{\psi(x)}{x^2}=+\infty,$$

which in particular entails that

$$(2.35)\qquad \forall\, R>0\quad \exists\, C_R>0\ :\ Rx^2\leq\psi(x)+C_R\qquad \forall\, x\geq 0.$$

Remark 2.3. Owing to (2.32)-(2.33), there exist $\mathbf{F}_\infty\in W'$, $A_\infty\in L^2(\Gamma_c)$ such that

$$(2.36)\qquad \mathbf{F}(t)\to\mathbf{F}_\infty\ \text{ in } W',\quad A(t)\to A_\infty\ \text{ in } L^2(\Gamma_c)\quad \text{as } t\to+\infty.$$

To check the claim for $\mathbf{F}$ (the one for A can be proved in the same way), we first of all observe that for any increasing sequence $\{s_m\}\subset[0,+\infty)$ with $s_m\nearrow+\infty$ as $m\nearrow\infty$, there holds for all $j>0$

$$\|\mathbf{F}(s_{m+j})-\mathbf{F}(s_m)\|_{W'}\leq\int_{s_m}^{s_{m+j}}\|\mathbf{F}_t\|_{W'}\leq\int_{s_m}^{+\infty}\|\mathbf{F}_t\|_{W'}\to 0\ \text{ as } m\nearrow\infty,$$

due to the fact that $\mathbf{F}_t\in L^1(0,+\infty;W')$. This shows that $\{\mathbf{F}(s_m)\}$ is a Cauchy sequence, and thus converges as $m\nearrow\infty$ to some $\mathbf{F}_\infty$ in W'. A similar argument shows that the limit $\mathbf{F}_\infty$ does not depend on the sequence $\{s_m\}$, so that the convergence to $\mathbf{F}_\infty$ holds as $t\to+\infty$. Moreover, we easily conclude the inequality

$$\|\mathbf{F}(t)-\mathbf{F}_\infty\|_{W'}\leq\int_t^{+\infty}\|\mathbf{F}_t\|_{W'}\qquad\forall\, t>0.$$

Theorem 2. *Assume* (2.5)-(2.6), (2.11), *and* (2.32)–(2.34). *Then, for any pair of initial data* $(\mathbf{u}_0, \chi_0)$ *complying with* (2.14)–(2.15), *the* ω*-limit set* $\omega(\mathbf{u}_0, \chi_0)$ *is a non empty, compact and connected subset of* $H \times L^2(\Gamma_c)$. *Moreover, for any* $(\mathbf{u}_\infty, \chi_\infty) \in \omega(\mathbf{u}_0, \chi_0)$ *we have*

$$u_\infty \in W \text{ and } a(\mathbf{u}_\infty, \mathbf{v}) + \int_{\Gamma_c} (\chi_\infty \mathbf{u}_\infty + \boldsymbol{\eta}_\infty) \cdot \mathbf{v} = {}_{W'}\langle \mathbf{F}_\infty, \mathbf{v} \rangle_W \qquad \forall\, \mathbf{v} \in W,\ \boldsymbol{\eta}_\infty \in (H^{-1/2}(\Gamma_c))^3,\ \boldsymbol{\eta}_\infty \in \alpha(\mathbf{u}_\infty), \tag{2.37}$$

$$\chi_\infty \in H^2(\Gamma_c) \quad \text{and} \quad \begin{cases} -\Delta\chi_\infty + \xi_\infty = w_s(\chi_\infty) - \frac{1}{2}|\mathbf{u}_\infty|^2 + A_\infty & \text{a.e. in } \Gamma_c, \\ \xi_\infty \in L^2(\Gamma_c),\ \xi_\infty \in \beta(\chi_\infty) & \text{a.e. in } \Gamma_c, \\ \partial_{\mathbf{n}_s}\chi_\infty = 0 & \text{a.e. in } \partial\Gamma_c. \end{cases} \tag{2.38}$$

Remark 2.4. As mentioned in the introduction, in general we cannot deduce that system (2.37)-(2.38) has a unique solution (actually, in some cases it is straightforward proved that there is not a unique solution). Hence, it is not possible to deduce directly from Theorem 2 that $\omega(\mathbf{u}_0, \chi_0)$ is a singleton and that the solution trajectories thus converge to a unique equilibrium. Anyhow, the ensuing Propositions 2.5 (and 2.7) provide sufficient conditions for such a convergence result to hold.

Proposition 2.5. *Under the assumptions* (2.5), (2.11), *and* (2.32)–(2.33), *suppose that*

$$\beta = \partial I_{[0,1]}, \tag{2.39}$$

$$\mathbf{F}(t) \to \mathbf{0} \quad \text{as } t \to +\infty, \tag{2.40}$$

and that either (2.41) *or* (2.42) *below hold:*

$$\exists\, a_\infty > 0 : \quad A_\infty(x) + \min_{r\in[0,1]} w_s(r) \ge a_\infty \qquad \text{for a.e. } x \in \Gamma_c, \tag{2.41}$$

$$\exists\, b_\infty > 0 : \quad A_\infty(x) + \max_{r\in[0,1]} w_s(r) \le -b_\infty \qquad \text{for a.e. } x \in \Gamma_c. \tag{2.42}$$

Then, for any pair $(\mathbf{u}_\infty, \chi_\infty) \in \omega(\mathbf{u}_0, \chi_0)$ *we have*

$$\mathbf{u}_\infty \equiv \mathbf{0} \qquad \text{in } \Omega, \tag{2.43}$$

$$\chi_\infty(x) = \begin{cases} 1 & \text{if (2.41) holds,} \\ 0 & \text{if (2.42) holds.} \end{cases} \qquad \forall\, x \in \Gamma_c.$$

In both cases, we have the following convergences of the solution trajectory $(\mathbf{u}(t), \chi(t))_{t\geq 0}$ *as* $t \to \infty$*:*

$$
\begin{aligned}
(2.44) \qquad & \mathbf{u}(t) \to \mathbf{0} \ \text{ in } H, \quad \mathbf{u}(t) \rightharpoonup \mathbf{0} \ \text{ in } W, \\
& \qquad\qquad \chi(t) \to \chi_\infty \ \text{ in } L^2(\Gamma_c), \quad \chi(t) \rightharpoonup \chi_\infty \ \text{ in } H^1(\Gamma_c).
\end{aligned}
$$

Remark 2.6. Let us briefly comment on the physical meaning of Proposition 2.5. In the case when at the limit there is not any external volume force, the deformations tend to zero. Thus, at the limit they do not play a role in the damage of the adhesion which turns out to be governed only by the relation between the external damage source A_∞ (we think a damage source to be negative as it induces the damage parameter χ to go to 0) and the residual cohesion w_s. Thus, if the residual cohesion in the body is sufficiently strong with respect to the damage source, the adhesion turns out to be completely undamaged, while in the case when the damage source A_∞ is stronger we get that the adhesion is not active. In particular, we are able to control the behaviour of the structure at the limit, just in terms of the external damage source and the cohesion of the glue. Hence, one may observe that the diffusive character of the equations leads to u_∞, χ_∞ be constant and the value of these constants does not depend on initial data: this is mainly due to the reversibility of the process.

In the same direction of Proposition 2.5 (cf. also Remark 2.6), we recover a similar result also with different assumptions on these data, as it is specified by the following Proposition.

Proposition 2.7. *Assume* (2.5)–(2.6), (2.11), (2.34), *and*

$$
(2.45) \qquad
\begin{aligned}
& \mathbf{f} \in L^2(0, +\infty; H), \\
& \mathbf{g} \in L^2(0, +\infty; (H^{-1/2}(\Gamma_c))^3), \\
& A \in L^2(0, +\infty; L^2(\Gamma_c)).
\end{aligned}
$$

Then, for any $(\mathbf{u}_0, \chi_0)$ *complying with* (2.14)–(2.15) *the set* $\omega(\mathbf{u}_0, \chi_0)$ *is non empty, compact and connected, and any pair* $(\mathbf{u}_\infty, \chi_\infty) \in \omega(\mathbf{u}_0, \chi_0)$ *fulfils*

$$
(2.46) \qquad
\begin{aligned}
& \mathbf{u}_\infty \equiv \mathbf{0} \quad \text{in } \Omega, \\
& \chi_\infty \in H^2(\Gamma_c) \quad \text{and} \\
& \begin{cases}
-\Delta \chi_\infty + \xi_\infty = w_s(\chi_\infty) & \text{a.e. in } \Gamma_c, \\
\qquad \text{with } \xi_\infty \in L^2(\Gamma_c), \ \xi_\infty \in \beta(\chi_\infty) & \text{a.e. in } \Gamma_c, \\
\partial_{\mathbf{n}_s} \chi_\infty = 0 & \text{a.e. in } \partial\Gamma_c.
\end{cases}
\end{aligned}
$$

Furthermore, under the additional hypothesis (2.39) *and assuming that*

$$\min_{r\in[0,1]} w_s(r) > 0, \tag{2.47}$$

then $\chi_\infty \equiv 1$ *on* Γ_c*, and the convergences* (2.44) *hold.*

In the end, let us collect here some properties which shall be useful in the sequel. We recall that, by Sobolev's embedding theorem, there exists a positive constant c such that

$$\|\mathbf{v}\|_{L^4(\Gamma_c)} + \|\mathbf{v}\|_{H^{1/2}(\Gamma_c)} \le c\|\mathbf{v}\|_W \quad \forall\, \mathbf{v} \in W\,. \tag{2.48}$$

Moreover, let us point out for later convenience that for all $\zeta \in L^2(\Gamma_c)$ ($\zeta \in L^p(\Gamma_c)$, respectively, with $p > 4/3$), and $\mathbf{v} \in (L^4(\Gamma_c))^3$ ($\mathbf{v} \in (L^{q_p}(\Gamma_c))^3$, resp., with q_p fulfilling $1 = 1/p + 1/q_p + 1/4$), the product $\zeta\mathbf{v}$ is in $(H^{-1/2}(\Gamma_c))^3$, and

$$\begin{aligned} \|\zeta\mathbf{v}\|_{H^{-1/2}(\Gamma_c)} &\le \|\zeta\|_{L^2(\Gamma_c)}\,\|\mathbf{v}\|_{L^4(\Gamma_c)},\\ \|\zeta\mathbf{v}\|_{H^{-1/2}(\Gamma_c)} &\le \|\zeta\|_{L^p(\Gamma_c)}\,\|\mathbf{v}\|_{L^{q_p}(\Gamma_c)}. \end{aligned} \tag{2.49}$$

Finally, we shall use the Young inequality

$$ab \le (\delta/2)a^2 + (2\delta)^{-1}b^2 \quad \forall\, a,b \in \mathbb{R}, \quad \delta > 0\,. \tag{2.50}$$

Hence, we warn that, in the following proofs, we shall employ the same symbols c, C for different constants, even in the same formula, for the sake of simplicity.

3. WELL-POSEDNESS AND CONTINUOUS DEPENDENCE

3.1. Proof of Proposition 2.2 (and uniqueness of the solution). Preliminarily, let us state the following a priori estimates on the components $(\mathbf{u},\chi)$ of the solution to Problem 2.1.

Lemma 3.1. *Assume* (2.5)–(2.6) *and* (2.8)–(2.9), (2.11)–(2.12), *and* (2.14)–(2.15). *Then, there exists a positive constant* S*, only depending on* M, C_a, C_b, L_s, T, $|\Omega|$ *and* $|\Gamma_c|$*, such that for any solution* $(\mathbf{u},\chi,\boldsymbol{\eta},\xi)$ *of Problem 2.1 there holds*

$$\begin{aligned} &\|\mathbf{u}\|_{L^\infty(0,T;W)} + \|\chi\|_{L^2(0,T;H^1(\Gamma_c))\cap L^\infty(0,T;L^2(\Gamma_c))}\\ &\qquad \le S\left(1 + \|\mathbf{u}_0\|_W + \|\chi_0\|_{L^2(\Gamma_c)} + \|\mathbf{F}\|_{L^2(0,T;W')} + \|A\|_{L^2(0,T;L^2(\Gamma_c))}\right). \end{aligned} \tag{3.1}$$

Proof. Let us choose $\mathbf{v} = \mathbf{u}$ in (2.20) and test (2.22) by χ; adding the resulting relations and integrating on the time interval $[0,t]$, for $t \in [0,T]$, we obtain

$$\frac{1}{2}b(\mathbf{u}(t),\mathbf{u}(t)) + \int_0^t a(\mathbf{u},\mathbf{u}) + \int_0^t \int_{\Gamma_c} \boldsymbol{\eta}\cdot\mathbf{u} + \frac{3}{2}\int_0^t\int_{\Gamma_c} \chi|\mathbf{u}|^2$$
$$+ \frac{1}{2}\|\chi(t)\|^2_{L^2(\Gamma_c)} + \int_0^t \|\nabla\chi\|^2_{L^2(\Gamma_c)} + \int_0^t\int_{\Gamma_c} \xi\chi$$
$$= \frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0) + \int_0^t {}_{W'}\langle \mathbf{F},\mathbf{u}\rangle_W + \frac{1}{2}\|\chi_0\|^2_{L^2(\Gamma_c)} + \int_0^t\int_{\Gamma_c} (w_s(\chi)+A)\chi.$$

Owing to (2.5) and (2.6), the third, the fourth and the seventh integral terms on the left-hand side of the above inequality are non negative. As for the right-hand side, the first summand is estimated by (2.2) and (2.14), whereas the remaining integral terms can be controlled by means of (2.8)–(2.12) and inequality (2.50). Thus, after recalling that w_s is Lipschitz and applying the Hölder inequality, we find (cf. (2.2)–(2.4))

$$\text{(3.2)}\quad \frac{C_b}{2}\|\mathbf{u}(t)\|^2_W + \frac{C_a}{2}\int_0^t \|\mathbf{u}\|^2_W + \frac{1}{2}\|\chi(t)\|^2_{L^2(\Gamma_c)} + \int_0^t \|\nabla\chi\|^2_{L^2(\Gamma_c)}$$
$$\le c\left(1 + \|\mathbf{u}_0\|^2_W + \|\chi_0\|^2_{L^2(\Gamma_c)} + \|\mathbf{F}\|^2_{L^2(0,T;W')} + \|A\|^2_{L^2(0,T;L^2(\Gamma_c))}\right) + c\int_0^t \|\chi\|^2_{L^2(\Gamma_c)},$$

where the constant c only depends on L_s, C_a, M, T, and $|\Gamma_c|$. Hence, estimate (3.1) follows by a trivial application of the Gronwall Lemma (see e.g. [7, Lemma A.4]). □

Now, let us come back to the proof of the continuous dependence result. Given two solutions $(\mathbf{u}_1,\chi_1,\boldsymbol{\eta}_1,\xi_1)$ and $(\mathbf{u}_2,\chi_2,\boldsymbol{\eta}_2,\xi_2)$ of Problem 2.1, respectively corresponding to the data $(\mathbf{u}_0^1,\chi_0^1,\mathbf{F}_1,A_1)$ and $(\mathbf{u}_0^2,\chi_0^2,\mathbf{F}_2,A_2)$, we introduce the following notation:

$$\widetilde{\mathbf{u}_0} := \mathbf{u}_0^1 - \mathbf{u}_0^2,\quad \widetilde{\chi_0} := \chi_0^1 - \chi_0^2,\quad \tilde{\mathbf{F}} := \mathbf{F}_1 - \mathbf{F}_2,\quad \tilde{A} := A_1 - A_2,$$
$$\tilde{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2,\quad \tilde{\chi} := \chi_1 - \chi_2,\quad \tilde{\boldsymbol{\eta}} := \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2,\quad \tilde{\xi} := \xi_1 - \xi_2.$$

We subtract (2.20) written for $(\mathbf{u}_2,\chi_2,\boldsymbol{\eta}_2,\mathbf{F}_2)$ from (2.20) written for $(\mathbf{u}_1,\chi_1,\boldsymbol{\eta}_1,\mathbf{F}_1)$, we test the resulting relation by $\tilde{\mathbf{u}}$ and integrate on the interval $[0,t]$. Recalling (2.2)–(2.4) and arguing as in the proof of Lemma 3.1, we end up with

$$
\begin{aligned}
&\frac{C_b}{2}\|\tilde{\mathbf{u}}(t)\|_W^2 + C_a\|\tilde{\mathbf{u}}\|^2_{L^2(0,t;W)} + \int_0^t\int_{\Gamma_c}\tilde{\boldsymbol{\eta}}\cdot\tilde{\mathbf{u}} \\
&\quad\le \frac{1}{2}b(\widetilde{\mathbf{u}_0},\widetilde{\mathbf{u}_0}) - \int_0^t\int_{\Gamma_c}\chi_2\,(\tilde{\mathbf{u}})^2 - \int_0^t\int_{\Gamma_c}\tilde{\chi}\,\mathbf{u}_1\,\tilde{\mathbf{u}} + \int_0^t {}_{W'}\langle\tilde{\mathbf{F}},\tilde{\mathbf{u}}\rangle_W \\
&\quad\le c\left(\|\widetilde{\mathbf{u}_0}\|_W^2 + \int_0^t\|\tilde{\mathbf{F}}\|^2_{W'}\right) + \frac{1}{2}\int_0^t\|\tilde{\mathbf{u}}\|_W^2 + \int_0^t\|\tilde{\chi}\|_{L^2(\Gamma_c)}\|\mathbf{u}_1\|_{L^4(\Gamma_c)}\|\tilde{\mathbf{u}}\|_{L^4(\Gamma_c)}\,,
\end{aligned}
$$

where the last inequality follows from the Hölder inequality and (2.50), from (2.2) and from the fact that $\chi_2 \ge 0$ a.e. on $(0,T)\times\Gamma_c$, due to (2.6). Owing to (2.48), to (2.5), and to the estimate (3.1) for $\|\mathbf{u}_1\|_{L^\infty(0,T;L^4(\Gamma_c))}$, we infer

$$
\frac{C_b}{2}\|\tilde{\mathbf{u}}(t)\|_W^2 \le c\left(\|\widetilde{\mathbf{u}_0}\|_W^2 + \|\tilde{\mathbf{F}}\|^2_{L^2(0,T;W')}\right) + C_1\int_0^t\|\tilde{\chi}\|^2_{L^2(\Gamma_c)} + c\int_0^t\|\tilde{\mathbf{u}}\|_W^2, \tag{3.3}
$$

where the constant C_1 only depends on the data $(\mathbf{u}_0^1,\chi_0^1,\mathbf{f}_1,\mathbf{g}_1)$ through (3.1).

On the other hand, let us consider the difference of (2.22) written for $(\mathbf{u}_1,\chi_1,\xi_1,A_1)$ and (2.22) for $(\mathbf{u}_2,\chi_2,\xi_2,A_2)$, multiply it by $\tilde{\chi}$ and integrate on $[0,t]\times\Gamma_c$. Taking (2.11) into account as well, we get

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{\chi}(t)\|^2_{L^2(\Gamma_c)} + \int_0^t\|\nabla\tilde{\chi}\|^2_{L^2(\Gamma_c)} + \int_0^t\int_{\Gamma_c}\tilde{\xi}\tilde{\chi} \\
&\quad\le \frac{1}{2}\|\widetilde{\chi_0}\|^2_{L^2(\Gamma_c)} - \frac{1}{2}\int_0^t\int_{\Gamma_c}(\mathbf{u}_1+\mathbf{u}_2)\cdot\tilde{\mathbf{u}}\tilde{\chi} + \int_0^t\int_{\Gamma_c}(w_s(\chi_1)-w_s(\chi_2)+\tilde{A})\tilde{\chi} \\
&\quad\le \frac{1}{2}\|\widetilde{\chi_0}\|^2_{L^2(\Gamma_c)} + C_2\int_0^t\|\tilde{\chi}\|_{L^2(\Gamma_c)}\|\tilde{\mathbf{u}}\|_W + \|\tilde{A}\|^2_{L^2(0,T;L^2(\Gamma_c))} + c\int_0^t\|\tilde{\chi}\|^2_{L^2(\Gamma_c)}\,,
\end{aligned}
\tag{3.4}
$$

where again the constant C_2 depends on the Sobolev's embedding constant (2.48) and on the estimate (3.1) for $\|\mathbf{u}_1+\mathbf{u}_2\|_{L^\infty(0,T;W)}$, and c depends in particular on w_s.

Adding (3.3) and (3.4) and using (2.6) and (2.50), we easily conclude that there exist two positive constants K_1 and K_2, only depending on the data of the problem and on the constant m in (2.29), such that

$$
\begin{aligned}
&\|\tilde{\mathbf{u}}(t)\|_W^2 + \|\tilde{\chi}(t)\|^2_{L^2(\Gamma_c)} \\
&\qquad\le K_1\left(\|\widetilde{\mathbf{u}_0}\|_W^2 + \|\widetilde{\chi_0}\|^2_{L^2(\Gamma_c)} + \|\tilde{\mathbf{F}}\|^2_{L^2(0,T;W')} + \|\tilde{A}\|^2_{L^2(0,T;L^2(\Gamma_c))}\right) \\
&\qquad\qquad + K_2\int_0^t\left(\|\tilde{\chi}\|^2_{L^2(\Gamma_c)} + \|\tilde{\mathbf{u}}\|_W^2\right).
\end{aligned}
\tag{3.5}
$$

Thus, by the Gronwall Lemma we have

$$
\begin{aligned}
&\|\tilde{\mathbf{u}}(t)\|_W^2 + \|\tilde{\chi}(t)\|_{L^2(\Gamma_c)}^2 \\
&\quad \le K_1 \left(\|\widetilde{\mathbf{u}_0}\|_W^2 + \|\widetilde{\chi_0}\|_{L^2(\Gamma_c)}^2 + \|\tilde{\mathbf{F}}\|_{L^2(0,T;W')}^2 + \|\tilde{A}\|_{L^2(0,T;L^2(\Gamma_c))}^2 \right) \exp(TK_2),
\end{aligned}
$$

and, recalling (3.4), the estimate for $\|\chi_1 - \chi_2\|_{L^2(0,T;H^1(\Gamma_c))}$ ensues as well.

A trivial consequence of (2.30) is that, when $\widetilde{\mathbf{u}_0} = \mathbf{0}$, $\tilde{\mathbf{F}} = \mathbf{0}$, $\widetilde{\chi_0} = 0$ and $\tilde{A} = 0$, then $\mathbf{u}_1 = \mathbf{u}_2$ and $\chi_1 = \chi_2$. A comparison in (2.20) and (2.22) also yields $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$ and $\xi_1 = \xi_2$ a.e., and the uniqueness statement in Theorem 1 is proved. □

3.2. Proof of existence. The proof of Theorem 1 will be carried out by means of a contraction argument: namely, we will show that a suitably introduced operator $\mathcal{T}$ is contractive, at least in some interval $(0, \hat{T})$, and thus has a (unique) fixed point. It is straightforward to check that this fixed point yields the (unique by Proposition 2.2) solution to Problem 2.1 in $(0, \hat{T})$. Actually, we are proving a global in time existence result, as due to suitable a priori estimates on the solution, we can extend it to the whole interval $(0, T)$ (cf. Remark 3.4).

Construction of the solution operator $\mathcal{T}$. Let us fix $0 < \hat{T} \le T$ (to be specified later), a constant $R > 0$, and an index p with $2 < p < 4$; let us consider the space

$$
\mathcal{Y} := \left\{ \chi \in L^2(0, \hat{T}; L^p(\Gamma_c)) \ : \ \|\chi\|_{L^2(0,\hat{T};L^p(\Gamma_c))} \le R \right\}. \tag{3.6}
$$

In order to define the operator $\mathcal{T}$, we need the following well-posedness results for the two auxiliary problems related to Problem 2.1.

Proposition 3.2. *For any $\hat{\chi} \in L^2(0, \hat{T}; L^p(\Gamma_c))$ and any $\mathbf{u}_0 \in W$, $\mathbf{u}_0|_{\Gamma_c} \in \mathrm{dom}(\varphi)$ there exists a unique pair $(\mathbf{u}, \boldsymbol{\eta}) \in H^1(0, \hat{T}; W) \times L^2(0, \hat{T}; (H^{-1/2}(\Gamma_c))^3)$ fulfilling*

$$
b(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} (\hat{\chi}\mathbf{u} + \boldsymbol{\eta}) \cdot \mathbf{v} = {}_{W'}\langle \mathbf{F}, \mathbf{v} \rangle_W \tag{3.7}
$$

$$
\forall\, \mathbf{v} \in W \ \text{ a.e. in } \ (0, \hat{T}), \quad \boldsymbol{\eta} \in \alpha(\mathbf{u}) \ \text{ a.e. in } \ (0, \hat{T}),
$$

$$
\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega. \tag{3.8}
$$

Moreover, there exist two positives constant Λ_1 and Λ_2, depending on the data of the problem, such that

$$
\|\mathbf{u}(t)\|_W^2 \le \Lambda_1 \left(\|\mathbf{u}_0\|_W^2 + \|\mathbf{F}\|_{L^2(0,T;W')}^2 \right) \exp(\Lambda_2(1 + \|\hat{\chi}\|_{L^2(0,\hat{T};L^p(\Gamma_c))})) \quad \forall\, t \in [0, \hat{T}]. \tag{3.9}
$$

Proposition 3.3. *For any* $\hat{\mathbf{u}} \in L^4(0,\hat{T};(L^4(\Gamma_c))^3)$ *and any* $\chi_0 \in H^1(\Gamma_c)$ *such that* $\psi(\chi_0) \in L^1(\Gamma_c)$ *there exists a unique pair* (χ,ξ)*, with* $\chi \in L^2(0,\hat{T};H^2(\Gamma_c)) \cap C^0(0,\hat{T};H^1(\Gamma_c)) \cap H^1(0,\hat{T};L^2(\Gamma_c))$*, and* $\xi \in L^2(0,\hat{T};L^2(\Gamma_c))$*, fulfilling*

$$
\chi_t - \Delta\chi + \xi = w_s(\chi) - \frac{1}{2}|\hat{\mathbf{u}}|^2 + A \quad \text{a.e. in } \Gamma_c \times (0,\hat{T}), \tag{3.10}
$$

$$
\xi \in \beta(\chi) \quad \text{a.e. in } \Gamma_c \times (0,\hat{T}),
$$

$$
\partial_{\mathbf{n}_s}\chi = 0 \quad \text{a.e. in } \partial\Gamma_c \times (0,\hat{T}), \tag{3.11}
$$

$$
\chi(\cdot,0) = \chi_0 \quad \text{a.e. in } \Gamma_c\,. \tag{3.12}
$$

Furthermore, there exists a positive constant Λ_3*, only depending on the data of the problem, such that for any* t *there holds*

$$
\begin{aligned}
&\|\chi(t)\|^2_{H^1(\Gamma_c)} \\
&\quad \le \Lambda_3\left(1 + \|\chi_0\|^2_{H^1(\Gamma_c)} + \|\psi(\chi_0)\|_{L^1(\Gamma_c)} + \|A\|^2_{L^2(0,T;L^2(\Gamma_c))} + \|\hat{\mathbf{u}}\|^4_{L^4(0,\hat{T};L^4(\Gamma_c))}\right)
\end{aligned} \tag{3.13}
$$

Remark 3.4. Thanks to Proposition 3.2, we may introduce the operator

$$
\mathcal{T}_1 : L^2(0,\hat{T};L^p(\Gamma_c)) \to H^1(0,\hat{T};W)
$$

which associates with a given $\hat{\chi}$ the unique solution $\mathbf{u}$ of (3.7)–(3.8). In the same way, it follows from Proposition 3.3 that the solution operator $\mathcal{T}_2$ associated with (3.10–3.12) is well defined on $L^4(0,\hat{T};(L^4(\Gamma_c))^3)$. Since $H^1(0,\hat{T};W) \subset L^4(0,\hat{T};(L^4(\Gamma_c))^3)$, the composition

$$
\mathcal{T} := \mathcal{T}_2 \circ \mathcal{T}_1 : L^2(0,\hat{T};L^p(\Gamma_c)) \to L^2(0,\hat{T};L^p(\Gamma_c))
$$

is well defined.

In the sequel, we shall check that, for a sufficiently small $\hat{T}$, $\mathcal{T}$ indeed maps $\mathcal{Y}$ into $\mathcal{Y}$, and that it is a contraction mapping on $\mathcal{Y}$, hence admitting a unique fixed point. Now, it is immediate to see that any fixed point χ of $\mathcal{T}$ yields a solution $(\mathbf{u} = \mathcal{T}_1(\chi), \chi, \boldsymbol{\eta}, \xi)$ of Problem 2.1 on the interval $(0,\hat{T})$. Thus, we shall deduce the existence of a local-in-time solution to Problem 2.1, which we shall eventually extend to the *whole* interval $(0,T)$.

Preliminarily to proving Propositions 3.2 and 3.3, let us consider for any $\varepsilon > 0$ the Yosida approximation (cf. [3, Chap. II.1.2]) $\alpha_\varepsilon : (H^{1/2}(\Gamma_c))^3 \to (H^{-1/2}(\Gamma_c))^3$ of the graph α. Standard results in the theory of maximal monotone operators ensure that there exists a Fréchet differentiable functional $\varphi_\varepsilon : (H^{1/2}(\Gamma_c))^3 \to \mathbb{R}$ such that $\alpha_\varepsilon = \mathrm{D}\varphi_\varepsilon$. Moreover, φ_ε Mosco-converges to φ as $\varepsilon \searrow 0$ (see, e.g., [1]), and

$$
0 \le \varphi_\varepsilon(\mathbf{u}) \le \varphi(\mathbf{u}) \quad \forall \mathbf{u} \in \mathrm{dom}(\varphi). \tag{3.14}
$$

Proof of Proposition 3.2. Since we are indeed going to obtain a *global* existence result for the initial boundary-value problem (3.7–3.8), to simplify notation we shall directly work on the time interval $(0, T)$ instead of $(0, \hat{T})$.

In [4, Prop. 4.2] it was proved that, given $\hat{\chi} \in L^2(0, T; L^p(\Gamma_c))$, for any $\varepsilon > 0$ there exists a unique $\mathbf{u}_\varepsilon \in H^1(0, T; W)$ fulfilling $\mathbf{u}_\varepsilon(\cdot, 0) = \mathbf{u}_0$ almost everywhere in Ω and

$$b(\mathbf{u}_{\varepsilon t}, \mathbf{v}) + a(\mathbf{u}_\varepsilon, \mathbf{v}) + \int_{\Gamma_c} (\hat{\chi}\mathbf{u}_\varepsilon + \alpha_\varepsilon(\mathbf{u}_\varepsilon)) \cdot \mathbf{v} = {}_{W'}\langle \mathbf{F}, \mathbf{v} \rangle_W \quad \forall\, \mathbf{v} \in W \text{ almost everywhere in } (0, T). \tag{3.15}$$

Let us choose $\mathbf{v} = \mathbf{u}_{\varepsilon t}$ in (3.15) and integrate in time the ensuing relation. Using (2.2), (2.3), (2.4) and applying the chain rule for the functional φ_ε we get

$$\begin{aligned} & C_b \int_0^t \|\mathbf{u}_{\varepsilon t}\|_W^2 + \frac{C_a}{2} \|\mathbf{u}_\varepsilon(t)\|_W^2 + \varphi_\varepsilon(\mathbf{u}_\varepsilon(t)) \\ & \le \varphi_\varepsilon(\mathbf{u}_0) + \frac{1}{2} a(\mathbf{u}_0, \mathbf{u}_0) + \int_0^t \|\hat{\chi}\|_{L^2(\Gamma_c)} \|\mathbf{u}_\varepsilon\|_{L^4(\Gamma_c)} \|\mathbf{u}_{\varepsilon t}\|_{L^4(\Gamma_c)} + \int_0^t \|\mathbf{F}\|_{W'} \|\mathbf{u}_{\varepsilon t}\|_W \\ & \le \varphi(\mathbf{u}_0) + \frac{M}{2} \|\mathbf{u}_0\|_W^2 + \frac{1}{C_b} \int_0^t \|\mathbf{F}\|_{W'}^2 + C \int_0^t \|\hat{\chi}\|_{L^2(\Gamma_c)}^2 \|\mathbf{u}_\varepsilon\|_W^2 + \frac{C_b}{2} \int_0^t \|\mathbf{u}_{\varepsilon t}\|_W^2 . \end{aligned}$$

Note that the latter inequality is also due to (3.14), to the Hölder and Young inequalities, and to (2.48). Hence, a straightforward application of the Gronwall Lemma gives that there exists a constant C such that

$$\|\mathbf{u}_\varepsilon\|_{H^1(0,T;W)} \le C \quad \forall\, \varepsilon > 0. \tag{3.16}$$

Combining (3.16) with the estimate (see (2.49))

$$\|\hat{\chi}\mathbf{u}_\varepsilon\|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \le \|\hat{\chi}\|_{L^2(0,T;L^2(\Gamma_c))} \|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^4(\Gamma_c))}, \tag{3.17}$$

we conclude that $\{\hat{\chi}\mathbf{u}_\varepsilon\}$ is bounded in $L^2(0, T; (H^{-1/2}(\Gamma_c))^3)$. Therefore, a comparison in (3.15) yields that $\{\alpha_\varepsilon(\mathbf{u}_\varepsilon)\}$ is also bounded in $L^2(0, T; (H^{-1/2}(\Gamma_c))^3)$.

Standard weak compactness results ensure that there exist a vanishing sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ and a pair $(\mathbf{u}, \boldsymbol{\eta}) \in H^1(0, T; W) \times L^2(0, T; (H^{-1/2}(\Gamma_c))^3)$ such that

$$\begin{aligned} \mathbf{u}_{\varepsilon_k} &\rightharpoonup \mathbf{u} && \text{in } H^1(0, T; W), \\ \alpha_{\varepsilon_k}(\mathbf{u}_{\varepsilon_k}) &\rightharpoonup \boldsymbol{\eta} && \text{in } L^2(0, T; (H^{-1/2}(\Gamma_c))^3). \end{aligned} \tag{3.18}$$

Moreover, recalling that the embedding $W \subset (L^q(\Gamma_c))^3$ is compact for $1 \le q < 4$ and due to (a version of) the Lions-Aubin theorem (cf. [27, Thm. 4, Cor. 5]), as well as to the Ascoli-Arzelà theorem in the framework of the weak topology of W, we deduce that

$$(3.19)\qquad \begin{aligned} \mathbf{u}_{\varepsilon_k} &\to \mathbf{u} && \text{in } L^2(0,T;(L^q(\Gamma_c))^3) \text{ for } 1 \le q < 4, \\ \mathbf{u}_{\varepsilon_k}(t) &\rightharpoonup \mathbf{u}(t) && \text{in } W \text{ for all } t \in [0,T]. \end{aligned}$$

Using the first convergence of (3.19) with the index q_p fulfilling $1 = 1/p + 1/q_p + \frac{1}{4}$ and recalling the second of (2.49), we have that

$$(3.20)\qquad \begin{aligned} &\|\widehat{\chi}(\mathbf{u}_{\varepsilon_k} - \mathbf{u})\|_{L^1(0,T;H^{-1/2}(\Gamma_c))} \\ &\qquad \le \|\widehat{\chi}\|_{L^2(0,T;L^p(\Gamma_c))} \|\mathbf{u}_{\varepsilon_k} - \mathbf{u}\|_{L^2(0,T;L^{q_p}(\Gamma_c))} \to 0, \quad \text{as } k \to \infty\,. \end{aligned}$$

Thanks to (3.18)–(3.20), the pair $(\mathbf{u}, \boldsymbol{\eta})$ fulfils

$$(3.21)\qquad b(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} (\widehat{\chi}\mathbf{u} + \boldsymbol{\eta}) \cdot \mathbf{v} = {}_{W'}\langle \mathbf{F}, \mathbf{v} \rangle_W \qquad \forall\, \mathbf{v} \in W \quad \text{a.e. in } (0,T).$$

Therefore, it remains to prove that $\boldsymbol{\eta} \in \alpha(\mathbf{u})$. As α induces a maximal monotone graph in $L^2(0,T;(H^{1/2}(\Gamma_c))^3) \times L^2(0,T;(H^{-1/2}(\Gamma_c))^3)$, this will follow (see [3, Lemma II.1.3]) from the inequality

$$(3.22)\qquad \limsup_{k \to \infty} \int_0^T \int_{\Gamma_c} \alpha_{\varepsilon_k}(\mathbf{u}_{\varepsilon_k}) \cdot \mathbf{u}_{\varepsilon_k} \le \int_0^T \int_{\Gamma_c} \boldsymbol{\eta} \cdot \mathbf{u}.$$

Indeed, for all $t \in (0,T]$ we have

$$\begin{aligned} &\limsup_{k \to \infty} \int_0^t \int_{\Gamma_c} \alpha_{\varepsilon_k}(\mathbf{u}_{\varepsilon_k}) \cdot \mathbf{u}_{\varepsilon_k} \quad \le \\ &\le \limsup_{k \to \infty} \left(-\int_0^t a(\mathbf{u}_{\varepsilon_k}, \mathbf{u}_{\varepsilon_k}) - \frac{1}{2} b(\mathbf{u}_{\varepsilon_k}(t), \mathbf{u}_{\varepsilon_k}(t)) \right) \\ &\qquad + \frac{1}{2} b(\mathbf{u}_0, \mathbf{u}_0) + \lim_{k \to \infty} \left(-\int_0^t \int_{\Gamma_c} \widehat{\chi}\mathbf{u}_{\varepsilon_k} \cdot \mathbf{u}_{\varepsilon_k} + \int_0^t {}_{W'}\langle \mathbf{F}, \mathbf{u}_{\varepsilon_k} \rangle_W \right) \\ &\le -\int_0^t a(\mathbf{u}, \mathbf{u}) - \frac{1}{2} b(\mathbf{u}(t), \mathbf{u}(t)) + \frac{1}{2} b(\mathbf{u}_0, \mathbf{u}_0) - \int_0^t \int_{\Gamma_c} \widehat{\chi}\mathbf{u} \cdot \mathbf{u} + \int_0^t {}_{W'}\langle \mathbf{F}, \mathbf{u} \rangle_W \\ &= \int_0^t \int_{\Gamma_c} \boldsymbol{\eta} \cdot \mathbf{u}, \end{aligned}$$

where the first inequality in the chain above follows from choosing $\mathbf{v} = \mathbf{u}_{\varepsilon_k}$ in (3.15), integrating in time, and applying the chain rule for the form b. The second inequality is due to the convergences (3.18)–(3.20), and the last equality is due to (3.21). Thus, (3.22) is proved, and we conclude that the pair $(\mathbf{u}, \boldsymbol{\eta})$ is a solution of (3.7–3.8).

As far as the uniqueness issue is concerned, we point out that a continuous dependence result on the data $(\mathbf{u}_0, \mathbf{f}, \mathbf{g})$, yielding uniqueness of the solutions, may be proved for (3.7–3.8) by arguing exactly in the same way as in the proof of [4, Prop. 4.2], to which we refer the reader.

Finally, estimate (3.9) is obtained by testing (3.7) by the element $\mathbf{v} = \mathbf{u}$. Upon integrating in time and using (2.2), (2.3), (2.4), and (2.5), we end up with

$$\frac{C_b}{2}\|\mathbf{u}(t)\|_W^2 + C_a\int_0^t \|\mathbf{u}\|_W^2 \le b(\mathbf{u}_0,\mathbf{u}_0) - \int_0^t\int_{\Gamma_c} \widehat{\chi}|\mathbf{u}|^2 + \int_0^t {}_{W'}\langle \mathbf{F},\mathbf{u}\rangle_W$$
$$\le M\|\mathbf{u}_0\|_W^2 + C\int_0^t \|\widehat{\chi}\|_{L^p(\Gamma_c)}\|\mathbf{u}\|_W^2$$
$$+ \frac{1}{2C_a}\int_0^t \|\mathbf{F}\|_{W'}^2 + \frac{C_a}{2}\int_0^t \|\mathbf{u}\|_W^2,$$

where again the constant C encompasses some embedding constants. By the Gronwall Lemma, (3.9) follows. ◻

Sketch of the proof of Proposition 3.3. Since $\hat{\mathbf{u}} \in L^4(0,T;(L^4(\Gamma_c))^3)$, the term $-\frac{1}{2}|\hat{\mathbf{u}}|^2 + A$ on the right-hand side of (3.10) belongs to $L^2(0,T;L^2(\Gamma_c))$ (cf. (2.12)). Therefore, the well-posedness of the system (3.10–3.12) follows from, e.g., the abstract result of [12, Lemma 3.3], relying on the theory of maximal monotone operators in Hilbert spaces, see [7]. As far as the estimate (3.13), it is straightforward obtained by multiplying (3.10) by χ_t and integrating in time over $(0,t)$. We do not enter the details and refer to the *First a priori estimate* below (cf.(3.27)). Thus, after exploiting the Young inequality and (2.11), we get for a suitable c

$$\|\chi_t\|_{L^2(0,t;L^2(\Gamma_c))}^2 + \frac{1}{2}\|\nabla\chi(t)\|_{L^2(\Gamma_c)}^2 + \int_\Omega \psi(\chi(t)) \tag{3.23}$$
$$\le \frac{1}{2}\|\nabla\chi_0\|_{L^2(\Gamma_c)}^2 + \int_\Omega \psi(\chi_0) + \frac{1}{2}\|\chi_t\|_{L^2(0,t;L^2(\Gamma_c))}^2$$
$$+ c\left(1 + \|\hat{\mathbf{u}}\|_{L^4(0,\hat{T};L^4(\Gamma_c)^3)}^4 + \|A\|_{L^2(0,\hat{T};L^2(\Gamma_c))}^2 + \int_0^t \|\chi\|_{L^2(\Gamma_c)}^2\right),$$

from which (cf. (3.28)), applying the Gronwall Lemma, we eventually get (3.13). ◻

Proof of Theorem 1.

Existence of a local solution. First of all, we show that there exists $\widehat{T} \in (0, T]$ for which the operator $\mathcal{T}$ maps $\mathcal{Y}$ into itself. Indeed, combining (3.9) and (3.13) we get that for any $\widehat{\chi} \in L^2(0, \widehat{T}; L^p(\Gamma_c))$

$$\|\mathcal{T}(\widehat{\chi})\|^2_{L^\infty(0,\widehat{T};L^p(\Gamma_c))} \leq \Lambda_4\Big(1 + \|\chi_0\|^2_{H^1(\Gamma_c)} + \|\psi(\chi_0)\|_{L^1(\Gamma_c)} + \|A\|^2_{L^2(0,T;L^2(\Gamma_c))} + \|\mathbf{u}_0\|^4_W + \|\mathbf{F}\|^4_{L^2(0,T;W')}\Big),$$

where the positive constant Λ_4 depends on the data of the problem, on R, and also encompasses the embedding constants of the Sobolev inclusions $W \subset L^4(\Gamma_c)^3$ and $H^1(\Gamma_c) \subset L^p(\Gamma_c)$. Hence, choosing $\widehat{T}$ sufficiently small one obtains

$$\|\mathcal{T}(\widehat{\chi})\|_{L^2(0,\widehat{T};L^p(\Gamma_c))} \leq R.$$

In order to prove that $\mathcal{T}$ is a contraction, let us fix $\widehat{\chi}_1, \widehat{\chi}_2 \in \mathcal{Y}$ and, accordingly, let $\mathbf{u}_i := \mathcal{T}_1(\widehat{\chi}_i)$, $i = 1, 2$, and let $\chi_i := \mathcal{T}_2(\mathbf{u}_i) = \mathcal{T}(\widehat{\chi}_i)$, $i = 1, 2$. Then, we may repeat the same computations leading to (3.3) (compare with the proof of Proposition 2.2), with the only difference that in this case $\widetilde{\mathbf{F}} = \mathbf{0}$, $\widetilde{\mathbf{u}_0} = \mathbf{0}$ and we cannot exploit the positivity of $\widehat{\chi}_i$. Thus, recalling the regularity of $\widehat{\chi}_i$, we deduce

$$\begin{aligned}
&\frac{C_b}{2}\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_W + C_a\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(0,t;W)} \\
&\quad \leq \int_0^t \|\widehat{\chi}_1 - \widehat{\chi}_2\|_{L^2(\Gamma_c)}\|\mathbf{u}_1\|_{L^4(\Gamma_c)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^4(\Gamma_c)} + \int_0^t \|\widehat{\chi}_2\|_{L^2(\Gamma_c)}\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^4(\Gamma_c)} \\
&\quad \leq \int_0^t \|\widehat{\chi}_1 - \widehat{\chi}_2\|^2_{L^2(\Gamma_c)} + C_3\int_0^t (\|\widehat{\mathbf{u}}_1\|^2_{L^4(\Gamma_c)} + \|\widehat{\chi}_2\|_{L^2(\Gamma_c)})\|\mathbf{u}_1 - \mathbf{u}_2\|^2_W,
\end{aligned}$$

where the constant C_3 depends on the Sobolev embedding (2.48). Thus, using the Gronwall lemma we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_W \leq \mathcal{K}_1\int_0^t \|\widehat{\chi}_1 - \widehat{\chi}_2\|^2_{L^2(\Gamma_c)}, \tag{3.24}$$

where $\mathcal{K}_1$ depends on C_b, $\widehat{T}$, C_3, on the data $\mathbf{u}_0$ and $\mathbf{F}$ through the a priori estimate (3.9) for $\|\mathbf{u}_1\|^2_{L^1(0,\widehat{T};W)}$, and on R through the norm $\|\widehat{\chi}_2\|_{L^1(0,\widehat{T};L^2(\Gamma_c))}$.

On the other hand, arguing as in the derivation of (3.4), we have that

$$(3.25)\quad \begin{aligned} &\frac{1}{2}\|\chi_1(t)-\chi_2(t)\|^2_{L^2(\Gamma_c)} + \int_0^t \|\nabla(\chi_1-\chi_2)\|^2_{L^2(\Gamma_c)} \\ &\leq \frac{1}{2}\int_0^t \|\mathbf{u}_1-\mathbf{u}_2\|_{L^4(\Gamma_c)}\|\mathbf{u}_1+\mathbf{u}_2\|_{L^4(\Gamma_c)}\|\chi_1-\chi_2\|_{L^2(\Gamma_c)} + L_s\int_0^t \|\chi_1-\chi_2\|^2_{L^2(\Gamma_c)} \\ &\leq \frac{C_4}{2}\int_0^t \|\mathbf{u}_1-\mathbf{u}_2\|_W\|\chi_1-\chi_2\|_{L^2(\Gamma_c)} + L_s\int_0^t \|\chi_1-\chi_2\|^2_{L^2(\Gamma_c)} \\ &\leq \frac{1}{2}\int_0^t \|\mathbf{u}_1-\mathbf{u}_2\|^2_W + \left(\frac{C_4^2}{8}+L_s\right)\int_0^t \|\chi_1-\chi_2\|^2_{L^2(\Gamma_c)}, \end{aligned}$$

where the constant C_4 has the same dependence of C_3 on the data of the problem. Again by the Gronwall Lemma, in view of (3.24) we find (here $\mathcal{K}_2$ depends in particular on C_4, $\hat{T}$, and L_s)

$$\begin{aligned} \|\chi_1(t)-\chi_2(t)\|^2_{L^2(\Gamma_c)} &\leq \mathcal{K}_2\int_0^t \|\mathbf{u}_1-\mathbf{u}_2\|^2_W \\ &\leq \mathcal{K}_1\mathcal{K}_2 t\|\hat{\chi}_1-\hat{\chi}_2\|^2_{L^2(0,\hat{T};L^p(\Gamma_c))}, \end{aligned}$$

(for simplicity, hereafter we shall neglect the constant of the embedding $L^p(\Gamma_c)\subset L^2(\Gamma_c)$), so that

$$\|\chi_1-\chi_2\|_{L^2(0,t;L^2(\Gamma_c))} \leq (\mathcal{K}_1\mathcal{K}_2)^{1/2}\, t\, \|\hat{\chi}_1-\hat{\chi}_2\|_{L^2(0,\hat{T};L^p(\Gamma_c))} \quad \text{for } 0<t\leq\hat{T}.$$

Furthermore, performing a comparison in (3.25), we deduce

$$(3.26)\quad \begin{aligned} \|\chi_1-\chi_2\|_{L^2(0,\hat{T};L^p(\Gamma_c))} &\leq c\|\chi_1-\chi_2\|_{L^2(0,\hat{T};H^1(\Gamma_c))} \\ &\leq \mathcal{K}_3(\hat{T})^{1/2}\|\hat{\chi}_1-\hat{\chi}_2\|_{L^2(0,\hat{T};L^p(\Gamma_c))}, \end{aligned}$$

for a suitable constant $\mathcal{K}_3$, depending on $\mathcal{K}_1$, $\mathcal{K}_2$, and on the constant of the embedding $H^1(\Gamma_c)\subset L^p(\Gamma_c)$. Choosing a possibly smaller $\hat{T}$, we finally conclude that $\mathcal{T}$ is a contraction. Therefore, $\mathcal{T}$ admits a unique fixed point χ, providing a solution $(\mathbf{u},\chi,\boldsymbol{\eta},\xi)$ to Problem 2.1 on the time interval $[0,\hat{T}]$.

Extension to of the local solution. Now, it remains to extend the latter local solution to the whole interval $[0,T]$. To this aim, we derive the following *global in time* a priori estimates on the quadruple $(\mathbf{u},\chi,\boldsymbol{\eta},\xi)$.

First estimate. We choose $\mathbf{v} := \mathbf{u}_t$ in (2.20), test (2.22) by χ_t, add the resulting relations and integrate on the interval $(0,t)$, $t\in(0,\hat{T}]$. Owing to (2.2)-(2.4), (2.5), and to (2.7) (using the chain rule [10, Lemma 4.1] for the functionals

φ and ψ), we readily conclude

$$
\begin{aligned}
(3.27)\quad & C_b \int_0^t \|\mathbf{u}_t\|_W^2 + \frac{C_a}{2}\|\mathbf{u}(t)\|_W^2 + \varphi(\mathbf{u}(t)) + \int_0^t \int_{\Gamma_c} \chi \mathbf{u} \cdot \mathbf{u}_t \\
& \quad + \int_0^t \|\chi_t\|_{L^2(\Gamma_c)}^2 + \frac{1}{2}\|\nabla \chi(t)\|_{L^2(\Gamma_c)}^2 + \int_{\Gamma_c} \psi(\chi(t)) \\
& \le \frac{M}{2}\|\mathbf{u}_0\|_W^2 + \varphi(\mathbf{u}_0) + \frac{1}{2C_b}\int_0^t \|\mathbf{F}\|_{W'}^2 + \frac{C_b}{2}\int_0^t \|\mathbf{u}_t\|_W^2 \\
& \quad + \frac{1}{2}\|\chi_0\|_{H^1(\Gamma_c)}^2 + \int_{\Gamma_c} \psi(\chi_0) - \frac{1}{2}\int_0^t \int_{\Gamma_c} |\mathbf{u}|^2 \chi_t \\
& \quad + \frac{1}{2}\int_0^t \|\chi_t\|_{L^2(\Gamma_c)}^2 + \|A\|_{L^2(0,t;L^2(\Gamma_c))}^2 + c\left(1 + \int_0^t \|\chi\|_{L^2(\Gamma_c)}^2\right),
\end{aligned}
$$

where we have also suitably used the Young inequality (2.50) as well as (2.11) (the constant c on the right-hand side of (3.27) depends only on L_s, T, and $|\Gamma_c|$). In order to estimate the last integral, we recall that there exists a positive constant C_5 depending only on T such that for any χ in $H^1(0,T;L^2(\Gamma_c))$ there holds

$$
(3.28)\quad \|\chi(t)\|_{L^2(\Gamma_c)}^2 \le C_5\left(\|\chi(0)\|_{L^2(\Gamma_c)}^2 + \int_0^t \|\chi_t(s)\|_{L^2(\Gamma_c)}^2 \, \mathrm{d}s\right) \quad \forall\, t \in (0,T].
$$

Moreover, an integration by parts leads to

$$
\begin{aligned}
(3.29)\quad -\frac{1}{2}\int_0^t \int_{\Gamma_c} |\mathbf{u}|^2 \chi_t &= \int_0^t \int_{\Gamma_c} \chi \mathbf{u}_t \cdot \mathbf{u} + \frac{1}{2}\int_{\Gamma_c} |\mathbf{u}_0|^2 \chi_0 - \frac{1}{2}\int_{\Gamma_c} |\mathbf{u}(t)|^2 \chi(t) \\
&\le \int_0^t \int_{\Gamma_c} \chi \mathbf{u}_t \cdot \mathbf{u} + c\|\chi_0\|_{L^2(\Gamma_c)}\|\mathbf{u}_0\|_W^2,
\end{aligned}
$$

the latter inequality following from the fact that, since $\chi \in \mathrm{dom}(\beta) \subseteq [0,+\infty)$, $\chi \ge 0$ a.e. on $\Gamma_c \times (0,T)$, and from (2.48). Combining (3.27), (3.28) and (3.29), and recalling that both φ and ψ take positive values, we obtain

$$
\begin{aligned}
(3.30)\quad & \frac{C_b}{2}\int_0^t \|\mathbf{u}_t\|_W^2 + \frac{C_a}{2}\|\mathbf{u}(t)\|_W^2 + \frac{1}{2}\int_0^t \|\chi_t\|_{L^2(\Gamma_c)}^2 + \frac{1}{2}\|\nabla\chi(t)\|_{L^2(\Gamma_c)}^2 \\
& \quad \le C_6\left(1 + \|\mathbf{u}_0\|_W^4 + \varphi(\mathbf{u}_0) + \|\chi_0\|_{H^1(\Gamma_c)}^2 + \|\psi(\chi_0)\|_{L^1(\Gamma_c)}\right) \\
& \qquad + C_6\left(\|\mathbf{F}\|_{L^2(0,T,W')}^2 + \|A\|_{L^2(0,T;L^2(\Gamma_c))}^2 + \int_0^t \|\chi_t\|_{L^2(0,s;L^2(\Gamma_c))}^2 \, \mathrm{d}s\right) \\
& \qquad \forall\, t \in (0,\hat{T}],
\end{aligned}
$$

for a positive constant C_6 depending on M, C_a, C_b, L_s, T, $|\Omega|$ and $|\Gamma_c|$. Again by the Gronwall Lemma, there exists a positive constant C_7 such that

$$\|\mathbf{u}\|_{H^1(0,t;W)} + \|\chi\|_{L^\infty(0,t;H^1(\Gamma_c))\cap H^1(0,t;L^2(\Gamma_c))} \le C_7 \quad \forall\, t \in (0,\hat{T}]. \tag{3.31}$$

Second estimate. By comparison in (2.22), we conclude that

$$\|-\Delta\chi + \xi\|_{L^2(0,t;L^2(\Gamma_c))} \le c \quad \forall\, t \in (0,\hat{T}]$$

for some positive constant c. Using the monotonicity of β, by which

$$\|-\Delta\chi(t) + \xi(t)\|^2_{L^2(\Gamma_c)} \ge \|-\Delta\chi(t)\|^2_{L^2(\Gamma_c)} + \|\xi(t)\|^2_{L^2(\Gamma_c)} \quad \text{for a.e. } t \in (0,\hat{T}), \tag{3.32}$$

as well as standard elliptic regularity results, we infer that there exists $C_8 > 0$ such that

$$\|\xi\|_{L^2(0,t;L^2(\Gamma_c))} + \|\chi\|_{L^2(0,t;H^2(\Gamma_c))} \le C_8 \quad \forall\, t \in (0,\hat{T}]. \tag{3.33}$$

Third estimate. Owing to (3.31) we also have that (compare with (3.17))

$$\|\chi\mathbf{u}\|_{L^\infty(0,t;H^{-1/2}(\Gamma_c))} \le c,$$

so that by comparison in (2.20) we infer

$$\|\boldsymbol{\eta}\|_{L^2(0,t;H^{-1/2}(\Gamma_c))} \le C_9 \quad \forall\, t \in (0,\hat{T}]. \tag{3.34}$$

Collecting (3.31)–(3.34), we conclude that

$$\begin{aligned}\|\mathbf{u}\|_{H^1(0,t;W)} &+ \|\chi\|_{L^2(0,t;H^2(\Gamma_c))\cap L^\infty(0,t;H^1(\Gamma_c))\cap H^1(0,t;L^2(\Gamma_c))} \\ &+ \|\boldsymbol{\eta}\|_{L^2(0,t;H^{-1/2}(\Gamma_c))} + \|\xi\|_{L^2(0,t;L^2(\Gamma_c))} \le C \quad \forall\, t \in (0,\hat{T}],\end{aligned} \tag{3.35}$$

where $C := C_7 + C_8 + C_9$ does not depend on $\hat{T}$. A standard prolongation argument then enables us to extend the solution on the interval $[0,T]$.

Further regularity of the solution. Under the additional assumptions (2.25)–(2.26), the further regularity (2.27) may be proved by means of the following estimate. Proceeding formally, we differentiate (2.22) with respect to time, we multiply the resulting equation by χ_t and we integrate over $\Gamma_c \times (0,t)$. Such a procedure could be indeed made rigorous by replacing β by its Yosida approximation β_ε (which is Lipschitz continuous) and passing to the limit in the regularization parameter ε. Hence, for simplicity in the next lines we will formally write $\beta'(\chi)$ instead of $\beta'_\varepsilon(\chi_\varepsilon)$. We get

$$
\begin{aligned}
(3.36)\quad & \frac{1}{2}\|\chi_t(t)\|^2_{L^2(\Gamma_c)} + \|\nabla\chi_t\|^2_{L^2(0,t;L^2(\Gamma_c))} + \int_0^t\!\!\int_{\Gamma_c} \beta'(\chi)(\chi_t)^2 \\
&\le \frac{1}{2}\|\chi_t(0)\|^2_{L^2(\Gamma_c)} + L_s\int_0^t \|\chi_t\|^2_{L^2(\Gamma_c)} \\
&\qquad + c\int_0^t \|\mathbf{u}\|_W\,\|\mathbf{u}_t\|_W\,\|\chi_t\|_{L^2(\Gamma_c)} + \int_0^t \|A_t\|_{L^2(\Gamma_c)}\,\|\chi_t\|_{L^2(\Gamma_c)} \\
&\le \frac{1}{2}\|\chi_t(0)\|^2_{L^2(\Gamma_c)} + L_s\int_0^t \|\chi_t\|^2_{L^2(\Gamma_c)} \\
&\qquad + \int_0^t \left(c\|\mathbf{u}\|_{L^\infty(0,t;W)}\|\mathbf{u}_t\|_W + \|A_t\|_{L^2(\Gamma_c)}\right)\|\chi_t\|_{L^2(\Gamma_c)}\,.
\end{aligned}
$$

We note that the third term on the left-hand side of (3.36) is non-negative due to the monotonicity of β. On the other hand, to deal with the right-hand side of (3.36) we may recall the previous estimate (3.35), recover the initial value of χ_t from (2.22) on account of (2.25)–(2.26), and apply to (3.36) a generalized version of Gronwall Lemma (see, e.g., [2]). We end up with deducing the following upper bound

$$\|\chi\|_{W^{1,\infty}(0,T;L^2(\Gamma_c))\cap H^1(0,T;H^1(\Gamma_c))} \le c\,.$$

Finally, a comparison in (2.22) easily yields that

$$\|-\Delta\chi + \xi\|_{L^\infty(0,T;L^2(\Gamma_c))} \le c$$

and hence (cf. (3.32)–(3.33))

$$\|\chi\|_{L^\infty(0,T;H^2(\Gamma_c))} + \|\xi\|_{L^\infty(0,T;L^2(\Gamma_c))} \le c\,.$$

Thus, the regularity specified in (2.27) is proved. □

Remark 3.5. Let us point out that the sign constraint on the solution component χ, due to the hypothesis $\mathrm{dom}(\beta) \subseteq [0,+\infty)$, has been substantially exploited in the above derivation of global estimates on the pair $(\mathbf{u},\chi)$ (cf. the inequality in (3.29)). Nonetheless, by a slight modification in the proof of (3.31) we might replace the assumption $\mathrm{dom}(\beta) \subseteq [0,+\infty)$ with the requirement that $\mathrm{dom}(\beta)$ be a bounded interval.

4. Analysis of the long-time behaviour

Preliminarily, we need the following result (cf. with Lemma 3.1).

Lemma 4.1. *Assume (2.5), (2.6), (2.11), and (2.32)–(2.34), with T arbitrarily fixed. Then, there exists a positive constant $\mathcal{J}$, depending on M, C_a, C_b, w_s, $|\Omega|$ and $|\Gamma_c|$, but not on T, such that, for any pair of initial data $(\mathbf{u}_0, \chi_0)$ complying with (2.14)–(2.15), the associated solution $(\mathbf{u}, \chi)$ of Problem 2.1 fulfills for all $t \in (0, +\infty)$*

$$(4.1)\quad \|\mathbf{u}(t)\|_W^2 + \int_0^t \|\mathbf{u}_t\|_W^2 + \int_0^t \|\chi_t\|_{L^2(\Gamma_c)}^2 + \|\chi(t)\|_{H^1(\Gamma_c)}^2 + \|\psi(\chi(t))\|_{L^1(\Gamma_c)}$$
$$\leq \mathcal{J}\Big(\|\mathbf{u}_0\|_W^4 + \varphi(\mathbf{u}_0) + \|\chi_0\|_{H^1(\Gamma_c)}^2 + \|\mathbf{F}\|_{L^\infty(0,+\infty;W')}^2 + \|\mathbf{F}_t\|_{L^1(0,+\infty;W')}^2$$
$$+ \int_{\Gamma_c} \psi(\chi_0) + \|A\|_{L^\infty(0,+\infty;L^2(\Gamma_c))}^2 + \|A_t\|_{L^1(0,+\infty;L^2(\Gamma_c))}^2 + 1\Big).$$

Proof. In order to prove (4.1), we repeat the same estimate as in the proof of Theorem 1. Namely, we take $\mathbf{v} := \mathbf{u}_t$ in (2.20), multiply (2.22) by χ_t, add the resulting relations and integrate in time. Hence, arguing as in (3.27)–(3.29), we obtain

$$C_b \int_0^t \|\mathbf{u}_t\|_W^2 + \frac{C_a}{2}\|\mathbf{u}(t)\|_W^2 + \varphi(\mathbf{u}(t)) + \int_0^t \|\chi_t\|_{L^2(\Gamma_c)}^2 + \frac{1}{2}\|\nabla\chi(t)\|_{L^2(\Gamma_c)}^2 + \int_{\Gamma_c} \psi(\chi(t))$$
$$\leq \frac{M}{2}\|\mathbf{u}_0\|_W^2 + \varphi(\mathbf{u}_0) + \frac{1}{2}\|\chi_0\|_{H^1(\Gamma_c)}^2 + \int_{\Gamma_c} \psi(\chi_0) + c\|\chi_0\|_{L^2(\Gamma_c)}\|\mathbf{u}_0\|_W^2$$
$$+ \int_0^t {}_{W'}\langle \mathbf{F}, \mathbf{u}_t\rangle_W + \int_0^t \int_{\Gamma_c} w_s(\chi)\chi_t + \int_0^t \int_{\Gamma_c} A\chi_t,$$

Now, note that

$$(4.2)\quad \int_0^t {}_{W'}\langle \mathbf{F}, \mathbf{u}_t\rangle_W = {}_{W'}\langle \mathbf{F}(t), \mathbf{u}(t)\rangle_W - {}_{W'}\langle \mathbf{F}(0), \mathbf{u}_0\rangle_W - \int_0^t {}_{W'}\langle \mathbf{F}_t, \mathbf{u}\rangle_W$$
$$\leq \left(\frac{1}{2} + \frac{1}{C_a}\right)\|\mathbf{F}\|_{L^\infty(0,+\infty;W')}^2 + \frac{1}{2}\|\mathbf{u}_0\|_W^2 + \frac{C_a}{4}\|\mathbf{u}(t)\|_W^2 + \int_0^t \|F_t\|_{W'}\|\mathbf{u}\|_W$$

Moreover, thanks to (2.13) we have

$$(4.3)\quad \int_0^t \int_{\Gamma_c} w_s(\chi)\chi_t = \int_{\Gamma_c} \widehat{w_s}(\chi(t)) - \int_{\Gamma_c} \widehat{w_s}(\chi_0)$$
$$\leq 2C_L|\Gamma_c| + K_L\|\chi_0\|_{L^2(\Gamma_c)}^2 + K_L\|\chi(t)\|_{L^2(\Gamma_c)}^2.$$

Further, again applying the integration by parts formula and inequality (3.28) one deduces

$$
\begin{aligned}
(4.4)\quad \int_0^t \int_{\Gamma_c} A\chi_t \le{}& \frac{1}{4K_L}\|A(t)\|^2_{L^2(\Gamma_c)} + K_L\|\chi(t)\|^2_{L^2(\Gamma_c)} \\
&+ \frac{1}{2}\|A(0)\|^2_{L^2(\Gamma_c)} + \frac{1}{2}\|\chi_0\|^2_{L^2(\Gamma_c)} + \int_0^t \|A_t\|_{L^2(\Gamma_c)}\|\chi\|_{L^2(\Gamma_c)} \\
\le{}& \left(\frac{1}{2} + \frac{1}{4K_L}\right)\|A\|^2_{L^\infty(0,+\infty;L^2(\Gamma_c))} + \frac{1}{2}\|\chi_0\|^2_{L^2(\Gamma_c)} \\
&+ K_L\|\chi(t)\|^2_{L^2(\Gamma_c)} + \int_0^t \|A_t\|_{L^2(\Gamma_c)}\|\chi\|_{L^2(\Gamma_c)}.
\end{aligned}
$$

On the other hand, due to (2.35) there exists a positive constant C, only depending on K_L, such that

$$
(4.5)\qquad \int_{\Gamma_c} \psi(\chi(t)) \ge \left(2K_L + \frac{1}{2}\right)\int_{\Gamma_c} \chi(t)^2 - C.
$$

Collecting (4.2)–(4.5), we finally deduce

$$
\begin{aligned}
(4.6)\quad & C_b\int_0^t \|\mathbf{u}_t\|^2_W + \frac{C_a}{4}\|\mathbf{u}(t)\|^2_W + \int_0^t \|\chi_t\|^2_{L^2(\Gamma_c)} \\
&\qquad + \frac{1}{2}\left(\|\chi(t)\|^2_{L^2(\Gamma_c)} + \|\nabla\chi(t)\|^2_{L^2(\Gamma_c)}\right) \\
&\le C_{10}\Big(1 + \|A\|^2_{L^\infty(0,+\infty;L^2(\Gamma_c))} + \|\chi_0\|^2_{H^1(\Gamma_c)} + \|\mathbf{u}_0\|^4_W + \varphi(\mathbf{u}_0) \\
&\qquad + \|\psi(\chi_0)\|_{L^1(\Gamma_c)} + \|\mathbf{F}\|^2_{L^\infty(0,+\infty;W')}\Big) \\
&\qquad + \int_0^t \|F_t\|_{W'}\|\mathbf{u}\|_W + c\int_0^t \|A_t\|_{L^2(\Gamma_c)}\|\chi\|_{H^1(\Gamma_c)},
\end{aligned}
$$

where the constant C_{10} is independent of T and c depends on the continuous embedding of $H^1(\Gamma_c)$ in $L^2(\Gamma_c)$. Then, by the Gronwall Lemma there exists $C_{11} > 0$ such that

$$
\begin{aligned}
(4.7)\quad \|\mathbf{u}(t)\|^2_W + \|\chi(t)\|^2_{H^1(\Gamma_c)} \le{}& C_{11}(1 + \|A\|^2_{L^\infty(0,+\infty;L^2(\Gamma_c))} + \|\chi_0\|^2_{H^1(\Gamma_c)} \\
&+ \|\mathbf{u}_0\|^4_W + \varphi(\mathbf{u}_0) + \|\psi(\chi_0)\|_{L^1(\Gamma_c)} + \|\mathbf{F}\|^2_{L^\infty(0,+\infty;W')} \\
&+ \|F_t\|^2_{L^1(0,+\infty;W')} + \|A_t\|^2_{L^1(0,+\infty;L^2(\Gamma_c))}) \qquad \forall t > 0.
\end{aligned}
$$

Inserting this estimate in (4.6), after using the Hölder inequality to deal with

$$\int_0^t \|F_t\|_{W'}\|\mathbf{u}\|_W \le \|F_t\|_{L^1(0,+\infty;W')}\|\mathbf{u}\|_{L^\infty(0,+\infty;W)},$$
$$\int_0^t \|A_t\|_{L^2(\Gamma_c)}\|\chi\|_{H^1(\Gamma_c)} \le \|A_t\|_{L^1(0,+\infty;L^2(\Gamma_c))}\|\chi\|_{L^\infty(0,+\infty;H^1(\Gamma_c))},$$

we deduce (4.1) in the end. □

Remark 4.2. It can be checked by easily modifying the above estimates that, if (2.45) holds, estimate (4.1) is still valid, up to replacing the term $\|\mathbf{F}\|^2_{L^\infty(0,+\infty;W')} + \|\mathbf{F}_t\|^2_{L^1(0,+\infty;W')}$ on the right-hand side with $\|\mathbf{F}\|^2_{L^2(0,+\infty;W')}$, and analogously for A.

Proof of Theorem 2. It follows from estimate (4.1) that the trajectories $\{(\mathbf{u}(t),\chi(t)),\, t \ge 0\}$ is bounded in $W\times H^1(\Gamma_c)$, and hence it is relatively compact in $H\times L^2(\Gamma_c)$. Therefore, the set $\omega(\mathbf{u}_0,\chi_0)$ (see (2.31)) is non empty and compact in $H\times L^2(\Gamma_c)$. Furthermore, since the solution pair $(\mathbf{u},\chi)$ is in $C^0([0,+\infty);W\times H^1(\Gamma_c))$, $\omega(\mathbf{u}_0,\chi_0)$ is connected thanks to a well-known result in the theory of dynamical systems, see e.g. [21].

In order to prove claims (2.37) and (2.38), let us fix $(\mathbf{u}_\infty,\chi_\infty)\in\omega(\mathbf{u}_0,\chi_0)$. By the definition (2.31) of $\omega(\mathbf{u}_0,\chi_0)$, there exists an increasing sequence $\{t_n\}\subset[0,+\infty)$ such that $t_n \nearrow +\infty$ as $n\to\infty$ and $\mathbf{u}(t_n)\to\mathbf{u}_\infty$ in H, $\chi(t_n)\to\chi_\infty$ in $L^2(\Gamma_c)$. Hence, for a.e. $t\in(0,+\infty)$ let us introduce the functions

$$\mathbf{u}_n(t) := \mathbf{u}(t+t_n),\quad \chi_n(t) := \chi(t+t_n),\quad \boldsymbol{\eta}_n(t) := \boldsymbol{\eta}(t+t_n),$$
$$\xi_n(t) := \xi(t+t_n),\quad \mathbf{F}_n(t) := \mathbf{F}(t+t_n),\quad A_n(t) := A(t+t_n).$$

Of course, $\boldsymbol{\eta}_n(t)\in\alpha(\mathbf{u}_n(t))$ and $\xi_n(t)\in\beta(\chi_n(t))$ for a.e. $t\in(0,+\infty)$ and for all $n\in\mathbb{N}$. Clearly, for all $n\in\mathbb{N}$

$$\|\mathbf{F}_n\|_{L^\infty(0,+\infty;W')} + \|A_n\|_{L^\infty(0,+\infty;L^2(\Gamma_c))} \le \|\mathbf{F}\|_{L^\infty(0,+\infty;W')} + \|A\|_{L^\infty(0,+\infty;L^2(\Gamma_c))}, \tag{4.8}$$

and, due to Remark 2.3, we have

$$\mathbf{F}_n(t)\to\mathbf{F}_\infty \text{ in } W',\quad A_n(t)\to A_\infty \text{ in } L^2(\Gamma_c)\quad \forall t\in[0,T]. \tag{4.9}$$

Moreover, (4.1) yields that there exists a positive constant c_0 such that for all $n\in\mathbb{N}$

$$\|\mathbf{u}_n\|_{L^\infty(0,+\infty;W)} + \|\mathbf{u}_{nt}\|_{L^2(0,+\infty;W)} + \|\chi_n\|_{L^\infty(0,+\infty;H^1(\Gamma_c))} + \|\chi_{nt}\|_{L^2(0,+\infty;L^2(\Gamma_c))} + \|\psi(\chi_n)\|_{L^\infty(0,+\infty;L^1(\Gamma_c))} \le c_0. \tag{4.10}$$

Trivially, for all $T > 0$ the quadruple $(\mathbf{u}_n, \chi_n, \boldsymbol{\eta}_n, \xi_n)$ fulfils on the interval $(0,T)$ system (2.20)–(2.24), with $\mathbf{F}$ and A replaced by $\mathbf{F}_n$ and A_n, and complies with the initial conditions

$$\mathbf{u}_n(0) = \mathbf{u}(t_n) \quad \text{in } \Omega, \qquad \chi_n(0) = \chi(t_n) \quad \text{in } \Gamma_c. \tag{4.11}$$

It follows from (4.10) that there exists a constant $c_1(T)$, depending on T, such that

$$\|\mathbf{u}_n\|_{L^4(0,T;W)} + \|\chi_n\|_{L^2(0,T;H^1(\Gamma_c))} \le c_1(T) \quad \forall\, n \in \mathbb{N}. \tag{4.12}$$

Multiplying (2.22) by $-\Delta\chi_n + \xi_n$ and integrating on $(0,t)$, $t \in (0,T]$, we get

$$\begin{aligned}
&\frac{1}{2}\|\nabla\chi_n(t)\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c} \psi(\chi_n(t)) + \int_0^t \|-\Delta\chi_n + \xi_n\|^2_{L^2(\Gamma_c)} \\
&= \frac{1}{2}\|\nabla\chi(t_n)\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c} \psi(\chi(t_n)) \\
&\qquad + \int_0^t \int_{\Gamma_c} \left(w_s(\chi_n) - \frac{1}{2}|\mathbf{u}_n|^2 + A_n \right)(-\Delta\chi_n + \xi_n) \\
&\le C + \frac{3}{4}\|-\Delta\chi_n + \xi_n\|^2_{L^2(0,t;L^2(\Gamma_c))} + L_s^2\|\chi_n\|^2_{L^2(0,t;L^2(\Gamma_c))} \\
&\qquad + c\|\mathbf{u}_n\|^4_{L^4(0,t;W)} + \|A_n\|^2_{L^2(0,t;L^2(\Gamma_c))}
\end{aligned}$$

where we have used (4.11) and the chain rule [7, Lemma III.3.3] for the functional ψ to deduce the first equality. Then, we have exploited the Lipschitz continuity of w_s and estimated $\|\chi(t_n)\|_{H^1(\Gamma_c)}$ and $\|\psi(\chi(t_n))\|_{L^1(\Gamma_c)}$ by means of (4.1). Also taking into account the monotonicity inequality (3.32) and estimate (4.8), by standard regularity results we conclude that there exists a positive constant $c_2(T)$, depending on T, such that

$$\|\chi_n\|_{L^2(0,T;H^2(\Gamma_c))} + \|\xi_n\|_{L^2(0,T;L^2(\Gamma_c))} \le c_2(T) \quad \forall\, n \in \mathbb{N}. \tag{4.13}$$

Finally, exploiting (4.8), (4.10), and (3.17) (which yields that the sequence $\{\chi_n\mathbf{u}_n\}$ is bounded in $L^\infty(0,+\infty;(H^{-1/2}(\Gamma_c))^3)$), arguing by comparison in (2.20) we conclude that

$$\|\boldsymbol{\eta}_n\|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \le c_3(T) \quad \forall\, n \in \mathbb{N}. \tag{4.14}$$

Thus, standard weak-compactness arguments, the compactness results [27, Thm. 4, Cor. 5], and the Ascoli-Arzelà theorem in the framework of the weak topologies of W and $H^1(\Gamma_c)$, ensure that there exist subsequences of $\{\mathbf{u}_n\}$, $\{\chi_n\}$,

$\{\boldsymbol{\eta}_n\}$, and $\{\xi_n\}$ (which we do not relabel), and a quadruple $(\overline{\mathbf{u}}, \overline{\chi}, \overline{\boldsymbol{\eta}}, \overline{\xi})$ for which the following convergences hold as $n \to \infty$ (compare with (3.18)–(3.19) in the proof of Proposition 3.2):

$$
(4.15)\qquad \left|\begin{array}{l}
\mathbf{u}_n \rightharpoonup \overline{\mathbf{u}} \ \text{ in } H^1(0,T;W),\\
\mathbf{u}_n \to \overline{\mathbf{u}} \ \text{ in } L^2(0,T;(L^q(\Gamma_c))^3) \text{ for all } 1 \le q < 4,\\
\mathbf{u}_n(t) \rightharpoonup \overline{\mathbf{u}}(t) \ \text{ in } W \text{ for all } t \in [0,T],\\
\chi_n \rightharpoonup^* \overline{\chi} \ \text{ in } L^2(0,T;H^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap H^1(0,T;L^2(\Gamma_c)),\\
\chi_n \to \overline{\chi} \ \text{ in } C^0(0,T;L^2(\Gamma_c)),\\
\chi_n(t) \rightharpoonup \overline{\chi}(t) \ \text{ in } H^1(\Gamma_c) \text{ for all } t \in [0,T],\\
\boldsymbol{\eta}_n \rightharpoonup \overline{\boldsymbol{\eta}} \ \text{ in } L^2(0,T;(H^{-1/2}(\Gamma_c))^3),\\
\xi_n \rightharpoonup \overline{\xi} \ \text{ in } L^2(0,T;L^2(\Gamma_c)).
\end{array}\right.
$$

As a byproduct, by the Lipschitz continuity of w_s we have

$$
w_s(\chi_n) \to w_s(\overline{\chi}) \quad \text{in } \ C^0(0,T;L^2(\Gamma_c)),
$$

as well as the enhanced convergences (with respect to (2.31))

$$
(4.16)\qquad \mathbf{u}(t_n) = \mathbf{u}_n(0) \rightharpoonup \overline{\mathbf{u}}(0) \text{ in } \ W, \quad \chi(t_n) = \chi_n(0) \rightharpoonup \overline{\chi}(0) \text{ in } \ H^1(\Gamma_c).
$$

Arguing as to obtain (3.20), we conclude that $\chi_n\mathbf{u}_n \to \overline{\chi}\overline{\mathbf{u}}$ in $L^2(0,T;(H^{-1/2}(\Gamma_c))^3)$ (and weakly-star in $L^\infty(0,T;(H^{-1/2}(\Gamma_c)))^3)$. Moreover, by the strong-weak closedness of the graph of the operator induced by β on $L^2(0,T;L^2(\Gamma_c))$, combining the convergences for $\{\chi_n\}$ and for $\{\xi_n\}$ we have that

$$
(4.17)\qquad \overline{\xi} \in \beta(\overline{\chi}) \quad \text{almost everywhere in } \ \Gamma_c \times (0,T).
$$

Convergences (4.9) and (4.15) enable us to pass to the limit in the equations (2.20)–(2.24), and we infer that, besides (4.17), the quadruple $(\overline{\mathbf{u}}, \overline{\chi}, \overline{\boldsymbol{\eta}}, \overline{\xi})$ fulfills

$$
(4.18)\qquad b(\overline{\mathbf{u}}_t, \mathbf{v}) + a(\overline{\mathbf{u}}, \mathbf{v}) + \int_{\Gamma_c} (\overline{\chi}\overline{\mathbf{u}} + \overline{\boldsymbol{\eta}}) \cdot \mathbf{v} = {}_{W'}\langle \mathbf{F}_\infty, \mathbf{v}\rangle_W \qquad \forall\, \mathbf{v} \in W \ \text{ a.e. in } \ (0,T),
$$

$$
(4.19)\qquad \begin{array}{ll}
\overline{\chi}_t - \Delta\overline{\chi} + \overline{\xi} = w_s(\overline{\chi}) - \dfrac{1}{2}|\overline{\mathbf{u}}|^2 + A_\infty & \text{a.e. in } \Gamma_c \times (0,T),\\
\partial_{\mathbf{n}_s}\overline{\chi} = 0 & \text{a.e. in } \ \partial\Gamma_c \times (0,T).
\end{array}
$$

In order to conclude that

$$
(4.20)\qquad \overline{\boldsymbol{\eta}} \in \alpha(\overline{\mathbf{u}}) \quad \text{in } L^2(0,T;(H^{-1/2}(\Gamma_c))^3),
$$

it is sufficient to show (cf. with the proof of Proposition 3.2) that

$$\limsup_{n\to\infty} \int_0^T \int_{\Gamma_c} \boldsymbol{\eta}_n \cdot \mathbf{u}_n \le \int_0^T \int_{\Gamma_c} \overline{\boldsymbol{\eta}} \cdot \overline{\mathbf{u}}.$$

The latter inequality may be checked exactly in the same way as in the proof of Proposition 3.2, i.e. arguing on the equations (2.20) and (4.18).

Thanks to the estimate (4.1) for $\|\mathbf{u}_t\|_{L^2(0,+\infty;W)}$, we have

$$\int_0^T \|\mathbf{u}_{nt}\|_W^2 = \int_{t_n}^{t_n+T} \|\mathbf{u}_t\|_W^2 \le \int_{t_n}^{+\infty} \|\mathbf{u}_t\|_W^2 \to 0 \tag{4.21}$$

as $n \nearrow \infty$, so that $\mathbf{u}_{nt} \to \mathbf{0}$ in $L^2(0,T;W)$. In the same way, we have $\chi_{nt} \to 0$ in $L^2(0,T;L^2(\Gamma_c))$. Thus, $\overline{\mathbf{u}}_t = \mathbf{0}$ a.e. in Ω, so that $\overline{\mathbf{u}}$ is constant in time in Ω. From (4.11), (4.15) and (4.16) we infer

$$\overline{\mathbf{u}}(t) = \overline{\mathbf{u}}(0) = \lim_{n\to\infty} \mathbf{u}(t_n) = \mathbf{u}_\infty \quad \forall\, t \in [0,T].$$

In the same way, (4.1), (4.11), (4.15) and (4.16) give

$$\overline{\chi}(t) = \chi_\infty \quad \forall\, t \in [0,T].$$

A comparison in (4.18)–(4.20) provides

$$\overline{\boldsymbol{\eta}}(t) \equiv \eta_\infty \in \alpha(\mathbf{u}_\infty), \quad \overline{\xi}(t) \equiv \xi_\infty \in \beta(\chi_\infty) \quad \text{for a.e. } t \in (0,T).$$

Eventually, (4.18)–(4.20) yield the limit system (2.37)–(2.38). From the latter equation we immediately deduce $-\Delta\chi_\infty \in L^2(\Gamma_c)$, hence $\chi_\infty \in H^2(\Gamma_c)$, which concludes the proof. □

Proof of Proposition 2.5. First of all, let us point out that, as $\mathbf{F}_\infty = \mathbf{0}$, every pair $(\mathbf{u}_\infty, \chi_\infty) \in \omega(\mathbf{u}_0, \chi_0)$ actually satisfies (2.38) coupled with

$$a(\mathbf{u}_\infty, \mathbf{v}) + \int_{\Gamma_c} (\chi_\infty \mathbf{u}_\infty + \boldsymbol{\eta}_\infty) \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \in W, \quad \boldsymbol{\eta}_\infty \in \alpha(\mathbf{u}_\infty).$$

Let us now choose $\mathbf{v} = \mathbf{u}_\infty$ in the above relation: by (2.3), we gather

$$C_a \|\mathbf{u}_\infty\|_W^2 + \int_{\Gamma_c} \chi_\infty |\mathbf{u}_\infty|^2 + \int_{\Gamma_c} \boldsymbol{\eta}_\infty \cdot \mathbf{u}_\infty \le 0.$$

Taking into account (2.5) and the fact that $\chi_\infty \ge 0$ a.e. in Γ_c (since $\chi_\infty \in \mathrm{dom}(\beta)$ a.e. in Γ_c), we infer from the above inequality that $\|\mathbf{u}_\infty\|_W^2 = 0$, whence (2.43). Hence, χ_∞ satisfies

$$-\Delta\chi_\infty + \xi_\infty = w_s(\chi_\infty) + A_\infty \quad \text{a.e. in } \Gamma_c. \tag{4.22}$$

Suppose that (2.41) holds. Then, let us test (4.22) by $(\chi_\infty - 1)$. Due to (2.41), we obtain

$$\text{(4.23)} \quad \int_{\Gamma_c} |\nabla(\chi_\infty - 1)|^2 + \int_{\Gamma_c} \xi_\infty(\chi_\infty - 1) = \int_{\Gamma_c} (w_s(\chi_\infty) + A_\infty)(\chi_\infty - 1) \le 0.$$

We note that

$$\xi_\infty(x)(\chi_\infty(x) - 1) \ge 0 \quad \text{for a.e. } x \in \Gamma_c,$$

since $\xi_\infty \neq 0$ if and only if $\chi_\infty = 0$ or $\chi_\infty = 1$ and, in the former case, $\xi_\infty \le 0$. As a consequence, we deduce from (4.23) that $\|\nabla(\chi_\infty - 1)\|_{L^2(\Gamma_c)} = 0$, whence there exists $r' \le 0$ such that $(\chi_\infty - 1) \equiv r'$ on Γ_c. Suppose that $r' < 0$. Hence, $\chi_\infty < 1$ on Γ_c. On the other hand, integrating (4.22) over Γ_c and using (2.41) and that $\partial_{\mathbf{n}_s}\chi_\infty = 0$ on $\partial\Gamma_c$, we get

$$\int_{\Gamma_c} \xi_\infty = \int_{\Gamma_c} (w_s(\chi_\infty) + A_\infty) \ge a_\infty |\Gamma_c| > 0,$$

which leads to a contradiction. Thus, we conclude that $\chi_\infty \equiv 1$ on Γ_c.

Assume now (2.42). Let us test (4.22) by χ_∞. Arguing by monotonicity and taking into account (2.42), we get

$$\int_{\Gamma_c} |\nabla\chi_\infty|^2 \le \int_{\Gamma_c} (w_s(\chi_\infty) + A_\infty)\chi_\infty \le 0,$$

whence $\nabla\chi_\infty \equiv \mathbf{0}$ on Γ_c and

$$\exists\, s' \in [0,1] : \quad \chi_\infty \equiv s' \text{ on } \Gamma_c.$$

Now, integrating (4.22) on Γ_c we get

$$\int_{\Gamma_c} \xi_\infty = \int_{\Gamma_c} (w_s(\chi_\infty) + A_\infty) \le -b_\infty |\Gamma_c| < 0,$$

so that necessarily $s' = 0$.

Finally, (2.44) is an easy consequence of the definition (2.31) of ω-limit, of the convergences (4.16), and of the fact that $\omega(\mathbf{u}_0, \chi_0)$ is a singleton. □

Sketch of the proof of Proposition 2.7. Due to Remark 4.2 our argument can be easily adapted to treat the problem under the alternative assumption (2.45). Indeed, it is possible to prove that for any pair of initial data $(\mathbf{u}_0, \chi_0)$ the ω-limit set $\omega(\mathbf{u}_0, \chi_0)$ is non empty, compact and connected in $H \times L^2(\Gamma_c)$. Next, one shows that any pair $(\mathbf{u}_\infty, \chi_\infty) \in \omega(\mathbf{u}_0, \chi_0)$ fulfils (2.46) by suitably adapting the proof of Theorem 2: in particular, arguing as in (4.21) one checks that (cf. the notation (4.8))

$$\mathbf{F}_n \to \mathbf{0} \text{ in } L^2(0,T;W'), \quad A_n \to 0 \text{ in } L^2(0,T;L^2(\Gamma_c)).$$

Then, arguing on the limit system (2.46) one combines (2.39) and (2.47) with the argument developed in the proof of Proposition 2.5 to prove that $\chi_\infty \equiv 1$. □

REFERENCES

[1] H. ATTOUCH, *Variational Convergence for Functions and Operators*, Pitman, London, 1984.
[2] C. BAIOCCHI, *Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert*, Ann. Mat. Pura Appl. (4) **76** (1967), 233–304.
[3] V. BARBU, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1976.
[4] E. BONETTI, G. BONFANTI, and R. ROSSI, *Global existence for a contact problem with adhesion*, Quaderno del Seminario Matematico di Brescia, N. 16 (2005), Math. Meth. Appl. Sci. (in press).
[5] E. BONETTI and G. SCHIMPERNA, *Local existence for Frémond's model of damage in elastic materials*, Contin. Mech. Thermodyn. **16** (2004), 319–335.
[6] G. BONFANTI, M. FRÉMOND, and F. LUTEROTTI, *Global solution to a nonlinear system for irreversible phase changes*, Adv. Math. Sci. Appl. **10** (2000), 1–24.
[7] H. BRÉZIS, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North Holland Math. Studies, vol. 5, North-Holland, Amsterdam, 1973.
[8] O. CHAU, J.R. FERNÁNDEZ, M. SHILLOR, and M. SOFONEA, *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, J. Comput. Appl. Math. **159** (2003), 431–465.
[9] O. CHAU, M. SHILLOR, and M. SOFONEA, *Dynamic frictionless contact with adhesion*, J. Appl. Math. Phys. (ZAMP) **55** (2004), 32–47.
[10] P. COLLI, *On some doubly nonlinear evolution equations in Banach spaces*, Japan J. Indust. Appl. Math. **9** (1992), 181–203.
[11] P. COLLI, G. GILARDI, P. LAURENÇOT, and A. NOVICK-COHEN, *Existence and long-time behavior of the conserved phase-field system with memory*, Discrete Contin. Dynam. Systems **5** (1995), 375–390.
[12] P. COLLI and P. LAURENÇOT, *Weak solutions to the Penrose-Fife phase field model for a class of admissible heat flux laws*, Phys. D **111** (1998), 311–334.
[13] E. FEIREISL and G. SCHIMPERNA, *Large time behaviour of solutions to Penrose-Fife phase change models*, Math. Methods Appl. Sci. **28** (2005), 2117–2132.
[14] J.R. FERNÁNDEZ, M. SHILLOR, and M. SOFONEA, *Analysis and numerical simulations of a dynamic contact problem with adhesion*, Math. Comput. Model. **37** (2003), 1317–1333.
[15] M. FRÉMOND, *Equilibre des structures qui adhèrent à leur support*, C.R. Acad. Sci. Paris **295** (1982), 913–916.
[16] ______, *Adhèrence des solides*, J. Méc. Théor. Appl. **6** (1987), 383–407.
[17] ______, *Non-smooth Thermomechanics*, Springer-Verlag, Berlin, 2002.
[18] M. FRÉMOND and B. NEDJAR, *Damage, gradient of damage and principle of virtual power*, Internat. J. Solids Structures **33** (1996), 1083–1103.
[19] P. GERMAIN, *Mécanique des milieux continus*, Masson, Paris, 1973.
[20] M. GRASSELLI, H. PETZELTOVÁ, and G. SCHIMPERNA, *Long-time behavior of solutions to the Caginalp system with singular potential*, Z. Anal. Anwend. **25** (2006), 51–72.
[21] A. HARAUX, *Systèmes Dynamiques Dissipatifs et Applications*, RMA Res. Notes Appl. Math., vol. 17, Masson, Paris, 1991.
[22] P. KREJČÍ and S. ZHENG, *Pointwise asymptotic convergence of solutions for a phase separation model*, Discrete Contin. Dyn. Syst. **16** (2006), 1–18.

[23] J.L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier–Villars, Paris, 1969.

[24] N. POINT, *Unilateral contact with adherence*, Math. Methods Appl. Sci. **10** (1998), 367–381.

[25] M. RAOUS, L. CANGÉMI, and M. COCU, *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Methods Appl. Mech. Eng. **177** (1999), 383–399.

[26] P. RYBKA and K.-H. HOFFMANN, *Convergence of solutions to Cahn-Hilliard equation*, Comm. Partial Differential Equations **24** (1999), 1055-1077.

[27] J. SIMON, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.

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