EXISTENCE RESULTS FOR A COUPLED VISCOPLASTIC-DAMAGE MODEL IN THERMOVISCOELASTICITY

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Abstract. In this paper we address a model coupling viscoplasticity with damage in thermoviscoelasticity. The associated PDE system consists of the momentum balance with viscosity and inertia for the displacement variable, at small strains, of the plastic and damage flow rules, and of the heat equation. It has a strongly nonlinear character and in particular features quadratic terms on the right-hand side of the heat equation and of the damage flow rule, which have to be handled carefully. We propose two weak solution concepts for the related initial-boundary value problem, namely 'entropic' and 'weak energy' solutions. Accordingly, we prove two existence results by passing to the limit in a carefully devised time discretization scheme. Finally, in the case of a *prescribed* temperature profile, and under a strongly simplifying condition, we provide a continuous dependence result, yielding uniqueness of weak energy solutions.

2010 Mathematics Subject Classification: 35Q74, 74H20, 74C05, 74C10, 74F05. 74R05.

Key words and phrases: Thermoviscoelastoplasticity, Damage, Entropic Solutions, Time Discretization.

Dedicated to Tomáš Roubíček on the occasion of his 60th birthday

1. Introduction

This paper focuses on two phenomena related to inelastic behavior in materials, namely damage and plasticity. Damage can be interpreted as a degradation of the elastic properties of a material due to the failure of its microscopic structure. Such macroscopic mechanical effects take their origin from the formation of cracks and cavities at the microscopic scale. They may be described in terms of an internal variable, the damage parameter, on which the elastic modulus depends, in such a way that stiffness decreases with ongoing damage. Plasticity produces residual deformations that remain after complete unloading.

Recently, models combining plasticity with damage have been proposed in the context of geophysical modeling [RSV13, RV16] and, more in general, within the study of the thermomechanics of damageable materials under diffusion [RT15]. Perfect plasticity is featured in [RSV13, RV16], where the evolution of the damage variable is governed by viscosity, i.e. it is *rate-dependent.* Conversely, in [RT15] damage evolves rate-independently, while the evolution of plasticity is rate-dependent. In a different spirit, a fully rate-independent system for the evolution of the damage parameter, coupled with a tensorial variable which stands for the transformation strain arising during damage evolution, is analyzed in [BRRT16]. Finally, let us mention the coupled elastoplastic damage model from [AMV14, AMV15], analyzed in [Cri16a, Cri16b, CL16]. While the first two papers deal with the fully rate-independent case in [Cri16a], in [CL16], the model is regularized by adding viscosity to the damage flow rule, while keeping the evolution of the plastic tensor rate-independent. A vanishing-viscosity analysis is then carried out, leading to an alternative solution concept ('rescaled quasistatic viscosity evolution') for the rate-independent elastoplastic damage system. A common feature in [RSV13, RV16, Cri16a, CL16] is that the plastic yield surface, and thus the plastic dissipation potential, depends on the damage variable.

In this paper we aim to bring temperature into the picture. Thermoplasticity models, both in the case of rate-independent evolution of the plastic variable and of rate-dependent one, have been the object of several studies, cf. e.g. [KS97, KSS02, KSS03, BR08, BR11, Rou13b, HMS17, Ros16]. In recent years, there has also been a growing literature on the analysis of (rate-independent or rate-dependent) damage models with thermal effects: We quote among others [BB08, Rou10, RR14, HR15, RR15, LRTT14]. As for models coupling plasticity and damage with temperature, one of the examples illustrating the general theory developed in [RT15] concerns geophysical models of lithospheres in short time scales, which couple the (small-strain) momentum balance,

Date: December 29, 2016.

The author has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

damage, *rate-dependendent* plasticity, the heat equation, as well as the porosity and the water concentration variables. Here we shall neglect the latter two variables and tackle the weak solvability and the existence of solutions, for a (*fully rate-dependent*) viscoplastic (gradient) damage model, with viscosity and inertia in the momentum balance (the first according to Kelvin-Voigt rheology), and with thermal effects encompassed through the heat equation, whereas in [RT15] the enthalpy equation was analyzed, after a transformation of variables. We plan to address the vanishing-viscosity and inertia analysis for our model, and discuss the weak solution concept thus obtained, in a future contribution.

In what follows, we shortly comment on the model. Then, we illustrate the mathematical challenges posed by its analysis, motivate a suitable regularization, and introduce the two solution concepts, for the original system and its regularized version, for which we will prove two existence results.

1.1. **The thermoviscoelastoplastic damage system.** The PDE system, posed in $\Omega \times (0,T)$, where the reference configuration Ω is a bounded, open, Lipschitz domain in $\mathbb{R}^d$, $d \in \{2,3\}$, and $(0,T)$ is a given time interval, consists of

- the kinematic admissibility condition
$$E(u) = e + p \qquad \text{in } \Omega \times (0,T), \tag{1.1a}$$
which provides a decomposition of the linearized strain tensor $E(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ into the sum of the elastic and plastic strains e and p. In fact, $e \in \mathbb{M}^{d\times d}_{\mathrm{sym}}$ (the space of symmetric $(d\times d)$-matrices), while $p \in \mathbb{M}^{d\times d}_{\mathrm{D}}$ (the space of symmetric $(d\times d)$-matrices with null trace).
- The momentum balance
$$\begin{aligned} &\rho\ddot{u} - \mathrm{div}\sigma = F && \text{in } \Omega \times (0,T), \\ \text{with the stress given by}\quad &\sigma = \mathbb{D}(z)\dot{e} + \mathbb{C}(z)e - \mathbb{C}(z)\mathbb{E}\vartheta && \text{in } \Omega \times (0,T) \end{aligned} \tag{1.1b}$$
according to Kelvin-Voigt rheology for materials subject to thermal expansion (with $\mathbb{E}$ the matrix of the thermal expansion coefficients). Here, F is a given body force. Observe that both the elasticity and viscosity tensors $\mathbb{C}$ and $\mathbb{D}$ depend on the damage parameter z, but we restrict to *incomplete* damage. Namely, the tensors $\mathbb{C}$ and $\mathbb{D}$ are definite positive uniformly w.r.t. z, meaning that the system retains its elastic properties even when damage is maximal.
- The damage flow rule for z
$$\partial\mathrm{R}(\dot{z}) + \dot{z} + A_{\mathrm{s}}(z) + W'(z) \ni -\tfrac{1}{2}\mathbb{C}'(z)e : e + \vartheta \qquad \text{in } \Omega \times (0,T), \tag{1.1c}$$
where the dissipation potential (density)
$$\mathrm{R} : \mathbb{R} \to [0,+\infty] \ \text{ defined by } \ \mathrm{R}(\eta) := \begin{cases} |\eta| & \text{if } \eta \leq 0, \\ +\infty & \text{otherwise,} \end{cases}$$
encompasses the unidirectionality in the evolution of damage. We denote by $\partial\mathrm{R} : \mathbb{R} \rightrightarrows \mathbb{R}$ its subdifferential in the sense of convex analysis. As in several other damage models, we confine ourselves to a *gradient theory*. However, along the lines of [KRZ13] and [CL16], we adopt a special choice of the gradient regularization, i.e. through the s-Laplacian operator A_{s}, with $\mathrm{s} > \frac{d}{2}$. This technical condition ensures the compact embedding $H^{\mathrm{s}}(\Omega) \Subset \mathrm{C}^0(\overline{\Omega})$ for the associated Sobolev-Slobodeckij space $H^{\mathrm{s}}(\Omega) := W^{\mathrm{s},2}(\Omega)$. Furthermore, a key role in the analysis, especially for the regularized system with the flow rule (1.3c) ahead, will be played by the *linearity* of the operator A_{s}. As in [CL16], the term $W'(z)$ will have a singularity at $z = 0$, which will enable us to prove the positivity of the damage variable. Combining this with the unidirectionality constraint $\dot{z} \leq 0$ a.e. in $\Omega \times (0,T)$, we will ultimately infer that all z emanating from an initial datum $z_0 \leq 1$ take values in the physically admissible interval $[0,1]$.
- The flow rule for the plastic tensor reads
$$\partial_{\dot{p}}\mathrm{H}(z,\vartheta;\dot{p}) + \dot{p} \ni \sigma_{\mathrm{D}} \qquad \text{in } \Omega \times (0,T), \tag{1.1d}$$
with σ_{D} the deviatoric part of the stress tensor σ. Here, the plastic dissipation potential (density) H depends on the plastic strain rate $\dot{p}$ but also (on the space variable x), on the temperature, and on the damage variable; the symbol $\partial_{\dot{p}}\mathrm{H}$ denotes its convex subdifferential w.r.t. the plastic rate. Typically, one may assume that the plastic yield surface decreases as damage increases, although this monotonicity

property will not be needed for the analysis developed in this paper. Observe that (1.1d) is in fact a viscous regularization of the flow rule

$$\partial_{\dot p}\mathrm{H}(z,\vartheta;\dot p) \ni \sigma_{\mathrm{D}} \qquad \text{in } \Omega\times(0,T)$$

of perfect plasticity.

- The heat equation

$$\dot\vartheta - \mathrm{div}(\kappa(\vartheta)\nabla\vartheta) = G + \mathbb{D}\dot e : \dot e - \vartheta\mathbb{C}(z)\mathbb{E} : \dot e + \mathrm{R}(\dot z) + |\dot z|^2 - \vartheta\dot z + \mathrm{H}(z,\vartheta;\dot p) + \dot p : \dot p \qquad \text{in } \Omega\times(0,T), \quad (1.1e)$$

with $\kappa \in \mathrm{C}^0(\mathbb{R}^+)$ the heat conductivity coefficient and G a given, positive, heat source.

We will supplement system (1.1) with the boundary conditions

$$u = w \qquad \text{on } \Gamma_{\mathrm{Dir}}\times(0,T), \qquad \sigma n = f \qquad \text{on } \Gamma_{\mathrm{Neu}}\times(0,T), \qquad (1.2a)$$

$$\partial_n z = 0 \qquad \text{on } \partial\Omega\times(0,T), \qquad \kappa(\vartheta)\nabla\vartheta n = g \qquad \text{on } \partial\Omega\times(0,T), \qquad (1.2b)$$

where n is the outward unit normal to $\partial\Omega$, e Γ_{Dir}, Γ_{Neu} the Dirichlet/Neumann parts of the boundary, respectively, and with initial conditions.

In fact, system (1.1) can be seen as the extension of the model considered in [CL16], featuring the static momentum balance and a mixed rate-dependent/rate-independent character in the evolution laws for the damage/plastic variables, respectively, to the case where viscosity is included in the plastic flow rule and in the momentum balance, the latter also with inertia, and the evolution of temperature is also encompassed.

System (1.1, 1.2) can be rigorously derived following, e.g., the thermomechanical modeling approach by M. Frémond [Fré02, Chap. 12]. In this way one can also verify its compliance with the first and second principle of Thermodynamics, hence its thermodynamical consistency.

1.2. **Analytical challenges and weak solution concepts.** Despite the fact that the 'viscous' plastic flow rule (1.1d) does not bring all the technical difficulties attached to perfect plasticity (cf. [DMDM06], see also [Rou13b] for the coupling with temperature), the analysis of system (1.1, 1.2) still poses some mathematical difficulties. Namely,

(1): The overall nonlinear character of (1.1) and, in particular, the quadratic terms on the right-hand side of the damage flow rule (1.1c), and on the right-hand side of the heat equation (1.1e). Both sides are, thus, only estimated in $L^1(\Omega\times(0,T))$ as soon as $\dot e$, $\dot z$, and $\dot p$ are estimated in $L^2(\Omega\times(0,T);\mathbb{M}^{d\times d}_{\mathrm{sym}})$, $L^2(\Omega)$, and $L^2(\Omega\times(0,T);\mathbb{M}^{d\times d}_{\mathrm{D}})$, respectively, as guaranteed by the *dissipative estimates* associated with (1.1).

Observe that the particular character of the momentum balance, where the elasticity and the viscosity contributions only involve the elastic part of the strain e and its rate $\dot e$, instead of the full strain $E(u)$ and strain rate $E(\dot u)$, does not allow for elliptic regularity arguments which could at least enhance the spatial regularity/summability of the right-hand side of the damage flow rule.

(2): Another obstacle is given by the presence of the *unbounded* maximal monotone operator $\partial\mathrm{R}$ in the damage flow rule. Because of this, no comparison estimates can be performed. In particular, a pointwise formulation of (1.1c) would require a separate estimate of the terms $A_{\mathrm{s}}(z)$ and of (a selection in) $\partial\mathrm{R}(\dot z)$. This cannot be obtained by standard monotonicity arguments due to the nonlocal character of the operator A_{s}.

All of these issues shall be reflected in the weak solution concept for system (1.1, 1.2) proposed in the forthcoming Definition 2.3 and referred to as 'entropic solution'. This solvability notion consists of the so-called *entropic formulation* of the heat equation, and of a weak formulation of the damage flow rule, in the spirit of the *Karush-Kuhn-Tucker* conditions. The entropic formulation originates from the work by E. Feireisl in fluid mechanics [Fei07] and has been first adapted to the context of phase transition systems in [FPR09], and later extended to damage models in [RR15]. It is given by an *entropy inequality*, formally obtained by dividing the heat equation by ϑ and testing the resulting relation by a sufficiently regular, *positive* test function (cf. the calculations at the beginning of Section 2.3), combined with a *total energy inequality.* The weak formulation of the damage flow rule has been first proposed in the context of damage modeling in [HK11, HK13]: the subdifferential inclusion for damage is replaced by a one-sided variational inequality, with test functions reflecting the sign constraint imposed by the dissipation potential R, joint with a (mechanical) energy-dissipation inequality also incorporating contributions from the momentum balance and the plastic flow rule.

Clearly, one of the analytical advantages of the entropy inequality for the heat equation, and of the one-sided inequality for the damage flow rule, is that the troublesome quadratic terms on the right-hand sides of (1.1c) and of (1.1e) feature as multiplied by a negative test function, cf. (2.34a) and (2.37) ahead, respectively. This allows for upper semicontinuity arguments in the limit passage in suitable approximations of such inequalities. Instead, the total and mechanical energy inequalities can be obtained by lower semicontinuity techniques.

We will also consider a *regularized version* of system (1.1, 1.2), where the damage flow rule features the additional term $A_{\mathrm{s}}(\dot z)$, modulated by a positive constant ν, and, accordingly, the term $\nu a_{\mathrm{s}}(\dot z, \dot z)$ (with a_{s} the bilinear form associated with A_{s}) occurs on the right-hand side of the heat equation. This leads to the *regularized* thermoviscoelastoplastic damage system

$$E(u) = e + p \qquad \text{in } \Omega \times (0,T), \tag{1.3a}$$

$$\rho \ddot u - \mathrm{div}\sigma = F \qquad \text{in } \Omega \times (0,T), \tag{1.3b}$$

$$\sigma = \mathbb{D}(z)\dot e + \mathbb{C}(z)e - \mathbb{C}(z)\mathbb{E}\vartheta \qquad \text{in } \Omega \times (0,T),$$

$$\partial \mathrm{R}(\dot z) + \dot z + \nu A_{\mathrm{s}}(\dot z) + A_{\mathrm{s}}(z) + W'(z) \ni -\tfrac12 \mathbb{C}'(z)e : e + \vartheta \qquad \text{in } \Omega \times (0,T), \tag{1.3c}$$

$$\partial_{\dot p}\mathrm{H}(z,\vartheta;\dot p) + \dot p \ni \sigma_{\mathrm{D}} \qquad \text{in } \Omega \times (0,T), \tag{1.3d}$$

$$\begin{aligned} \dot\vartheta - \mathrm{div}(\kappa(\vartheta)\nabla\vartheta) = {}& G + \mathbb{D}\dot e : \dot e - \vartheta\mathbb{C}(z)\mathbb{E} : \dot e \\ & + \mathrm{R}(\dot z) + |\dot z|^2 + \bar\nu a_{\mathrm{s}}(\dot z,\dot z) - \vartheta\dot z + \mathrm{H}(z,\vartheta;\dot p) + \dot p : \dot p \end{aligned} \qquad \text{in } \Omega \times (0,T), \tag{1.3e}$$

with $\bar\nu = \nu/|\Omega|$, supplemented with the boundary conditions

$$u = w \quad \text{on } \Gamma_{\mathrm{Dir}} \times (0,T), \qquad \sigma n = f \quad \text{on } \Gamma_{\mathrm{Neu}} \times (0,T), \tag{1.4a}$$

$$\partial_n z = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \partial_n \dot z = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \kappa(\vartheta)\nabla\vartheta n = g \quad \text{on } \partial\Omega \times (0,T). \tag{1.4b}$$

For this regularized system we will be able to show the existence of an enhanced type of solution. It features a conventional weak formulation of the heat equation with suitable test functions, and the formulation of the damage flow rule as a subdifferential inclusion in $H^{\mathrm{s}}(\Omega)^*$. To obtain the latter, a key role is played by the regularizing term $A_{\mathrm{s}}(\dot z)$, ensuring that $\dot z \in H^{\mathrm{s}}(\Omega)$ a.e. in $(0,T)$. Thanks to this feature, it is admissible to test the subdifferential inclusion rendering (1.3c) by $\dot z$ itself. This is at the core of the validity of the total energy *balance*. That is why, also in accordance with the nomenclature from [RR15, Ros16], we shall refer to these enhanced solutions as *weak energy solutions*.

1.3. **Our results.** We will prove the existence of *entropic* solutions, see **Theorem 2.5**, and of *weak energy* solutions, see **Theorem 2.6**, to (the Cauchy problems for) systems (1.1, 1.2) and (1.3, 1.4), respectively, by passing to the limit in a time-discretization scheme carefully devised in such a way as to ensure the validity of discrete versions of the entropy and energy inequalities along suitable interpolants of the discrete solutions. One of our standing assumptions will be a suitable growth of the heat conductivity coefficient κ, namely

$$\kappa(\vartheta) \sim \vartheta^{\mu} \quad \text{for some } \mu > 1. \tag{1.5}$$

Mimicking the calculations from [FPR09, RR15], we will exploit (1.5) in a key way to derive an estimate for ϑ in $L^2(0,T;H^1(\Omega))$ by testing the (discrete) heat equation by a suitable negative power of ϑ.

Under a more restrictive condition on μ, in fact depending on the space dimension d, (i.e., $\mu \in (1,2)$ if $d=2$, $\mu \in (1,\frac53)$ if $d=3$), we will also be able to obtain a BV-in-time estimate for ϑ, with values in a suitable dual space, which will be at the core of the proof of the enhanced formulation of the heat equation for the weak energy solutions to system (1.3, 1.4). Concerning the physical interpretation of our growth conditions, we refer to [Kle12] for a discussion of experimental findings suggesting that a class of polymers exhibit a subquadratic growth for κ.

Finally, with **Proposition 2.7** we will provide a continuous dependence estimate, yielding uniqueness, for the weak energy solutions to (1.3, 1.4) in the case of a *prescribed* temperature profile, and with a plastic dissipation potential *independent* of the state variables z and ϑ.

Plan of the paper. In Section 2 we fix all our assumptions, motivate and state our two weak solvability notions for systems (1.1, 1.2) & (1.3, 1.4), and finally give the existence Theorems 2.5 & 2.6, and the continuous dependence result Proposition 2.7. In Section 3 we set up a common time discretization scheme for systems (1.1, 1.2) and (1.3, 1.4), and prove the existence of discrete solutions, while Section 4 is devoted to the derivation of all the a priori estimates on the approximate solutions, obtained by interpolation of the discrete ones. In

Section 5 we conclude the proofs of Thms. 2.5 & 2.6 by passing to the time-continuous limit, while in Section 6 we perform the proof of Prop. 2.7.

We conclude by fixing some notation that shall be used in the paper.

Notation 1.1 (General notation)**.** Throughout the paper, $\mathbb{R}^+$ shall stand for $(0,+\infty)$. For a given $z \in \mathbb{R}$, we will use the notation $(z)^+$ for its positive part $\max\{z,0\}$. We will denote by $\mathbb{M}^{d\times d}$ ($\mathbb{M}^{d\times d\times d\times d}$) the space of $(d{\times}d)$ $((d{\times}d{\times}d{\times}d)$, respectively) matrices. We will consider $\mathbb{M}^{d\times d}$ endowed with the Frobenius inner product $A : B := \sum_{ij} a_{ij} b_{ij}$ for two matrices $A = (a_{ij})$ and $B = (b_{ij})$, which induces the matrix norm $|\cdot|$. Therefore, we will often write $|A|^2$ in place of $A : A$. The symbol $\mathbb{M}^{d\times d}_{\mathrm{sym}}$ stands for the subspace of symmetric matrices, and $\mathbb{M}^{d\times d}_{\mathrm{D}}$ for the subspace of symmetric matrices with null trace. We recall that $\mathbb{M}^{d\times d}_{\mathrm{sym}} = \mathbb{M}^{d\times d}_{\mathrm{D}} \oplus \mathbb{R}\mathbb{I}$ ($\mathbb{I}$ denoting the identity matrix), since every $\eta \in \mathbb{M}^{d\times d}_{\mathrm{sym}}$ can be written as $\eta = \eta_{\mathrm{D}} + \frac{\mathrm{tr}(\eta)}{d}\mathbb{I}$ with η_{D} the orthogonal projection of η into $\mathbb{M}^{d\times d}_{\mathrm{D}}$. We will refer to η_{D} as the deviatoric part of η.

For a given Banach space X, the symbol $\langle\cdot,\cdot\rangle_X$ will stand for the duality pairing between X^* and X; if X is a Hilbert space, $(\cdot,\cdot)_X$ will denote its inner product. For simpler notation, we shall often write $\|\cdot\|_X$ both for the norm on X, and on the product space $X \times \ldots \times X$. With the symbol $\overline{B}_{1,X}(0)$ we will denote the closed unitary ball in X. We shall use the symbols

$$\text{(i) } \mathrm{B}([0,T];X), \qquad \text{(ii) } \mathrm{C}^0_{\mathrm{weak}}([0,T];X), \qquad \text{(iii) } \mathrm{BV}([0,T];X)$$

for the spaces of functions from $[0,T]$ with values in X that are defined at *every* $t \in [0,T]$ and (i) are measurable; (ii) are *weakly* continuous on $[0,T]$; (iii) have bounded variation on $[0,T]$.

Finally, throughout the paper we will denote various positive constants depending only on known quantities by the symbols c, c', C, C', whose meaning may vary even within the same line. Furthermore, the symbols I_i, $i = 0,1,...$, will be used as place-holders for several integral terms (or sums of integral terms) occurring in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, the symbol I_1 will have different meanings.

2. **Setup and main results for the thermoviscoelastoplastic damage system**

After fixing the setup for our analysis in Section 2.1, in Sec. 2.2 we motivate the notion of 'weak energy' solution to system (1.3, 1.4) by unveiling its underlying energetics. This concept is then precisely fixed in Definition 2.1. Sec. 2.3 is devoted to the introduction of the considerably weaker concept of 'entropic' solutions. Our existence theorems are stated in Sec. 2.4, while in Sec. 2.5 we confine the discussion to the case of a given temperature profile, and give a continuous dependence result for weak energy solutions.

2.1. **Setup.**

The reference configuration. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with Lipschitz boundary; we set $Q := \Omega \times (0,T)$. The boundary $\partial\Omega$ is given by

$$\begin{gathered}\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \partial\Gamma \quad \text{ with } \Gamma_{\mathrm{Dir}},\, \Gamma_{\mathrm{Neu}},\, \partial\Gamma \text{ pairwise disjoint,} \\ \Gamma_{\mathrm{Dir}} \text{ and } \Gamma_{\mathrm{Neu}} \text{ relatively open in } \partial\Omega, \text{ and } \partial\Gamma \text{ their relative boundary in } \partial\Omega, \\ \text{with Hausdorff measure } \mathcal{H}^{d-1}(\partial\Gamma) = 0.\end{gathered} \tag{2.Ω}$$

We will denote by $|\Omega|$ the Lebesgue measure of Ω. On the Dirichlet part Γ_{Dir}, assumed with $\mathcal{H}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0$, we shall prescribe the displacement, while on Γ_{Neu} we will impose a Neumann condition on the displacement. The trace of a function v on Γ_{Dir} or Γ_{Neu} shall be still denoted by the symbol v.

Sobolev spaces, the s*-Laplacian, Korn's inequality.* In what follows, we will use the notation $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d) := \{u \in H^1(\Omega;\mathbb{R}^d) \,:\, u|_{\mathrm{Dir}} = 0\}$. The symbol $W^{1,p}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$, $p > 1$, shall denote the analogous $W^{1,p}$-space. Further, we will use the notation

$$W^{1,p}_+(\Omega) := \left\{\zeta \in W^{1,p}(\Omega) \,:\, \zeta(x) \geq 0 \quad \text{for a.a. } x \in \Omega\right\}, \qquad \text{and analogously for } W^{1,p}_-(\Omega). \tag{2.1}$$

Throughout the paper, we shall extensively resort to Korn's inequality (cf. [GS86]): for every $1 < p < \infty$ there exists a constant $C_K = C_K(\Omega,p) > 0$ such that there holds

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \leq C_K \|E(u)\|_{L^p(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \qquad \text{for all } u \in W^{1,p}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\,. \tag{2.2}$$

We will denote by

$$H^{\mathrm{s}}(\Omega) \text{ the Sobolev–Slobodeckij space } W^{\mathrm{s},2}(\Omega), \text{ with } \mathrm{s} \in \left(\tfrac{d}{2}, 2\right).$$

We will also use the notation $H^{\mathrm{s}}_+(\Omega) = \{z \in H^{\mathrm{s}}(\Omega) : z \geq 0 \text{ in } \Omega\}$ and the analogously defined notation $H^{\mathrm{s}}_-(\Omega)$. We recall that $H^{\mathrm{s}}(\Omega)$ is a Hilbert space, with inner product $(z_1, z_2)_{H^{\mathrm{s}}(\Omega)} := (z_1, z_2)_{L^2(\Omega)} + a_{\mathrm{s}}(z_1, z_2)$, where the bilinear form $a_{\mathrm{s}}(\cdot, \cdot)$ is defined by

$$a_{\mathrm{s}}(z_1, z_2) := \iint_{\Omega\times\Omega} \frac{(\nabla z_1(x) - \nabla z_1(y))\,(\nabla z_2(x) - \nabla z_2(y))}{|x-y|^{d+2(\mathrm{s}-1)}}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{2.3}$$

We denote by $A_{\mathrm{s}} : H^{\mathrm{s}}(\Omega) \to H^{\mathrm{s}}(\Omega)^*$ the associated operator

$$\langle A_{\mathrm{s}}(z), v\rangle_{H^{\mathrm{s}}(\Omega)} := a_{\mathrm{s}}(z, v) \qquad \text{for all } v \in H^{\mathrm{s}}(\Omega)\,. \tag{2.4}$$

Kinematic admissibility and stress. Given a function $w \in H^1(\Omega;\mathbb{R}^d)$, we say that a triple (u, e, p) is *kinematically admissible with boundary datum* w, and write $(u, e, p) \in \mathcal{A}(w)$, if

$$u \in H^1(\Omega;\mathbb{R}^d), \quad e \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \quad p \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}), \tag{2.5a}$$

$$E(u) = e + p \quad \text{a.e. in } \Omega, \tag{2.5b}$$

$$u = w \quad \text{on } \Gamma_{\mathrm{Dir}}. \tag{2.5c}$$

As for the elasticity and viscosity tensors, we will suppose that

$$\mathbb{C},\, \mathbb{D} \in \mathrm{C}^0(\overline{\Omega}\times\mathbb{R};\mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}}))\,, \text{ and}$$

$$\exists\, C^1_{\mathbb{C}},\, C^2_{\mathbb{C}},\, C^1_{\mathbb{D}},\, C^2_{\mathbb{D}} > 0\ \forall x \in \Omega\ \forall z \in \mathbb{R}\ \forall A \in \mathbb{M}^{d\times d}_{\mathrm{sym}} : \quad \begin{cases} C^1_{\mathbb{C}}|A|^2 \leq \mathbb{C}(x,z)A : A \leq C^2_{\mathbb{C}}|A|^2, \\ C^1_{\mathbb{D}}|A|^2 \leq \mathbb{D}(x,z)A : A \leq C^2_{\mathbb{D}}|A|^2, \end{cases} \tag{2.$(\mathbb{C},\mathbb{D})_1$}$$

where $\mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}})$ denotes the space of linear operators from $\mathbb{M}^{d\times d}_{\mathrm{sym}}$ to $\mathbb{M}^{d\times d}_{\mathrm{sym}}$. Furthermore, we will suppose that for every $x \in \Omega$ the map $z \mapsto \mathbb{C}(x, z)$ is continuously differentiable on $\mathbb{R}$ and fulfills

$$\mathbb{C}'(x,0) = 0 \quad \text{for all } x \in \Omega \quad \text{ and } \quad \forall\, C_Z > 0\ \exists\, L_{\mathbb{C}} > 0\ \forall\, x \in \Omega\ :\ |z| \leq C_Z\ \Rightarrow\ |\mathbb{C}'(x,z)| \leq L_{\mathbb{C}}\,. \tag{2.$(\mathbb{C},\mathbb{D})_2$}$$

Finally, for technical reasons (cf. Remark 3.2 later on) it will be convenient to require that the map $z \mapsto \mathbb{C}(x, z)$ is convex, i.e. for every $A \in \mathbb{M}^{d\times d}_{\mathrm{sym}}$ there holds

$$\begin{aligned} \mathbb{C}(x, (1-\theta)z_1 + \theta z_2)A : A \leq (1-\theta)\mathbb{C}(x, z_1)A : A + \theta\mathbb{C}(x, z_2)A : A \\ \text{for all } \theta \in [0,1],\ x \in \Omega,\ z_1,\ z_2 \in \mathbb{R}. \end{aligned} \tag{2.$(\mathbb{C},\mathbb{D})_3$}$$

It follows from the convexity (2.$(\mathbb{C},\mathbb{D})_3$) that

$$\mathbb{C}'(x, z_1)(z_1 - z_2)A : A \geq \mathbb{C}(x, z_1)A : A - \mathbb{C}(x, z_2)A : A \qquad \text{for all } x \in \Omega,\ z_1,\ z_2 \in \mathbb{R},\ A \in \mathbb{M}^{d\times d}_{\mathrm{sym}}\,, \tag{2.6a}$$

whence $(\mathbb{C}'(x, z_1) - \mathbb{C}'(x, z_2))(z_1 - z_2)A : A \geq 0$. In particular, due to the first of (2.$(\mathbb{C},\mathbb{D})_3$), we find that

$$\mathbb{C}'(x, z)A : A \geq 0 \quad \text{for all } x \in \Omega,\ z \in [0, +\infty),\ A \in \mathbb{M}^{d\times d}_{\mathrm{sym}}\,. \tag{2.6b}$$

Finally, we also suppose that the thermal expansion tensor fulfills

$$\mathbb{E} \in L^\infty(\Omega;\mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}}))\,. \tag{2.$\mathbb{E}$}$$

Observe that with (2.$(\mathbb{C},\mathbb{D},\mathbb{E})$) and (2.$\mathbb{E}$) we encompass in our analysis the case of an anisotropic and inhomogeneous material.

External heat sources. For the volume and boundary heat sources G and g we require

$$G \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)^*), \qquad G \geq 0 \quad \text{a.e. in } Q\,, \tag{2.G_1}$$

$$g \in L^1(0,T;L^2(\partial\Omega)), \qquad g \geq 0 \quad \text{a.e. in } (0,T)\times\partial\Omega\,. \tag{2.G_2}$$

Indeed, the positivity of G and g is crucial for obtaining the strict positivity of the temperature ϑ.

Body force and traction. Our basic conditions on the volume force F and the assigned traction f are

$$F \in L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*), \qquad f \in L^2(0,T;H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)^*), \tag{2.L$_1$}$$

where $H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)$ is the space of functions $\gamma \in H^{1/2}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)$ such that there exists $\tilde\gamma \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$ with $\tilde\gamma = \gamma$ in Γ_{Neu}.

For technical reasons, in order to allow for a non-zero traction f, we will need to additionally require a *uniform safe load* type condition. Observe that this kind of assumption usually occurs in the analysis of perfectly plastic systems. In the present context, it will play a pivotal role in the derivation of the *First a priori estimate* for the approximate solutions constructed by time discretization, cf. the proof of Proposition 4.3 later on as well as [Ros16, Rmk. 4.4] for more detailed comments. Namely, we impose that there exists a function $\varrho : [0,T] \to L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$, with $\varrho \in W^{1,1}(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))$ and $\varrho_{\mathrm{D}} \in L^1(0,T;L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))$, solving for almost all $t \in (0,T)$ the following elliptic problem

$$-\operatorname{div}(\varrho(t)) = F(t) \text{ in } \Omega, \qquad \varrho(t)\nu = f(t) \text{ on } \Gamma_{\mathrm{Neu}}\,. \tag{2.L$_2$}$$

When not explicitly using (2.L$_2$), to shorten notation we will incorporate the volume force F and the traction f into the induced total load, namely the function $\mathcal{L} : (0,T) \to H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*$ given at $t \in (0,T)$ by

$$\langle \mathcal{L}(t), u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} := \langle F(t), u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \langle f(t), u\rangle_{H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)} \qquad \text{for all } u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d), \tag{2.7}$$

which fulfills $\mathcal{L} \in L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*)$ in view of (2.L$_1$).

Dirichlet loading. We will suppose that the hard device w to which the body is subject on Γ_{Dir} is the trace on Γ_{Dir} of a function, denoted by the same symbol, fulfilling

$$w \in L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)) \cap W^{2,1}(0,T;H^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))\,. \tag{2.w}$$

Also condition (2.w) will be used in the proof of Prop. 4.3 ahead; we again refer to [Ros16, Rmk. 4.4] for more comments.

The weak formulation of the momentum balance. The variational formulation of (1.1b), supplemented with the boundary conditions (1.2a), reads for almost all $t \in (0,T)$

$$\begin{aligned}\rho\int_\Omega \ddot u(t) v\,\mathrm{d}x + \int_\Omega \big(\mathbb{D}(z(t))\dot e(t) + \mathbb{C}(z(t))e(t) - \vartheta(t)\mathbb{C}(z(t))\mathbb{E}\big) : E(v)\,\mathrm{d}x = {}& \langle \mathcal{L}(t), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \\ & \forall\, v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\,.\end{aligned} \tag{2.8}$$

We will often use the short-hand notation $-\operatorname{div}_{\mathrm{Dir}}$ for the elliptic operator defined by

$$\langle -\operatorname{div}_{\mathrm{Dir}}(\sigma), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} := \int_\Omega \sigma : E(v)\,\mathrm{d}x \qquad \text{for all } v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\,. \tag{2.9}$$

The plastic dissipation potential. Our assumptions on the multifunction $K : \Omega \times \mathbb{R} \times \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}}$ involve the notions of measurability, lower semicontinuity, and upper semicontinuity for general multifunctions. For such concepts and the related results, we refer, e.g., to [CV77]. Hence, we suppose that

$$\begin{aligned} &K : \Omega \times \mathbb{R} \times \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}} && \text{is measurable w.r.t. the variables } (x,\vartheta,z), \\ &K(x,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}} && \text{is continuous for almost all } x \in \Omega. \end{aligned} \tag{2.K$_1$}$$

Furthermore, we require that

$$\begin{aligned} &K(x,z,\vartheta) \text{ is a convex and compact set in } \mathbb{M}^{d\times d}_{\mathrm{D}} \text{ for all } (z,\vartheta) \in \mathbb{R}\times\mathbb{R}^+ \text{ and for almost all } x \in \Omega, \\ &\exists\, 0 < c_r < C_R \quad \text{for a.a. } x \in \Omega,\ \forall\, z \in \mathbb{R},\ \forall\, \vartheta \in \mathbb{R}^+ : \quad B_{c_r}(0) \subset K(x,z,\vartheta) \subset B_{C_R}(0). \end{aligned} \tag{2.K$_2$}$$

Therefore, the support function associated with the multifunction K, i.e.

$$\mathrm{H} : \Omega \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty) \quad \text{defined by } \mathrm{H}(x,z,\vartheta;\dot p) := \sup_{\pi \in K(x,z,\vartheta)} \pi : \dot p \tag{2.10}$$

is positive, with $\mathrm{H}(x,z,\vartheta;\cdot) : \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty)$ convex and 1-positively homogeneous for almost all $x \in \Omega$ and for all $(z,\vartheta) \in \mathbb{R}\times\mathbb{R}^+$. By the first of (2.K$_1$), the function $\mathrm{H} : \Omega \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty)$ is measurable. Moreover, by the second of (2.K$_1$), in view of [CV77, Thms. II.20, II.21] (cf. also [Sol09, Prop. 2.4]) the function

$$\mathrm{H}(x,\cdot,\cdot;\cdot) : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty) \text{ is (jointly) lower semicontinuous,} \tag{2.11a}$$

for almost all $x \in \Omega$, i.e. H is a *normal integrand*, and

$$\mathrm{H}(x,\cdot,\cdot;\dot p) : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous for every } \dot p \in \mathbb{M}_{\mathrm{D}}^{d\times d}. \tag{2.11b}$$

Finally, it follows from the second of (2.K$_2$) that for almost all $x \in \Omega$ and for all $(z,\vartheta,\dot p) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{M}_{\mathrm{D}}^{d\times d}$ there holds

$$c_r|\dot p| \leq \mathrm{H}(x,z,\vartheta;\dot p) \leq C_R|\dot p|, \tag{2.12a}$$
$$\partial_{\dot p}\mathrm{H}(x,z,\vartheta;\dot p) \subset \partial_{\dot p}\mathrm{H}(x,z,\vartheta;0) = K(x,z,\vartheta) \subset B_{C_R}(0)\,. \tag{2.12b}$$

Finally, we also introduce the *plastic dissipation potential* $\mathcal{H} : L^1(\Omega)\times L^1(\Omega;\mathbb{R}^+)\times L^1(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$ given by

$$\mathcal{H}(z,\vartheta;\dot p) := \int_\Omega \mathrm{H}(x,z(x),\vartheta(x);\dot p(x))\,\mathrm{d}x\,. \tag{2.13}$$

From now on, throughout the paper we will most often omit the x-dependence of the tensors $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$, and of the dissipation density H.

Nonlinearities in the damage flow rule. Along the footsteps of [CL16], we will suppose that

$$W \in \mathrm{C}^2(\mathbb{R}^+) \text{ is bounded from below and fulfills } z^{2d}W(z) \to +\infty \text{ as } z \downarrow 0. \tag{2.W$_1$}$$

The latter coercivity condition will play a key role in the proof that the damage variable z takes values in the feasible interval $[0,1]$. In this way, we will not have to include the indicator term $I_{[0,1]}$ in the potential energy. This will greatly simplify the analysis of the damage flow rule.

Furthermore, we shall require that

$$\exists\,\lambda_W > 0\ \forall\, z \in \mathbb{R}^+ : \quad W''(z) \geq -\lambda_W\,. \tag{2.W$_2$}$$

Observe that (2.W$_2$) is equivalent to imposing that the function $z \mapsto W(z) + \frac{\lambda_W}{2}|z|^2 =: \beta(z)$ is convex. Therefore, we have the *convex/concave* decomposition

$$W(z) = \beta(z) - \frac{\lambda_W}{2}|z|^2 \text{ with } \beta \in \mathrm{C}^2(\mathbb{R}^+), \text{ convex, and fulfilling } z^{2d}\beta(z) \to +\infty \text{ as } z \downarrow 0. \tag{2.14}$$

Let us point out that (2.14) will be expedient in devising the time discretization scheme for the (regularized) thermoviscoelastoplastic damage system, in such a way that its solutions comply with the discrete version of the total energy inequality. We refer to Remark 3.2 ahead for more comments.

Cauchy data. We will supplement the thermoviscoelastoplastic damage system with initial data

$$u_0 \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d),\ \dot u_0 \in L^2(\Omega;\mathbb{R}^d), \tag{2.15a}$$
$$e_0 \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \quad p_0 \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \quad \text{such that } (u_0,e_0,p_0) \in \mathcal{A}(w(0)), \tag{2.15b}$$
$$z_0 \in H^{\mathrm{s}}(\Omega) \text{ with } W(z_0) \in L^1(\Omega) \text{ and } z_0(x) \leq 1 \text{ for every } x \in \Omega\,, \tag{2.15c}$$
$$\begin{aligned} &\vartheta_0 \in L^1(\Omega), \text{ fulfilling the strict positivity condition } \exists\,\vartheta_* > 0 : \inf_{x\in\Omega}\vartheta_0(x) \geq \vartheta_*, \\ &\qquad\qquad \text{and such that } \log(\vartheta_0) \in L^1(\Omega). \end{aligned} \tag{2.15d}$$

In the remainder of this section, we shall suppose that the functions $\mathbb{C},\ldots,W$, the data $G,\ldots,w$, and the initial data $(u_0,\dot u_0,e_0,p_0,z_0,\vartheta_0)$ fulfill the conditions stated in Section 2.1. We will first address the weak solvability of the regularized system in Sec. 2.2, and then turn to examining the non-regularized one in Sec. 2.3. We will then state our existence results for both in Sec. 2.4.

2.2. **Energetics and weak solvability for the (regularized) thermoviscoelastoplastic damage system.** Prior to stating the precise notion of weak solution for the regularized thermoviscoelastoplastic damage system in Definition 2.1 ahead, we formally derive the mechanical & total energy balances associated with systems (1.1, 1.2) and (1.3, 1.4) (in the ensuing discussion, we shall take the parameter $\nu \geq 0$).

The mechanical and total energy balances. The free energy of the system is given by

$$\mathcal{E}(\vartheta,u,e,p,z)=\mathcal{E}(\vartheta,e,z):=\mathcal{F}(\vartheta)+\mathcal{Q}(e,z)+\mathcal{G}(z)\quad\text{with}\quad\begin{cases}\mathcal{F}(\vartheta):=\int_\Omega\vartheta\,\mathrm{d}x,\\ \mathcal{Q}(e,z):=\int_\Omega\frac12\mathbb{C}(z)e{:}e\,\mathrm{d}x\\ \mathcal{G}(z):=\frac12 a_{\mathrm{s}}(z,z)+\int_\Omega W(z)\,\mathrm{d}x\,.\end{cases}\tag{2.16}$$

The total energy balance can be (formally) obtained by testing the momentum balance (1.3b) by $(\dot u-\dot w)$, the damage flow rule (1.3c) by $\dot z$, the plastic flow rule (1.3d) by $\dot p$, the heat equation (1.3e) by 1, adding the resulting relations and integrating in space and over a generic interval $(s,t)\subset(0,T)$.

Indeed, the tests of (1.3b), (1.3c), and (1.3d) yield

$$\begin{aligned}&\frac{\rho}{2}\int_\Omega|\dot u(t)|^2\,\mathrm{d}x+\int_s^t\int_\Omega(\mathbb{D}(z)\dot e+\mathbb{C}(z)e-\vartheta\mathbb{C}(z)\mathbb{E}):E(\dot u)\,\mathrm{d}x\,\mathrm{d}r+\nu\int_s^t a_{\mathrm{s}}(\dot z,\dot z)\,\mathrm{d}r\\ &\quad+\int_s^t\int_\Omega\left(\mathrm{R}(\dot z)+|\dot z|^2\right)\mathrm{d}x\,\mathrm{d}r+\frac12 a_{\mathrm{s}}(z(t),z(t))+\int_\Omega W(z(t))\,\mathrm{d}x+\int_s^t\int_\Omega\tfrac12\dot z\mathbb{C}'(z)e{:}e\,\mathrm{d}x\,\mathrm{d}r\\ &\quad-\int_s^t\int_\Omega\vartheta\dot z\,\mathrm{d}x\,\mathrm{d}r+\int_s^t\int_\Omega\left(\mathrm{H}(z,\vartheta;\dot p)+|\dot p|^2\right)\mathrm{d}x\,\mathrm{d}r\\ &=\frac{\rho}{2}\int_\Omega|\dot u(s)|^2\,\mathrm{d}x+\int_s^t\langle\mathcal{L},\dot u-\dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r+\int_s^t\int_\Omega(\mathbb{D}(z)\dot e+\mathbb{C}(z)e-\vartheta\mathbb{C}(z)\mathbb{E}):E(\dot w)\,\mathrm{d}x\,\mathrm{d}r\\ &\quad+\rho\left(\int_\Omega\dot u(t)\dot w(t)\,\mathrm{d}x-\int_\Omega\dot u_0\dot w(0)\,\mathrm{d}x-\int_s^t\int_\Omega\dot u\ddot w\,\mathrm{d}x\,\mathrm{d}r\right)+\int_s^t\int_\Omega\sigma_{\mathrm{D}}:\dot p\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned}\tag{2.17}$$

Now, taking into account that $E(\dot u)=\dot e+\dot p$ by the kinematical admissibility condition, rearranging some terms one has that

$$\begin{aligned}\int_s^t\int_\Omega(\mathbb{D}(z)\dot e+\mathbb{C}(z)e-\vartheta\mathbb{C}(z)\mathbb{E}):E(\dot u)\,\mathrm{d}x\,\mathrm{d}r&=\int_s^t\int_\Omega(\mathbb{D}(z)\dot e:\dot e+\mathbb{C}(z)\dot e:e)\,\mathrm{d}x\,\mathrm{d}r-\int_s^t\int_\Omega\vartheta\mathbb{C}(z)\mathbb{E}:\dot e\,\mathrm{d}x\,\mathrm{d}r\\ &\quad+\int_s^t\int_\Omega(\mathbb{D}(z)\dot e+\mathbb{C}(z)e-\vartheta\mathbb{C}(z)\mathbb{E}):\dot p\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned}$$

We substitute this in (2.17) and note that $\int_s^t\int_\Omega(\mathbb{D}(z)\dot e+\mathbb{C}(z)e-\vartheta\mathbb{C}(z)\mathbb{E}):\dot p\,\mathrm{d}x\,\mathrm{d}r=\int_s^t\int_\Omega\sigma_{\mathrm{D}}:\dot p\,\mathrm{d}x\,\mathrm{d}r$ since $\dot p\in\mathbb{M}^{d\times d}_{\mathrm{D}}$, so that the last term on the right-hand side of (2.17) cancels out. Furthermore, by the chain rule we have that

$$\int_s^t\int_\Omega\left(\mathbb{C}(z)\dot e:e+\tfrac12\dot z\mathbb{C}'(z)e{:}e\right)\mathrm{d}x\,\mathrm{d}r=\int_\Omega\tfrac12\mathbb{C}(z(t))e(t){:}e(t)\,\mathrm{d}x-\int_\Omega\tfrac12\mathbb{C}(z(s))e(s){:}e(s)\,\mathrm{d}x\,.$$

Collecting all of the above calculations, we obtain the *mechanical energy balance*, featuring the kinetic, dissipated, and mechanical energies

$$\begin{aligned}&\underbrace{\frac{\rho}{2}\int_\Omega|\dot u(t)|^2\,\mathrm{d}x}_{\text{kinetic}}+\underbrace{\int_s^t\int_\Omega\left(\mathbb{D}(z)\dot e:\dot e+\mathrm{R}(\dot z)+|\dot z|^2+\mathrm{H}(z,\vartheta;\dot p)+|\dot p|^2\right)\mathrm{d}x\,\mathrm{d}r+\nu\int_s^t a_{\mathrm{s}}(\dot z,\dot z)\,\mathrm{d}r}_{\text{dissipated}}\\ &+\underbrace{\mathcal{Q}(e(t),z(t))+\mathcal{G}(z(t))}_{\text{mechanical}}\\ &=\frac{\rho}{2}\int_\Omega|\dot u(s)|^2\,\mathrm{d}x+\mathcal{Q}(e(s),z(s))+\mathcal{G}(z(s))+\int_s^t\langle\mathcal{L},\dot u-\dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r+\int_s^t\int_\Omega(\vartheta\mathbb{C}(z)\mathbb{E}:\dot e+\vartheta\dot z)\,\mathrm{d}x\,\mathrm{d}r\\ &\quad+\rho\left(\int_\Omega\dot u(t)\dot w(t)\,\mathrm{d}x-\int_\Omega\dot u(s)\dot w(s)\,\mathrm{d}x-\int_s^t\int_\Omega\dot u\ddot w\,\mathrm{d}x\,\mathrm{d}r\right)+\int_s^t\int_\Omega\sigma:E(\dot w)\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned}\tag{2.18}$$

Let us highlight that (2.18) will also have a significant role for our analysis.

Hence, we sum (2.18) with the heat equation (1.3e) tested by 1 and integrated in time and space. We observe the cancelation of some terms, in particular noting that

$$\bar\nu\int_s^t\int_\Omega a_{\mathrm{s}}(\dot z,\dot z)\cdot1\,\mathrm{d}x\,\mathrm{d}r=\nu\int_s^t a_{\mathrm{s}}(\dot z,\dot z)\,\mathrm{d}r\,.$$

All in all, we conclude the *total energy balance*

$$\begin{aligned} &\frac{\rho}{2}\int_\Omega |\dot u(t)|^2 \,\mathrm{d}x + \mathcal{E}(\vartheta(t),e(t),z(t)) \\ &= \frac{\rho}{2}\int_\Omega |\dot u(s)|^2 \,\mathrm{d}x + \mathcal{E}(\vartheta(s),e(s),z(s)) + \int_s^t \langle \mathcal{L}, \dot u - \dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \int_s^t\int_\Omega G \,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\partial\Omega} g \,\mathrm{d}S\,\mathrm{d}r \\ &\quad + \rho\left(\int_\Omega \dot u(t)\dot w(t)\,\mathrm{d}x - \int_\Omega \dot u_0 \dot w(0)\,\mathrm{d}x - \int_s^t\int_\Omega \dot u \ddot w \,\mathrm{d}x\,\mathrm{d}r\right) + \int_s^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}r\,. \end{aligned} \tag{2.19}$$

Weak energy solutions for the regularized system. With the following definition (where the conditions from Sec. 2.1 are tacitly assumed), we fix the properties of the weak solution concept for the regularized thermoviscoplastic damage system. Let us mention in advance that, in addition to the conventional weak formulations of the momentum balance and of the heat equation (in the latter case, with test functions with suitable regularity and summability properties), we will require the validity of the plastic flow rule *pointwise* (almost everywhere) in $\Omega\times(0,T)$, and that of the damage flow rule as a subdifferential inclusion in $H^{\mathrm{s}}(\Omega)^*$. It will be in fact possible to obtain the latter by exploiting the additional, strongly regularizing term $A_{\mathrm{s}}(\dot z)$ in the damage flow rule (1.3c). In this connection, we now introduce the dissipation potential, defined on $H^{\mathrm{s}}(\Omega)$ and induced by R, namely

$$\mathcal{R} : H^{\mathrm{s}}(\Omega) \to [0,+\infty], \qquad \mathcal{R}(\dot z) := \int_\Omega \mathrm{R}(\dot z)\,\mathrm{d}x, \tag{2.20}$$

and its 'viscous' regularization, where $\mathcal{R}$ is augmented by the (squared) $L^2(\Omega)$-norm

$$\mathcal{R}_2 : H^{\mathrm{s}}(\Omega) \to [0,+\infty], \qquad \mathcal{R}_2(\dot z) := \int_\Omega \mathrm{R}(\dot z)\,\mathrm{d}x + \frac{1}{2}\|\dot z\|^2_{L^2(\Omega)}. \tag{2.21}$$

We will denote by $\partial\mathcal{R} : H^{\mathrm{s}}(\Omega) \rightrightarrows H^{\mathrm{s}}(\Omega)^*$ and $\partial\mathcal{R}_2 : H^{\mathrm{s}}(\Omega) \rightrightarrows H^{\mathrm{s}}(\Omega)^*$ the the convex analysis subdifferentials of $\mathcal{R}$ and $\mathcal{R}_2$, respectively.

Definition 2.1 (Weak energy solutions to the regularized thermoviscoelastoplastic damage system)**.** *Given initial data $(u_0, \dot u_0, e_0, z_0, p_0, \vartheta_0)$ fulfilling* (2.15)*, we call a quintuple (u,e,z,p,ϑ) a* weak energy solution *to the Cauchy problem for system (1.3, 1.4), supplemented with the boundary conditions (1.2a, 1.4), if*

$$u \in H^1(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*), \tag{2.22a}$$

$$e \in H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{2.22b}$$

$$z \in L^\infty(0,T;H^{\mathrm{s}}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{2.22c}$$

$$p \in H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})), \tag{2.22d}$$

$$\vartheta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \tag{2.22e}$$

$$z \in H^1(0,T;H^{\mathrm{s}}(\Omega)), \tag{2.22f}$$

$$\vartheta \in W^{1,1}(0,T;W^{1,\infty}(\Omega)^*) \text{ and } \kappa(\vartheta)\nabla\vartheta \in L^1(Q;\mathbb{R}^d), \tag{2.22g}$$

(u,e,z,p,ϑ) comply with the initial conditions

$$u(0,x) = u_0(x), \quad \dot u(0,x) = \dot u_0(x) \qquad \text{for a.a. } x\in\Omega, \tag{2.23a}$$

$$e(0,x) = e_0(x) \qquad \text{for a.a. } x\in\Omega, \tag{2.23b}$$

$$z(0,x) = z_0(x) \qquad \text{for a.a. } x\in\Omega, \tag{2.23c}$$

$$p(0,x) = p_0(x) \qquad \text{for a.a. } x\in\Omega, \tag{2.23d}$$

$$\vartheta(0) = \vartheta_0 \qquad \text{in } W^{1,\infty}(\Omega)^*, \tag{2.23e}$$

and with

- *the* kinematic admissibility *condition*

$$(u(t,x), e(t,x), p(t,x)) \in \mathcal{A}(w(t,x)) \qquad \text{for a.a. } (t,x)\in Q; \tag{2.24}$$

- *the weak formulation* (2.8) *of the* momentum balance (1.3b)*;*
- *the feasibility and unidirectionality constraints*

$$z\in[0,1] \text{ and } \dot z \le 0 \text{ a.e. in } \Omega\times(0,T) \tag{2.25}$$

and the subdifferential inclusion for damage evolution

$$\partial\mathcal{R}_2(\dot z)+\nu A_{\mathrm{s}}(\dot z)+A_{\mathrm{s}}(z)+W'(z)\ni-\frac12\mathbb{C}'(z)e:e+\vartheta \qquad \text{in } H^{\mathrm{s}}(\Omega)^* \text{ a.e. in } (0,T); \tag{2.26}$$

- *the* plastic flow rule

$$\partial_{\dot p}\mathrm{H}(z,\vartheta;\dot p)+\dot p\ni(\mathbb{D}(z)\dot e+\mathbb{C}(z)e-\vartheta\mathbb{C}(z)\mathbb{E})_{\mathrm{D}} \qquad \text{a.e. in } \Omega\times(0,T), \tag{2.27}$$

- *the* strict positivity *of ϑ:*

$$\exists\,\bar\vartheta>0 \text{ for a.a. } (t,x)\in Q:\quad \vartheta(t,x)>\bar\vartheta; \tag{2.28}$$

and the weak formulation of the heat equation (1.3e) *for every test function $\varphi\in W^{1,\infty}(\Omega)$:*

$$\begin{aligned}&\langle\dot\vartheta,\varphi\rangle_{W^{1,\infty}(\Omega)}+\int_\Omega\kappa(\vartheta)\nabla\vartheta\nabla\varphi\,\mathrm{d}x\\&=\int_\Omega\big(G+\mathbb{D}(z)\dot e{:}\dot e-\vartheta\mathbb{C}(z)\mathbb{E}{:}\dot e+\mathrm{R}(\dot z)+|\dot z|^2+\bar\nu a_{\mathrm{s}}(\dot z,\dot z)+\mathrm{H}(z,\vartheta;\dot p)+|\dot p|^2\big)\,\varphi\,\mathrm{d}x+\int_{\partial\Omega}g\varphi\,\mathrm{d}S\,.\end{aligned} \tag{2.29}$$

Since the 'viscous' contribution $\dot z\mapsto\frac12\|\dot z\|^2_{L^2(\Omega)}$ is differentiable on $H^{\mathrm{s}}(\Omega)$, by the sum rule we have that $\partial\mathcal{R}_2(\dot z)=\partial\mathcal{R}(\dot z)+J(\dot z)$, where $J:L^2(\Omega)\to H^{\mathrm{s}}(\Omega)^*$ is the embedding operator. Nevertheless, as the spaces $(H^{\mathrm{s}}(\Omega),L^2(\Omega),H^{\mathrm{s}}(\Omega)^*)$ form a Hilbert triple, in what follows we will omit the symbol J. Therefore, (2.26) rewrites as

$$\begin{cases}\omega+\dot z+\nu A_{\mathrm{s}}(\dot z)+A_{\mathrm{s}}(z)+W'(z)=-\frac12\mathbb{C}'(z)e:e+\vartheta,\\ \omega\in\partial\mathcal{R}(\dot z)\end{cases} \qquad \text{in } H^{\mathrm{s}}(\Omega)^* \text{ a.e. in } (0,T)\,. \tag{2.30}$$

We refer to the previously developed calculations, leading to the mechanical and total energy balances (2.18) and (2.19), for the proof of the following result.

Lemma 2.2. *Let (u,e,z,p,ϑ) be a* weak energy solution *to (the Cauchy problem for) system (1.3, 1.4). Then, for every $0\le s\le t\le T$ the functions (u,e,z,p,ϑ) comply with the mechanical energy balance* (2.18) *and with*

$$\begin{aligned}&\frac{\rho}{2}\int_\Omega|\dot u(t)|^2\,\mathrm{d}x+\langle\vartheta(t),1\rangle_{W^{1,\infty}(\Omega)}+\mathcal{Q}(e(t),z(t))+\mathcal{G}(z(t))\\&=\frac{\rho}{2}\int_\Omega|\dot u(s)|^2\,\mathrm{d}x+\langle\vartheta(s),1\rangle_{W^{1,\infty}(\Omega)}+\mathcal{Q}(e(s),z(s))+\mathcal{G}(z(s))\\&\quad+\int_s^t\langle\mathcal{L},\dot u-\dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}+\int_s^t\int_\Omega G\,\mathrm{d}x\,\mathrm{d}r+\int_s^t\int_{\partial\Omega}g\,\mathrm{d}S\,\mathrm{d}r\\&\quad+\rho\left(\int_\Omega\dot u(t)\dot w(t)\,\mathrm{d}x-\int_\Omega\dot u_0\dot w(0)\,\mathrm{d}x-\int_s^t\int_\Omega\dot u\ddot w\,\mathrm{d}x\,\mathrm{d}r\right)+\int_s^t\int_\Omega\sigma:E(\dot w)\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned} \tag{2.31}$$

Observe that, since $\vartheta\in L^\infty(0,T;L^1(\Omega))$, there holds $\langle\vartheta(t),1\rangle_{W^{1,\infty}(\Omega)}=\int_\Omega\vartheta(t)\,\mathrm{d}x=\mathcal{F}(\vartheta(t))$ for almost all $t\in(0,T)$ and for $t=0$, and in that case (2.31) coincides with the total energy balance (2.19).

2.3. **Entropic solutions for the thermoviscoelastoplastic damage system.** For the (Cauchy problem associated with the) non-regularized system (1.1, 1.2), we will be able to prove an existence result only for a solution concept containing much less information than that from Def. 2.1. In particular, we will notably weaken the formulations of the heat equation (1.1e), given in terms of an entropy inequality joint with the total energy inequality, and of the damage and plastic flow rules. In order to motivate Definition 2.3 ahead, we develop some preliminary considerations on the weak formulation of the damage and plastic flow rules, and on the entropy inequality. The latter will be formally obtained from the heat equation (1.1e) assuming the strict positivity of the temperature ϑ, which shall be rigorously proved (cf. Prop. 3.3 ahead).

The entropy inequality. It can be formally obtained by multiplying the heat equation (1.1e) by φ/ϑ, with φ a smooth and *positive* test function. Integrating in space and over a generic interval $(s,t) \subset (0,T)$ leads to the identity

$$\begin{aligned} &\int_s^t \int_\Omega \partial_t \log(\vartheta)\varphi \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_\Omega \Big(\kappa(\vartheta)\nabla\log(\vartheta)\nabla\varphi - \kappa(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta)\nabla\vartheta\Big)\,\mathrm{d}x\,\mathrm{d}r \\ &= \int_s^t \int_\Omega \left(G + \mathbb{D}(z)\dot e : \dot e - \vartheta\mathbb{C}(z)\mathbb{E} : \dot e + \mathrm{R}(\dot z) + |\dot z|^2 - \vartheta\dot z + \mathrm{H}(z,\vartheta;\dot p) + \dot p : \dot p\right)\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r \\ &\quad + \int_s^t \int_{\partial\Omega} g\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r \end{aligned} \tag{2.32}$$

The entropy inequality (2.37) ahead is indeed the $\geq$-estimate in (2.32), with *positive* test functions, and where the first integral on the left-hand side is integrated in time.

Weak solvability of the damage and plastic flow rules. Setting $\xi := \vartheta - \frac{1}{2}\mathbb{C}'(z)e : e - \dot z - A_{\mathrm{s}}(z) - W'(z)$, the subdifferential inclusion (1.1c) for damage evolution reformulates as $\xi \in \partial\mathrm{R}(\dot z)$ in $\Omega\times(0,T)$. By the 1-homogeneity of R, the latter is in turn equivalent to the system of inequalities

$$\xi\zeta \leq \mathrm{R}(\zeta) \qquad \text{in } \Omega\times(0,T) \qquad \text{for all } \zeta \in \mathrm{dom}(\mathrm{R}) = (-\infty,0], \tag{2.33a}$$

$$\xi\dot z \geq \mathrm{R}(\dot z) \qquad \text{in } \Omega\times(0,T)\,. \tag{2.33b}$$

Therefore, the damage flow rule (1.1c) could be formulated in terms of the constraints (2.25), of the integrated (in space) version of (2.33a) with test functions $\zeta \in H^{\mathrm{s}}_{-}(\Omega)$, and of the integrated (in space and time) version of (2.33b), which can be interpreted as an *energy-dissipation* inequality. Namely,

- the *one-sided* variational inequality

$$a_{\mathrm{s}}(z(t),\zeta) + \int_\Omega \left(\mathrm{R}(\zeta) + \dot z(t)\zeta + W'(z(t))\zeta + \tfrac{1}{2}\mathbb{C}'(z(t))e : e\zeta - \vartheta(t)\zeta\right)\mathrm{d}x \geq 0 \quad \text{for all } \zeta \in H^{\mathrm{s}}_{-}(\Omega) \text{ for a.a. } t\in(0,T); \tag{2.34a}$$

- the energy-dissipation inequality for damage evolution

$$\int_s^t \int_\Omega \left(\mathrm{R}(\dot z) + |\dot z|^2\right)\mathrm{d}x\,\mathrm{d}r + \nu\int_s^t a_{\mathrm{s}}(\dot z,\dot z)\,\mathrm{d}r + \mathcal{G}(z(t)) \leq \mathcal{G}(z(s)) + \int_s^t \int_\Omega \dot z\left(-\tfrac{1}{2}\mathbb{C}'(z)e{:}e + \vartheta\right)\mathrm{d}x, \tag{2.34b}$$

  on sub-intervals $(s,t) \subset (0,T)$.

To our knowledge this formulation, inspired by the Karush-Kuhn-Tucker conditions, was first proposed in the context of damage in [HK11].

Analogously, the plastic flow rule (1.1d) can be weakly formulated in terms of the system of inequalities

$$\mathcal{H}(z(t),\vartheta(t);\omega) \geq \int_\Omega (\sigma_{\mathrm{D}}(t) - \dot p(t)) : \omega\,\mathrm{d}x \quad \text{for all } \omega \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \text{ for a.a. } t\in(0,T), \tag{2.35a}$$

$$\int_s^t \mathcal{H}(z(r),\vartheta(r);\dot p(r)) + \int_s^t \int_\Omega |\dot p(r)|^2\,\mathrm{d}x\,\mathrm{d}r \leq \int_s^t \int_\Omega \sigma_{\mathrm{D}}(r) : \dot p(r)\,\mathrm{d}x\,\mathrm{d}r \tag{2.35b}$$

on sub-intervals $(s,t) \subset (0,T)$.

Entropic solutions for the non-regularized system. We are now in the position to give our (extremely) weak solution concept for the initial-boundary value problem associated with system (1.1), where, in addition to the entropic formulation of the heat equation, the damage and plastic flow rules shall be formulated only through the Kuhn-Tucker type variational inequalities (2.34a) and (2.35a), combined with the *mechanical energy inequality* (2.36) ahead. Observe that the latter corresponds to the *sum* of the energy-dissipation inequalities for damage and plastic evolution (2.34b) and (2.35b), with the weak formulation of the momentum balance tested by $(\dot u - \dot w)$ and integrated in time.

Definition 2.3 (Entropic solution to the thermoviscoelastoplastic damage system)**.**
Given initial data $(u_0,\dot u_0,e_0,z_0,p_0,\vartheta_0)$ *fulfilling* (2.15)*, we call a quintuple* (u,e,z,p,ϑ) *an* entropic solution *to the Cauchy problem for system (1.1, 1.2), if* (u,e,z,p,ϑ) *enjoy the summability and regularity properties* (2.22a)–(2.22e)*,* (u,e,z,p) *comply with the initial conditions* (2.23a)–(2.23d)*, and if there hold*

- *the* kinematic admissibility *condition* (2.24)*;*
- *the weak* momentum balance (2.8)*;*

- *the feasibility and unidirectionality constraints* (2.25), *joint with the* one-sided variational inequality (2.34a) *for damage evolution;*
- *the* variational inequality (2.35a) *for plastic evolution;*
- *the mechanical energy inequality*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega \left(\mathbb{D}(z)\dot e:\dot e + \mathrm{R}(\dot z) + |\dot z|^2 + \mathrm{H}(z,\vartheta;\dot p) + |\dot p|^2\right)\mathrm{d}x\,\mathrm{d}r + \mathcal{Q}(e(t),z(t)) + \mathcal{G}(z(t))\\
&\leq \frac{\rho}{2}\int_\Omega |\dot u(0)|^2\,\mathrm{d}x + \mathcal{Q}(e(0),z(0)) + \mathcal{G}(z(0))\\
&\quad + \int_0^t \langle \mathcal{L}, \dot u - \dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r + \int_0^t\int_\Omega (\vartheta\mathbb{C}(z)\mathbb{E}:\dot e + \vartheta\dot z)\,\mathrm{d}x\,\mathrm{d}r\\
&\quad + \rho\left(\int_\Omega \dot u(t)\dot w(t)\,\mathrm{d}x - \int_\Omega \dot u(0)\dot w(0)\,\mathrm{d}x - \int_0^t\int_\Omega \dot u\ddot w\,\mathrm{d}x\,\mathrm{d}r\right) + \int_0^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}r,
\end{aligned}
\tag{2.36}
$$

for every $t \in (0,T]$*;*
- *the* strict positivity (2.28) *of* ϑ *and the* entropy inequality

$$
\begin{aligned}
&\int_s^t\int_\Omega \log(\vartheta)\dot\varphi\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_\Omega \left(\kappa(\vartheta)\nabla\log(\vartheta)\nabla\varphi - \kappa(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta)\nabla\vartheta\right)\mathrm{d}x\,\mathrm{d}r\\
&\leq \int_\Omega \log(\vartheta(t))\varphi(t)\,\mathrm{d}x - \int_\Omega \log(\vartheta(s))\varphi(s)\,\mathrm{d}x\\
&\quad - \int_s^t\int_\Omega \left(G + \mathbb{D}(z)\dot e:\dot e - \vartheta\mathbb{C}(z)\mathbb{E}:\dot e + \mathrm{R}(\dot z) + |\dot z|^2 - \vartheta\dot z + \mathrm{H}(z,\vartheta;\dot p) + \dot p:\dot p\right)\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r\\
&\quad - \int_s^t\int_{\partial\Omega} g\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{2.37}
$$

for all φ *in* $L^\infty([0,T];W^{1,\infty}(\Omega))\cap H^1(0,T;L^{6/5}(\Omega))$ *with* $\varphi\geq 0$*, for almost all* $t\in(0,T)$*, almost all* $s\in(0,t)$*, and for* $s=0$ *(with* $\log(\vartheta(0))$ *to be understood as* $\log(\vartheta_0)$*);*
- *the* total energy inequality

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \mathcal{E}(\vartheta(t),e(t),z(t))\\
&\leq \frac{\rho}{2}\int_\Omega |\dot u(0)|^2\,\mathrm{d}x + \mathcal{E}(\vartheta(0),e(0),z(0)) + \int_0^t \langle \mathcal{L}, \dot u - \dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r + \int_0^t\int_\Omega G\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_{\partial\Omega} g\,\mathrm{d}S\,\mathrm{d}r\\
&\quad + \rho\left(\int_\Omega \dot u(t)\dot w(t)\,\mathrm{d}x - \int_\Omega \dot u_0\dot w(0)\,\mathrm{d}x - \int_s^t\int_\Omega \dot u\ddot w\,\mathrm{d}x\,\mathrm{d}r\right) + \int_0^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{2.38}
$$

for almost all $t\in(0,T)$*, almost all* $s\in(0,t)$*, and for* $s=0$ *(with* $\vartheta(0)=\vartheta_0$*).*

Remark 2.4. A few comments on the above definition are in order:

(1) While for weak energy solutions it is possible to a posteriori deduce the validity of mechanical and total energy *balances* via suitable tests, here the upper energy inequalities (2.36) and (2.38) have to be both claimed, as neither of them follows from the other items of the definition.

(2) Observe that, subtracting the weak momentum balance (2.8), (legally) tested by $(\dot u - \dot w)$ and integrated in time, from the mechanical energy inequality (2.36), it would be possible to deduce the *joint energy-dissipation inequality for damage and plastic* evolution

$$
\begin{aligned}
&\int_0^t \left(\mathcal{R}_2(\dot z(r)) + \mathcal{H}(z(r),\vartheta(r);\dot p(r))\right)\mathrm{d}r + \int_0^t\int_\Omega |\dot p(r)|^2\,\mathrm{d}x\,\mathrm{d}r + \mathcal{G}(z(t))\\
&\leq \mathcal{G}(z(0)) + \int_0^t\int_\Omega \dot z\left(-\tfrac{1}{2}\mathbb{C}'(z)e{:}e + \vartheta\right)\mathrm{d}x + \int_0^t\int_\Omega \sigma_{\mathrm{D}}(r):\dot p(r)\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{2.39}
$$

only under the validity of the chain rule

$$
\mathcal{Q}(e(t),z(t)) - \mathcal{Q}(e(0),z(0)) = \int_0^t\int_\Omega \left(\frac{1}{2}\mathbb{C}'(z)\dot z e{:}e + \mathbb{C}(z)\dot e{:}e\right)\mathrm{d}x\,\mathrm{d}r\,. \tag{2.40}
$$

However, (2.40) can be only formally written: indeed, observe that the summability properties $z\in H^1(0,T;L^2(\Omega))$ and $e\in L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))$ do not ensure that $\mathbb{C}'(z)\dot z e{:}e\in L^1(Q)$.

That is why, in Definition 2.3 we only claim the validity of the *full* inequality (2.36), which shall be obtained via lower semicontinuity arguments, by passing to the limit in a discrete version of it.

(3) Combining the information that $\vartheta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ with the strict positivity (2.28) we infer that $\log(\vartheta)$ itself belongs to $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$. The regularity and summability requirements $\varphi \in L^\infty(0,T;W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$ on every admissible test function for the entropy inequality (2.37) in fact guarantee the integrals $\iint \log(\vartheta)\dot\varphi \,\mathrm{d}x\,\mathrm{d}r$ and $\int_\Omega \log(\vartheta)\varphi\,\mathrm{d}x$ are well defined since, in particular, $L^{6/5}(\Omega))$ is the dual of $L^6(\Omega)$, which is the smallest Lebesgue space into which $H^1(\Omega)$ embeds in $d=3$. Furthermore, with (2.37) we are also tacitly claiming the summability properties

$$\kappa(\vartheta)|\nabla \log(\vartheta)|^2\varphi \in L^1(Q), \qquad \kappa(\vartheta)\nabla\log(\vartheta) \in L^1(Q)\,.$$

(4) We refer to [RR15, Rmk. 2.6] for some discussion on the consistency between the entropic (consisting of the entropy and total energy inequalities) and the classical formulations of the heat equation.

2.4. **Existence results.** We start by stating the existence of *entropic* solutions to the (non-regularized) system (1.1, 1.2) under a mild growth condition on the thermal conductivity κ. Observe that, with (2.42) below we will exhibit a precise lower bound for the temperature in terms of quantities related to the material tensors $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$. For shorter notation, in the statement below, as well as in Thm. 2.6 ahead, we shall write $(2.(\mathbb{C},\mathbb{D},\mathbb{E}))$ in place of $(2.(\mathbb{C},\mathbb{D})_1)$–$(2.(\mathbb{C},\mathbb{D})_3)$ and $(2.\mathbb{E})$, (2.G) in place of $(2.\mathrm{G}_1)$, $(2.\mathrm{G}_2)$, and analogously for (2.L), (2.K), and (2.W).

Theorem 2.5. *Let $\nu=0$. Assume $(2.\Omega)$, $(2.(\mathbb{C},\mathbb{D},\mathbb{E}))$, (2.G), (2.L), (2.w), (2.K), and (2.W). In addition, suppose that*

$$\begin{aligned} &\text{the function } \kappa:\mathbb{R}^+\to\mathbb{R}^+ \text{ is continuous and} \\ &\exists\, c_0,\,c_1>0 \quad \exists\,\mu>1 \quad \forall\vartheta\in\mathbb{R}^+: \quad c_0(1+\vartheta^\mu)\le\kappa(\vartheta)\le c_1(1+\vartheta^\mu)\,. \end{aligned} \tag{2.κ_1}$$

Then, for every $(u_0,\dot u_0,e_0,z_0,p_0,\vartheta_0)$ satisfying (2.15) there exists an entropic *solution (u,e,z,p,ϑ) to the Cauchy problem for system (1.1, 1.2) such that, in addition,*

(1) there exists $\zeta_*>0$ such that

$$z(x,t)\in[\zeta_*,1] \qquad \text{for all } (x,t)\in Q; \tag{2.41}$$

(2) ϑ complies with the positivity property

$$\vartheta(x,t)\ge\bar\vartheta := \left(C_*T+\frac{1}{\vartheta_*}\right)^{-1} \qquad \text{for almost all } (x,t)\in Q, \tag{2.42}$$

where $\vartheta_*>0$ is from (2.15d) and $C_* := \frac{\overline{C}^2|\mathbb{E}|^2}{2C_{\mathbb{D}}^1}$ with $\overline{C}=\max_{z\in[0,1]}|\mathbb{C}(z)|$ and $C_{\mathbb{D}}^1>0$ from (2.6);

(3) there holds $\log(\vartheta)\in L^\infty(0,T;L^p(\Omega))$ for all $p\in[1,\infty)$,

$$\begin{aligned} &\kappa(\vartheta)\nabla\log(\vartheta)\in L^{1+\bar\delta}(Q;\mathbb{R}^d) \text{ with } \bar\delta=\frac{\alpha}{\mu} \text{ and } \alpha\in[0\vee(2-\mu),1), \text{ and} \\ &\kappa(\vartheta)\nabla\log(\vartheta)\in L^1(0,T;X) \quad \text{with } X=\begin{cases} L^{2-\eta}(\Omega;\mathbb{R}^d) & \text{for all } \eta\in(0,1] \quad \text{if } d=2, \\ L^{3/2-\eta}(\Omega;\mathbb{R}^d) & \text{for all } \eta\in(0,1/2] \quad \text{if } d=3, \end{cases} \end{aligned} \tag{2.43}$$

(where $0\vee(2-\mu)=\max\{0,(2-\mu)\}$), so that the entropy inequality (2.37) in fact holds for all positive test functions $\varphi\in L^\infty([0,T];W^{1,d+\epsilon}(\Omega))\cap H^1(0,T;L^{6/5}(\Omega))$, for every $\epsilon>0$.

Under a more restrictive growth condition on κ, we are able to establish the existence of *weak energy* solutions for the regularized thermoviscoelastoplastic damage system. Let us also point out that we will be able to enhance the temporal regularity of the temperature and obtain a variational formulation of the heat equation with a wider class of test functions. We will also show the validity of the entropy inequality (2.37): this is a result on its own, as (2.37) cannot be inferred from the weak formulation (2.29) of the heat equation, not even in the enhanced form established with Thm. 2.6.

Theorem 2.6. *Let $\nu>0$. Assume $(2.\Omega)$, $(2.(\mathbb{C},\mathbb{D},\mathbb{E}))$, (2.G), (2.L), (2.w), (2.K), and (2.W). In addition to $(2.\kappa_1)$, suppose that*

$$\begin{cases} \mu\in(1,2) & \text{if } d=2, \\ \mu\in\left(1,\tfrac{5}{3}\right) & \text{if } d=3. \end{cases} \tag{2.κ_2}$$

Then, for every $(u_0, \dot{u}_0, e_0, z_0, p_0, \vartheta_0)$ *satisfying* (2.15) *there exists a* weak energy *solution* (u, e, z, p, ϑ) *to the Cauchy problem for system (1.3, 1.4) such that, in addition,* z *fulfills* (2.41), ϑ *complies with the positivity property* (2.42), *and with*

$$\hat{\kappa}(\vartheta) \in L^{1+\tilde{\delta}}(Q) \text{ for some } \tilde{\delta} \in \left(0, \tfrac{1}{3}\right), \tag{2.44}$$

cf. (5.40) *ahead, where* $\hat{\kappa}$ *is a primitive of* κ. *Therefore,* (2.29) *in fact holds for all test functions* $\varphi \in W^{1,q_{\tilde{\delta}}}(\Omega)$, *with* $q_{\tilde{\delta}} = 1 + \frac{1}{\tilde{\delta}} > 4$. *Ultimately,* ϑ *has the enhanced regularity* $\vartheta \in W^{1,1}(0,T; W^{1,q_{\tilde{\delta}}}(\Omega)^*)$.

Finally, (u, e, z, p, ϑ) *comply with the entropy inequality* (2.37) *associated with the "regularized"* heat equation (1.3e) *(i.e., featuring the additional term* $-\int_s^t \int_\Omega \bar{\nu} a_{\mathrm{s}}(\dot{z}, \dot{z}) \frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r$ *on the right-hand side), for almost all* $t \in (0, T)$ *and almost all* $s \in (0, t)$.

The proof of Theorems 2.5 & 2.6 shall be developed throughout Secs. 3–5 by passing to the limit in a carefully devised time discretization scheme, along the footsteps of the analysis previously developed in [RR15, Ros16].

As we will illustrate in Remark 5.3, it would also be possible to prove the existence of entropic solutions to the (Cauchy problem for the) thermoviscoelastoplastic system (1.1, 1.2) by an alternative method. Namely, we could pass to the limit as the regularization parameter $\nu \downarrow 0$ in the weak energy formulation of the regularized system, featuring a family $(\kappa_\nu)_\nu$ of thermal conductivities fulfilling $(2.\kappa_2)$ and suitably converging as $\nu \downarrow 0$ to a function κ that only complies with $(2.\kappa_1)$. However, to avoid overburdening the paper, we have chosen not to develop this asymptotic analysis.

2.5. **Continuous dependence on the external and initial data in the case of a prescribed temperature profile.** Let us now confine the discussion to the regularized system to the case the temperature profile is prescribed. Namely we consider the the PDE system consisting of the momentum balance (1.3b), of the regularized damage flow rule (1.3c), and of the plastic flow rule (1.3d), with a *given* temperature

$$\Theta \in L^2(Q; \mathbb{R}^+). \tag{2.45}$$

In this context, weak energy solutions fulfill the weak momentum balance (2.8), the subdifferential inclusion for damage evolution (2.26) (i.e.,(2.30)), and the pointwise formulation (2.27) of the plastic flow rule.

We aim at providing a continuous dependence estimate for weak energy solutions in terms of the initial and external data, in particular obtaining their uniqueness. To this end we shall have to introduce a further, quite strong simplification. Namely, we shall assume that the plastic dissipation potential H neither depends on the temperature, nor on the damage variable, and we thus restrict to a functional

$$\mathrm{H} : \Omega \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0, +\infty) \text{ lower semicontinuous, convex, 1-positively homog., and fulfilling (2.12a).} \tag{2.46}$$

Indeed, while the dependence of H on the fixed temperature profile could be kept, proving continuous dependence/uniqueness results in the case of a *state-dependent* dissipation potential is definitely more arduous (cf. e.g. [BKS04], [MR07] for some results in the context of abstract hysteresis/rate-independent systems), and outside the scope of the present contribution. Finally, for technical reasons we will have to strengthen the regularity of $\mathbb{C}$ and $\mathbb{D}$ and require that

$$\begin{aligned} &\forall\, C_Z > 0\ \exists\, \tilde{L}_{\mathbb{C}} > 0\ \forall\, x \in \Omega\ :\ |z_1|, |z_2| \le C_Z\ \Rightarrow\ |\mathbb{C}'(x, z_1) - \mathbb{C}'(x, z_2)| \le \tilde{L}_{\mathbb{C}} |z_1 - z_2|\,, \\ &\forall\, C_Z > 0\ \exists\, L_{\mathbb{D}} > 0\ \forall\, x \in \Omega\ :\ |z_1|, |z_2| \le C_Z\ \Rightarrow\ |\mathbb{D}(x, z_1) - \mathbb{D}(x, z_2)| \le L_{\mathbb{D}} |z_1 - z_2|\,. \end{aligned} \tag{2.$(\mathbb{C},\mathbb{D})_4$}$$

In this context we have the following result, where we now write $(2.(\mathbb{C}, \mathbb{D}, \mathbb{E}))$ as a place-holder for $(2.(\mathbb{C}, \mathbb{D})_1)+ (2.(\mathbb{C}, \mathbb{D})_2) + (2.(\mathbb{C}, \mathbb{D})_3) + (2.(\mathbb{C}, \mathbb{D})_4) + (2.\mathbb{E})$.

Proposition 2.7. *Let* $\nu > 0$. *Assume* $(2.\Omega)$, $(2.(\mathbb{C}, \mathbb{D}, \mathbb{E}))$, (2.W), *and* (2.46). *Let* (F_i, f_i, w_i) *and* $(u_i^0, \dot{u}_i^0, z_i^0, p_i^0)$, *for* $i = 1, 2$, *be two sets of external and initial data for the regularized viscoplastic damage system (1.3b, 1.3c, 1.3d), with boundary conditions* (1.4) *and with given temperature profiles* $\Theta_i \in L^2(Q; \mathbb{R}^+)$, $i = 1, 2$. *Suppose that the data* (F_i, f_i, w_i) *and* $(u_i^0, \dot{u}_i^0, z_i^0, p_i^0)$ *comply with* (2.L), (2.w), *and* (2.15). *Let* (u_i, e_i, z_i, p_i), $i = 1, 2$, *be corresponding weak energy solutions to the initial-boundary value problem for system (1.3b, 1.3c, 1.3d).*

Set $P := \max_{i=1,2}\{\|e_i\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|z_i\|_{L^\infty(\Omega)}\}$*. Then, there exists a positive constant* C_P *depending on* P *such that*

$$
\begin{aligned}
&\|u_1-u_2\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^1(\Omega;\mathbb{R}^d))} + \|e_1-e_2\|_{H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \\
&\quad + \|z_1-z_2\|_{H^1(0,T;H^s(\Omega))} + \|p_1-p_2\|_{H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \\
&\leq C_P\Big(\|u_1^0-u_2^0\|_{H^1(\Omega;\mathbb{R}^{d\times d})} + \|\dot u_1^0-\dot u_2^0\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|e_1^0-e_2^0\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|z_1^0-z_2^0\|_{H^s(\Omega)} \\
&\quad + \|p_1^0-p_2^0\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|F_1-F_2\|_{L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*)} + \|f_1-f_2\|_{L^2(0,T;H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)^*)} \\
&\quad + \|w_1-w_2\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{2,1}(0,T;L^2(\Omega;\mathbb{R}^d))} + \|\Theta_1-\Theta_2\|_{L^2(Q)}\Big)
\end{aligned}
\tag{2.47}
$$

In particular, the initial boundary value problem for the regularized viscoplastic damage system with prescribed temperature admits a unique solution.

3. Time discretization of the thermoviscoelastoplastic damage system(s)

In all of the results of this section, we will tacitly assume all of the conditions listed in Section 2.1.

3.1. **The time discrete scheme.** We will consider a unified discretization scheme for both the regularized thermoviscoelastoplastic damage system (1.3, 1.4) and for system (1.1, 1.2). Therefore, within this section, the parameter ν modulating the viscous regularizing contribution to (1.3c) shall be considered as $\underline{\nu \geq 0}$.

Given a partition of $[0,T]$ with constant time-step $\tau>0$ and nodes $t_\tau^k := k\tau$, $k=0,\ldots,K_\tau$, we approximate the data F, f, G, and g by local means:

$$
F_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} F(s)\,\mathrm{d}s\,, \quad f_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} g(s)\,\mathrm{d}s\,, \quad G_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} G(s)\,\mathrm{d}s\,, \quad g_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} g(s)\,\mathrm{d}s \tag{3.1}
$$

for all $k=1,\ldots,K_\tau$. From the terms F_τ^k and f_τ^k one then defines the elements $\mathcal{L}_\tau^k$, which are the local-mean approximations of $\mathcal{L}$. Hereafter, given elements $(v_\tau^k)_{k=1,\ldots,K_\tau}$, we will use the notation

$$
\mathrm{D}_{k,\tau}(v) := \frac{v_\tau^k - v_\tau^{k-1}}{\tau}, \qquad \mathrm{D}^2_{k,\tau}(v) := \frac{v_\tau^k - 2v_\tau^{k-1} + v_\tau^{k-2}}{\tau^2}. \tag{3.2}
$$

We construct discrete solutions to the (regularized) thermoviscoelastoplastic system by recursively solving an elliptic system, cf. the forthcoming Problem 3.1, where the weak formulation of the discrete heat equation features the function space

$$
X_\theta := \{\theta \in H^1(\Omega)\ :\ \kappa(\theta)\nabla\theta\nabla v \in L^1(\Omega) \text{ for all } v \in H^1(\Omega)\}, \tag{3.3}
$$

and, for $k \in \{1,\ldots,K_\tau\}$, the elliptic operator

$$
A^k : X_\theta \to H^1(\Omega)^* \text{ defined by } \langle A^k(\theta), v\rangle_{H^1(\Omega)} := \int_\Omega \kappa(\theta)\nabla\theta\nabla v\,\mathrm{d}x - \int_{\partial\Omega} g_\tau^k v\,\mathrm{d}S\,. \tag{3.4}
$$

Furthermore, for technical reasons (cf. Remark 3.2 ahead), we shall add the regularizing term $-\tau\mathrm{div}(|e_\tau^k|^{\gamma-2}e_\tau^k)$ to the discrete momentum equation, as well as $\tau|p_\tau^k|^{\gamma-2}p_\tau^k$ to the discrete plastic flow rule, respectively. We will take $\gamma>4$. That is why, we will seek for discrete solutions with $e_\tau^k \in L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ and $p_\tau^k \in L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$, giving $E(u_\tau^k) \in L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ by the kinematic admissibility condition and thus, via Korn's inequality (2.2), $u_\tau^k \in W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$.

Because of these regularizations, it will be necessary to supplement the discrete system with approximate initial data

$$
(e_\tau^0)_\tau \subset L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \qquad \text{such that } \lim_{\tau\downarrow 0}\tau^{1/\gamma}\|e_\tau^0\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} = 0 \text{ and } e_\tau^0 \to e_0 \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \tag{3.5a}
$$

$$
(p_\tau^0)_\tau \subset L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \qquad \text{such that } \lim_{\tau\downarrow 0}\tau^{1/\gamma}\|p_\tau^0\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} = 0 \text{ and } p_\tau^0 \to p_0 \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}). \tag{3.5b}
$$

By consistency with the kinematic admissibility condition at time $t=0$, we will also approximate the initial datum u_0 with a family $(u_\tau^0)_\tau \subset W^{1,\gamma}(\Omega;\mathbb{R}^d)$ such that

$$
(u_\tau^0)_\tau \subset W^{1,\gamma}(\Omega;\mathbb{R}^d) \text{ such that } \lim_{\tau\downarrow 0}\tau^{1/\gamma}\|u_\tau^0\|_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} = 0 \text{ and } u_\tau^0 \to u_0 \text{ in } H^1(\Omega;\mathbb{R}^d). \tag{3.5c}
$$

The data $(u_\tau^0)_\tau$ may be constructed by a perturbation technique. In connection with the regularization of the discrete momentum balance, we will have to approximate the Dirichlet loading w by a family $(w_\tau)_\tau \subset \mathbf{W} \cap W^{1,1}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d))$, where we have used the place-holder $\mathbf{W} := L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)) \cap W^{2,1}(0,T;H^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$. We will require that

$$w_\tau \to w \text{ in } \mathbf{W} \text{ as } \tau \downarrow 0, \qquad \text{as well as} \qquad \exists\, \alpha_w \in \left(0, \frac{1}{\gamma}\right) \text{ s.t. } \sup_{\tau>0} \tau^{\alpha_w} \|E(\dot{w}_\tau)\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \leq C < \infty\,. \tag{3.6}$$

We will then consider the discrete data

$$w_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} w_\tau(s)\,\mathrm{d}s\,.$$

For technical reasons related to the proof of Prop. 3.3 (cf. (3.11) ahead), it will be expedient to replace the argument of the elasticity tensor $\mathbb{C}$ with its positive part. We will proceed in this way in the thermal expansion terms contributing to the momentum balance and to the heat equation. Since we will ultimately prove that the discrete damage solutions are confined to the admissible interval $[0,1]$, cf. (4.9) in Prop. 4.3 ahead, the restriction to the positive part in the argument of $\mathbb{C}$ will "disappear" in the end.

Finally, in the discrete version of the damage flow rule (where we will stay with the notation (2.30)), we will resort to the convex-concave decomposition $W(z) = \beta(z) - \frac{\lambda_W}{2}|z|^2$ from (2.14), with $\lambda_W > 0$ and $\beta \in \mathrm{C}^2(\mathbb{R}^+)$ convex. For shorter notation, in what follows we will use the place-holders

$$\begin{aligned}\mathbf{X} &:= W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d) \times L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \times H^{\mathrm{s}}(\Omega) \times L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \times H^1(\Omega),\\ \mathbf{B} &:= W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d) \times L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \times L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \times H^1(\Omega)\end{aligned}$$

to indicate the state spaces for the solutions to system (3.8) below.

Problem 3.1. *Let $\gamma > 4$. Using the notation (3.2) and starting from data*

$$u_\tau^0 := u_\tau^0, \qquad u_\tau^{-1} := u_0 - \tau \dot{u}_\tau^0, \qquad e_\tau^0 := e_\tau^0, \qquad z_\tau^0 := z_\tau^0, \qquad p_\tau^0 := p_\tau^0, \qquad \vartheta_\tau^0 := \vartheta_0, \tag{3.7}$$

for all $k = 1, \ldots, K_\tau$, given $(u_\tau^{k-1}, e_\tau^{k-1}, z_\tau^{k-1}, p_\tau^{k-1}, \vartheta_\tau^{k-1}) \in \mathbf{X}$, find $z_\tau^k \in H^{\mathrm{s}}(\Omega)$ fulfilling

- *the discrete damage flow rule*

$$\begin{aligned}&\omega_\tau^k + \mathrm{D}_{k,\tau}(z) + \nu A_{\mathrm{s}}(\mathrm{D}_{k,\tau}(z)) + A_{\mathrm{s}}(z_\tau^k) + \beta'(z_\tau^k) - \lambda_W z_\tau^{k-1}\\ &\qquad = -\tfrac{1}{2}\mathbb{C}'(z_\tau^k)e_\tau^{k-1} : e_\tau^{k-1} + \vartheta_\tau^{k-1} \quad \text{in } H^{\mathrm{s}}(\Omega)^* \quad \text{with } \omega_\tau^k \in \partial\mathrm{R}(\mathrm{D}_{k,\tau}(z)).\end{aligned} \tag{3.8a}$$

Given $(u_\tau^{k-1}, e_\tau^{k-1}, z_\tau^{k-1}, p_\tau^{k-1}, \vartheta_\tau^{k-1}) \in \mathbf{X}$ and $z_\tau^k \in H^{\mathrm{s}}(\Omega)$, find $(u_\tau^k, e_\tau^k, p_\tau^k, \vartheta_\tau^k) \in \mathbf{B}$ fulfilling

- *the kinematic admissibility $(u_\tau^k, e_\tau^k, p_\tau^k) \in \mathcal{A}(w_\tau^k)$ (in the sense of (2.5));*
- *the discrete momentum balance*

$$\rho\int_\Omega \mathrm{D}^2_{k,\tau}(u) v\,\mathrm{d}x + \int_\Omega \sigma_\tau^k : E(v)\,\mathrm{d}x = \langle \mathcal{L}_\tau^k, v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \quad \text{for all } v \in W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d), \tag{3.8b}$$

where we have used the place-holder $\sigma_\tau^k := \mathbb{D}(z_\tau^k)\mathrm{D}_{k,\tau}(e) + \mathbb{C}(z_\tau^k)e_\tau^k + \tau|e_\tau^k|^{\gamma-2}e_\tau^k - \vartheta_\tau^k\mathbb{C}((z_\tau^k)^+)\mathbb{E}$;
- *the discrete plastic flow rule*

$$\zeta_\tau^k + \mathrm{D}_{k,\tau}(p) + \tau|p_\tau^k|^{\gamma-2}p_\tau^k = (\sigma_\tau^k)_{\mathrm{D}} \quad \text{with } \zeta_\tau^k \in \partial_{\dot{p}}\mathrm{H}(z_\tau^k, \vartheta_\tau^{k-1}; \mathrm{D}_{k,\tau}(p)), \qquad \text{a.e. in } \Omega; \tag{3.8c}$$

- *$\vartheta_\tau^k \in X_\theta$ and the discrete heat equation*

$$\begin{aligned}\mathrm{D}_{k,\tau}(\vartheta) + A^k(\vartheta_\tau^k) = {} & G_\tau^k + \mathbb{D}(z_\tau^k)\mathrm{D}_{k,\tau}(e) : \mathrm{D}_{k,\tau}(e) - \vartheta_\tau^k\mathbb{C}((z_\tau^k)^+)\mathbb{E} : \mathrm{D}_{k,\tau}(e)\\ &+ \mathrm{R}(\mathrm{D}_{k,\tau}(z)) + |\mathrm{D}_{k,\tau}(z)|^2 + \bar{\nu} a_{\mathrm{s}}(\mathrm{D}_{k,\tau}(z), \mathrm{D}_{k,\tau}(z)) - \vartheta_\tau^{k-1}\mathrm{D}_{k,\tau}(z)\\ &+ \mathrm{H}(z_\tau^k, \vartheta_\tau^{k-1}; \mathrm{D}_{k,\tau}(p)) + |\mathrm{D}_{k,\tau}(p)|^2 \quad \text{in } H^1(\Omega)^*;\end{aligned} \tag{3.8d}$$

Remark 3.2. The discrete system (3.8) has been designed in such a way as to ensure the validity of a discrete total energy inequality, cf. (4.7) ahead. The latter will be proved by exploiting suitable cancellations of the various terms contributing to (3.8), as well as the convex-concave decomposition (2.14) of W in the discrete damage flow rule (3.8a), where the contribution $\beta'(z_\tau^k)$ from the convex part has been kept implicit, while the term $-\lambda_W z_\tau^{k-1}$ related to the concave part is explicit. The convexity of $z \mapsto \mathbb{C}(z)$ will also be a key ingredient in the proof of (4.7), cf. the calculations in the proof of Lemma 3.6.

Several terms in (3.8) have been kept *implicit*, not only towards the validity of (4.7), but also in view of the strict positivity property (3.9) ahead for the discrete temperature. Our proof of (3.9) requires that ϑ_τ^k is implicit in the thermal expansion coupling term on the r.h.s. of the discrete heat equation, cf. the calculations leading to (3.10), which also rely on the truncation of the elasticity tensor $\mathbb{C}$. Therefore it has to be implicit in the corresponding terms in the discrete momentum balance and in the discrete plastic flow rule, which cannot be thus decoupled one from another. Instead, still compatibly with the proof of (3.9), the discrete damage flow rule is decoupled from the other equations. This will greatly simplify the proof of existence of solutions to (3.8).

Because of this implicit character of the thermoviscoplastic subsystem, in order to prove the existence of solutions (to an approximate version of it), we will have to resort to a (nonconstructive) existence result, of fixed point type, for elliptic systems involving pseudo-monotone coercive operators. The regularizing terms $-\tau\mathrm{div}(|e_\tau^k|^{\gamma-2}e_\tau^k)$ and $\tau|p_\tau^k|^{\gamma-2}p_\tau^k$, guaranteeing enhanced integrability properties, have to ensure the coercivity of the pseudo-monotone operator underlying the (approximate versions of the) discrete momentum balance, plastic flow rule, and heat equation. These terms will vanish in the limit as $\tau\downarrow 0$. Let us point out that, thanks to them, the right-hand side of the discrete heat equation is indeed an element in $H^1(\Omega)^*$. Thus all the calculations performed in the proof of Proposition 4.3 and involving suitable tests of the discrete heat equation will be rigorous.

Proposition 3.3 (Existence of discrete solutions)**.** *Under the growth condition* $(2.\kappa_1)$ *on* κ*, Problem 3.1 admits a solution* $\{(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k)\}_{k=1}^{K_\tau}$. *Furthermore, any solution to Problem 3.1 fulfills*

$$\vartheta_\tau^k \geq \bar\vartheta > 0 \qquad \textit{for all } k=1,\ldots,K_\tau \textit{ with } \bar\vartheta \textit{ from } (2.42)\,. \tag{3.9}$$

3.2. **Proof of Proposition 3.3.** First of all, let us point out that the strict positivity (3.9) ensues by the very same argument developed in the proof of [RR15, Lemma 4.4]. We shortly recapitulate it: From the discrete heat equation (3.8d) we deduce the variational inequality

$$\int_\Omega \mathrm{D}_{k,\tau}(\vartheta) w \,\mathrm{d}x + \int_\Omega \kappa(\vartheta_\tau^k)\nabla\vartheta_\tau^k\nabla w\,\mathrm{d}x \geq -C_* \int_\Omega (\vartheta_\tau^k)^2 w\,\mathrm{d}x \qquad \text{for all } w\in H^1_+(\Omega)\,. \tag{3.10}$$

with $C_* = \frac{\overline{C}^2|\mathbb{E}|^2}{2C_{\mathbb{D}}^1}$. To establish (3.10), we estimate

$$\begin{aligned} \mathbb{D}(z_\tau^k)\mathrm{D}_{k,\tau}(e):\mathrm{D}_{k,\tau}(e) - \vartheta_\tau^k\mathbb{C}((z_\tau^k)^+)\mathbb{E}:\mathrm{D}_{k,\tau}(e) &\geq C_{\mathbb{D}}^1|\mathrm{D}_{k,\tau}(e)|^2 - \overline{C}|\mathbb{E}||\vartheta_\tau^k||\mathrm{D}_{k,\tau}(e)| \\ &\geq \frac{C_{\mathbb{D}}^1}{2}|\mathrm{D}_{k,\tau}(e)|^2 - C_*|\vartheta_\tau^k|^2\,. \end{aligned} \tag{3.11}$$

For this, we have used: (1) the coercivity of $\mathbb{D}$ from $(2.(\mathbb{C},\mathbb{D})_1)$, (2) the fact that $0\leq (z_\tau^k)^+\leq 1$ in Ω (due to $z_\tau^k \leq z_\tau^{k-1}\leq .... \leq z_0\leq 1$ by the unidirectionality enforced by the dissipation potential R and condition (2.15c) on z_0), so that $|\mathbb{C}((z_\tau^k)^+)|\leq \overline{C} := \max_{z\in[0,1]}|\mathbb{C}(z)|$, and (3) Young's inequality. We also take into account the positivity of all the other terms on the right-hand side of (3.8d): in particular, note that

$$-\,\vartheta_\tau^{k-1}\mathrm{D}_{k,\tau}(z)\geq 0 \qquad \text{a.e. in } \Omega \tag{3.12}$$

as we may suppose by induction that $\vartheta_\tau^{k-1}>0$ a.e. in Ω, whereas $\mathrm{D}_{k,\tau}(z)\leq 0$ by unidirectionality. In view of (3.10), we may compare the elements $(\vartheta_\tau^k)_{k=1}^{K_\tau}$ with the decreasing sequence $(\theta^k)_{k=1}^{K_\tau}$, recursively defined by

$$\frac{\theta^k-\theta^{k-1}}{\tau} = -C_*(\theta^k)^2, \qquad \theta^0 := \vartheta_* > 0\,,$$

and conclude, on the one hand, that $\vartheta_\tau^k\geq\theta^k$ for all $k=1,\ldots,K_\tau$. On the other hand, the argument from [RR15, Lemma 4.4] yields that $\theta^k\geq\ldots\geq\theta^{K_\tau}\geq\bar\vartheta$ for all $k=1,\ldots,K_\tau$, which leads to (3.9).

In proving the *existence* of solutions to Problem 3.1, we will perform the following steps:

Step 1: While the discrete damage flow rule will be solved by a variational argument, we will approximate the discrete thermoviscoplastic subsystem by truncating the heat conductivity coefficient in the elliptic operator. On the one hand, this will allow us to apply the existence result from [Rou05], based on the theory of pseudomonotone operators, for proving the existence of solutions. On the other hand, due to this truncation we will no longer be able to exploit the growth of κ in order to handle the thermal expansion term on the r.h.s. of (3.8d). Therefore, in this term we will replace ϑ_τ^k by its truncation $\mathcal{T}_M(\vartheta_\tau^k)$. We shall do the same for the corresponding coupling terms in the discrete momentum balance and plastic flow rule.

Step 2**:** Existence of solutions to the approximate discrete thermoviscoplastic subsystem.
Step 3**:** A priori estimates on the discrete solutions, uniform with respect to the truncation parameter M
Step 4**:** Limit passage as $M \to \infty$.

In completing Steps 2–4 we will most often have to adapt analogous arguments developed in [RR15, Ros16], to which we will refer for all details.

Step 1**: The approximate discrete system** will feature the truncation operator

$$\mathcal{T}_M : \mathbb{R} \to \mathbb{R}, \qquad \mathcal{T}_M(r) := \begin{cases} -M & \text{if } r < -M, \\ r & \text{if } |r| \leq M, \\ M & \text{if } r > M, \end{cases} \tag{3.13}$$

where we suppose that $M \in \mathbb{N} \setminus \{0\}$. We thus introduce the truncated heat conductivity

$$\kappa_M(r) := \kappa(\mathcal{T}_M(r)) := \begin{cases} \kappa(-M) & \text{if } r < -M, \\ \kappa(r) & \text{if } |r| \leq M, \\ \kappa(M) & \text{if } r > M, \end{cases} \tag{3.14}$$

and, accordingly, the approximate elliptic operator

$$A_M^k : H^1(\Omega) \to H^1(\Omega)^* \text{ defined by } \quad \langle A_M^k(\theta), v \rangle_{H^1(\Omega)} := \int_\Omega \kappa_M(\theta) \nabla\theta \nabla v \, \mathrm{d}x - \int_{\partial\Omega} g_\tau^k v \, \mathrm{d}S. \tag{3.15}$$

We are now in the position to introduce the approximate discrete system (3.16). For notational simplicity, we will omit to indicate the dependence of the solution quintuple on the index M.

Problem 3.4. *Let $\gamma > 4$. Starting from the discrete Cauchy data* (3.7)*, for all $k = 1, \ldots, K_\tau$, given $(u_\tau^{k-1}, e_\tau^{k-1}, z_\tau^{k-1}, p_\tau^{k-1}, \vartheta_\tau^{k-1}) \in \mathbf{X}$, find $z_\tau^k \in H^{\mathrm{s}}(\Omega)$ fulfilling the discrete damage flow rule* (3.8a)*. Given $(u_\tau^{k-1}, e_\tau^{k-1}, z_\tau^{k-1}, p_\tau^{k-1}, \vartheta_\tau^{k-1}) \in \mathbf{X}$ and $z_\tau^k \in H^{\mathrm{s}}(\Omega)$, find $(u_\tau^k, e_\tau^k, p_\tau^k, \vartheta_\tau^k) \in \mathbf{B}$ fulfilling*

- *the kinematic admissibility $(u_\tau^k, e_\tau^k, p_\tau^k) \in \mathcal{A}(w_\tau^k)$;*
- *the approximate discrete momentum balance*

$$\rho \int_\Omega \mathrm{D}_{k,\tau}^2(u) v \, \mathrm{d}x + \int_\Omega \sigma_{M,\tau}^k : E(v) \, \mathrm{d}x = \langle \mathcal{L}_\tau^k, v \rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \quad \text{for all } v \in W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d), \tag{3.16a}$$

with the place-holder $\sigma_{M,\tau}^k := \mathbb{D}(z_\tau^k)\mathrm{D}_{k,\tau}(e) + \mathbb{C}(z_\tau^k)e_\tau^k + \tau|e_\tau^k|^{\gamma-2}e_\tau^k - \mathcal{T}_M(\vartheta_\tau^k)\mathbb{C}((z_\tau^k)^+)\mathbb{E}$;
- *the approximate discrete plastic flow rule*

$$\zeta_\tau^k + \mathrm{D}_{k,\tau}(p) + \tau|p_\tau^k|^{\gamma-2}p_\tau^k \ni (\sigma_{M,\tau}^k)_{\mathrm{D}}, \quad \text{with } \zeta_\tau^k \in \partial_{\dot p}\mathrm{H}(z_\tau^k, \vartheta_\tau^{k-1}; \mathrm{D}_{k,\tau}(p)) \qquad \text{a.e. in } \Omega; \tag{3.16b}$$

- *the approximate discrete heat equation*

$$\begin{aligned} \mathrm{D}_{k,\tau}(\vartheta) + A_M^k(\vartheta_\tau^k) = {} & G_\tau^k + \mathbb{D}(z_\tau^k)\mathrm{D}_{k,\tau}(e) : \mathrm{D}_{k,\tau}(e) - \mathcal{T}_M(\vartheta_\tau^k)\mathbb{C}((z_\tau^k)^+)\mathbb{E} : \mathrm{D}_{k,\tau}(e) \\ & + \mathrm{R}(\mathrm{D}_{k,\tau}(z)) + |\mathrm{D}_{k,\tau}(z)|^2 + \bar\nu a_{\mathrm{s}}(\mathrm{D}_{k,\tau}(z), \mathrm{D}_{k,\tau}(z)) - \vartheta_\tau^{k-1}\mathrm{D}_{k,\tau}(z) \\ & + \mathrm{H}(z_\tau^k, \vartheta_\tau^{k-1}; \mathrm{D}_{k,\tau}(p)) + |\mathrm{D}_{k,\tau}(p)|^2 \quad \text{in } H^1(\Omega)^*. \end{aligned} \tag{3.16c}$$

Step 2**: Existence of solutions to system** (3.16)**:** we have the following result.

Lemma 3.5. *Under the growth condition* (2.κ_1)*, there exists $\bar\tau > 0$ such that for $0 < \tau < \bar\tau$ and for every $k = 1, \ldots, K_\tau$ there exists a solution $(u_\tau^k, e_\tau^k, z_\tau^k, p_\tau^k, \vartheta_\tau^k) \in \mathbf{X}$ to system* (3.16)*. Furthermore, for any solution $(u_\tau^k, e_\tau^k, z_\tau^k, p_\tau^k, \vartheta_\tau^k)$ the function ϑ_τ^k complies with the positivity property* (3.9)*.*

Proof. The positivity property (3.9) (note that the constant providing the lower bound for ϑ_τ^k is independent of the truncation parameter M), follows from the analogue of estimate (3.10), with the same comparison argument. Let us now address the existence of solutions.

First of all, we find a solution z_τ^k to (3.8a) via the minimum problem

$$\begin{aligned} & \mathrm{Min}_{z \in H^{\mathrm{s}}(\Omega)} \left(\int_\Omega \mathrm{R}(z - z_\tau^{k-1}) \, \mathrm{d}x + \frac{1}{2\tau} \int_\Omega |z - z_\tau^{k-1}|^2 \, \mathrm{d}x + \frac{\nu}{2\tau} a_{\mathrm{s}}(z - z_\tau^{k-1}, z - z_\tau^{k-1}) + \mathscr{F}_k(z) \right), \\ & \quad \text{with } \mathscr{F}_k(z) := \frac{1}{2} a_{\mathrm{s}}(z,z) + \int_\Omega \left(\beta(z) - \lambda_W z_\tau^{k-1} z + \tfrac{1}{2}\mathbb{C}(z) e_\tau^{k-1} : e_\tau^{k-1} - \vartheta_\tau^{k-1} z \right) \mathrm{d}x \,. \end{aligned} \tag{3.17}$$

With the direct method in the calculus of variations, it is easy to check that (3.17), whose Euler-Lagrange equation is (3.8a), has a solution z_τ^k.

Let us now briefly address the solvability of the approximate discrete thermoviscoplastic system (3.16), for fixed $k \in \{1, \ldots, K_\tau\}$ with z_τ^k given. To this end, we reformulate system (3.16) in the form

$$\partial\Psi_k(u_\tau^k - w_\tau^k, p_\tau^k, \vartheta_\tau^k) + \mathscr{A}_k(u_\tau^k - w_\tau^k, p_\tau^k, \vartheta_\tau^k) \ni \mathscr{B}_k \qquad \text{in } \mathbf{V}^*, \tag{3.18}$$

with the dissipation potential $\Psi_k : \mathbf{V} \to [0, +\infty)$ defined by $\Psi_k(\tilde{u}, p, \vartheta) = \Psi_k(p) := \mathrm{H}(z_\tau^k, \vartheta_\tau^{k-1}; p - p_\tau^{k-1})$, the space $\mathbf{V} := W^{1,\gamma}_{\mathrm{Dir}}(\Omega; \mathbb{R}^d) \times L^\gamma(\Omega; \mathbb{M}^{d\times d}_{\mathrm{D}}) \times H^1(\Omega)$, and the operator $\mathscr{A}_k : \mathbf{V} \to \mathbf{V}^*$ given component-wise by

$$\begin{aligned}
\mathscr{A}_k^1(\tilde{u}, p, \vartheta) := \rho(\tilde{u} - w_\tau^k) - \mathrm{div}_{\mathrm{Dir}}\Big(&\tau\mathbb{D}(z_\tau^k)\left(E(\tilde{u} + w_\tau^k) - p\right) + \tau^2\mathbb{C}(z_\tau^k)\left(E(\tilde{u} + w_\tau^k) - p\right) \\
&+ \tau^3 \left|E(\tilde{u} + w_\tau^k) - p\right|^{\gamma-2}\left(E(\tilde{u} + w_\tau^k) - p\right) \\
&- \tau^2 \mathfrak{T}_M(\vartheta)\mathbb{C}((z_\tau^k)^+)\mathbb{E}\Big),
\end{aligned} \tag{3.19a}$$

$$\begin{aligned}
\mathscr{A}_k^2(\tilde{u}, p, \vartheta) := p + \tau^2|p|^{\gamma-2}p - \Big(&\mathbb{D}(z_\tau^k)\left(E(\tilde{u} + w_\tau^k) - p\right) + \tau\mathbb{C}(z_\tau^k)\left(E(\tilde{u} + w_\tau^k) - p\right) \\
&+ \tau^2\left|E(\tilde{u} + w_\tau^k) - p\right|^{\gamma-2}\left(E(\tilde{u} + w_\tau^k) - p\right) - \tau\mathfrak{T}_M(\vartheta)\mathbb{C}((z_\tau^k)^+)\mathbb{E}\Big)_{\mathrm{D}},
\end{aligned} \tag{3.19b}$$

$$\begin{aligned}
\mathscr{A}_k^3(\tilde{u}, p, \vartheta) := \vartheta + A_M^k(\vartheta) &- \frac{1}{\tau}\mathbb{D}(z_\tau^k)\left(E(\tilde{u} + w_\tau^k) - p\right) : \left(E(\tilde{u} + w_\tau^k) - p\right) \\
&- \frac{2}{\tau}\mathbb{D}(z_\tau^k)\left(E(\tilde{u} + w_\tau^k) - p\right) : e_\tau^{k-1} + \mathfrak{T}_M(\vartheta)\mathbb{C}((z_\tau^k)^+)\mathbb{E}\left(E(\tilde{u} + w_\tau^k) - p - e_\tau^{k-1}\right) \\
&- \mathrm{H}(z_\tau^k, \vartheta_\tau^{k-1}; p - p_\tau^{k-1}) - \frac{1}{\tau}|p|^2 - \frac{2}{\tau}p : p_\tau^{k-1},
\end{aligned} \tag{3.19c}$$

where $-\mathrm{div}_{\mathrm{Dir}}$ is defined by (2.9), while the vector $\mathscr{B}_k \in \mathbf{B}^*$ on the right-hand side of (3.18) has components

$$\mathscr{B}_k^1 := \mathcal{L}_\tau^k + 2\rho u_\tau^{k-1} - \rho u_\tau^{k-1} - \mathrm{div}_{\mathrm{Dir}}(\tau\mathbb{D}(z_\tau^k)e_\tau^{k-1}), \tag{3.20a}$$

$$\mathscr{B}_k^2 := p_\tau^{k-1} - (\mathbb{D}(z_\tau^k)e_\tau^{k-1})_{\mathrm{D}}, \tag{3.20b}$$

$$\begin{aligned}
\mathscr{B}_k^3 :=& G_\tau^k + \frac{1}{\tau}\mathbb{D}(z_\tau^{k-1})e_\tau^{k-1} : e_\tau^{k-1} \\
&+ \mathrm{R}(z_\tau^k - z_\tau^{k-1}) + \frac{1}{\tau}|z_\tau^k - z_\tau^{k-1}|^2 + \frac{\bar{\nu}}{\tau}a_{\mathrm{s}}(z_\tau^k - z_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}) + \frac{1}{\tau}|p_\tau^{k-1}|^2\,.
\end{aligned} \tag{3.20c}$$

The arguments therefore reduce to proving the existence of a solution to the abstract subdifferential inclusion (3.18). This follows from the very arguments developed in the proof of [Ros16, Lemma 3.4], to which we refer for all details. Let us only mention that the latter proof is in turn based on the existence result [Rou13a, Cor. 5.17] for elliptic systems featuring coercive pseudomonotone operators. In order to check coercivity of the operator $\mathscr{A}_k : \mathbf{V} \to \mathbf{V}^*$, we show that

$$\begin{aligned}
&\exists\, c,\, C > 0\ \forall\, (\tilde{u}, p, \vartheta) \in \mathbf{V}\ : \\
&\langle \mathscr{A}_k(\tilde{u}, p, \vartheta), (\tilde{u}, p, \vartheta)\rangle_{\mathbf{V}} = \langle \mathscr{A}_k^1(\tilde{u}, p, \vartheta), \tilde{u}\rangle_{W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \int_\Omega \mathscr{A}_k^2(\tilde{u}, p, \vartheta) : p\,\mathrm{d}x + \langle \mathscr{A}_k^3(\tilde{u}, p, \vartheta), \vartheta\rangle_{H^1(\Omega)} \\
&\qquad\qquad \geq c\|(\tilde{u}, p, \vartheta)\|_{\mathbf{V}}^2 - C\,.
\end{aligned}$$

The calculations for [Ros16, Lemma 3.4] show the key role of the regularizing terms $-\tau\mathrm{div}(|e_\tau^k|^{\gamma-2}e_\tau^k)$ and $\tau|p_\tau^k|^{\gamma-2}p_\tau^k$, added to the discrete momentum equation and plastic flow rule, for proving the above estimate. □

Step 3: A priori estimates on the solutions of system (3.16)**:** in order to pass to the limit as $M \to +\infty$ in Problem 3.4, for fixed $k \in \{1, \ldots, K_\tau\}$ and z_τ^k solving the discrete damage flow rule (3.8a), we need to establish suitable a priori estimates on a family $(u_{M,\tau}^k, e_{M,\tau}^k, p_{M,\tau}^k, \vartheta_{M,\tau}^k)_M$ of solutions to system (3.16). Along the footsteps of [Ros16], we shall derive them from a discrete version of the total energy inequality (2.38), cf. (3.22) below, featuring the discrete free energy (recall that the energy functional $\mathcal{E}$ was defined in (2.16))

$$\mathcal{E}_\tau(\vartheta, e, z, p) := \mathcal{E}(\vartheta, e, z) + \frac{\tau}{\gamma}\int_\Omega (|e|^\gamma + |p|^\gamma)\,\mathrm{d}x\,. \tag{3.21}$$

Lemma 3.6. *Assume* (2.κ_1). *Let* $k \in \{1,\dots,K_\tau\}$ *and* $\tau \in (0,\bar\tau)$ *be fixed. Let* $z_\tau^k \in H^{\mathrm{s}}(\Omega)$ *solve* (3.8a)*. Then, the solution quadruple* $(\vartheta^k_{M,\tau}, u^k_{M,\tau}, e^k_{M,\tau}, p^k_{M,\tau})$ *to* (3.16) *satisfies*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega \left|\frac{u^k_{M,\tau}-u_\tau^{k-1}}{\tau}\right|^2 \mathrm{d}x + \mathcal{E}_\tau(\vartheta^k_{M,\tau}, e^k_{M,\tau}, z^k_\tau, p^k_{M,\tau})\\
&\le \frac{\rho}{2}\int_\Omega \left|\frac{u^{k-1}_\tau-u_\tau^{k-2}}{\tau}\right|^2 \mathrm{d}x + \mathcal{E}_\tau(\vartheta^{k-1}_\tau, e^{k-1}_\tau, z^{k-1}_\tau, p^{k-1}_\tau) + \tau\int_\Omega G^k_\tau\,\mathrm{d}x + \tau\int_{\partial\Omega} g^k_\tau\,\mathrm{d}x\\
&\quad + \tau\,\langle \mathcal{L}^k_\tau, \frac{u^k_{M,\tau}-u^{k-1}_\tau}{\tau} - \mathrm{D}_{k,\tau}(w)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \tau\int_\Omega \sigma^k_{M,\tau} : E(\mathrm{D}_{k,\tau}(w))\,\mathrm{d}x\\
&\quad + \rho\int_\Omega \left(\frac{u^k_{M,\tau}-u^{k-1}_\tau}{\tau} - \mathrm{D}_{k-1,\tau}(u)\right)\mathrm{D}_{k,\tau}(w)\,\mathrm{d}x\,.
\end{aligned}
\tag{3.22}
$$

Moreover, there exists a constant $C>0$ *such that for all* $M>0$

$$\|\vartheta^k_{M,\tau}\|_{L^1(\Omega)} + \|u^k_{M,\tau}\|_{L^2(\Omega;\mathbb{R}^d)} + \|e^k_{M,\tau}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \le C, \tag{3.23a}$$

$$\tau^{1/\gamma}\|u^k_{M,\tau}\|_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} + \tau^{1/\gamma}\|e^k_{M,\tau}\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \tau^{1/\gamma}\|p^k_{M,\tau}\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \le C, \tag{3.23b}$$

$$\|\vartheta^k_{M,\tau}\|_{H^1(\Omega)} \le C, \tag{3.23c}$$

$$\|\zeta^k_\tau\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \le C, \tag{3.23d}$$

where ζ^k_τ *is a selection in* $\partial_{\dot p}\mathrm{H}(z^k_\tau, \vartheta^{k-1}_\tau; p^k_{M,\tau} - p^{k-1}_\tau)$ *fulfilling* (3.16b).

Proof. Inequality (3.22) follows by testing (3.8a) by $z^k_\tau - z^{k-1}_\tau$, (3.16a) by $(u^k_{M,\tau} - w^k_\tau) - (u^{k-1}_\tau - w^{k-1}_\tau)$, (3.16b) by $p^k_{M,\tau} - p^{k-1}_{M,\tau}$, and by multiplying (3.16c) by τ and integrating it in space. We add the resulting relations. We develop the following estimates for the terms arising from the test of the momentum balance (3.16a)

$$
\begin{aligned}
&\frac{\rho}{\tau^2}\int_\Omega \left(u^k_{M,\tau} - u^{k-1}_\tau - (u^{k-1}_\tau - u^{k-2}_\tau)\right)(u^k_{M,\tau} - u^{k-1}_\tau)\,\mathrm{d}x\\
&\ge \frac{\rho}{2}\left\|\frac{u^k_{M,\tau}-u^{k-1}_\tau}{\tau}\right\|^2_{L^2(\Omega)} - \frac{\rho}{2}\left\|\frac{u^{k-1}_\tau-u^{k-2}_\tau}{\tau}\right\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.24a}
$$

$$
\begin{aligned}
&\int_\Omega \mathbb{D}(z^k_\tau)\left(\frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau}\right) : E(u^k_{M,\tau} - u^{k-1}_\tau)\,\mathrm{d}x\\
&= \tau\int_\Omega \mathbb{D}(z^k_\tau)\frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau} : \frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau}\,\mathrm{d}x + \int_\Omega \mathbb{D}(z^k_\tau)\frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau} : (p^k_{M,\tau} - p^{k-1}_\tau)\,\mathrm{d}x,
\end{aligned}
\tag{3.24b}
$$

$$
\begin{aligned}
&\int_\Omega \mathbb{C}(z^k_\tau)e^k_{M,\tau} : E(u^k_{M,\tau} - u^{k-1}_\tau)\,\mathrm{d}x\\
&= \int_\Omega \mathbb{C}(z^k_\tau)e^k_{M,\tau} : (e^k_{M,\tau} - e^{k-1}_\tau) + \mathbb{C}(z^k_\tau)e^k_{M,\tau} : (p^k_{M,\tau} - p^{k-1}_\tau)\,\mathrm{d}x\\
&\ge \int_\Omega \left(\tfrac12\mathbb{C}(z^k_\tau)e^k_{M,\tau} : e^k_{M,\tau} - \tfrac12\mathbb{C}(z^k_\tau)e^{k-1}_\tau : e^{k-1}_\tau + \mathbb{C}(z^k_\tau)e^k_{M,\tau} : (p^k_{M,\tau} - p^{k-1}_\tau)\right)\mathrm{d}x\,,
\end{aligned}
\tag{3.24c}
$$

$$
\begin{aligned}
&\int_\Omega |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau} : E(u^k_{M,\tau} - u^{k-1}_\tau)\,\mathrm{d}x\\
&= \int_\Omega |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau} : (e^k_{M,\tau} - e^{k-1}_\tau)\,\mathrm{d}x + \int_\Omega |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau} : (p^k_{M,\tau} - p^{k-1}_\tau)\,\mathrm{d}x\\
&\ge \int_\Omega \left(\tfrac1\gamma|e^k_{M,\tau}|^\gamma - \tfrac1\gamma|e^{k-1}_\tau|^\gamma + |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau} : (p^k_{M,\tau} - p^{k-1}_\tau)\right)\mathrm{d}x\,,
\end{aligned}
\tag{3.24d}
$$

where have exploited the kinematic admissibility condition in (3.24b) and (3.24c), as well as elementary convexity inequalities to establish estimates (3.24a), (3.24c), and (3.24d). As for the terms arising from the test of the discrete damage flow rule (3.16b), we observe that

$$\langle \omega_\tau^k, z_\tau^k - z_\tau^{k-1}\rangle_{H^s(\Omega)} \geq \tau \mathcal{R}(\mathrm{D}_{k,\tau}(z)), \tag{3.25a}$$

$$\begin{aligned} & \int_\Omega \mathrm{D}_{k,\tau}(z)(z_\tau^k - z_\tau^{k-1})\,\mathrm{d}x + \nu\,\langle A_{\mathrm{s}}(\mathrm{D}_{k,\tau}(z)), z_\tau^k{-}z_\tau^{k-1}\rangle_{H^s(\Omega)} \\ & = \frac{1}{\tau}|z_\tau^k{-}z_\tau^{k-1}|^2 + \frac{\nu}{\tau} a_{\mathrm{s}}(z_\tau^k{-}z_\tau^{k-1}, z_\tau^k{-}z_\tau^{k-1}), \end{aligned} \tag{3.25b}$$

$$\langle A_{\mathrm{s}}(z_\tau^k), z_\tau^k{-}z_\tau^{k-1}\rangle_{H^s(\Omega)} \geq \frac{1}{2} a_{\mathrm{s}}(z_\tau^k, z_\tau^k) - \frac{1}{2} a_{\mathrm{s}}(z_\tau^{k-1}, z_\tau^{k-1}), \tag{3.25c}$$

$$\begin{aligned} & \int_\Omega \left(\beta'(z_\tau^k)(z_\tau^k{-}z_\tau^{k-1}) {-} \lambda_W z_\tau^{k-1}(z_\tau^k{-}z_\tau^{k-1})\right)\mathrm{d}x \\ & \geq \int_\Omega \left(\beta(z_\tau^k){-}\beta(z_\tau^{k-1})\right)\mathrm{d}x - \lambda_W \int_\Omega \left(\tfrac{1}{2}|z_\tau^k|^2 {-} \tfrac{1}{2}|z_\tau^{k-1}|^2\right)\mathrm{d}x = \int_\Omega \left(W'(z_\tau^k){-}W'(z_\tau^{k-1})\right)\mathrm{d}x\,, \end{aligned} \tag{3.25d}$$

$$\int_\Omega \tfrac{1}{2}\mathbb{C}'(z_\tau^k)(z_\tau^k{-}z_\tau^{k-1}) e_\tau^{k-1}{:}e_\tau^{k-1}\,\mathrm{d}x \geq \int_\Omega \tfrac{1}{2}\mathbb{C}(z_\tau^k) e_\tau^{k-1}{:}e_\tau^{k-1}\,\mathrm{d}x - \int_\Omega \tfrac{1}{2}\mathbb{C}(z_\tau^{k-1}) e_\tau^{k-1}{:}e_\tau^{k-1}\,\mathrm{d}x\,, \tag{3.25e}$$

again by convexity arguments, also relying on $(2.(\mathbb{C},\mathbb{D})_3)$. In particular, in (3.25d) we have exploited the convex-concave decomposition (2.14) of W. We now observe several cancellations. Indeed, the two terms on the right-hand side of (3.24b) respectively cancel with the second term on the r.h.s. of the heat equation (3.16c), multiplied by τ, and with the analogous term deriving from (3.16b), tested by $p_{M,\tau}^k - p_\tau^{k-1}$. As for (3.24c), the second term on its r.h.s. cancels out with the first summand on the r.h.s. of (3.25e); the third term on its r.h.s. cancels with the one deriving from (3.16b), and so does the third term on the r.h.s. of (3.24d). Also the terms on the r.h.s. of (3.25a) and (3.25b) cancel out with the right-hand side of (3.16c) multiplied by τ: in particular, observe that

$$\nu\,\langle A_{\mathrm{s}}(\mathrm{D}_{k,\tau}(z)), z_\tau^k{-}z_\tau^{k-1}\rangle_{H^s(\Omega)} = \bar{\nu}\tau \int_\Omega a_{\mathrm{s}}(\mathrm{D}_{k,\tau}(z), \mathrm{D}_{k,\tau}(z))\,\mathrm{d}x\,.$$

In fact, with the exception of τG_τ^k, all the terms on the r.h.s. of (3.16c) cancel out. Thus, straightforward calculations lead to (3.22).

We refer to the proofs of [Ros16, Lemma 3.5] and [RR15, Lemma 4.4] for all the detailed calculations leading to estimates (3.23): let us only mention that one has to first test the discrete heat equation (3.8d) by $\mathcal{T}_M(\vartheta_{M,\tau}^k)$ and then by $\vartheta_{M,\tau}^k$, and exploit the growth properties of κ. □

Step 4: Limit passage as $M \to +\infty$: We again refer to [Ros16] (cf. Lemma 3.6 therein) for the proof of the following result on the limit passage in the approximate discrete thermoviscoplastic subsystem (3.16). Let us only mention here that the strong convergences (3.26a)–(3.26c) arise from standard lim sup-arguments, developed by testing the discrete momentum balance by $u_{M,\tau}^k - w_\tau^k$ and the discrete plastic flow rule by $p_{M,\tau}^k$.

Lemma 3.7. *Let $k \in \{1,\ldots,K_\tau\}$ and $\tau \in (0,\bar{\tau})$ be fixed. Under the growth condition $(2.\kappa_1)$, there exist a (not relabeled) subsequence of $(u_{M,\tau}^k, e_{M,\tau}^k, p_{M,\tau}^k, \vartheta_{M,\tau}^k)_M$ and of $(\zeta_{\tau,M}^k)_M$, a quadruple $(u_\tau^k, e_\tau^k, p_\tau^k, \vartheta_\tau^k) \in \mathbf{B}$, with $\vartheta_\tau^k \in X_\theta$, and a function $\zeta_\tau^k \in L^\infty(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$, such that the following convergences hold as $M \to \infty$*

$$u_{M,\tau}^k \to u_\tau^k \qquad \text{in } W_{\mathrm{Dir}}^{1,\gamma}(\Omega;\mathbb{R}^d), \tag{3.26a}$$

$$e_{M,\tau}^k \to e_\tau^k \qquad \text{in } L^\gamma(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d}), \tag{3.26b}$$

$$p_{M,\tau}^k \to p_\tau^k \qquad \text{in } L^\gamma(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d}), \tag{3.26c}$$

$$\zeta_{\tau,M}^k \overset{*}{\rightharpoonup} \zeta_\tau^k \qquad \text{in } L^\infty(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d}), \tag{3.26d}$$

$$\vartheta_{M,\tau}^k \rightharpoonup \vartheta_\tau^k \qquad \text{in } H^1(\Omega), \tag{3.26e}$$

and the quintuple $(u_\tau^k, e_\tau^k, p_\tau^k, \zeta_\tau^k, \vartheta_\tau^k)$ fulfills system (3.8b, 3.8c, 3.8d).

With this result, we conclude the proof of Proposition 3.3. ■

4. A priori estimates

Again, throughout this section we will tacitly assume all of the conditions listed in Section 2.1. We start by fixing some notation for the approximate solutions.

Notation 4.1 (Interpolants)**.** For a given Banach space X and a K_τ-tuple $(\mathfrak{h}_\tau^k)_{k=0}^{K_\tau} \subset X$, we introduce the left-continuous and right-continuous piecewise constant, and the piecewise linear interpolants

$$\left.\begin{array}{lll}\overline{\mathfrak{h}}_\tau : (0,T] \to X & \text{defined by} & \overline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^k, \\ \underline{\mathfrak{h}}_\tau : (0,T] \to X & \text{defined by} & \underline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^{k-1}, \\ \mathfrak{h}_\tau : (0,T] \to X & \text{defined by} & \mathfrak{h}_\tau(t) := \frac{t-t_\tau^{k-1}}{\tau}\mathfrak{h}_\tau^k + \frac{t_\tau^k - t}{\tau}\mathfrak{h}_\tau^{k-1}\end{array}\right\} \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k],$$

setting $\overline{\mathfrak{h}}_\tau(0) = \underline{\mathfrak{h}}_\tau(0) = \mathfrak{h}_\tau(0) := \mathfrak{h}_\tau^0$. We also introduce the piecewise linear interpolant of the values $\{\mathrm{D}_{k,\tau}(\mathfrak{h}) = \frac{\mathfrak{h}_\tau^k - \mathfrak{h}_\tau^{k-1})}{\tau}\}_{k=1}^{K_\tau}$ (i.e. the values taken by the piecewise constant function $\dot{\mathfrak{h}}_\tau$), viz.

$$\widehat{\mathfrak{h}}_\tau : (0,T) \to X \;\; \text{defined by} \;\; \widehat{\mathfrak{h}}_\tau(t) := \frac{(t - t_\tau^{k-1})}{\tau}\mathrm{D}_{k,\tau}(\mathfrak{h}) + \frac{(t_\tau^k - t)}{\tau}\mathrm{D}_{k-1,\tau}(\mathfrak{h}) \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k].$$

Note that $\partial_t \widehat{\mathfrak{h}}_\tau(t) = \mathrm{D}^2_{k,\tau}(\mathfrak{h})$ for $t \in (t_\tau^{k-1}, t_\tau^k]$.

Furthermore, we denote by $\bar{\mathsf{t}}_\tau$ and by $\underline{\mathsf{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition, i.e. $\bar{\mathsf{t}}_\tau(t) := t_\tau^k$ if $t_\tau^{k-1} < t \le t_\tau^k$ and $\underline{\mathsf{t}}_\tau(t) := t_\tau^{k-1}$ if $t_\tau^{k-1} \le t < t_\tau^k$. Clearly, for every $t \in [0,T]$ we have $\bar{\mathsf{t}}_\tau(t) \downarrow t$ and $\underline{\mathsf{t}}_\tau(t) \uparrow t$ as $\tau \to 0$.

It follows from conditions (2.G$_1$), (2.G$_2$), and (2.L$_1$) that the piecewise constant interpolants $(\overline{G}_\tau)_\tau$, $(\overline{g}_\tau)_\tau$, and $(\overline{\mathcal{L}}_\tau)_\tau$ of the values G_τ^k, g_τ^k, and $\mathcal{L}_\tau^k$, cf. (3.1), fulfill as $\tau \downarrow 0$

$$\overline{G}_\tau \to G \text{ in } L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)^*), \tag{4.1a}$$

$$\overline{g}_\tau \to g \text{ in } L^1(0,T;L^2(\partial\Omega)), \tag{4.1b}$$

$$\overline{\mathcal{L}}_\tau \to \mathcal{L} \text{ in } L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*). \tag{4.1c}$$

Furthermore, it follows from (2.w) and (3.6) that

$$\begin{aligned} &\overline{w}_\tau \to w \quad \text{in } L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)), \qquad w_\tau \to w \quad \text{in } W^{1,p}(0,T;H^1(\Omega;\mathbb{R}^d)) \text{ for all } 1 \le p < \infty, \\ &\qquad\qquad \widehat{w}_\tau \to w \quad \text{in } W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^d) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^d)), \\ &\qquad\qquad \sup_{\tau>0} \tau^{\alpha_w} \|E(\dot{w}_\tau)\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \le C < \infty \text{ with } \alpha_w \in (0,\tfrac{1}{\gamma})\,. \end{aligned} \tag{4.1d}$$

We now reformulate system (3.8) in terms of the interpolants of the discrete solutions $(u_\tau^k, e_\tau^k, z_\tau^k, p_\tau^k, \vartheta_\tau^k)_{k=1}^{K_\tau}$. Therefore, we have for almost all $t \in (0,T)$

$$\rho \int_\Omega \partial_t \widehat{u}_\tau(t) v \,\mathrm{d}x + \int_\Omega \overline{\sigma}_\tau(t) : E(v)\,\mathrm{d}x = \langle \overline{\mathcal{L}}_\tau(t), v \rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \quad \text{for all } v \in W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d) \tag{4.2a}$$

with the notation $\overline{\sigma}_\tau(t) := \mathbb{D}(\overline{z}_\tau(t))\dot{e}_\tau(t) + \mathbb{C}(\overline{z}_\tau(t))\overline{e}_\tau(t) + \tau|\overline{e}_\tau(t)|^{\gamma-2}\overline{e}_\tau(t) - \overline{\vartheta}_\tau(t)\overline{\mathbb{C}}((\overline{z}_\tau(t))^+)\mathbb{E}$,

$$\overline{\omega}_\tau(t) + \dot{z}_\tau(t) + \nu A_{\mathrm{s}}(\dot{z}_\tau(t)) + A_{\mathrm{s}}(\overline{z}_\tau(t)) + \beta'(\overline{z}_\tau(t)) - \lambda_W \underline{z}_\tau(t) = -\frac{1}{2}\mathbb{C}'(\overline{z}_\tau)\underline{e}_\tau : \underline{e}_\tau + \underline{\vartheta}_\tau \quad \text{in } H^{\mathrm{s}}(\Omega)^* \tag{4.2b}$$

with $\overline{\omega}_\tau(t) \in \partial\mathcal{R}(\dot{z}_\tau(t))$ in $H^{\mathrm{s}}(\Omega)^*$,

$$\overline{\zeta}_\tau(t) + \dot{p}_\tau(t) + \tau|\overline{p}_\tau(t)|^{\gamma-2}\overline{p}_\tau(t) \ni (\overline{\sigma}_\tau(t))_{\mathrm{D}} \quad \text{a.e. in } \Omega \tag{4.2c}$$

with $\overline{\zeta}_\tau(t) \in \partial_{\dot{p}}\mathrm{H}(\overline{z}_\tau(t), \underline{\vartheta}_\tau(t); \dot{p}_\tau(t))$ a.e. in Ω,

$$\begin{aligned} \partial_t \vartheta_\tau(t) + \mathcal{A}^{\frac{\bar{\mathsf{t}}_\tau(t)}{\tau}}(\overline{\vartheta}_\tau(t)) = \overline{G}_\tau(t) &+ \mathbb{D}(\overline{z}_\tau(t))\dot{e}_\tau(t) : \dot{e}_\tau(t) - \overline{\vartheta}_\tau(t)\mathbb{C}((\overline{z}_\tau(t))^+)\mathbb{E} : \dot{e}_\tau(t) \\ &+ \mathrm{R}(\dot{z}_\tau(t)) + |\dot{z}_\tau(t)|^2 + \bar{\nu} a_{\mathrm{s}}(\dot{z}_\tau(t), \dot{z}_\tau(t)) - \underline{\vartheta}_\tau(t)\dot{z}_\tau(t) \\ &+ \mathrm{H}(\overline{z}_\tau(t), \underline{\vartheta}_\tau(t); \dot{p}_\tau(t)) + |\dot{p}_\tau(t)|^2 \quad \text{in } H^1(\Omega)^*, \end{aligned} \tag{4.2d}$$

cf. (3.4) for the definition of the operator $A^{\frac{\bar{\mathsf{t}}_\tau(t)}{\tau}}$.

Our next result collects the discrete versions of the entropy, total energy, and mechanical energy inequalities satisfied by the approximate solutions. In order to state the discrete entropy inequality (4.6) below, we need to construct suitable approximations of the test functions for the limiting entropy inequality (2.37). Following

[Ros16], we will in fact approximate *positive* test functions φ with $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega))\cap H^1(0,T;L^{6/5}(\Omega))$, which is a slightly stronger temporal regularity than that required by Def. 2.3. We set

$$\varphi_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \varphi(s)\,\mathrm{d}s \qquad \text{for } k=1,\ldots,K_\tau, \tag{4.3}$$

and consider the piecewise constant and linear interpolants $\overline{\varphi}_\tau$ and φ_τ of the values $(\varphi_\tau^k)_{k=1}^{K_\tau}$. We can show that

$$\overline{\varphi}_\tau \to \varphi \quad \text{in } L^\infty(0,T;W^{1,\infty}(\Omega)) \text{ and } \quad \partial_t\varphi_\tau \to \partial_t\varphi \quad \text{in } L^2(0,T;L^{6/5}(\Omega)). \tag{4.4}$$

We are now in the position to give the discrete versions of the entropy and energy inequalities in which we will pass to the limit to conclude the existence of *weak energy* solutions to the regularized thermoviscoplastic system. The discrete total energy inequality (4.7) follows by adding up (3.21). The proof of the other two inequalities can be obtained by trivially adapting the arguments for [RR15, Prop. 4.8] and [Ros16, Lemma 4.2]. Their proof relies on the following *discrete by-part integration* formula, which we recall for later use, holding for all K_τ-uples $\{\mathfrak{h}_\tau^k\}_{k=0}^{K_\tau} \subset B$, $\{v_\tau^k\}_{k=0}^{K_\tau} \subset B^*$ in a given Banach space B:

$$\sum_{k=1}^{K_\tau} \tau\,\langle v_\tau^k, \mathrm{D}_{k,\tau}(\mathfrak{h})\rangle_B = \langle v_\tau^{K_\tau}, \mathfrak{h}_\tau^{K_\tau}\rangle_B - \langle v_\tau^0, \mathfrak{h}_\tau^0\rangle_B - \sum_{k=1}^{K_\tau} \tau\,\langle \mathrm{D}_{k,\tau}(v), \mathfrak{h}_\tau^{k-1}\rangle_B\,. \tag{4.5}$$

Lemma 4.2 (Discrete entropy, mechanical, and total energy inequalities)**.** *The interpolants of the discrete solutions* $(u_\tau^k, e_\tau^k, z_\tau^k, p_\tau^k, \vartheta_\tau^k)_{k=1}^{K_\tau}$ *to Problem 3.1 fulfill*

- *the* discrete *entropy inequality*

$$\begin{aligned}
&\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \log(\underline{\vartheta}_\tau(r))\dot{\varphi}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r - \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \kappa(\overline{\vartheta}_\tau(r))\nabla\log(\overline{\vartheta}_\tau(r))\nabla\overline{\varphi}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\le \int_\Omega \log(\overline{\vartheta}_\tau(t))\overline{\varphi}_\tau(t)\,\mathrm{d}x - \int_\Omega \log(\overline{\vartheta}_\tau(s))\overline{\varphi}_\tau(s)\,\mathrm{d}x - \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \kappa(\overline{\vartheta}_\tau(r))\frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\nabla\log(\overline{\vartheta}_\tau(r))\nabla\overline{\vartheta}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \Big(\overline{G}_\tau(r) + \mathbb{D}(\overline{z}_\tau(r))\dot{e}_\tau(r){:}\dot{e}_\tau(r) - \overline{\vartheta}_\tau(r)\mathbb{C}((\overline{z}_\tau(r))^+)\mathbb{E}{:}\dot{e}_\tau(r) + \mathrm{H}(\overline{z}_\tau(r),\underline{\vartheta}_\tau(r);\dot{p}_\tau(r)) \\
&\qquad\qquad + |\dot{p}_\tau(r)|^2 + \mathrm{R}(\dot{z}_\tau(r)) + |\dot{z}_\tau(r)|^2 + \bar{\nu}a_{\mathrm{s}}(\dot{z}_\tau(r),\dot{z}_\tau(r)) - \underline{\vartheta}_\tau(r)\dot{z}_\tau(r)\Big)\frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_{\partial\Omega} \overline{g}_\tau(r)\frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\,\mathrm{d}S\,\mathrm{d}r
\end{aligned} \tag{4.6}$$

for all $0 \le s \le t \le T$ *and for all* $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega))\cap H^1(0,T;L^{6/5}(\Omega))$ *with* $\varphi \ge 0$*;*

- *the* discrete *total energy inequality for all* $0 \le s \le t \le T$*, viz.*

$$\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\widehat{u}_\tau(\bar{\mathsf{t}}_\tau(t))|^2\,\mathrm{d}x + \mathcal{E}_\tau(\overline{\vartheta}_\tau(t),\overline{e}_\tau(t),\overline{z}_\tau(t),\overline{p}_\tau(t)) \\
&\le \frac{\rho}{2}\int_\Omega |\widehat{u}_\tau(\bar{\mathsf{t}}_\tau(s))|^2\,\mathrm{d}x + \mathcal{E}_\tau(\overline{\vartheta}_\tau(s),\overline{e}_\tau(s),\overline{z}_\tau(s),\overline{p}_\tau(s)) + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \langle\overline{\mathcal{L}}_\tau(r), \dot{u}_\tau(r) - \dot{w}_\tau(r)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r \\
&\quad + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\left(\int_\Omega \overline{G}_\tau\,\mathrm{d}x + \int_{\partial\Omega}\overline{g}_\tau\,\mathrm{d}S\right)\mathrm{d}r \\
&\quad + \rho\left(\int_\Omega \dot{u}_\tau(t)\dot{w}_\tau(t)\,\mathrm{d}x - \int_\Omega \dot{u}_\tau(s)\dot{w}_\tau(s)\,\mathrm{d}x - \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \dot{u}_\tau(r-\tau)\partial_t\widehat{w}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r\right) \\
&\quad + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \overline{\sigma}_\tau(r){:}E(\dot{w}_\tau(r))\,\mathrm{d}x\,\mathrm{d}r
\end{aligned} \tag{4.7}$$

with the discrete total energy functional $\mathcal{E}_\tau$ *from* (3.21)*;*

- *the* discrete *mechanical energy inequality for all* $0 \le s \le t \le T$*, featuring the energy functionals* $\mathcal{Q}$ *and* $\mathcal{G}$ *from* (2.16)

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\widehat{u}_\tau(\bar{\mathsf{t}}_\tau(t))|^2\,\mathrm{d}x \\
&+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \left(\mathbb{D}(\overline{z}_\tau(r))\dot{e}_\tau(r){:}\dot{e}_\tau(r)+\mathrm{R}(\dot{z}_\tau(r))+|\dot{z}_\tau(r)|^2+\mathrm{H}(\overline{z}_\tau(r),\underline{\vartheta}_\tau(r);\dot{p}_\tau(r))+|\dot{p}_\tau(r)|^2\right)\mathrm{d}x\,\mathrm{d}r \\
&+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\nu a_{\mathrm{s}}(\dot{z}_\tau(r),\dot{z}_\tau(r))\,\mathrm{d}r+\mathcal{Q}(\overline{e}_\tau(t),\overline{z}_\tau(t))+\mathcal{G}(\overline{z}_\tau(t))+\frac{\tau}{\gamma}\int_\Omega\left(|\overline{e}_\tau(t)|^\gamma+|\overline{p}_\tau(t)|^\gamma\right)\mathrm{d}x \\
&\le\frac{\rho}{2}\int_\Omega|\widehat{u}_\tau(\bar{\mathsf{t}}_\tau(s))|^2\,\mathrm{d}x+\mathcal{Q}(\overline{e}_\tau(s),\overline{z}_\tau(s))+\mathcal{G}(\overline{z}_\tau(s))+\frac{\tau}{\gamma}\int_\Omega\left(|\overline{e}_\tau(s)|^\gamma+|\overline{p}_\tau(s)|^\gamma\right)\mathrm{d}x \\
&+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\langle\overline{\mathcal{L}}_\tau(r),\dot{u}_\tau(r){-}\dot{w}_\tau(r)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega\overline{\vartheta}_\tau(r)\mathbb{C}(\overline{z}_\tau(r))\mathbb{E}{:}\dot{e}_\tau\,\mathrm{d}x\,\mathrm{d}r \\
&+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega\underline{\vartheta}_\tau(r)\dot{z}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r+\rho\int_\Omega\dot{u}_\tau(t)\dot{w}_\tau(t)\,\mathrm{d}x-\rho\int_\Omega\dot{u}_\tau(s)\dot{w}_\tau(s)\,\mathrm{d}x \\
&-\rho\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega\dot{u}_\tau(r{-}\tau)\partial_t\widehat{w}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega\overline{\sigma}_\tau(r){:}E(\dot{w}_\tau(r))\,\mathrm{d}x\,\mathrm{d}r\,.
\end{aligned}
\tag{4.8}
$$

In fact, observe that $\widehat{u}_\tau(\bar{\mathsf{t}}_\tau(t))=\dot{u}_\tau(t)$ at almost all $t\in(0,T)$ (i.e., at $t\in[0,T]\setminus\{t^1_\tau,\ldots,t^{K_\tau}_\tau\}$, where $\dot{u}_\tau(t)$ is defined. Using the interpolant $\widehat{u}_\tau$ in the discrete energy inequalities (4.7) and (4.8) allows us to write them at *every* couple of time instants $0\le s\le t\le T$.

The main result of this section collects all the a priori estimates on the approximate solutions.

Proposition 4.3. *Assume* (2.κ_1)*. Then, there exists a constant* $\zeta_*>0$ *such that*

$$\overline{z}_\tau(x,t),\ z_\tau(x,t)\in[\zeta_*,1]\quad\text{for every } (x,t)\in\Omega\times[0,T] \text{ and all } \tau>0 \tag{4.9}$$

and there exists a constant $S>0$ *such that for all* $\tau>0$ *the following estimates hold*

$$\|\overline{u}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))}\le S, \tag{4.10a}$$
$$\|u_\tau\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))}+\|u_\tau\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))}\le S, \tag{4.10b}$$
$$\|\widehat{u}_\tau\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))}+\|\widehat{u}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))}+\|\widehat{u}_\tau\|_{W^{1,\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*)}\le S, \tag{4.10c}$$
$$\|\overline{e}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))}\le S, \tag{4.10d}$$
$$\|e_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))}\le S, \tag{4.10e}$$
$$\tau^{1/\gamma}\|\overline{e}_\tau\|_{L^\infty(0,T;L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))}\le S, \tag{4.10f}$$
$$\|\overline{z}_\tau\|_{L^\infty(0,T;H^s(\Omega))}\le S, \tag{4.10g}$$
$$\|z_\tau\|_{H^1(0,T;L^2(\Omega)}\le S, \tag{4.10h}$$
$$\|z_\tau\|_{H^1(0,T;H^s(\Omega))}\le S\qquad\text{if } \nu>0, \tag{4.10i}$$
$$\|\overline{\omega}_\tau\|_{L^2(0,T;H^s(\Omega)^*)}\le S\qquad\text{if } \nu>0, \tag{4.10j}$$
$$\|\overline{p}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))}\le S, \tag{4.10k}$$
$$\|p_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))}\le S, \tag{4.10l}$$
$$\tau^{1/\gamma}\|\overline{p}_\tau\|_{L^\infty(0,T;L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))}\le S, \tag{4.10m}$$
$$\|\overline{\zeta}_\tau\|_{L^\infty(Q;\mathbb{M}^{d\times d}_{\mathrm{D}})}\le S, \tag{4.10n}$$
$$\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))}+\|\overline{\vartheta}_\tau\|_{L^2(0,T;H^1(\Omega))}\le S, \tag{4.10o}$$
$$\|\log(\overline{\vartheta}_\tau)\|_{L^\infty(0,T;L^p(\Omega))}+\|\log(\overline{\vartheta}_\tau)\|_{L^2(0,T;H^1(\Omega))}\le S\quad\text{for all } 1\le p<\infty, \tag{4.10p}$$
$$\|(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))}+\|(\overline{\vartheta}_\tau)^{(\mu-\alpha)/2}\|_{L^2(0,T;H^1(\Omega))}\le C\quad\text{for all } \alpha\in[0\vee(2-\mu),1), \tag{4.10q}$$
$$\sup_{\varphi\in W^{1,d+\epsilon}(\Omega),\ \|\varphi\|_{W^{1,d+\epsilon}(\Omega)}\le1}\mathrm{Var}(\langle\log(\overline{\vartheta}_\tau),\varphi\rangle_{W^{1,d+\epsilon}(\Omega)};[0,T])\le S\quad\text{for every } \epsilon>0 \tag{4.10r}$$

where we recall that $\gamma > 4$ *and refer to* (5.6) *ahead for the definition of* $\mathrm{Var}(\langle \log(\overline{\vartheta}_\tau), \varphi\rangle_{W^{1,d+\epsilon}(\Omega)}; [0,T])$*). Furthermore, if* κ *fulfills* (2.κ_2)*, there holds in addition*

$$\sup_{\tau>0} \|\vartheta_\tau\|_{\mathrm{BV}([0,T];W^{1,\infty}(\Omega)^*)} \le S. \tag{4.10s}$$

We will not develop all the calculations leading to estimates (4.10), but rather only give a sketch of the proof, referring to the proofs of [RR15, Prop. 4.10] and [Ros16, Prop. 4.3] for all details. Nevertheless, let us mention in advance the main ingredients of the various calculations:

(1) The starting point in the derivation of the a priori estimates is the discrete total energy inequality (4.7). Indeed, conditions (2.L_1), (2.L_2), and (2.w) allow us to suitably estimate the terms on the right-hand side of (4.7), based on the calculations from the proof of [Ros16, Prop. 4.3]. We thus deduce from (4.7) the uniform energy bound

$$\sup_{t\in[0,T]} \mathcal{E}_\tau(\overline{\vartheta}_\tau(t), \overline{e}_\tau(t), \overline{z}_\tau(t), \overline{p}_\tau(t)) \le S. \tag{4.11}$$

In view of the coercivity properties of the discrete energy $\mathcal{E}_\tau$, (4.11) yields estimates (4.10d), (4.10f), (4.10m), and the first of (4.10o). We also establish (4.10g). We further infer a bound for the kinetic energy term in (4.7), which leads to the first of (4.10c).

(2) Estimate (4.11) in particular implies that $\sup_{t\in[0,T]} \int_\Omega \beta(\overline{z}_\tau(t))\,\mathrm{d}x \le S$. Exploiting the coercivity condition $z^{2d}\beta(z) \to +\infty$ as $z \downarrow 0$, which in turn originates from (2.W_1), and repeating an argument from [CL16, Lemma 3.3], we thus conclude the feasibility condition (4.9).

(3) The crucial estimate for ϑ in $L^2(0,T;H^1(\Omega))$ ensues from testing the discrete heat equation (4.2d) by a suitable negative power of $\overline{\vartheta}_\tau$, as suggested in [FPR09], cf. also the proof of [RR15, Prop. 4.10].

(4) The *dissipative estimates* (4.10b), (4.10e), (4.10h) & (4.10i), as well as (4.10l), derive from the discrete mechanical energy inequality (4.8). We then establish (4.10a).

(5) The total variation-type estimate (4.10r) is deduced from the discrete entropy inequality (4.6) with the very same calculations as in the proofs of [RR15, Prop. 4.10] and [Ros16, Prop. 4.3].

(6) The enhanced BV-estimate (4.10s) derives from a comparison argument in the discrete heat equation (4.2d), again following the proofs of [RR15, Prop. 4.10] and [Ros16, Prop. 4.3].

We will now give a sketch of the proof of Proposition 4.3:

Step 1: First a priori estimate. We write the discrete total energy inequality (4.7) for $s = 0$ and $t \in (0,T]$. We estimate the terms on its right-hand side resorting to conditions (2.15) and (3.5) on the initial data $(u_\tau^0, e_\tau^0, z_0, p_\tau^0, \vartheta_0)$, which ensure a bound for the kinetic energy term $\int_\Omega |\dot{u}_\tau(0)|^2\,\mathrm{d}x$, as well as the estimate $\sup_{\tau>0} \mathcal{E}_\tau(\vartheta_0, e_\tau^0, z_0, p_\tau^0) \le C$. We use the safe load condition (2.L_2), condition (2.w) on the Dirichlet loading, as well as estimates (4.1a), (4.1b), and (4.1d) to handle all the other terms on the r.h.s. of (4.7), arguing in the very same way as throughout the proof of [Ros16, Prop. 4.3]. In turn, we observe that, due to the coercivity property (2.$(\mathbb{C},\mathbb{D})_1$), and by the (strict) positivity (3.9) of $\overline{\vartheta}_\tau$, there holds

$$\begin{aligned}\mathcal{E}_\tau(\overline{\vartheta}_\tau(t), \overline{e}_\tau(t), \overline{z}_\tau(t), \overline{p}_\tau(t)) \ge c\Big(&\|\overline{\vartheta}_\tau(t)\|_{L^1(\Omega)} + \|\overline{e}_\tau(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + a_{\mathrm{s}}(\overline{z}_\tau(t), \overline{z}_\tau(t)) + \int_\Omega W(\overline{z}_\tau(t))\,\mathrm{d}x \\ &+ \tau\|\overline{e}_\tau(t)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \tau\|\overline{p}_\tau(t)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\Big) - C\end{aligned}$$

Combining these facts, and repeating the very same calculations from the proof of [Ros16, Prop. 4.3], we arrive at the following estimate

$$
\begin{aligned}
&\int_\Omega |\dot{u}_\tau(t)|^2 \,\mathrm{d}x + \|\overline{\vartheta}_\tau(t)\|_{L^1(\Omega)} + \|\overline{e}_\tau(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \tau\|\overline{e}_\tau(t)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \\
&\qquad + a_{\mathrm{s}}(\overline{z}_\tau(t),\overline{z}_\tau(t)) + \int_\Omega W(\overline{z}_\tau(t))\,\mathrm{d}x + \tau\|\overline{p}_\tau(t)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \\
&\leq C + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\dot{\varrho}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \|\underline{e}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|(\overline{\varrho}_\tau(r))_{\mathrm{D}}\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\|\overline{\vartheta}_\tau(r)\|_{L^1(\Omega)}\,\mathrm{d}r \\
&\quad + C\int_0^{\bar{\mathsf{t}}_\tau(t)-\tau} \|\partial_t\widehat{w}_\tau(s+\tau)\|_{L^2(\Omega;\mathbb{R}^d)}\|\dot{u}_\tau(s)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \Big(\|E(\partial_t\widehat{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|E(\dot{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\Big)\|\overline{e}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \\
&\quad + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|E(\dot{w}_\tau(r))\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\|\overline{\vartheta}_\tau(r)\|_{L^1(\Omega)}\,\mathrm{d}r\,,
\end{aligned}
\tag{4.12}
$$

where $\overline{\varrho}_\tau$ and ϱ_τ denote the piecewise constant/linear interpolants of the local means of the safe load function ϱ. Then, taking into account that W is bounded from below (cf. (2.W$_1$)), and applying the Gronwall Lemma, we arrive at the energy bound (4.11), joint with the estimate $\|\dot{u}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C$. Estimates (4.10b)(2), (4.10c)(2), (4.10d), (4.10f), (4.10m), and (4.10o)(1) ensue. Observe that the latter implies the first bound in (4.10p), taking into account the elementary estimate

$$\forall\, p \in [1,\infty)\ \exists\, C_p > 0\ \forall\, \theta > 0 : \quad |\log(\theta)|^p \leq \theta + \frac{1}{\theta} + C_p\,.$$

and the previously proved (2.42).We also infer that

$$\sup_{t\in[0,T]} a_{\mathrm{s}}(\overline{z}_\tau(t),\overline{z}_\tau(t)) + \sup_{t\in[0,T]}\left|\int_\Omega W(\overline{z}_\tau(t))\,\mathrm{d}x\right| \leq C\,. \tag{4.13}$$

Observe that $0 < \overline{z}_\tau \leq 1$, with the upper bound due to the fact that, by unidirectionality of the evolution of damage, for every $x \in \Omega$ we have $\overline{z}_\tau(x,t) \leq \overline{z}_\tau(x,0) = z_0(x) \leq 1$, where the last inequality ensues from (2.15c). Then, in view of (4.13), we conclude (4.10g). Finally, estimate (4.10n) follows from the fact that $\overline{\zeta}_\tau \in \partial_{\dot{p}}\mathrm{H}(\overline{z}_\tau, \underline{\vartheta}_\tau; \dot{p}_\tau) \subset B_{C_R}(0)$ a.e. in $\Omega\times(0,T)$ by (2.12b).

Step 2: ad (4.9)**.** The upper bound $\overline{z}_\tau \leq 1$ follows from To obtain the lower bound, we repeat the argument from the proof of [CL16, Lemma 3.3]: due to the coercivity condition (2.W$_1$), for every $M > 0$ there exists $\tilde{\zeta} > 0$ such that for all $0 < z \leq \tilde{\zeta}$ there holds $W(z) \geq Mz^{-2d}$. Now, by contradiction suppose that $\overline{z}_\tau$ does not comply with the lower bound in (4.9). Then, corresponding to $\tilde{\zeta}$, there exist $\tilde{\tau} > 0$, $t \in [0,T]$, and $x \in \Omega$ such that $\overline{z}_{\tilde{\tau}}(x,t) < \frac{\tilde{\zeta}}{2}$. We now use that $\overline{z}_{\tilde{\tau}}(\cdot,t) \in H^{\mathrm{s}}(\Omega) \subset \mathrm{C}^{0,1/2}(\overline{\Omega})$, which yields that

$$\exists\, \tilde{C} > 0\ \ \forall\, x,y \in \Omega : \quad |\overline{z}_{\tilde{\tau}}(x,t) - \overline{z}_{\tilde{\tau}}(y,t)| \leq \tilde{C}|x-y|^{1/2}\,,$$

to deduce that $\overline{z}_{\tilde{\tau}}(y,t) < \tilde{\zeta}$ for every $y \in B_{\tilde{r}}(0)$, with $\tilde{r} = \left(\frac{\delta}{2\tilde{C}}\right)^2$. Hence

$$\int_\Omega W(\overline{z}_{\tilde{\tau}}(t))\,\mathrm{d}x \geq M\int_{B_{\tilde{r}}(0)} (\overline{z}_{\tilde{\tau}}(t))^{-2d}\,\mathrm{d}x \geq M(\tilde{\delta})^{-2d}|B_{\tilde{r}}(0)| = M\frac{\omega_d}{(2\tilde{C})^{2d}},$$

with ω_d the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Since M is arbitrary, this contradicts estimate (4.13). We thus conclude the lower bound for the piecewise constant interpolant $\overline{z}_\tau$. The analogous statement immediately follows for the interpolant z_τ, since it is given by a convex combination of the values of $\overline{z}_\tau$.

Step 3: Second a priori estimate. We test the discrete heat equation (3.8d) by $(\vartheta_\tau^k)^{\alpha-1}$, with $\alpha\in(0,1)$, thus obtaining

$$\begin{aligned} & \int_\Omega \Big(G_\tau^k + \mathbb{D}(z_\tau^k)\mathrm{D}_{k,\tau}(e):\mathrm{D}_{k,\tau}(e) + \mathrm{R}(\mathrm{D}_{k,\tau}(z)) + |\mathrm{D}_{k,\tau}(z)|^2 + \bar\nu a_{\mathrm{s}}(\mathrm{D}_{k,\tau}(z),\mathrm{D}_{k,\tau}(z)) \\ & \qquad\qquad + \mathrm{H}(z_\tau^k,\vartheta_\tau^{k-1};\mathrm{D}_{k,\tau}(p)) + |\mathrm{D}_{k,\tau}(p)|^2 \Big)(\vartheta_\tau^k)^{\alpha-1}\,\mathrm{d}x \\ & - \int_\Omega \kappa(\vartheta_\tau^k)\nabla\vartheta_\tau^k\nabla(\vartheta_\tau^k)^{\alpha-1}\,\mathrm{d}x + \int_{\partial\Omega} g_\tau^k(\vartheta_\tau^k)^{\alpha-1}\,\mathrm{d}S \\ & \leq \int_\Omega \left(\frac{1}{\alpha}\frac{(\vartheta_\tau^k)^\alpha - (\vartheta_\tau^{k-1})^\alpha}{\tau} + \vartheta_\tau^k\mathbb{C}(z_\tau^k)\mathbb{E}:\mathrm{D}_{k,\tau}(e)(\vartheta_\tau^k)^{\alpha-1} + \vartheta_\tau^{k-1}\mathrm{D}_{k,\tau}(z)(\vartheta_\tau^k)^{\alpha-1}\right)\mathrm{d}x, \end{aligned} \tag{4.14}$$

where we have used the concavity of the function $\psi(\vartheta)=\frac{1}{\alpha}\vartheta^\alpha$, leading to the estimate $(\vartheta_\tau^k - \vartheta_\tau^{k-1})(\vartheta_\tau^k)^{\alpha-1} \leq \psi(\vartheta_\tau^k)-\psi(\vartheta_\tau^{k-1})$. Note that we have omitted the positive part $(z_\tau^k)^+$ in the argument of $\mathbb{C}$ in the thermal expansion term, since $z_\tau^k>0$ on Ω by virtue of (4.9). Therefore, multiplying by τ, summing over the index k, neglecting some positive terms on the left-hand side of (4.14) and observing that $\vartheta_\tau^{k-1}\mathrm{D}_{k,\tau}(z)(\vartheta_\tau^k)^{\alpha-1}\leq 0$ a.e. in Ω since ϑ_τ^{k-1}, $\vartheta_\tau^k>0$ while $\mathrm{D}_{k,\tau}(z)\leq 0$, we obtain for all $t\in(0,T]$

$$\begin{aligned} & \frac{4(1-\alpha)}{\alpha^2}\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \kappa(\overline{\vartheta}_\tau)|\nabla((\overline{\vartheta}_\tau)^{\alpha/2})|^2\,\mathrm{d}x\,\mathrm{d}s + C_{\mathbb{D}}^1\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\dot e_\tau|^2(\overline{\vartheta}_\tau)^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s \\ & \leq \frac{1}{\alpha}\int_\Omega (\overline{\vartheta}_\tau(t))^\alpha\,\mathrm{d}x - \frac{1}{\alpha}\int_\Omega (\vartheta_0)^\alpha\,\mathrm{d}x + \int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \overline{\vartheta}_\tau(t)\mathbb{C}(\overline{z}_\tau)\mathbb{E}:\dot e_\tau(t)(\overline{\vartheta}_\tau(t))^{\alpha-1}\,\mathrm{d}x \doteq I_1+I_2+I_3\,. \end{aligned} \tag{4.15}$$

Next, we use that

$$I_1 \leq \frac{1}{\alpha}\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + C \leq C$$

via Young's inequality, using that $\alpha\in(0,1)$, and taking into account the previously obtained bound (4.10o)(1). We have $I_2\leq 0$, whereas for I_3 we observe that

$$\exists\, C>0\ \forall\,(x,t)\in\Omega\times[0,T]\quad |\mathbb{C}(\overline{z}_\tau(x,t))|\leq C\,. \tag{4.16}$$

by the continuity of the function $\mathbb{C}$, combined with the previously obtained property (4.9). Therefore, also taking into account that $\mathbb{E}\in L^\infty(\Omega;\mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}}))$, we obtain

$$I_3 \leq \frac{C_{\mathbb{D}}^1}{4}\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\dot e_\tau|^2(\overline{\vartheta}_\tau)^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s + C\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega (\overline{\vartheta}_\tau)^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}s\,. \tag{4.17}$$

Absorbing the first term on the right-hand side of (4.17) into the left-hand side of (4.15) and taking into account the coercivity condition (2.κ_1) on κ, we infer from (4.15) that

$$c\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\nabla(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}|^2\,\mathrm{d}x\,\mathrm{d}s \leq C + C\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega (\overline{\vartheta}_\tau)^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}s\,. \tag{4.18}$$

From now on, we can repeat the calculations developed in [RR15, (3.8)–(3.12)] (and based on techniques from [FPR09]), for the analogous estimate. We refer to [RR15] for all the detailed calculations, leading to an estimate for $(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}$ in $L^2(0,T;H^1(\Omega))$, i.e. the first of (4.10q). From this bound, relying on the constraint $\alpha\geq 2-\mu$ (cf. again [RR15] for all details) we deduce estimate (4.10o)(2). The latter in turns yields (4.10p)(2). The second of (4.10q) ensues from $(\overline{\vartheta}_\tau)^{(\mu-\alpha)/2}\leq(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}+1$ a.e. in Q and from

$$\int_\Omega |\nabla(\overline{\vartheta}_\tau)^{(\mu-\alpha)/2}|^2\,\mathrm{d}x = C\int_\Omega (\overline{\vartheta}_\tau)^{\mu-\alpha-2}|\nabla\overline{\vartheta}_\tau|^2\,\mathrm{d}x \leq \frac{C}{\bar\vartheta^{2\alpha}}\int_\Omega (\overline{\vartheta}_\tau)^{\mu+\alpha-2}|\nabla\overline{\vartheta}_\tau|^2\,\mathrm{d}x \leq C$$

where we have used that $\alpha\geq 0$, the strict positivity of $\overline{\vartheta}_\tau$, and the previously obtained estimate for $(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}$.

Step 4: Third a priori estimate. We consider the mechanical energy inequality (4.8) written for $s=0$. Since most of the terms on its right-hand side can be handled by the very same calculations developed for the right-hand side terms of (4.7), we refer to the proof of [Ros16, Prop. 4.3] for all details and only mention how to estimate the third and the fourth integral terms. We use that

$$\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \overline{\vartheta}_\tau\mathbb{C}(\overline{z}_\tau)\mathbb{E}:\dot e_\tau\,\mathrm{d}x\,\mathrm{d}r \leq \delta\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\dot e_\tau|^2\,\mathrm{d}x\,\mathrm{d}r + C_\delta\|\overline{\vartheta}_\tau\|^2_{L^2(0,T;L^2(\Omega))}$$

via Young's inequality, the previously obtained bound (4.16), and the fact that $|\mathbb{E}(x)| \le C$, and that

$$\int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \underline{\vartheta}_\tau \dot{z}_\tau \,\mathrm{d}x\,\mathrm{d}r \le \delta \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\dot{z}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}r + C_\delta \|\underline{\vartheta}_\tau\|^2_{L^2(0,T;L^2(\Omega))}$$

with the constant $\delta > 0$ chosen in such a way as to absorb the terms $\iint |\dot{e}_\tau|^2$ and $\iint |\dot{z}_\tau|^2$ into the left-hand side of (4.8). Since $\|\overline{\vartheta}_\tau\|_{L^2(Q)}, \|\underline{\vartheta}_\tau\|_{L^2(Q)} \le C$ thanks to (4.10o), we conclude a uniform bound for all the terms on the right-hand side of (4.8). Therefore, estimates (4.10e), (4.10h), (4.10i), and (4.10l) ensue. We then obtain (4.10b)(1) and (4.10c)(1) via kinematic admissibility. Furthermore, (4.10l) clearly implies estimate (4.10k), and then (4.10a) again by kinematic admissibility.

It follows from (4.10d), (4.10e), (4.10f), and (4.10o), also taking into account the bound (4.16) and its analogue for $\mathbb{D}(\overline{z}_\tau)$, that the stresses $(\overline{\sigma}_\tau)_\tau$ are uniformly bounded in $L^{\gamma/(\gamma-1)}(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. Therefore, also taking into account (4.1c), a comparison argument in the discrete momentum balance (4.2a) yields (4.10c)(3).

Finally, estimate (4.10j) ensues from a comparison argument in (4.2b).

Step 5: Fourth a priori estimate. Let us now shortly sketch the argument for (4.10r). The very same calculations as in the proof of [RR15, Prop. 4.10] lead us to deduce, from the discrete entropy inequality (4.6) written on the generic sub-interval $[\mathsf{s}_{i-1}, \mathsf{s}_i]$ of a partition $0 = \mathsf{s}_0 < \mathsf{s}_1 < \ldots < \mathsf{s}_J = T$ of $[0,T]$, the following estimate for the total variation of $\log(\overline{\vartheta}_\tau)$:

$$\begin{aligned}&\sum_{i=1}^J \Big| \langle \log(\overline{\vartheta}_\tau(\mathsf{s}_i)) - \log(\overline{\vartheta}_\tau(\mathsf{s}_{i-1})), \varphi \rangle_{W^{1,d+\epsilon}(\Omega)} \Big| \\ &\le \sum_{i=1}^J \int_\Omega (\log(\overline{\vartheta}_\tau(\mathsf{s}_i)) - \log(\overline{\vartheta}_\tau(\mathsf{s}_{i-1})))|\varphi|\,\mathrm{d}x + \Lambda_{i,\tau}(|\varphi|) + |\Lambda_{i,\tau}(\varphi^+)| + |\Lambda_{i,\tau}(\varphi^-)|\end{aligned} \tag{4.19}$$

for all $\varphi \in W^{1,d+\epsilon}(\Omega)$, with $\epsilon > 0$ arbitrary. Here we have used the place-holder

$$\begin{aligned}\Lambda_{i,\tau}(\varphi) :=& \int_{\bar{\mathsf{t}}_\tau(\mathsf{s}_{i-1})}^{\bar{\mathsf{t}}_\tau(\mathsf{s}_i)} \int_\Omega \kappa(\overline{\vartheta}_\tau) \nabla \log(\overline{\vartheta}_\tau) \nabla \varphi \,\mathrm{d}x\,\mathrm{d}r + \int_{\bar{\mathsf{t}}_\tau(\mathsf{s}_{i-1})}^{\bar{\mathsf{t}}_\tau(\mathsf{s}_i)} \int_\Omega \mathbb{C}(\overline{z}_\tau)\mathbb{E}{:}\dot{e}_\tau \varphi \,\mathrm{d}x\,\mathrm{d}r \\ &- \int_{\bar{\mathsf{t}}_\tau(\mathsf{s}_{i-1})}^{\bar{\mathsf{t}}_\tau(\mathsf{s}_i)} \int_\Omega \kappa(\overline{\vartheta}_\tau) \frac{\varphi}{\overline{\vartheta}_\tau} \nabla(\log(\overline{\vartheta}_\tau)) \nabla \overline{\vartheta}_\tau \,\mathrm{d}x\,\mathrm{d}r - \int_{\bar{\mathsf{t}}_\tau(\mathsf{s}_{i-1})}^{\bar{\mathsf{t}}_\tau(\mathsf{s}_i)} \int_{\partial\Omega} \overline{g}_\tau \frac{\varphi}{\overline{\vartheta}_\tau} \,\mathrm{d}S\,\mathrm{d}r \\ &- \int_{\bar{\mathsf{t}}_\tau(\mathsf{s}_{i-1})}^{\bar{\mathsf{t}}_\tau(\mathsf{s}_i)} \int_\Omega \left(\overline{G}_\tau + \mathbb{D}(\overline{z}_\tau)\dot{e}_\tau{:}\dot{e}_\tau + \mathrm{R}(\dot{z}_\tau) + |\dot{z}_\tau|^2 + \bar{\nu} a_{\mathrm{s}}(\dot{z}_\tau, \dot{z}_\tau) - \underline{\vartheta}_\tau \dot{z}_\tau + \mathrm{H}(\overline{z}_\tau, \underline{\vartheta}_\tau; \dot{p}_\tau) + |\dot{p}_\tau|^2 \right) \frac{\varphi}{\overline{\vartheta}_\tau} \,\mathrm{d}x\,\mathrm{d}r.\end{aligned} \tag{4.20}$$

The second, third, and fourth terms on the r.h.s. of (4.19) can be estimated by adapting the computations from the proof of [RR15, Prop. 4.10], taking into account the previously obtained bounds. All in all, from (4.19) we obtain that

$$\sum_{i=1}^J \Big| \langle \log(\overline{\vartheta}_\tau(\mathsf{s}_i)) - \log(\overline{\vartheta}_\tau(\mathsf{s}_{i-1})), \varphi \rangle_{W^{1,d+\epsilon}(\Omega)} \Big| \le \int_\Omega \left(\log(\overline{\vartheta}_\tau(T)) - \log(\vartheta_0) \right) |\varphi| \,\mathrm{d}x + C \le C$$

for every $\varphi \in W^{1,d+\epsilon}(\Omega)$ with $\|\varphi\|_{W^{1,d+\epsilon}(\Omega)} \le 1$, where for the last estimate we have used the bound for $\log(\overline{\vartheta}_\tau)$ in $L^\infty(0,T;L^1(\Omega))$ from (4.10p). Thus, (4.10r) ensues.

Step 6: Fifth a priori estimate: We now assume the stronger condition $(2.\kappa_2)$. We multiply the discrete heat equation (3.8d) by a test function $\varphi \in W^{1,\infty}(\Omega)$ and integrate in space. We thus obtain for a.a. $t \in (0,T)$

$$\left| \int_\Omega \dot{\vartheta}_\tau(t) \varphi \,\mathrm{d}x \right| \le \left| \int_\Omega \kappa(\overline{\vartheta}_\tau(t)) \nabla \overline{\vartheta}_\tau(t) \nabla \varphi \,\mathrm{d}x \right| + \left| \int_\Omega \overline{J}_\tau(t) \varphi \,\mathrm{d}x \right| + \left| \int_{\partial\Omega} \overline{g}_\tau(t) \varphi \,\mathrm{d}S \right| \doteq I_1 + I_2 + I_3 \,, \tag{4.21}$$

with $\overline{J}_\tau := \overline{G}_\tau + \mathbb{D}(\overline{z}_\tau)\dot{e}_\tau : \dot{e}_\tau - \overline{\vartheta}_\tau \mathbb{C}(\overline{z}_\tau)\mathbb{E} : \dot{e}_\tau + \mathrm{R}(\dot{z}_\tau) + |\dot{z}_\tau|^2 + \bar{\nu} a_{\mathrm{s}}(\dot{z}_\tau, \dot{z}_\tau) - \underline{\vartheta}_\tau \dot{z}_\tau + \mathrm{H}(\overline{z}_\tau, \underline{\vartheta}_\tau; \dot{p}_\tau) + |\dot{p}_\tau|^2$. With the very same calculations as in the proof of [RR15, Prop. 4.10], relying on (4.1a), (4.1b), on $(2.\kappa_2)$, and on the previously obtained estimates (4.10e), (4.10h), (4.10i), (4.10l), (4.10o), and (4.10q), we infer that

$$I_1 + I_2 + I_3 \le \mathcal{L}_\tau(t) \|\varphi\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}$$

for a family $(\mathcal{L}_\tau)_\tau$ that is uniformly bounded in $L^1(0,T)$. Hence, estimate (4.10s) follows. This concludes the proof of Proposition 4.3. ∎

5. **Proofs of Theorems 2.5 and 2.6**

We start by fixing the compactness properties of a family

$$(\overline{u}_{\tau_k}, u_{\tau_k}, \widehat{u}_{\tau_k}, \overline{e}_{\tau_k}, \underline{e}_{\tau_k}, e_{\tau_k}, \overline{z}_{\tau_k}, z_{\tau_k}, \overline{p}_{\tau_k}, p_{\tau_k}, \overline{\vartheta}_{\tau_k}, \underline{\vartheta}_{\tau_k}, \vartheta_{\tau_k}, \overline{\omega}_{\tau_k}, \overline{\zeta}_{\tau_k})_k,$$

of approximate solutions in the following result, where we again tacitly assume the validity of the conditions from Sec. 2.1. We will only distinguish the case where we only require (2.κ_1), from that where (2.κ_2) is also imposed and we are thus in the position to enhance the convergence properties of the temperature variables by virtue of the additional estimate (4.10s).

Lemma 5.1 (Compactness)**.** *Assume* (2.κ_1)*. Then, for any sequence $\tau_k \downarrow 0$ there exist a (not relabeled) subsequence and a seventuple $(u, e, z, p, \vartheta, \omega, \zeta)$ such that the following convergences hold*

$$u_{\tau_k} \overset{*}{\rightharpoonup} u \qquad \text{in } H^1(0,T;H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{5.1a}$$

$$\overline{u}_{\tau_k},\, \underline{u}_{\tau_k} \to u \qquad \text{in } L^\infty(0,T;H^{1-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{5.1b}$$

$$u_{\tau_k} \to u \qquad \text{in } \mathrm{C}^0([0,T];H^{1-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{5.1c}$$

$$\widehat{u}_{\tau_k}(\bar{\mathsf{t}}_{\tau_k}(t)) \rightharpoonup \dot{u}(t) \qquad \text{in } L^2(\Omega;\mathbb{R}^d) \text{ for every } t \in [0,T], \tag{5.1d}$$

$$\widehat{u}_{\tau_k}(\bar{\mathsf{t}}_{\tau_k}(t)) = \dot{u}_{\tau_k}(t) \rightharpoonup \dot{u}(t) \qquad \text{in } H^1(\Omega;\mathbb{R}^d) \text{ for almost all } t \in (0,T), \tag{5.1e}$$

$$\partial_t \widehat{u}_{\tau_k} \rightharpoonup \ddot{u} \qquad \text{in } L^{\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*), \tag{5.1f}$$

$$\overline{e}_{\tau_k},\, \underline{e}_{\tau_k} \overset{*}{\rightharpoonup} e \qquad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{5.1g}$$

$$e_{\tau_k} \rightharpoonup e \qquad \text{in } H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{5.1h}$$

$$e_{\tau_k} \to e \qquad \text{in } \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{5.1i}$$

$$\tau^\beta |\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k} \to 0 \qquad \text{in } L^\infty(0,T;L^{\gamma/(\gamma-1)}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})) \text{ for all } \beta > 1-\frac{1}{\gamma}, \tag{5.1j}$$

$$\overline{z}_{\tau_k}, z_{\tau_k} \overset{*}{\rightharpoonup} z \qquad \text{in } L^\infty(0,T;H^{\mathrm{s}}(\Omega)), \tag{5.1k}$$

$$z_{\tau_k} \rightharpoonup z \qquad \text{in } H^1(0,T;L^2(\Omega)), \tag{5.1l}$$

$$z_{\tau_k} \rightharpoonup z \qquad \text{in } H^1(0,T;H^{\mathrm{s}}(\Omega)) \text{ if } \nu > 0, \tag{5.1m}$$

$$\overline{z}_\tau,\, z_\tau \to z \qquad \text{in } L^\infty(0,T;\mathrm{C}^0(\overline{\Omega})), \tag{5.1n}$$

$$\overline{\omega}_{\tau_k} \rightharpoonup \omega \qquad \text{in } L^2(0,T;H^{\mathrm{s}}(\Omega)^*) \text{ if } \nu > 0, \tag{5.1o}$$

$$\overline{p}_{\tau_k} \overset{*}{\rightharpoonup} p \qquad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{5.1p}$$

$$p_{\tau_k} \rightharpoonup p \qquad \text{in } H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{5.1q}$$

$$p_{\tau_k} \to p \qquad \text{in } \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})), \tag{5.1r}$$

$$\tau^\beta |\overline{p}_{\tau_k}|^{\gamma-2}\overline{p}_{\tau_k} \to 0 \qquad \text{in } L^\infty(0,T;L^{\gamma/(\gamma-1)}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})) \text{ for all } \beta > 1-\frac{1}{\gamma}, \tag{5.1s}$$

$$\overline{\vartheta}_{\tau_k},\, \underline{\vartheta}_{\tau_k} \rightharpoonup \vartheta \qquad \text{in } L^2(0,T;H^1(\Omega)), \tag{5.1t}$$

$$\log(\overline{\vartheta}_{\tau_k}),\, \log(\underline{\vartheta}_{\tau_k}) \overset{*}{\rightharpoonup} \log(\vartheta) \qquad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)^*) \quad \text{for every } \epsilon > 0, \tag{5.1u}$$

$$\log(\overline{\vartheta}_{\tau_k}(t)),\, \log(\underline{\vartheta}_{\tau_k}(t)) \rightharpoonup \log(\vartheta(t)) \qquad \text{in } H^1(\Omega) \text{ for almost all } t \in (0,T), \tag{5.1v}$$

$$\overline{\vartheta}_{\tau_k},\, \underline{\vartheta}_{\tau_k} \to \vartheta \qquad \text{in } L^h(Q) \text{ for all } h \in [1,8/3) \text{ if } d=3 \text{ and all } h \in [1,3) \text{ if } d=2, \tag{5.1w}$$

$$\overline{\zeta}_{\tau_k} \overset{*}{\rightharpoonup} \zeta \qquad \text{in } L^\infty(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}). \tag{5.1x}$$

The functions z and ϑ also fulfill (2.41)*, with ζ_* from* (4.9)*, and*

$$\vartheta \in L^\infty(0,T;L^1(\Omega)) \text{ and } \vartheta \geq \bar{\vartheta} \text{ a.e. in } Q \tag{5.2}$$

with $\bar{\vartheta}$ from (3.9)*.*

Furthermore, under condition $(2.\kappa_2)$ *we also have* $\vartheta \in \mathrm{BV}([0,T];W^{1,\infty}(\Omega)^*)$, *and*

$$\overline{\vartheta}_{\tau_k},\, \underline{\vartheta}_{\tau_k} \to \vartheta \qquad \text{in } L^2(0,T;Y) \text{ for all } Y \text{ such that } H^1(\Omega) \Subset Y \subset W^{1,\infty}(\Omega)^*, \tag{5.3a}$$

$$\overline{\vartheta}_{\tau_k}(t),\, \underline{\vartheta}_{\tau_k}(t) \overset{*}{\rightharpoonup} \vartheta(t) \qquad \text{in } W^{1,\infty}(\Omega)^* \text{ for all } t \in [0,T]. \tag{5.3b}$$

Sketch of the proof. Convergences (5.1k)–(5.1o) follow from standard weak and strong compactness arguments, the latter based, e.g., on the Aubin-Lions type compactness tools from [Sim87].

Let us comment on (5.1d): It follows from estimate (4.10c) and the aforementioned compactness results that the sequence $(\widehat{u}_{\tau_k})_k$ admits a subsequence converging to some limit v, weakly* in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap W^{1,\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*)$, and strongly in $\mathrm{C}^0([0,T];X)$ for any space X with $L^2(\Omega) \Subset X$. Therefore, for every $t \in [0,T]$ we have that $\widehat{u}_{\tau_k}(\bar{\mathsf{t}}_{\tau_k}(t)) \to v(t)$ in X. We combine this with the information that $(\widehat{u}_{\tau_k})_k$ is bounded in $L^\infty(0,T;L^2(\Omega))$ to extend the latter statement to weak convergence in $L^2(\Omega)$. The identification $v = \dot{u}$ follows from the estimate

$$\|\widehat{u}_{\tau_k} - \dot{u}_{\tau_k}\|_{L^\infty(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*)} \le \tau_k^{1/\gamma} \|\partial_t \widehat{u}_{\tau_k}\|_{L^{\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*)} \le S{\tau_k}^{1/\gamma}.$$

As for (5.1e), we apply the compactness result stated in Theorem 5.2 below, with the choices $\ell_k = \widehat{u}_{\tau_k} \circ \bar{\mathsf{t}}_{\tau_k}$, $\mathbf{V} = H^1(\Omega;\mathbb{R}^d)$, and $\mathbf{Y} = W^{1,\gamma}(\Omega;\mathbb{R}^d)$. We thus deduce (cf. (5.8) ahead) that

$$\widehat{u}_{\tau_k}(\bar{\mathsf{t}}_{\tau_k}(t)) \rightharpoonup \dot{u}(t) \text{ in } H^1(\Omega;\mathbb{R}^d) \text{ for a.a. } t \in (0,T).$$

Hence, (5.1e) ensues, taking into account that $\widehat{u}_{\tau_k} \circ \bar{\mathsf{t}}_{\tau_k} \equiv \dot{u}_{\tau_k}$ a.e. in $(0,T)$.

For all the other convergence statements, the reader is referred to the proof of [Ros16, Lemma 4.6]: let us only mention that the pointwise convergences (5.1v) follow from Theorem 5.2, as well. □

We refer to [RR15] and [Ros16] (for a slight refinement of) the proof of the following compactness result.

Theorem 5.2. *Let* $\mathbf{V}$ *and* $\mathbf{Y}$ *be two (separable) reflexive Banach spaces such that* $\mathbf{V} \subset \mathbf{Y}^*$ *continuously. Let* $(\ell_k)_k \subset L^p(0,T;\mathbf{V}) \cap \mathrm{B}([0,T];\mathbf{Y}^*)$ *be bounded in* $L^p(0,T;\mathbf{V})$ *and suppose in addition that*

$$(\ell_k(0))_k \subset \mathbf{Y}^* \text{ is bounded,} \tag{5.4}$$

$$\exists\, C > 0 \quad \forall\, \varphi \in \overline{B}_{1,\mathbf{Y}}(0) \quad \forall\, k \in \mathbb{N}: \quad \mathrm{Var}(\langle \ell_k, \varphi \rangle_{\mathbf{Y}}; [0,T]) \le C, \tag{5.5}$$

where, for given $\ell \in \mathrm{B}([0,T];\mathbf{Y}^*)$ *and* $\varphi \in \mathbf{Y}$ *we set*

$$\mathrm{Var}(\langle \ell, \varphi \rangle_{\mathbf{Y}}; [0,T]) := \sup\Big\{ \sum_{i=1}^{J} |\langle \ell(\mathsf{s}_i), \varphi \rangle_{\mathbf{Y}} - \langle \ell(\mathsf{s}_{i-1}), \varphi \rangle_{\mathbf{Y}}| \, : \, 0 = \mathsf{s}_0 < \mathsf{s}_1 < \ldots < \mathsf{s}_J = T \Big\}. \tag{5.6}$$

Then, there exist a (not relabeled) subsequence $(\ell_k)_k$ *and a function* $\ell \in L^p(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{Y}^*)$ *such that as* $k \to \infty$

$$\ell_k \overset{*}{\rightharpoonup} \ell \quad \text{in } L^p(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{Y}^*), \tag{5.7}$$

$$\ell_k(t) \rightharpoonup \ell(t) \quad \text{in } \mathbf{V} \quad \text{for a.a. } t \in (0,T). \tag{5.8}$$

Furthermore, for almost all $t \in (0,T)$ *and any sequence* $(t_k)_k \subset [0,T]$ *with* $t_k \to t$ *there holds*

$$\ell_k(t_k) \rightharpoonup \ell(t) \qquad \text{in } \mathbf{Y}^*. \tag{5.9}$$

For expository reasons, in developing the proofs of our existence results we will reverse the order with which we have presented them. More precisely, we will start with the **proof of Theorem 2.6** and develop the existence of weak energy solutions and, in addition, establish the validity of the entropy inequality. Let us consider a null sequence $(\tau_k)_k$ and, correspondingly, a sequence

$$(\overline{u}_{\tau_k}, u_{\tau_k}, \widehat{u}_{\tau_k}, \overline{e}_{\tau_k}, \underline{e}_{\tau_k}, e_{\tau_k}, \overline{z}_{\tau_k}, z_{\tau_k}, \overline{p}_{\tau_k}, p_{\tau_k}, \overline{\vartheta}_{\tau_k}, \underline{\vartheta}_{\tau_k}, \vartheta_{\tau_k}, \overline{\omega}_{\tau_k}, \overline{\zeta}_{\tau_k})_k,$$

of solutions to the approximate thermoviscoelastoplastic damage system (4.2) along which convergences (5.1) to a seventuple $(u, e, z, p, \vartheta, \omega, \zeta)$ hold. Exploiting them we will pass to the limit in the time-discrete version of the momentum balance. In order to take the limit of the damage and plastic flow rules, and of the temperature equation, we will need to enhance the convergence properties of the approximate solutions. We will do so by establishing the validity of the mechanical energy balance. Therefrom we will strengthen some weak convergence properties, derived by compactness, via standard "limsup"-arguments. We will thus show that the quintuple (u, e, z, p, ϑ) is a weak energy solution of the (initial-boundary value problem for the) regularized thermoviscoelastoplastic damage system. We will also deduce the validity of the entropy inequality.

Step 0*: ad the initial conditions* (2.23) *and the kinematic admissibility* (2.24). We use the uniform convergences (5.1c), (5.1i), (5.1n), and (5.1r), as well as the pointwise convergences (5.1d), (5.3b), to pass to the limit in the discrete initial conditions (3.7), also taking into account convergences (3.5) for the approximate initial data $(e^0_{\tau_k}, p^0_{\tau_k})_k$. Also exploiting (4.1d) for $(\overline{w}_{\tau_k})_k$, we pass to the limit in the discrete version of the kinematic admissibility condition. We thus conclude (2.24).

Step 1*: ad the momentum balance* (2.8)*.* Combining convergence (5.1n) with the uniform continuity of the mappings $z \in [0,1] \mapsto \mathbb{C}(z)$, $\mathbb{D}(z)$, we infer that

$$\mathbb{C}(\overline{z}_{\tau_k}) \to \mathbb{C}(z), \quad \mathbb{D}(\overline{z}_{\tau_k}) \to \mathbb{D}(z) \quad \text{ in } L^\infty(0,T;\mathbb{M}^{d\times d}_{\mathrm{sym}}). \tag{5.10}$$

Therefore, in view of convergences (5.1g)–(5.1j) and (5.1t) we have that

$$\overline{\sigma}_{\tau_k} = \mathbb{D}(\overline{z}_{\tau_k})\dot{e}_{\tau_k} + \mathbb{C}(\overline{z}_{\tau_k})\overline{e}_{\tau_k} + \tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k} - \overline{\vartheta}_{\tau_k}\mathbb{C}(\overline{z}_{\tau_k})\mathbb{E} \rightharpoonup \sigma = \mathbb{D}(z)\dot{e} + \mathbb{C}(z)e - \vartheta\mathbb{C}(z)\mathbb{E} \tag{5.11}$$

in $L^{\gamma/(\gamma-1)}(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. With this stress convergence, with convergence (5.1f) for $(\partial_t \widehat{u}_{\tau_k})_k$, and with (4.1c) for $(\overline{\mathcal{L}}_{\tau_k})_k$, we pass to the limit in the integrated version of the discrete momentum balance (4.2a). With a localization argument we conclude that (u, e, z, ϑ) fulfill (2.8) with test functions in $W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$. Taking into account that $\sigma = \mathbb{D}(z)\dot{e} + \mathbb{C}(z)e - \vartheta\mathbb{C}(z)\mathbb{E} \in L^2(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))$, a comparison argument in (2.8) yields that $\ddot{u} \in L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*)$, whence (2.22a), and by density we conclude that (2.8) holds with test functions in $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$. We have thus established the momentum balance.

Step 2*: ad the entropy inequality* (2.37)*.* Let us fix a positive test function $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$ for (2.37), and approximate it with the discrete test functions from (4.3): their interpolants $\overline{\varphi}_\tau$, φ_τ converge to φ in the sense of (4.4). We now take the limit of the discrete entropy inequality (4.6), tested by $\overline{\varphi}_\tau$, φ_τ.

We pass to the limit of the first integral term on the left-hand side of (4.6) relying on convergence (5.1u) for $\log(\underline{\vartheta}_{\tau_k})$. For the second integral term, we prove that

$$\kappa(\overline{\vartheta}_{\tau_k})\nabla\log(\overline{\vartheta}_{\tau_k}) \rightharpoonup \kappa(\vartheta)\nabla\log(\vartheta) \qquad \text{in } L^{1+\bar\delta}(Q;\mathbb{R}^d) \text{ with } \bar\delta = \frac{\alpha}{\mu} \text{ and } \alpha \in [0\vee(2-\mu), 1), \tag{5.12}$$

by repeating the very same arguments from the proofs of [RR15, Thm. 1] and [Ros16, Thm. 1], to which we refer the reader for all details.

To take the limit in the right-hand side terms of (4.6), for the first two integrals we use the pointwise convergence (5.1v) at almost all $t \in (0,T)$ and almost all $s \in (0,t)$, combined with (4.4) for $(\overline{\varphi}_{\tau_k})$. A lower semicontinuity argument also based on the Ioffe theorem [Iof77] and on convergences (4.4), (5.1u), and (5.1w) gives that

$$\begin{aligned}\limsup_{k\to\infty}\left(-\int_{\bar{\mathsf{t}}_{\tau_k}(s)}^{\bar{\mathsf{t}}_{\tau_k}(t)}\int_\Omega \kappa(\overline{\vartheta}_{\tau_k})\frac{\overline{\varphi}_{\tau_k}}{\overline{\vartheta}_{\tau_k}}\nabla\log(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}r\right) &= -\liminf_{k\to\infty}\int_{\bar{\mathsf{t}}_{\tau_k}(s)}^{\bar{\mathsf{t}}_{\tau_k}(t)}\int_\Omega \kappa(\overline{\vartheta}_{\tau_k})\overline{\varphi}_{\tau_k}|\nabla\log(\overline{\vartheta}_{\tau_k})|^2\,\mathrm{d}x\,\mathrm{d}r \\ &\le -\int_s^t\int_\Omega \kappa(\vartheta)\varphi|\nabla\log(\vartheta)|^2\,\mathrm{d}x\,\mathrm{d}r.\end{aligned}$$

which allows us to deal with the third integral term on the r.h.s. of (4.6). For the limit passage in the fourth and fifth integral terms on the r.h.s. of (4.6), we preliminarily observe that

$$\frac{1}{\overline{\vartheta}_{\tau_k}} \to \frac{1}{\vartheta} \qquad \text{in } L^p(Q) \quad \text{for all } 1 \le p < \infty. \tag{5.13}$$

This follows from the pointwise convergence $\frac{1}{\overline{\vartheta}_{\tau_k}} \to \frac{1}{\vartheta}$ a.e. in Q, combined with the Dominated Convergence Theorem, since $\left|\frac{1}{\overline{\vartheta}_{\tau_k}}\right| \le \frac{1}{\bar\vartheta}$ by (2.42). Furthermore, since $\nabla\left(\frac{1}{\overline{\vartheta}_{\tau_k}}\right) = \frac{\nabla\overline{\vartheta}_{\tau_k}}{|\overline{\vartheta}_{\tau_k}|^2}$, combining (2.42) with estimate (4.10o) we infer that the sequence $\left(\frac{1}{\overline{\vartheta}_{\tau_k}}\right)_k$ is bounded in $L^2(0,T;H^1(\Omega))$. All in all, we have

$$\frac{1}{\overline{\vartheta}_{\tau_k}} \rightharpoonup \frac{1}{\vartheta} \qquad \text{in } L^2(0,T;H^1(\Omega)). \tag{5.14}$$

Therefore, we can now take the $\limsup_{k\to\infty}$ of the fourth integral combining convergences (4.1a) for $(\overline{G}_{\tau_k})_k$, (4.4) for $(\overline{\varphi}_{\tau_k})$, and (5.1h), (5.1q), (5.1l), (5.1m), (5.1n), (5.1w), (5.10), (5.13), and again resorting to the Ioffe theorem. Finally, the limit passage in the fifth integral term ensues from (4.1b) for $(\overline{h}_{\tau_k})_k$, (4.4), and (5.14). We thus conclude the validity (2.37), tested by functions $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$, on the interval (s,t) for almost all $t \in (0,T)$ and almost all $s \in (0,t)$.

With the very same argument as in the proof of [Ros16, Thm. 1], we establish the summability properties (2.43) for $\kappa(\vartheta)\nabla\log(\vartheta)$. In view of (2.43), the entropy inequality (2.37) makes sense for all positive test functions φ in $H^1(0,T;L^{6/5}(\Omega)) \cup L^\infty(0,T;W^{1,d+\epsilon}(\Omega))$ with $\epsilon > 0$. Therefore, with a density argument we conclude it for this larger test space.

Step 3: preliminary considerations for the damage and plastic flow rules. Convergences (5.1q)–(5.1s), (5.1x), and (5.11) ensure that the functions (ϑ, e, p, ζ) fulfill

$$\zeta + \dot{p} \ni \sigma_{\mathrm{D}} \qquad \text{a.e. in } Q. \tag{5.15}$$

To conclude the plastic flow rule (2.27), it thus remains to show that

$$\zeta \in \partial_{\dot{p}}\mathrm{H}(x,z,\vartheta;\dot{p}) \qquad \text{a.e. in } Q, \tag{5.16}$$

which is equivalent (cf. the considerations at the beginning of Sec. 2.3), to

$$\begin{cases} \zeta : \eta \le \mathrm{H}(x,z,\vartheta;\eta) & \text{for all } \eta \in \mathbb{M}^{d\times d}_{\mathrm{D}} \\ \zeta : \dot{p} \ge \mathrm{H}(x,z,\vartheta;\dot{p}) \end{cases} \qquad \text{a.e. in } Q,$$

by the 1-homogeneity of the functional $\dot{p} \mapsto \mathrm{H}(x,z,\vartheta;\dot{p})$. In fact, we will prove the (equivalent) relations

$$\iint_Q \zeta{:}\eta \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \mathcal{H}(z(t),\vartheta(t);\eta(t))\,\mathrm{d}t \quad \text{for all } \eta \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}), \tag{5.17a}$$

$$\iint_Q \zeta{:}\dot{p} \,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \mathcal{H}(z(t),\vartheta(t);\dot{p}(t))\,\mathrm{d}t, \tag{5.17b}$$

featuring the plastic dissipation potential $\mathcal{H}$ from (2.13). With this aim, we will pass to the limit in the inequalities satisfied at level k using the discrete subdifferential inclusion (4.2c), namely

$$\iint_Q \overline{\zeta}_{\tau_k}{:}\eta \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \mathcal{H}(\overline{z}_{\tau_k}(t),\underline{\vartheta}_{\tau_k}(t);\eta(t))\,\mathrm{d}t \qquad \text{for all } \eta \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}), \tag{5.18a}$$

$$\iint_Q \overline{\zeta}_{\tau_k}{:}\dot{p}_{\tau_k} \,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \mathcal{H}(\overline{z}_{\tau_k}(t),\underline{\vartheta}_{\tau_k}(t);\dot{p}_{\tau_k}(t))\,\mathrm{d}t. \tag{5.18b}$$

In order to take the limit of (5.18a), we use conditions (2.11) on the dissipation metric H. The strong convergences (5.1n) for $\overline{z}_{\tau_k}$ and (5.1w) for $\underline{\vartheta}_{\tau_k}$, combined with the continuity property (2.11b), ensure that for every test function $\eta \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ there holds $\mathrm{H}(\overline{z}_{\tau_k},\underline{\vartheta}_{\tau_k};\eta) \to \mathrm{H}(z,\vartheta;\eta)$ almost everywhere in Q. Using that $\mathrm{H}(\overline{z}_{\tau_k},\underline{\vartheta}_{\tau_k};\eta) \le C_R|\eta|$ a.e. in Q thanks to (2.12a), we conclude via the Dominated Convergence Theorem that

$$\lim_{k\to\infty} \iint_0^T \mathcal{H}(\overline{z}_{\tau_k}(t),\underline{\vartheta}_{\tau_k}(t);\eta(t))\,\mathrm{d}t = \int_0^T \mathcal{H}(z(t),\vartheta(t);\eta(t))\,\mathrm{d}t \quad \text{for every } \eta \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})\,. \tag{5.19}$$

The limit passage on the left-hand side of (5.18a) is guaranteed by convergence (5.1x) for $(\overline{\zeta}_{\tau_k})_k$. Hence, (5.17a) follows. As for (5.18b), we use (2.11a), the convexity of the map $\dot{p} \mapsto \mathrm{H}(z,\vartheta;\dot{p})$ and convergences (5.1n), (5.1q), and (5.1w), to conclude via the Ioffe theorem [Iof77] that

$$\liminf_{k\to\infty} \int_0^T \mathcal{H}(\overline{z}_{\tau_k}(t),\underline{\vartheta}_{\tau_k}(t);\dot{p}_{\tau_k}(t))\,\mathrm{d}t \ge \int_0^T \mathcal{H}(z(t),\vartheta(t);\dot{p}(t))\,\mathrm{d}t\,. \tag{5.20}$$

It thus remains to prove that

$$\limsup_{k\to\infty} \iint_Q \overline{\zeta}_{\tau_k}{:}\dot{p}_{\tau_k} \,\mathrm{d}x\,\mathrm{d}t \le \iint_Q \zeta{:}\dot{p}\,\mathrm{d}x\,\mathrm{d}t. \tag{5.21}$$

Analogously, in order to pass to the limit in the discrete damage flow rule and establish the subdifferential inclusion (2.26), we need to identify the weak limit ω of $(\overline{\omega}_{\tau_k})_k$ as an element of the subdifferential $\partial\mathcal{R}(\dot{z}) \subset H^{\mathrm{s}}(\Omega)^*$. We will also need to pass to the limit in the quadratic term $-\frac{1}{2}\mathbb{C}'(\overline{z}_{\tau_k})\underline{e}_{\tau_k} : \underline{e}_{\tau_k}$ on the right-hand side of (4.2b), which will require establishing a suitable strong convergence for $(\underline{e}_{\tau_k})_k$.

All of these issues will be addressed in the following steps: preliminarily, in Step 4 we will pass to the limit in the discrete mechanical energy inequality and obtain the upper mechanical energy estimate (2.36). Secondly, in Step 5 we will prove the one-sided damage variational inequality (2.34a). Finally, in Step 6 we will combine

(2.34a) with the mechanical energy inequality to establish the mechanical energy balance (2.18) and, moreover, the desired (5.21) together with further convergence properties. Hence, in Step 6 and 7 we will conclude the validity of the plastic and damage flow rules.

Step 4: ad the mechanical energy inequality. We pass to the limit in the discrete mechanical energy inequality (4.8), written on the interval $(0,t)$ with $t\in(0,T]$. As for the left-hand side, we use the pointwise weak convergence (5.1d) for $(\widehat{u}_{\tau_k}(\bar{\mathsf{t}}_{\tau_k}(\cdot))_k$ to take the $\liminf$ of the first integral. In view of convergences (5.1h) for $(\dot{e}_{\tau_k})_k$ and (5.10) for $(\mathbb{D}(\overline{z}_{\tau_k}))_k$ we also have

$$\liminf_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \mathbb{D}(\overline{z}_{\tau_k}(r))\dot{e}_{\tau_k}(r){:}\dot{e}_{\tau_k}(r)\,\mathrm{d}x\,\mathrm{d}r \geq \int_0^t\int_\Omega \mathbb{D}(z(r))\dot{e}(r){:}\dot{e}(r)\,\mathrm{d}x\,\mathrm{d}r\,.$$

We pass to the limit in the other dissipative contributions $\iint \mathrm{R}(\dot{z}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}r,\ldots,\iint|\dot{p}_{\tau_k}|^2\,\mathrm{d}x\,\mathrm{d}r$ invoking convergences (5.1l), (5.1m), (5.1q). We also resort to the previously established $\liminf$-inequality (5.20). We use that

$$\|\overline{e}_{\tau_k}-e_{\tau_k}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\leq \tau_k^{1/2}\|\dot{e}_{\tau_k}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\leq S\tau_k^{1/2} \tag{5.22}$$

to deduce from (5.1i) that

$$\overline{e}_{\tau_k}(t)\rightharpoonup e(t)\quad \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \text{ for every } t\in[0,T]. \tag{5.23}$$

In the same way, we prove that

$$\overline{z}_{\tau_k}(t)\rightharpoonup z(t)\quad \text{ in } H^{\mathrm{s}}(\Omega),\qquad \overline{z}_{\tau_k}(t)\to z(t)\quad \text{ in } \mathrm{C}^0(\overline{\Omega}) \text{ for every } t\in[0,T]. \tag{5.24}$$

Therefore, again via the Ioffe theorem we have that $\liminf_{k\to\infty}\mathcal{Q}(\overline{e}_{\tau_k}(t),\overline{z}_{\tau_k}(t))\geq \mathcal{Q}(e(t),z(t))$; by the lower semicontinuity of the functional $\mathcal{G}$ we also infer $\liminf_{k\to\infty}\mathcal{G}(\overline{z}_{\tau_k}(t))\geq\mathcal{G}(z(t))$ (where the functionals $\mathcal{Q}$ and $\mathcal{G}$ are from (2.16)).

In order to pass to the limit on the right-hand side of (4.8), we use convergences (3.5) to deal with the terms on the left-hand side involving the approximate initial data. Convergences (4.1c), (4.1d), and (5.1a) allow us to take the limit of the fifth integral term on the right-hand side. As for the sixth and seventh ones, we combine the strong convergences (5.1w) and (5.10) with the weak ones (5.1h), (5.1l). We handle the limit passage in the eighth, ninth, and tenth terms by using convergences (5.1a) and (5.1d) for $(\widehat{u}_{\tau_k}(\bar{\mathsf{t}}_{\tau_k}(\cdot))_k$, and (4.1d) for $(\dot{w}_{\tau_k})_k$ and $(\partial_t\widehat{w}_{\tau_k})_k$. Finally, the limit passage in the eleventh term results from (4.1d) combined with the stress convergence (5.11) and with (5.1j). In fact, we use that

$$\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k}:E(\dot{w}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}r=\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \tau^{1-\alpha_w}|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k}:\tau^{\alpha_w}E(\dot{w}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}r\to 0$$

as $k\to\infty$ thanks to (4.1d) and (5.1j). Therefore,

$$\lim_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \overline{\sigma}_{\tau_k}{:}E(\dot{w}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}t=\int_0^t\int_\Omega \sigma{:}E(\dot{w})\,\mathrm{d}x\,\mathrm{d}t\,. \tag{5.25}$$

We have thus established the mechanical energy inequality (2.36) on the interval $(0,t)$, for every $t\in(0,T]$. In fact, we have shown that

$$\begin{aligned}\text{l.h.s. of (2.36) on } (0,t) &\leq \liminf_{k\to\infty}\text{l.h.s. of (4.8) on } (0,t)\leq \limsup_{k\to\infty}\text{l.h.s. of (4.8) on } (0,t)\\ &\leq \lim_{k\to\infty}\text{ r.h.s. of (4.8) on } (0,t)=\text{r.h.s. of (2.36) on } (0,t)\,.\end{aligned} \tag{5.26}$$

Let us mention that the very same convergence arguments as in the above lines also give the upper total energy inequality on $(0,t)$.

Step 5: ad the one-sided variational inequality (2.34a). Using the 1-homogeneity of R (cf. again Sec. 2.2), we reformulate the discrete damage flow rule as the system

$$\mathcal{R}(\zeta)+\nu a_{\mathrm{s}}(\dot{z}_\tau,\zeta)+a_{\mathrm{s}}(\overline{z}_\tau,\zeta)+\int_\Omega(\dot{z}_\tau+\beta'(\overline{z}_\tau)-\lambda_W\underline{z}_\tau)\,\zeta\,\mathrm{d}x\geq\int_\Omega\left(-\tfrac12\mathbb{C}'(\overline{z}_\tau)\underline{e}_\tau{:}\underline{e}_\tau+\underline{\vartheta}_\tau\right)\zeta\,\mathrm{d}x\quad\text{for all }\zeta\in H^{\mathrm{s}}_-(\Omega), \tag{5.27a}$$

$$\mathcal{R}(\dot{z}_\tau)+\nu a_{\mathrm{s}}(\dot{z}_\tau,\dot{z}_\tau)+a_{\mathrm{s}}(\overline{z}_\tau,\dot{z}_\tau)+\int_\Omega\left(|\dot{z}_\tau|^2+\beta'(\overline{z}_\tau)\dot{z}_\tau-\lambda_W\underline{z}_\tau\dot{z}_\tau\right)\mathrm{d}x\leq\int_\Omega\left(-\tfrac12\mathbb{C}'(\overline{z}_\tau)\underline{e}_\tau{:}\underline{e}_\tau+\underline{\vartheta}_\tau\right)\dot{z}_\tau\,\mathrm{d}x\,. \tag{5.27b}$$

We now pass to the limit in (5.27a), integrated along the interval $(0,t)$, using convergences (5.1k)–(5.1n), as well as (5.1w). Let us only comment on the fact that, since $\beta \in \mathrm{C}^2(\mathbb{R}^+)$ and the functions $(\overline{z}_{\tau_k})_k$, taking values in the interval $[\zeta_*, 1]$ by (4.9), converge to z uniformly in Q, there holds

$$\beta'(\overline{z}_{\tau_k}) \to \beta'(z) \qquad \text{uniformly in } Q\,. \tag{5.28}$$

Moreover, by the Ioffe Theorem (recall that $\zeta \leq 0$ in Ω and the positivity property (2.6b) of $\mathbb{C}'$), we have

$$\liminf_{k\to\infty} \int_0^t \int_\Omega -\tfrac{1}{2}\mathbb{C}'(\overline{z}_{\tau_k})\underline{e}_{\tau_k}{:}\underline{e}_{\tau_k}\zeta \,\mathrm{d}x\,\mathrm{d}r \geq \int_0^t \int_\Omega -\tfrac{1}{2}\mathbb{C}'(z)e{:}e\zeta \,\mathrm{d}x\,\mathrm{d}r\,.$$

We have thus established

$$\int_0^t (\mathcal{R}(\zeta){+}\nu a_{\mathrm{s}}(\dot z,\zeta){+}a_{\mathrm{s}}(z,\zeta))\,\mathrm{d}r + \int_0^t\int_\Omega (\dot z{+}\beta'(z){-}\lambda_W z)\,\zeta\,\mathrm{d}x\,\mathrm{d}r \geq \int_0^t\int_\Omega \left(-\tfrac{1}{2}\mathbb{C}'(z)e{:}e{+}\vartheta\right)\zeta\,\mathrm{d}x\,\mathrm{d}r$$

for all $\zeta \in H^{\mathrm{s}}_-(\Omega)$. A localization argument yields the one-sided variational inequality (2.34a).

Step 6: enhanced convergence properties and plastic flow rule. Taking the test function $\zeta = \dot z$ in (2.34a), which is admissible since $\dot z(t) \in H^{\mathrm{s}}_-(\Omega)$ for almost all $t \in (0,T)$, and integrating in time, yields the converse of inequality (2.34b), namely

$$\int_0^t \left(\mathcal{R}(\dot z)\,\mathrm{d}r{+}\int_\Omega |\dot z|^2\,\mathrm{d}x\,\mathrm{d}r{+}\nu a_{\mathrm{s}}(\dot z,\dot z)\right)\mathrm{d}r + \mathcal{G}(z(t)) + \int_\Omega \tfrac{1}{2}\mathbb{C}'(z)e{:}e\dot z\,\mathrm{d}x\,\mathrm{d}r \geq \mathcal{G}(z(0)) + \int_0^t\int_\Omega \vartheta\dot z\,\mathrm{d}x\,\mathrm{d}r\,, \tag{5.29}$$

where we have used the chain rule identity

$$\int_0^t \left(a_{\mathrm{s}}(z,\dot z){+}\int_\Omega W'(z)\dot z\,\mathrm{d}x\right)\mathrm{d}r = \mathcal{G}(z(t)) - \mathcal{G}(z(0))\,.$$

We now consider (5.17a), with the test function $\eta = \chi_{(0,t)}\dot p$ (and $\chi_{(0,t)}$ the characteristic function of $(0,t)$), and deduce that

$$\begin{aligned}\int_0^t \mathcal{H}(z(r),\vartheta(r);\dot p(r))\,\mathrm{d}r &\geq \int_0^t\int_\Omega \zeta{:}\dot p\,\mathrm{d}x\,\mathrm{d}r\\ &= \int_0^t\int_\Omega (\mathbb{D}(z)\dot e{+}\mathbb{C}(z)e{-}\vartheta\mathbb{C}(z)\mathbb{E})\,{:}\dot p\,\mathrm{d}x\,\mathrm{d}r - \int_0^t\int_\Omega |\dot p|^2\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned} \tag{5.30}$$

Finally, we test the momentum balance by $\dot u - \dot w$, integrate on $(0,t)$, and add the resulting relation with (5.29) and (5.30). We observe that some terms cancel out and repeat the very same calculations as those leading to (2.18). We thus conclude that the converse of the mechanical energy inequality (2.36) holds. This establishes the mechanical energy *balance* (2.18) on the interval $(0,t)$.

Furthermore, we continue the chain of inequalities (5.26) and conclude that

$$\begin{aligned}\text{l.h.s. of (2.18) on } (0,t) &\leq \liminf_{k\to\infty} \text{l.h.s. of (4.8) on } (0,t) \leq \limsup_{k\to\infty} \text{l.h.s. of (4.8) on } (0,t)\\ &\leq \lim_{k\to\infty} \text{r.h.s. of (4.8) on } (0,t) = \text{r.h.s. of (2.18) on } (0,t) \overset{(!)}{=} \text{l.h.s. of (2.18) on } (0,t)\,.\end{aligned}$$

Therefore, all inequalities hold as equalities, the lower and upper limits coincide. Moreover, taking into account the lim inf-inequalities previously observed in Step 3, with a standard argument we conclude that *each* of the terms on the left-hand side of (4.8) does converge to its analogue on the left-hand side of (2.18). Thus, we have in particular

$$\lim_{k\to\infty}\int_\Omega |\dot{u}_{\tau_k}(t)|^2\,\mathrm{d}x = \int_\Omega |\dot{u}(t)|^2\,\mathrm{d}x\,, \tag{5.31a}$$

$$\lim_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega \mathbb{D}(\overline{z}_{\tau_k}(r))\dot{e}_{\tau_k}(r){:}\dot{e}_{\tau_k}(r)\,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_\Omega \mathbb{D}(z(r))\dot{e}(r){:}\dot{e}(r)\,\mathrm{d}x\,\mathrm{d}r\,, \tag{5.31b}$$

$$\lim_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)} \mathcal{R}(\dot{z}_{\tau_k}(r))\,\mathrm{d}r = \int_0^t \mathcal{R}(\dot{z}(r))\,\mathrm{d}x\,, \tag{5.31c}$$

$$\lim_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\dot{z}_{\tau_k}(r)|^2\,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_\Omega |\dot{z}(r)|^2\,\mathrm{d}x\,\mathrm{d}r\,, \tag{5.31d}$$

$$\lim_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)} \bar{\nu}a_{\mathrm{s}}(\dot{z}_{\tau_k}(r),\dot{z}_{\tau_k}(r))\,\mathrm{d}r = \int_0^t \bar{\nu}a_{\mathrm{s}}(\dot{z}(r),\dot{z}(r))\,\mathrm{d}r, \tag{5.31e}$$

$$\lim_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)} \mathcal{H}(\overline{z}_{\tau_k}(r),\underline{\vartheta}_{\tau_k}(r);\dot{p}_{\tau_k}(r))\,\mathrm{d}r = \int_0^t \mathcal{H}(z(r),\vartheta(r);\dot{p}(r))\,\mathrm{d}r\,, \tag{5.31f}$$

$$\lim_{k\to\infty}\int_0^{\bar{\mathsf{t}}_\tau(t)}\int_\Omega |\dot{p}_{\tau_k}(r)|^2\,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_\Omega |\dot{p}(r)|^2\,\mathrm{d}x\,\mathrm{d}r\,. \tag{5.31g}$$

In particular, from (5.31b), repeating the very same arguments from the proof of [LRTT14, Thm. 5.3], which substantially rely on the uniform positive definiteness of the tensor $\mathbb{D}$, we infer that

$$\dot{e}_{\tau_k} \to \dot{e} \qquad \text{in } L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})\,. \tag{5.32}$$

Then, $e_{\tau_k} \to e$ in $\mathrm{C}^0([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))$. In view of (5.22) we then infer that

$$\overline{e}_{\tau_k},\,\underline{e}_{\tau_k} \to e \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))\,. \tag{5.33}$$

Let us also record that (5.31d) & (5.31e) yield

$$\dot{z}_{\tau_k} \to \dot{z} \text{ in } L^2(Q) \text{ and } \dot{z}_{\tau_k} \to \dot{z} \text{ in } L^2(0,T;H^{\mathrm{s}}(\Omega)) \text{ if } \nu>0, \tag{5.34}$$

which gives, by dominated convergence,

$$\mathrm{R}(\dot{z}_{\tau_k}) \to \mathrm{R}(\dot{z}) \text{ in } L^1(Q). \tag{5.35}$$

Finally, (5.31g) gives

$$\dot{p}_{\tau_k} \to \dot{p} \text{ in } L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}})\,, \tag{5.36}$$

from which we deduce, taking into account convergences (5.1n), (5.1w), the continuity of H, and repeating the same arguments leading to (5.19), that

$$\mathrm{H}(\overline{z}_{\tau_k},\underline{\vartheta}_{\tau_k};\dot{p}_{\tau_k}) \to \mathrm{H}(z,\vartheta;\dot{p}) \text{ in } L^1(Q). \tag{5.37}$$

Furthermore, in view of the strong convergence (5.36), the lim sup-inequality (5.21) immediately follows. This establishes the plastic flow rule.

Step 7: ad the damage flow rule. It follows from (2.($\mathbb{C},\mathbb{D})_2$) and (5.1n) that $\mathbb{C}'(\overline{z}_{\tau_k}) \overset{*}{\rightharpoonup} \mathbb{C}'(z)$ in $L^\infty(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. Combining this with (5.33) we find that

$$\mathbb{C}'(\overline{z}_{\tau_k})\underline{e}_{\tau_k} : \underline{e}_{\tau_k} \rightharpoonup \mathbb{C}'(z)e : e \qquad \text{in } L^1(Q)\,.$$

Combining this with convergences (5.1k)–(5.1o), (5.1w), and (5.28), we are thus able to pass to the limit in an integrated version of the discrete damage flow rule (4.2b), with test functions in $L^\infty(0,T;H^{\mathrm{s}}(\Omega))$ (which embeds continuously into $L^\infty(Q)$). With a localization argument we then deduce that the quadruple (e,z,ϑ,ω) complies with

$$\omega + \dot{z} + \nu A_{\mathrm{s}}(\dot{z}) + A_{\mathrm{s}}(z) + W'(z) = -\frac{1}{2}\mathbb{C}'(z)e : e + \vartheta \qquad \text{in } H^{\mathrm{s}}(\Omega)^* \quad \text{a.e. in } (0,T). \tag{5.38}$$

In order to conclude the damage flow rule (2.26), it remains to show that $\omega(t) \in \partial\mathcal{R}(z(t))$ for almost all $t\in(0,T)$, with $\mathcal{R}$ from (2.20). In fact, this is equivalent to showing that

$$\omega \in \partial\overline{\mathcal{R}}(\dot{z}) \;\text{ with }\; \overline{\mathcal{R}} : L^2(0,T;H^{\mathrm{s}}(\Omega)) \to [0,+\infty] \;\text{ given by }\; \overline{\mathcal{R}}(v) := \int_0^T \mathcal{R}(v(t))\,\mathrm{d}t \tag{5.39}$$

and $\partial\overline{\mathcal{R}} : L^2(0,T;H^{\mathrm{s}}(\Omega)) \rightrightarrows L^2(0,T;H^{\mathrm{s}}(\Omega)^*)$ its subdifferential in the sense of convex analysis. Now, (5.39) directly follows by passing to the limit in its discrete version $\overline{\omega}_{\tau_k} \in \partial\overline{\mathcal{R}}(\dot{z}_{\tau_k})$, combining the weak convergence

(5.1o) for $\overline{\omega}_{\tau_k}$ with the strong one (5.34) for $\dot{z}_{\tau_k}$ and using the strong-weak closedness of the graph of $\partial\overline{\mathcal{R}}$. This concludes the proof of the damage flow rule (2.26).

Step 8: ad the temperature equation. We pass to the limit in the approximate temperature equation (4.2d), with the test functions from (4.3) approximating a fixed test function $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$. Using formula (4.5), we integrate by parts in time the term

$$\int_0^t \int_\Omega \dot{\vartheta}_{\tau_k}\overline{\varphi}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}s = -\int_0^t \int_\Omega \underline{\vartheta}_{\tau_k}\varphi_{\tau_k}\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \overline{\vartheta}_{\tau_k}(t)\overline{\varphi}_{\tau_k}(t)\,\mathrm{d}x - \int_\Omega \overline{\vartheta}_{\tau_k}(0)\overline{\varphi}_{\tau_k}(0)\,\mathrm{d}x$$

and for taking its limit we exploit convergences (4.4), as well as (5.1t) for $\underline{\vartheta}_{\tau_k}$ and the pointwise (5.3b) for $\overline{\vartheta}_{\tau_k}$. For the limit passage in the term $\int_0^t\int_\Omega \kappa(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k}\nabla\overline{\varphi}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t$, repeating the very same arguments from the proofs of [RR15, Thm. 2] and [Ros16, Thm. 2], which in turn rely on the growth (2.κ_2) and on estimates (4.10q), we show that

$$\kappa(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k} \rightharpoonup \kappa(\vartheta)\nabla\vartheta \quad \text{in } L^{1+\tilde{\delta}}(Q;\mathbb{R}^d), \text{ with} \quad \tilde{\delta} = \frac{2-3\mu+3\alpha}{3(\mu-\alpha+2)} \in \left(0,\frac{1}{3}\right) \tag{5.40}$$

(cf. (2.44)). Finally, observe that $\kappa(\vartheta)\nabla\vartheta = \nabla(\hat{\kappa}(\vartheta))$ thanks to [MM79]. Since $\hat{\kappa}(\vartheta)$ itself is a function in $L^{1+\tilde{\delta}}(Q)$ (for $d=3$, this follows from the fact that $\hat{\kappa}(\vartheta) \sim \vartheta^{\mu+1} \in L^{h/(\mu+1)}(Q)$ for every $1 \le h < \frac{8}{3}$), we conclude (2.44). The limit passage on the r.h.s. of the discrete heat equation (4.2d) results from (4.1a), (4.1b), and from the previously established strong convergences (5.3a), (5.10), (5.32), (5.34)–(5.37). All in all, we have established that the limit quintuple (u,e,p,z,ϑ) complies with

$$\begin{aligned}
&\langle\vartheta(t),\varphi(t)\rangle_{W^{1,\infty}(\Omega)} - \int_0^t\int_\Omega \vartheta\varphi_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \kappa(\vartheta)\nabla\vartheta\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_\Omega \vartheta_0\varphi(0)\,\mathrm{d}x + \int_0^t\int_\Omega \big(G{+}\mathbb{D}(z)\dot{e}{:}\dot{e}{-}\vartheta\mathbb{C}(z)\mathbb{E}{:}\dot{e}{+}\mathrm{R}(\dot{z}){+}|\dot{z}|^2{+}\bar{\nu}a_{\mathrm{s}}(\dot{z},\dot{z}){-}\vartheta\dot{z} + \mathrm{H}(z,\vartheta;\dot{p}){+}|\dot{p}|^2\big)\,\varphi\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\partial\Omega} h\varphi\,\mathrm{d}S\,\mathrm{d}s\,.
\end{aligned} \tag{5.41}$$

for all test functions $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$ and for all $t \in (0,T]$.

Clearly, testing (5.41) with functions $\varphi \in W^{1,\infty}(\Omega)$ independent of the time variable, and repeating the very same arguments from the proof of [Ros16, Thm. 2], we conclude that ϑ is in $W^{1,1}(0,T;W^{1,\infty}(\Omega)^*)$ and that it complies with (2.29). Again arguing as for [Ros16, Thm. 2], we can enlarge the space of test functions to $W^{1,q_{\tilde{\delta}}}(\Omega)$, with $q_{\tilde{\delta}} = 1+\frac{1}{\tilde{\delta}}$ and $\tilde{\delta}$ from (5.40), and improve the regularity of ϑ to $W^{1,1}(0,T;W^{1,q_{\tilde{\delta}}}(\Omega)^*)$. We have thus completed the proof of Theorem 2.6. ∎

Proof of Theorem 2.5 (Entropic solutions): It is immediate to check that Steps 0–5 of the proof of Thm. 2.6 do not rely on the more stringent growth condition (2.κ_2) on κ. Therefore, they carry over in the setting of Theorem 2.5. We thus immediately establish the validity of the kinematic admissibility, of the weak momentum balance, of the one-sided damage variational inequality (2.34a), of the entropy inequality, and of the upper mechanical energy estimate (2.36).

From (5.17a) and (5.15) we also infer the plastic variational inequality (2.35a). The upper total energy estimate (2.38) follows from passing to the limit in (4.7) with the very same arguments used for obtaining the mechanical energy inequality.

This concludes the proof. ∎

Remark 5.3. A few more comments on the proof of Theorem 2.5 are in order.

(1) Step 6 in the proof of Thm. 2.6 (and, consequently, Steps 7 & 8) does not carry over to the setting of Thm. 2.5. Indeed, testing the one-sided variational inequality for damage (2.34a) by $\dot{z}$ is no longer possible, as, setting $\nu = 0$ we have lost the information that $\dot{z} \in H^{\mathrm{s}}(\Omega)$. Hence, all the lim sup-arguments from Step 7, which strengthened the convergences of the approximate solutions, cannot be repeated.

(2) As already hinted at the end of Sec. 2.4, it would be possible to establish the existence of entropic solutions to the (non-regularized) thermoviscoelastoplastic damage system by passing to the limit as $\nu \downarrow 0$ in its regularized version, featuring a family $(\kappa_\nu)_\nu$ of heat conductivity functions converging to some κ that only complies with (2.κ_1). In fact, the a priori estimates (4.10a)–(4.10r), performed

on the time-discrete version of the regularized system, are inherited by the weak energy solutions: in particular, it is easy to check that the bounded variation estimate (4.10r) is preserved by lower semicontinuity arguments. Then, a close perusal of Steps 0–5 reveals that all the arguments performed for the time-discrete to continuous limit carry over to the limit passage $\nu \downarrow 0$.

6. Proof of Proposition 2.7

Throughout this section we will work under the strongly simplifying condition that H neither depends on the damage variable nor on the temperature. Let (u_i, e_i, z_i, p_i), $i = 1, 2$, be two (weak energy) solutions to the regularized viscolelastoplastic damage system with a given temperature profile Θ, supplemented with initial data $(u_i^0, e_i^0, z_i^0, p_i^0)$ and external data $(\mathcal{L}_i, w_i)$, $i = 1, 2$ (where $\mathcal{L}_i$ subsume the volume forces F_i and applied tractions f_i). We will use the place-holders

$$\tilde u := u_1 - u_2, \quad \tilde e := e_1 - e_2, \quad \tilde z := z_1 - z_2, \quad \tilde p := p_1 - p_2\,, \quad \tilde\sigma := \sigma_1 - \sigma_2\,,$$

where $\sigma_i := \mathbb{D}(z_i)\dot e_1 + \mathbb{C}(z_i)e_i - \Theta_i\mathbb{C}(z_i)\mathbb{E}$ for $i = 1, 2$. Throughout the proof, we will also often use the short hand $\|\cdot\|_{L^p}$ for $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{L^p(\Omega;\mathbb{R}^d)}$ and so forth.

We preliminarily observe that, at fixed $t \in (0, T)$ (which we omit) there holds

$$\begin{aligned}
\|\tilde\sigma\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} &\leq \|\mathbb{D}(z_1)\partial_t\tilde e\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|\left(\mathbb{D}(z_1)-\mathbb{D}(z_2)\right)\dot e_2\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|\mathbb{C}(z_1)\tilde e\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \\
&\quad + \|\left(\mathbb{C}(z_1)-\mathbb{C}(z_2)\right)e_2\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|\Theta_1\left(\mathbb{C}(z_1)-\mathbb{C}(z_2)\right)\mathbb{E}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|(\Theta_1-\Theta_2)\mathbb{C}(z_2)\mathbb{E}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \\
&\leq M_1\left(\|\partial_t\tilde e\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|\tilde e\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|\tilde z\|_{L^\infty(\Omega)} + \|\Theta_1\|_{L^2(\Omega)}\|\tilde z\|_{L^\infty(\Omega)} + \|\Theta_1-\Theta_2\|_{L^2(\Omega)}\right).
\end{aligned} \tag{6.1}$$

The last estimate follows from $(2.(\mathbb{C},\mathbb{D},\mathbb{E}))$, with a constant M_1 only depending on the norms $\|z_1\|_{L^\infty}$, $\|z_2\|_{L^\infty}$. In fact, $\|\mathbb{D}(z_1)\|_{L^\infty}$, $\|\mathbb{C}(z_1)\|_{L^\infty}$ are estimated by a constant depending on $\|z_1\|_{L^\infty}$. Analogously, thanks to $(2.(\mathbb{C},\mathbb{D})_4)$, the Lipschitz estimates $\|\mathbb{D}(z_1)-\mathbb{D}(z_2)\|_{L^\infty} \leq C\|z_1-z_2\|_{L^\infty}$ and $\|\mathbb{C}(z_1)-\mathbb{C}(z_2)\|_{L^\infty} \leq \tilde C\|z_1-z_2\|_{L^\infty}$ hold with constants depending on $\|z_1\|_{L^\infty}$, $\|z_2\|_{L^\infty}$.

In order to prove (2.47), we start by subtracting the weak momentum balance (2.8) for u_2 from that for u_1, test the resulting relation by $\partial_t\tilde u - \partial_t(w_1-w_2)$, and integrate on an arbitrary time interval $(0, t)$. Elementary calculations (cf. (2.17)) lead to

$$\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\partial_t\tilde u(t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega \tilde\sigma : E(\partial_t\tilde u)\,\mathrm{d}x\,\mathrm{d}r \\
&= \frac{\rho}{2}\int_\Omega |\dot u_1^0-\dot u_2^0|^2\,\mathrm{d}x + \int_0^t \langle \mathcal{L}_1-\mathcal{L}_2, \partial_t\tilde u-\partial_t(w_1-w_2)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r + \int_0^t\int_\Omega \tilde\sigma : E(\partial_t(w_1-w_2))\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \rho\left(\int_\Omega \partial_t\tilde u(t)\partial_t(w_1-w_2)(t)\,\mathrm{d}x - \int_\Omega(\dot u_1^0-\dot u_2^0)\partial_t(w_1-w_2)(0)\,\mathrm{d}x - \int_0^t\int_\Omega \partial_t\tilde u\,\partial_{tt}(w_1-w_2)\,\mathrm{d}x\,\mathrm{d}r\right) \\
&\doteq I_1 + I_2 + I_3 + I_4\,.
\end{aligned} \tag{6.2}$$

We estimate

$$\begin{aligned}
|I_1| &\leq C\|\dot u_1^0-\dot u_2^0\|^2_{L^2(\Omega;\mathbb{R}^d)}, \\
|I_2| &\leq \frac12\|\mathcal{L}_1-\mathcal{L}_2\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^d)^*)} + \frac12\|w_1-w_2\|^2_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))}, \\
|I_3| &\leq \eta\int_0^t\left(\|\partial_t\tilde e\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|\tilde e\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|\tilde z\|^2_{L^\infty(\Omega)} + \|\Theta_1\|^2_{L^2(\Omega)}\|\tilde z\|^2_{L^\infty(\Omega)}\right)\mathrm{d}r + C\|\Theta_1-\Theta_2\|^2_{L^2(Q)} \\
&\qquad + C_\eta\|w_1-w_2\|^2_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))}, \\
|I_4| &\leq C\|\dot u_1^0-\dot u_2^0\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{\rho}{8}\int_\Omega|\partial_t\tilde u(t)|^2\,\mathrm{d}x + C\|w_1-w_2\|^2_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} \\
&\qquad + \rho\int_0^t\|\partial_{tt}(w_1-w_2)\|_{L^2(\Omega;\mathbb{R}^d)}\|\partial_t\tilde u\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}r,
\end{aligned}$$

where the bound for I_3 follows from (6.1), and the constant $\eta > 0$ therein, on which $C_\eta > 0$ depends, will be chosen suitably. As for the second term on the left-hand side of (6.2), it follows from the kinematic admissibility condition that

$$\int_0^t\int_\Omega \tilde\sigma : E(\partial_t\tilde u)\,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_\Omega \tilde\sigma : \partial_t\tilde e\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_\Omega \tilde\sigma : \partial_t\tilde p\,\mathrm{d}x\,\mathrm{d}r \doteq I_5 + I_6\,.$$

We mention in advance that I_6 will cancel out with a term arising from the test of the plastic flow rules. As for I_5, we have that (cf. (6.1))

$$\begin{aligned} I_5 = & \int_0^t \int_\Omega \mathbb{D}(z_1)\partial_t\tilde{e} : \partial_t\tilde{e}\,\mathrm{d}x\,\mathrm{d}r + \int_0^t \int_\Omega (\mathbb{D}(z_1){-}\mathbb{D}(z_2))\,\dot{e}_2 : \partial_t\tilde{e}\,\mathrm{d}x\,\mathrm{d}r \\ & + \int_0^t \int_\Omega \mathbb{C}(z_1)\tilde{e} : \partial_t\tilde{e}\,\mathrm{d}x\,\mathrm{d}r + \int_0^t \int_\Omega (\mathbb{C}(z_1){-}\mathbb{C}(z_2))\,e_2 : \partial_t\tilde{e}\,\mathrm{d}x\,\mathrm{d}r \\ & + \int_0^t \int_\Omega \Theta_2(\mathbb{C}(z_2){-}\mathbb{C}(z_1))\mathbb{E} : \partial_t\tilde{e}\,\mathrm{d}x\,\mathrm{d}r + \int_0^t \int_\Omega (\Theta_2{-}\Theta_1)\mathbb{C}(z_1)\mathbb{E} : \partial_t\tilde{e}\,\mathrm{d}x\,\mathrm{d}r \\ \doteq\ & I_{5,1} + I_{5,2} + I_{5,3} + I_{5,4} + I_{5,5} + I_{5,6}\,. \end{aligned}$$

By the uniform positive definiteness of $\mathbb{D}$ we have

$$I_{5,1} \geq C_{\mathbb{D}}^1 \int_0^t \int_\Omega |\partial_t\tilde{e}|^2\,\mathrm{d}x\,\mathrm{d}r,$$

while the other terms, to be moved to the right-hand side of estimate (6.2), can be estimated with analogous arguments as for (6.1). Namely,

$$\begin{aligned} |I_{5,2}| & \leq M_2 \int_0^t \|\tilde{z}\|_{L^\infty(\Omega)} \|\partial_t\tilde{e}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \leq \varrho \int_0^t \|\partial_t\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C(\varrho, M_2) \int_0^t \|\tilde{z}\|^2_{L^\infty(\Omega)}\,\mathrm{d}r, \\ |I_{5,3}| & \leq M_2 \int_0^t \|\partial_t\tilde{e}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \|\tilde{e}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \leq \varrho \int_0^t \|\partial_t\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C(\varrho, M_2) \int_0^t \|\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r, \\ |I_{5,4}| & \leq M_2 \int_0^t \|\tilde{z}\|_{L^\infty(\Omega)} \|\partial_t\tilde{e}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \leq \varrho \int_0^t \|\partial_t\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C(\varrho, M_2) \int_0^t \|\tilde{z}\|^2_{L^\infty(\Omega)}\,\mathrm{d}r, \\ |I_{5,5}| & \leq M_2 \int_0^t \|\tilde{z}\|_{L^\infty(\Omega)} \|\Theta\|_{L^2(\Omega)} \|\partial_t\tilde{e}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \\ & \leq \varrho \int_0^t \|\partial_t\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C(\varrho, M_2) \int_0^t \|\Theta\|^2_{L^2(\Omega)} \|\tilde{z}\|^2_{L^\infty(\Omega)}\,\mathrm{d}r, \\ |I_{5,6}| & \leq \varrho \int_0^t \|\partial_t\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C(\varrho, M_2) \int_0^t \|\Theta_1{-}\Theta_2\|^2_{L^2(\Omega)}\,\mathrm{d}r. \end{aligned}$$

where the constant M_2 depends on $\|z_1\|_{L^\infty}$, $\|z_2\|_{L^\infty}$, and $\|e_2\|_{L^2}$. The positive constant ϱ (note that $C(\varrho, M_2)$ depends on ϱ and M_2), will be specified later.

Next, we subtract the subdifferential inclusion (2.30) for z_2 from that for z_1, test the resulting relation by $\partial_t\tilde{z}$, and integrate on $(0,t)$. We thus obtain

$$\begin{aligned} & \int_0^t \int_\Omega |\partial_t\tilde{z}|^2\,\mathrm{d}x\,\mathrm{d}r + \nu \int_0^t a_{\mathrm{s}}(\partial_t\tilde{z}, \partial_t\tilde{z})\,\mathrm{d}r + \frac{1}{2} a_{\mathrm{s}}(\tilde{z}(t), \tilde{z}(t)) \\ & = \frac{1}{2} a_{\mathrm{s}}(z_1^0{-}z_2^0, z_1^0{-}z_2^0) - \int_0^t \langle \omega_1{-}\omega_2, \partial_t\tilde{z}\rangle_{H^s(\Omega)}\,\mathrm{d}r + \int_0^t \int_\Omega (W'(z_2){-}W'(z_1))\,\partial_t\tilde{z}\,\mathrm{d}x\,\mathrm{d}r \\ & \quad + \int_0^t \int_\Omega \left(\tfrac{1}{2}\mathbb{C}'(z_2)|e_2|^2{-}\tfrac{1}{2}\mathbb{C}'(z_1)|e_1|^2\right) \partial_t\tilde{z}\,\mathrm{d}x\,\mathrm{d}r \doteq I_7 + I_8 + I_9 + I_{10}\,, \end{aligned} \tag{6.3}$$

with $\omega_i(t) \in \partial\mathcal{R}(\dot{z}_i(t))$ for almost all $t \in (0,T)$. Observe that $I_7 \leq \|z_1^0{-}z_2^0\|^2_{H^s(\Omega)}$, while $I_8 \leq 0$ by monotonicity and

$$\begin{aligned} |I_9| & \leq M_3 \int_0^t \|\tilde{z}\|_{L^2(\Omega)} \|\partial_t\tilde{z}\|_{L^2(\Omega)}\,\mathrm{d}r \leq \frac{1}{4} \int_0^t \|\partial_t\tilde{z}\|^2_{L^2(\Omega)}\,\mathrm{d}r + C \int_0^t \|\tilde{z}\|^2_{L^2(\Omega)}\,\mathrm{d}r\,, \\ I_{10} & = \int_0^t \int_\Omega \tfrac{1}{2}\mathbb{C}'(z_2)(e_2{+}e_1)\tilde{e}\partial_t\tilde{z}\,\mathrm{d}x\,\mathrm{d}r + \int_0^t \int_\Omega \tfrac{1}{2}\left(\mathbb{C}'(z_2){-}\mathbb{C}'(z_1)\right)|e_1|^2 \partial_t\tilde{z}\,\mathrm{d}x\,\mathrm{d}r \\ & \leq M_4 \int_0^t \|\tilde{e}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \|\partial_t\tilde{z}\|_{L^\infty(\Omega)}\,\mathrm{d}r + M_4 \int_0^t \|\tilde{z}\|_{L^\infty(\Omega)} \|\partial_t\tilde{z}\|_{L^\infty(\Omega)}\,\mathrm{d}r \\ & \leq \lambda \int_0^t \|\partial_t\tilde{z}\|^2_{L^\infty(\Omega)}\,\mathrm{d}r + C(\lambda, M_4) \int_0^t \|\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C(\lambda, M_4) \int_0^t \|\tilde{z}\|^2_{L^\infty(\Omega)}\,\mathrm{d}r. \end{aligned}$$

The constant M_3 depends on $\|z_1\|_{L^\infty}$, $\|z_2\|_{L^\infty}$, and on the constants ζ_*^1, $\zeta_*^2 > 0$ such that for $i = 1,2$ there holds $\zeta_*^i \leq z_i(x,t) \leq 1$ for every $(x,t) \in Q$, observing that the restriction of W to $[\min\{\zeta_*^1, \zeta_*^2\}, 1]$ is of class C^2.

The constant M_4 depends on $\|z_1\|_{L^\infty}$, $\|z_2\|_{L^\infty}$, $\|e_1\|_{L^2}$, and $\|e_2\|_{L^2}$. The positive constant λ will be specified later; $C(\lambda, M_4)$ depends on λ and M_4.

Finally, we subtract the pointwise plastic flow rule (2.27) for p_2 from that for p_1, test the resulting relation by $\partial_t\tilde{p}$, and integrate in time. This leads to

$$\int_0^t\int_\Omega |\partial_t\tilde{p}|^2 \,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_\Omega (\zeta_2-\zeta_1)\partial_t\tilde{p}\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_\Omega \tilde{\sigma}_\mathrm{D}\partial_t\tilde{p}\,\mathrm{d}x\,\mathrm{d}r \doteq I_{10} + I_6 \tag{6.4}$$

with $\zeta_i \in \partial\mathrm{H}(\dot{p}_i)$ a.e. in Q fulfilling the plastic flow rules for $i = 1, 2$. By monotonicity (it is crucial that H does not depend on the other variables), we clearly have $I_{10} \leq 0$, while the second integral coincides with I_6.

In the end, we sum up (6.2), (6.3), and (6.4). Taking into account all of the previous calculations, the cancellation of the last term on the right-hand side of (6.4) with that arising from the test of the momentum balance, and choosing the constants η, ϱ, and λ in such a way as to absorb the term $\iint_Q |\partial_t\tilde{e}|^2$ into its analogue on the left-hand side, and to absorb $\int_0^t \|\partial_t\tilde{z}\|^2_{L^\infty}$ into $\int_0^t a_\mathrm{s}(\partial_t\tilde{z}, \partial_t\tilde{z})\,\mathrm{d}r$, we obtain

$$\begin{aligned}
& c\left(\int_\Omega |\partial_t\tilde{u}(t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega |\partial_t\tilde{e}|^2\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_\Omega |\partial_t\tilde{z}|^2\,\mathrm{d}x\,\mathrm{d}r + \int_0^t a_\mathrm{s}(\partial_t\tilde{z}, \partial_t\tilde{z})\,\mathrm{d}r\right) \\
&\quad + \frac{1}{2}a_\mathrm{s}(\tilde{z}(t), \tilde{z}(t)) + \int_0^t\int_\Omega |\partial_t\tilde{p}|^2\,\mathrm{d}x\,\mathrm{d}r \\
&\leq C\Big(\|\dot{u}_1^0-\dot{u}_2^0\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|z_1^0-z_2^0\|^2_{H^\mathrm{s}(\Omega)} + \|\mathcal{L}_1-\mathcal{L}_2\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^d)^*)} + \|\Theta_1-\Theta_2\|^2_{L^2(Q)} \\
&\qquad\qquad + \|w_1-w_2\|^2_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))}\Big) \\
&\quad + C\int_0^t \|\tilde{e}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_\mathrm{sym})}\,\mathrm{d}r + C\int_0^t (1+\|\Theta_1\|^2_{L^2(\Omega)}+\|\Theta_2\|^2_{L^2(\Omega)})\|\tilde{z}\|^2_{H^\mathrm{s}(\Omega)}\,\mathrm{d}r \\
&\qquad\qquad + \rho\int_0^t \|\partial_{tt}(w_1-w_2)\|_{L^2(\Omega;\mathbb{R}^d)}\|\partial_t\tilde{u}\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}r,
\end{aligned}$$

where we have also used the continuous embedding $H^\mathrm{s}(\Omega) \subset L^\infty(\Omega)$. Hence, applying the Gronwall Lemma we obtain a continuous dependence estimate in terms of the norms

$$\|\partial_t\tilde{u}\|_{L^\infty(0,t;L^2(\Omega;\mathbb{R}^d))}\,,\ \|\partial_t\tilde{e}\|_{L^2(0,t;L^2(\Omega;\mathbb{M}^{d\times d}_\mathrm{sym}))}\,,\ \|\partial_t\tilde{z}\|_{L^2(0,t;H^\mathrm{s}(\Omega))}\,,\ \|\tilde{z}\|_{L^\infty(0,t;H^\mathrm{s}(\Omega))}\,,\ \|\partial_t\tilde{p}\|_{L^2(0,t;L^2(\Omega;\mathbb{M}^{d\times d}_\mathrm{sym}))}\,.$$

Then, estimate (2.47) follows by easy calculations. This concludes the proof. ∎

Acknowledgments. I am very grateful to the two anonymous referees for reading this paper very carefully and offering several suggestions that have helped improving it.

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